# ON CIRCUMFERENTIALLY MEAN $p$-VALENT FUNCTIONS

BY
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1. In the paper [4] was given a geometric proof of the precise bound for the third coefficient of normalized regular univalent functions in the unit circle, a result first proved by Löwner with his parametric method. The present paper has its inception in the observation that the proof of [4] with suitable technical modifications applies also to the case of circumferentially mean 1-valent functions in the sense of Biernacki [1]. This leads in a natural manner to the precise bound for the third coefficient of normalized regular $p$-valent functions in the unit circle with a $p$-fold zero at the origin.

2. We begin by proving a symmetrization result for certain doubly-connected Riemann domains. Let us denote by $\mathfrak{B}$ the class of doubly-connected Riemann domains $D$ covering the $w$-plane obtained by slitting a simply-connected Riemann surface, covering neither the origin nor the point at infinity, along a piece-wise analytic arc $\gamma$ on the open surface. Then we can obtain a corresponding circularly symmetrized domain $D^*$ in the following manner. Let the portion of the domain $D$ covering the circle $|w| = R$ have total angular Lebesgue measure $l(R)$ (which may be $+\infty$). Let $\mathfrak{D}$ be the Riemann surface swept out by the open arcs $-l(R)/2 < \Phi < l(R)/2$ where $R, \Phi$ are polar coordinates in the $w$-plane. Let $R_1 \leqq R \leqq R_2$ be the range of values for which $\gamma$ covers a point on $|w| = R$. Then $D^*$ is obtained by slitting $\mathfrak{D}$ along the arc $R_1 \leqq R \leqq R_2$, $\Phi = 0$. The symmetrization result in question is then

THEOREM 1. *Let $D$ be a Riemann domain in $\mathfrak{B}$ of module $M$ and let $D^*$, the corresponding symmetrized domain, have module $M^*$. Then*

$$M \leqq M^*.$$

The modules here are the usual ones, i.e. for the class of curves separating the boundary components [3]. For the proof we may suppose that the boundary component $\Gamma$ of $D$ other than $\gamma$ is an analytic curve since the result follows in general in the limit from this case. Let now $u(\mathfrak{w})$ be a function harmonic for $\mathfrak{w} \in D$ taking the continuous boundary values 1 on $\gamma$ and 0 on $\Gamma$. Let $u = u(\xi, \eta)$ be a surface lying over the $w$-plane, $w = \xi + i\eta$, such that $u(\xi, \eta) = u(\mathfrak{w})$ where $\mathfrak{w}$ covers $\xi + i\eta$. This surface will in general have self-intersections, nevertheless we can apply to it the process of circular symmetrization with respect to the half-plane $\xi > 0$, $\eta = 0$ [6, p. 194]. In this way we obtain a surface $u = u^*(\xi, \eta)$ likewise lying over the $w$-plane and which again may have self-intersections. The function $u^*$ has as natural domain of

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definition the surface $\mathfrak{D}$ and the subset of $\mathfrak{D}$ on which $0<u^*<1$ is precisely $D^*$. The standard argument now shows that the Dirichlet integral of $u$ over $D$ is not less than the Dirichlet integral of $u^*$ over $D^*$ [6, pp. 194, 185] and the inequality $M\leqq M^*$ follows by the Dirichlet principle and the familiar connection of the Dirichlet integral with the module [4, p. 512].

3. Let us denote by $F_p$, $p$ a positive integer, the class of functions regular in the unit circle, $|z|<1$, which have power series expansion about $z=0$,

$$f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots$$

and which are circumferentially mean $p$-valent. By the latter we mean that if $f$ maps $|z|<1$ on a Riemann surface $\mathfrak{R}$ covering the $w$-plane the total angular measure of the open arcs on $\mathfrak{R}$ covering $|w|=r$, all $r>0$, is at most $2\pi p$ [1].

In the paper [4] we constructed a continuous one-parameter family $f_t(z)$, $0\leqq t\leqq 1$, of normalized regular univalent functions in the unit circle associated with a pair of values $r_1$, $r_2$, $0<r_i<1$, $i=1, 2$, and having the following mapping properties. To each function $f_t(z)$ is associated a quadratic differential

