MONOMIAL GROUPS

BY

RALPH B. CROUCH

Introduction([1]). Let U be a set with n elements, where n is a finite cardinal. Let H be a fixed group. A monomial substitution y is a transformation that maps every x of the set U in a one-to-one fashion into an x of U multiplied by an element h_x of H. If an operation is defined between monomial substitutions by successive applications, the set of all monomial substitutions over H is a group which we denote by Σ_n. Ore [1]([2]) has studied this complete monomial group or symmetry over H. Among the results of the study are those of the following paragraph.

The subset V of elements of Σ_n that map each x of U onto an element of H multiplied by the same x form a normal subgroup of Σ_n, the basis group. The subset S of elements of Σ_n that map each x of U onto an x of U multiplied by the identity e of H form a subgroup of Σ_n, the permutation group. The symmetry splits over the basis group; $\Sigma_n = V \cup S$, $V \cap S = E$ where E is the identity of Σ_n. A complete solution to the problem of finding all representative groups in the splitting over the basis group is presented. All of the normal subgroups of Σ_n and all of the automorphisms of Σ_n are determined. The investigation is concluded with the study of imbedding an arbitrary group in a monomial group.

This paper generalizes the monomial group by removing the requirement that U be a finite set. Furthermore, the group H is arbitrary throughout the entire paper. If $o(U) = B = \aleph_u$, $u \geq 0$, where $o(U)$ means the number of elements of U, then a monomial substitution over an arbitrary fixed group H is defined as for the case where $o(U) = n < \aleph_0$. With an operation between monomial substitutions again defined as successive applications, the set of all monomial substitutions over H forms a group Σ_B.

The splitting of Σ_B over the basis group is discussed, and a complete solution for the determination of all representative groups in a very general case has been found. Corresponding theorems for various subgroups of Σ_B are also found. All of the normal subgroups of various subgroups of the symmetry have been determined. Some progress toward the determination of the automorphisms of the general monomial group has been made by showing that

the basis group is characteristic for some subgroups of Σ_B.

In addition, the subgroup $\Sigma_{n,A}$ of Σ_n that has elements which can be written as the product of elements of the basis group multiplied by elements from the alternating group on U is discussed. The problem of describing all representative groups in the splitting over the basis group is completely solved. All of the normal subgroups for $n \geqq 5$ are determined.

Since Σ_B splits over the basis group with a group isomorphic to the infinite symmetric group on U, some discussion of infinite permutation groups is given in Chapter I. In addition, the more elementary topics such as transformation, center, centralizer, etc. are discussed in Chapter I. The center of the symmetry is found; a normal form for elements of the symmetry is determined; and the centralizer of any element of the symmetry is found.

In Chapter II all representative groups for the splitting over the basis group are determined. Necessary and sufficient conditions for the symmetry to split regularly are found. In addition, the splitting of $\Sigma_{n,A}$ over the basis group is discussed, and a complete solution for constructing representative groups is given. For this group also, necessary and sufficient conditions for the group to split regularly are given.

In Chapter III all of the normal subgroups of various subgroups of Σ_B are determined. In one case the normal subgroups are less complicated than for the case where $o(U) = n$. All of the normal subgroups of $\Sigma_{n,A}$ are determined for $n \geqq 5$.

The final chapter is devoted to showing that the basis group is a characteristic subgroup for some of the subgroups of Σ_B. It is also shown that the basis group is a characteristic subgroup for $\Sigma_{n,A}$.

The paper leaves unanswered some questions corresponding to known results when $o(U) = n$.

The method of procedure used, in particular in Chapter III and Chapter IV, is similar to that used by Ore [1].

CHAPTER I. THE SYMMETRIES

1. **Definitions.** Let d be the cardinal of the set of integers. Let B be an infinite cardinal; B^+, the successor of B; U, a set such that $o(U) = B$, where $o(U)$ denotes the number of elements in U; and let C be such that $d \leqq C \leqq B^+$. Let s be a one-to-one transformation of U onto itself and let $U(s)$ be the set of x belonging to U such that s moves x. Denote by $\{U, C\}$ the set of s such that the number of elements x of U that s moves is less than C. The product of two transformations s and s' in $\{U, C\}$ is defined to be that transformation resulting from successive application of s and s' in the given order.

The groups $S(B, C)$ are called the *infinite symmetric groups*. Let I denote the identity of the groups.

If $o(U(s))$ is finite, then s may be considered as an element of the finite symmetric group on those objects. Let $A\{U, d\}$ be the subset of $\{U, d\}$ con-

sisting of those elements s which are in the alternating group $A(U(s))$ of $U(s)$.