$$d\zeta^2 = \pm \frac{(w-a)}{w^2(w-b)(w-c)}\, dw^2 \equiv Q(w)dw^2$$

where $b<0$, $c>0$ and $a<b$ or $c<a$ or for one particular value $t_0$ of $t$, $a$ coincides with the point at infinity, the factor in the numerator of $Q(w)$ being replaced by unity. The sign is to be chosen so that $d\zeta^2$ is positive on the real axis near the origin. By the trajectories of the quadratic differential we mean the maximal open arcs or topological circles on which $Q(w)dw^2>0$. The maximal open arcs or topological circles on which $Q(w)dw^2<0$ are called orthogonal trajectories. There are three trajectories which have a limiting end point at $a$, $T_1$ running out along the real axis to infinity and $T_2$, $T_3$ symmetric in the real axis and tending to the origin. In the particular case where $a$ coincides with the point at infinity, $t=t_0$, these degenerate to a single trajectory through the point at infinity. Now $f_t$ maps $|z|<1$ onto the $w$-plane slit along $T_1$ and along equilong arcs on the closures of $T_2$, $T_3$ having an end point at $w=a$. In particular the functions $f_t(z)$ have real coefficients in their power series expansions about the origin. Further $f_t$ maps $z=-r_1$ into $w=b$ and $z=r_2$ into $w=c$. Finally $f_0(z)=z(1-z)^{-2}$, $f_1(z)=z(1+z)^{-2}$ and the function $f_t(z)$ passes continuously from one to the other as $t$ runs over $0\leqq t\leqq 1$.

THEOREM 2. *If $f\in F_1$ and $|f(-r_1)|=|f_t(-r_1)|$ then*

$$|f(r_2)| \leqq |f_t(r_2)|.$$

It should be observed that for $f\in F_1$

$$r_1/(1+r_1)^2 \leqq |f(-r_1)| \leqq r_1/(1-r_1)^2$$

as was proved by Hayman [2] (actually for a slightly larger class of functions) so that for every $f\in F_1$ there is a function $f_t$ satisfying $|f(-r_1)| = |f_t(-r_1)|$, in fact precisely one. Hayman's result for the class $F_1$ can also be proved by a simple version of the present method.

For the proof of Theorem 2 we observe first that the orthogonal trajectories of $Q(w)dw^2$ in the neighborhood of $w=0$ are Jordan curves which approach circular form as they shrink down to the origin [7; 5]. Taking such an orthogonal trajectory $L$ sufficiently small, together with the trajectories $T_1$, $T_2$, $T_3$ and the segment $b\leqq\xi\leqq c$ on the real axis it will bound two quadrangles $R_1$ and $R_2$, each with a pair of opposite sides on $L$, $R_1$ having one further side along the segment $\sigma_1$ of the real axis from $L$ to $c$ described twice, $R_2$ having a corresponding side along the segment $\sigma_2$ of the real axis from $b$ to $L$ described twice. If $a<b$ the final side of $R_1$ is made up of arcs on $T_1$, $T_2$, $T_3$, the final side of $R_2$ of arcs on $T_2$, $T_3$; if $c<a$ the situation is reversed.

Regard now the mappings $w=f_t(z)$, $w'=f(z)$ and the composite mapping $w'(w)$ from the image domain of $|z|<1$ under $f_t$ into the $w'$-plane. The mapping $w'(w)$ carries $R_1$ and $R_2$ into quadrangles $R_1'$ and $R_2'$ which are not necessarily schlicht and $L$ into a closed curve $L'$. Let $R_i$, $i=1, 2$, have module $M_i$ for the class of curves joining the pair of opposite sides on $L$. Then $R_i'$ has the corresponding module also equal $M_i$, $i=1, 2$. We now choose $L$ so small that $w'(w)$ is univalent on $L$, i.e. $L'$ is a Jordan curve containing $w'=0$, that the image Riemann surface covers simply a circle $N$ centered at $w'=0$, containing $L'$ on the closed disc it bounds and touching $L'$ from outside, and that $N$ cuts off from $R_i'$, $i=1, 2$, just two domains, one at each end, leaving a quadrangle $R_i''$ with a pair of opposite sides on $N$, the other sides lying along the corresponding sides of $R_i'$. The module $M_i''$ of $R_i''$ for the class of curves joining the pair of opposite sides lying on $N$ satisfies $M_i''\geqq M_i$, $i=1, 2$.

Reflecting $R_i''$, $i=1, 2$, in the circle $N$ and joining it with its image across its boundary arcs on $N$ we obtain a doubly-connected domain $D_i$ in $\mathfrak{B}$. The boundary component of $D_i$ arising from the side of $R_i''$ corresponding to $\sigma_i$ and its reflection in $N$ plays the role of $\gamma$ of §2. The domain $D_i$ has module $M_i''/2$, $i=1, 2$, for the class of curves separating its boundary components. Applying to $D_i$ the symmetrization process of §2 we obtain a domain $D_i^*$ of not smaller module which in the present situation is schlicht. The circle $N$ divides $D_i^*$ into two quadrangles each with a pair of opposite sides on $N$ and of equal module for the classes of curves joining these sides. Let the quadrangle arising from $D_1^*$ and exterior to $N$ be denoted by $R_1^*$. Let the quadrangle obtained by rotating through 180° the quadrangle arising from $D_2^*$ and exterior to $N$ be denoted by $R_2^*$. The quadrangle $R_1^*$ has in addition to its sides on $N$ a side along a segment of the positive real axis from $N$ to a point of modulus $\geqq|f(r_2)|$ described twice. The quadrangle $R_2^*$ has in addition to its sides on $N$ a side along a segment of the negative real axis from $N$ to a point of modulus $\geqq|f(-r_1)|$ described twice. The quadrangle $R_i^*$,

$i=1, 2$, has module $M_i^* \geqq M_i'' \geqq M_i$ for the class of curves joining the pair of opposite sides on $N$. The two quadrangles $R_1^*$ and $R_2^*$ do not overlap since $f$ is in $F_1$.

Now the assumption $|f(r_2)| > |f_t(r_2)|$ leads to a contradiction by the same argument as in [4, pp. 519, 520]. This completes the proof of Theorem 2.

THEOREM 3. *If $f(z) \in F_1$ and $0 < r_1 \leqq r_2 < 1$, then*

$$|f(-r_1)| + |f(r_2)| \leqq r_2/(1 - r_2)^2 + r_1/(1 + r_1)^2,$$

*equality being attained for the function $z(1-z)^{-2}$.*

The proof is almost the same as for Theorem 4 in [4] which the present theorem extends. Because of its brevity it is repeated here. By Theorem 2, the maximum of $|f(-r_1)| + |f(r_2)|$ for $r_1, r_2$ fixed and $f$ varying in the family $F_1$ is attained for a univalent function with real coefficients. For such

$$|f(-r_1)| + |f(r_2)| = r_2 + a_2 r_2^2 + \cdots + r_1 - a_2 r_1^2 + \cdots .$$

For functions with real coefficients we have $|a_n| \leqq n$ and since $r_1 \leqq r_2$

$$\begin{aligned} |f(-r_1)| + |f(r_2)| &\leqq r_2 + 2r_2^2 + 3r_2^3 + \cdots + r_1 - 2r_1^2 + 3r_1^3 - \cdots \\ &\leqq r_2/(1 - r_2)^2 + r_1/(1 + r_1)^2. \end{aligned}$$

The statement concerning equality is evident.

COROLLARY 1. *If $f(z) \in F_1$, $0 < r_1 \leqq r_2 < 1$, $\theta$ real, then*

$$|f(-r_1 e^{i\theta})| + |f(r_2 e^{i\theta})| \leqq r_2/(1 - r_2)^2 + r_1/(1 + r_1)^2,$$

*equality being attained for the function $z(1 - ze^{-i\theta})^{-2}$.*

COROLLARY 2. *If $f(z) \in F_1$, $0 < r < 1$, then, for $|z_1| = r$*

$$|f(-z_1)| + |f(z_1)| \leqq 2r(1 + r^2)/(1 - r^2)^2,$$

*equality being attained for the function $z(1 - ze^{-i\theta})^{-2}$ with $\theta = \arg z_1$.*

COROLLARY 3. *If $f(z) \in F_1$, then $|a_3| \leqq 3$.*

Indeed, if $z = re^{i\theta}$, then

$$|f(-z) + f(z)| \leqq |f(-z)| + |f(z)| \leqq 2r(1 + r^2)/(1 - r^2)^2.$$

Thus

$$r(1 + \Re(a_3 z^2) + O(r^4)) \leqq r(1 + 3r^2 + O(r^4))$$

or

$$\Re(a_3 e^{2i\theta}) \leqq 3.$$

Proper choice of $\theta$ gives $|a_3| \leqq 3$.

It should be observed that the corresponding result does not hold for the class of areally mean 1-valent functions [8] as is shown by an unpublished result due to A. C. Schaeffer and D. C. Spencer.

THEOREM 4. *If $f(z) \in F_p$, then*

$$\left| \frac{a_{p+2}}{p} + \frac{1}{2p}\left(\frac{1}{p} - 1\right) a_{p+1}^2 \right| \leqq 3. \tag{1}$$

*This bound is sharp, equality being attained for the functions $z^p(1-ze^{i\theta})^{-2p}$, $\theta$ real.*

Indeed, with a suitable choice of determination, the function $(f(z))^{1/p}$ belongs to $F_1$ and has power series expansion about the origin

$$(f(z))^{1/p} = z + \frac{a_{p+1}}{p} z^2 + \left(\frac{a_{p+2}}{p} + \frac{1}{2p}\left(\frac{1}{p} - 1\right) a_{p+1}^2\right) z^3 + \cdots.$$

The inequality (1) then follows immediately by Corollary 3. The statement concerning equality is evident.

COROLLARY 4. *If $f(z) \in F_p$, then*

$$|a_{p+2}| \leqq 2p^2 + p.$$

*This bound is sharp, equality being attained for the functions $z^p(1-ze^{i\theta})^{-2p}$, $\theta$ real.*

Indeed $|a_{p+1}| \leqq 2p$ as was proved by Spencer [8] for the larger class of areally mean $p$-valent functions. Thus by inequality (1)

$$\frac{|a_{p+2}|}{p} \leqq 3 + \frac{1}{2p}\left(1 - \frac{1}{p}\right) 4p^2$$

or

$$|a_{p+2}| \leqq 2p^2 + p.$$

The statement concerning equality is evident.

In case $p > 1$ equality can be attained only for the extremal functions indicated, since it is only for these functions that equality can occur in Spencer's result.

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