The groups $A(B, d)$ are called the *infinite alternating groups*.

Every s of $S(B, B^+)$ determines a set of cycles of the form

$$c = \begin{pmatrix} x_1, & x_2, & \cdots, & x_{n-1}, & x_n \\ x_2, & x_3, & \cdots, & x_n, & x_1 \end{pmatrix} = (x_1, x_2, \cdots, x_n)$$

or

$$c = \begin{pmatrix} \cdots, & x_{-1}, & x_0, & x_1, & \cdots \\ \cdots, & x_0, & x_1, & x_2, & \cdots \end{pmatrix} = (\cdots, x_{-1}, x_0, x_1, \cdots).$$

A cycle with n distinct x's is called an n-cycle; $n = 1, 2, \cdots; d$. Conversely, a set of disjoint cycles which together contain all x of U determines an s of $S(B, B^+)$. It is customary to say that s is the "product" of its cycles and use corresponding notation. It must be remembered that this cyclic decomposition may involve an infinite number of cycles, however.

Now let H be some group, finite or infinite. Denote by e the identity of H. Let U be a set of order $B = \aleph_u$ for $u \geq 0$. For convenience the set U is well ordered.

A *monomial substitution* over H is a transformation of the form

(1)
$$y = \begin{pmatrix} \cdots, & x_\epsilon, & \cdots \\ \cdots, & h_\epsilon x_{i_\epsilon}, & \cdots \end{pmatrix}$$

where the mapping $x_\epsilon \rightarrow x_{i_\epsilon}$ is a one-to-one mapping of U onto itself and h_ϵ belongs to H. The h_ϵ will be called *factors* of y.

If y is given by (1) and y_1 is given by

(2)
$$y_1 = \begin{pmatrix} \cdots, & x_\epsilon, & \cdots \\ \cdots, & k_\epsilon x_{j_\epsilon}, & \cdots \end{pmatrix},$$

then the *product* yy_1 is defined by

(3)
$$yy_1 = \begin{pmatrix} \cdots, & x_\epsilon, & \cdots \\ \cdots, & h_\epsilon k_{i_\epsilon} x_{i_{j_\epsilon}}, & \cdots \end{pmatrix}.$$

By this definition of multiplication the set of monomial substitutions is a group that will be denoted by $\Sigma(H; B, B^+, B^+)$ and called the *monomial group* of H of degree B or, more simply, the *symmetry* of H. The reason for the complexity of the notation for the monomial group is to provide an adequate notation for various subgroups to be discussed later. The identity of the symmetry will be denoted by E.

If H consists only of the identity element, then $\Sigma(H; B, B^+, B^+)$ is the symmetric group on a set of elements of order B.

A *permutation* in $\Sigma(H; B, B^+, B^+)$ is a substitution of the form

(4) $$s = \begin{pmatrix} \cdots, & x_\epsilon, & \cdots \\ \cdots, & ex_{i_\epsilon}, & \cdots \end{pmatrix} = \begin{pmatrix} \cdots, & \epsilon, & \cdots \\ \cdots, & i_\epsilon, & \cdots \end{pmatrix}.$$

The set of permutations forms a subgroup of $\Sigma(H; B, B^+, B^+)$ which we will denote by $S(B, B^+)$ and call the *permutation subgroup* of $\Sigma(H; B, B^+, B^+)$. This is in conformity with earlier notation.

A *multiplication* in $\Sigma(H; B, B^+, B^+)$ is a substitution of the form

(5) $$v = \begin{pmatrix} \cdots, & x_\epsilon, & \cdots \\ \cdots, & h_\epsilon x_\epsilon, & \cdots \end{pmatrix} = \{ \cdots, h_\epsilon, \cdots \}.$$

The set of multiplications forms a normal subgroup of $\Sigma(H; B, B^+, B^+)$, denoted by $V(B, B^+)$, called the *basis group*.

The basis group is isomorphic to the strong direct product of B groups, each of which is isomorphic to H.

A *scalar* in $\Sigma(H; B, B^+, B^+)$ is a multiplication with each factor the same. Scalars will be denoted by $v = \{h\}$. A brief computation shows that scalars are the only elements that commute with all permutations.

The center $Z(\Sigma(H; B, B^+, B^+))$ of $\Sigma(H; B, B^+, B^+)$ is the set of all scalars $v = \{f\}$ where f belongs to the center of H. $Z(\Sigma(H; B, B^+, B^+))$ is isomorphic to the center of H.

A group G *splits* over a normal subgroup N if there exists a subgroup M of G such that $G = N \cup M$, $N \cap M = E$. The group M may be replaced by any of its conjugates and the relations will still hold. If for every subgroup T such that $G = N \cup T$, $N \cap T = E$ it follows that T is conjugate to M, then G *splits regularly* over N.

Any substitution y of $\Sigma(H; B, B^+, B^+)$ can be factored into a multiplication multiplied by a permutation. If y is as in (1), then

$$y = \{ \cdots, h_\epsilon, \cdots \} \begin{pmatrix} \cdots, & x_\epsilon, & \cdots \\ \cdots, & x_{i_\epsilon}, & \cdots \end{pmatrix}.$$

This shows that

(6) $\Sigma(H; B, B^+, B^+) = V(B, B^+) \cup S(B, B^+)$, $V(B, B^+) \cap S(B, B^+) = E$.

Let B, C, D be infinite cardinals such that $d \leq C \leq B^+$, $d \leq D \leq B^+$. Let $\Sigma(H; B, C, D)$ be the set of all $y = vs$ where v belongs to $V(B, B^+)$, s belongs to $S(B, B^+)$, and v has less than C nonidentity factors, s moves less than D of the x's. Then $\Sigma(H; B, C, D)$ is a subgroup of $\Sigma(H; B, B^+, B^+)$ that splits over its basis group.

The set $\Sigma_A(H; B, C, d)$ of all $y = vs$ of the form v has less than C non-identity factors and s belongs to $A(B, d)$ forms a subgroup of $\Sigma(H; B, B^+, B^+)$ that splits over its basis group.

Let $o(U) = n$, where n is a finite cardinal. Denote by $\Sigma(H; n, n+1, n+1)$ $= \Sigma_n(H)$ the symmetry over H and U. Then the set $\Sigma_A(H; n, n+1, n+1)$ $= \Sigma_{n,A}(H)$ of elements of $\Sigma_n(H)$ of the form $y = vs$, where s belongs to $A(n, n+1) = A_n$, is a subgroup that splits over its basis group, $V(n, n+1)$ $= V_n$.

2. **Cycles, transformations, and centralizers.** Let y be an arbitrary element of $\Sigma(H; B, B^+, B^+)$. It has been shown that y has a unique decomposition $y = vs$, where v belongs to $V(B, B^+)$, and s belongs to $S(B, B^+)$. Permutations may be decomposed uniquely into disjoint, commutative cycles of length n where $n = 1, 2, \cdots ; d$. This decomposition induces a decomposition of v such that to each cycle c_ϵ of s there corresponds a multiplication v_ϵ with all factors e in those positions corresponding to x that s does not move and factors the same as in v for the x that s moves. Thus $v_\epsilon c_\epsilon$ has one of the two forms

$$v_\epsilon c_\epsilon = \begin{pmatrix} x_1, & \cdots, & x_n \\ h_1 x_2, & \cdots, & h_n x_1 \end{pmatrix} \qquad \text{when } n < d$$

or

$$v_\epsilon c_\epsilon = \begin{pmatrix} \cdots, & x_{-1}, & x_0, & x_1, \cdots \\ \cdots, & h_{-1} x_0, & h_0 x_1, & h_1 x_2, \cdots \end{pmatrix} \qquad \text{when } n = d.$$

Therefore, y can be decomposed into the "product" of disjoint commutative cycles $v_\epsilon c_\epsilon$. It must be remembered that this decomposition may yield an infinite number of the $v_\epsilon c_\epsilon$.

Ore [1, p. 19] has investigated the result of transforming a finite cycle of an element of a monomial group. If c is a cycle of length n and of the form

$$c = \begin{pmatrix} x_1, & x_2, \cdots, & x_n \\ h_1 x_2, & h_2 x_3, \cdots, & h_n x_1 \end{pmatrix},$$

then the nth power of c is $\{\delta_1, \cdots, \delta_n\}$ where $\delta_1 = h_1 \cdots h_n$, $\delta_2 = h_2 \cdots h_n h_1, \cdots, \delta_n = h_n h_1 \cdots h_{n-1}$. These δ_i are called the *determinants* of c. Since the δ_i are conjugate, there exists a unique determinant class for each cycle. A necessary and sufficient condition for two finite cycles to be conjugate is that they have the same length and determinant class.

Theorem 1. *A necessary and sufficient condition that two cycles of length d be conjugate is that they leave the same number of x fixed.*

Proof. The condition is clearly necessary. Conversely, let

(7) $$c = \begin{pmatrix} \cdots, & x_{-1}, & x_0, & x_1, \cdots \\ \cdots, & h_{-1} x_0, & h_0 x_1, & h_1 x_2, \cdots \end{pmatrix}$$

and

(8) $$c' = \begin{pmatrix} \cdots , & x_{i_{-1}}, & x_{i_0}, & x_{i_1}, \cdots \\ \cdots , & r_{-1}x_{i_0}, & r_0 x_{i_1}, & r_1 x_{i_2}, \cdots \end{pmatrix}$$

where c, c' leave the same number of x fixed. There exists a y of $\Sigma(H; B, B^+, B^+)$ such that y has a cycle c_1 given by

(9) $$c_1 = \begin{pmatrix} \cdots , & x_{-1}, & x_0, & x_1, \cdots \\ \cdots , & k_{-1}x_{i_{-1}}, & k_0 x_{i_0}, & k_1 x_{i_1}, \cdots \end{pmatrix}$$

where k_0 is arbitrary in H and the remaining k_i satisfy

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad ,$$
$$k_{-2} = h_{-2}k_{-1}r_{-2}^{-1},$$
$$k_{-1} = h_{-1}k_0 r_{-1}^{-1},$$
$$k_1 = h_0^{-1} k_0 r_0,$$
$$k_2 = h_1^{-1} k_1 r_1,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad .$$

But $y^{-1}cy = c'$.

From the theorem just proved and the corresponding theorem proved by Ore [1, p. 19] it follows that:

THEOREM 2. *Two monomial substitutions y and y_1 are conjugate if and only if in their cyclic decomposition the finite cycles can be made to correspond in a one-to-one manner such that corresponding cycles have the same length and determinant class and the cardinal of the set of infinite cycles is the same for both y and y_1.*

Any infinite cycle c as in (7) can be trasfornmed into the normal form

$$c = \begin{pmatrix} \cdots , & x_{-1}, & x_0, & x_1, \cdots \\ \cdots , & x_0, & x_1, & x_2, \cdots \end{pmatrix}$$

by a proper choice of the factors of c_1 as given by (9). One sees that a transformation of cycles of length d into normal form is possible using a substitution involving only those x which the cycle moves. Ore [1, p. 20] has shown that any finite cycle can be transformed to the normal form

$$c = \begin{pmatrix} x_1, & \cdots , & x_{n-1}, & x_n \\ x_2, & \cdots , & x_n, & a x_1 \end{pmatrix}$$

where a is any element of the determinant class of c. This transformation involves only those variables which c moves. Therefore, all of the cycles of any substitution may be put in normal form by one transformation. This shows that any substitution y is conjugate to a "product" of cycles without

common variables where each cycle is in normal form.

Ore [1, pp. 20, 21] has found the centralizer of a substitution which has only a finite number of finite cycles in its decomposition. Let

$$c_a = \begin{pmatrix} x_1, & \cdots & , & x_{n-1}, & x_n \\ x_2, & \cdots & , & x_n & , & ax_1 \end{pmatrix}.$$

The centralizer F_{c_a} of c_a in the symmetry involving only the variables that c_a moves will be determined. Let D be the centralizer of a in H. Any element y of F_{c_a} is of the form

$$y = \begin{pmatrix} x_1, & \cdots, & x_{n-j+1}, & x_{n-j+2}, & \cdots, & x_n \\ kx_j, & \cdots, & kx_n & , & kax_1 & , \cdots, & kax_{j-1} \end{pmatrix}$$

where k is any element of D. The element y can be written $y=\{k\}c_a^{j-1}$ $=c_a^{j-1}\{k\}$. Therefore, since $c_a^n=\{a\}$ and a is in D, F_{c_a} is isomorphic to a cyclic extension of degree n of a group isomorphic to D.

It has already been demonstrated that a d-cycle c may be transformed such that

$$c = \begin{pmatrix} \cdots, & x_{-1}, & x_0, & x_1, & \cdots \\ \cdots, & x_0 & , & x_1, & x_2, & \cdots \end{pmatrix}.$$

It is clear that one need consider only the symmetry involving the variables that c moves. When c is transformed by

$$y = \begin{pmatrix} \cdots, & x_{-1}, & x_0, & x_1, & \cdots \\ \cdots, & k_{-1}x_{i_{-1}}, & k_0 x_{i_0}, & k_1 x_{i_1}, & \cdots \end{pmatrix},$$

a computation shows that

$$y^{-1}cy = \begin{pmatrix} \cdots, & x_{i_{-1}}, & x_{i_0}, & x_{i_1}, & \cdots \\ \cdots, & k_{-1}^{-1}k_0 x_{i_0}, & k_0^{-1}k_1 x_{i_1}, & k_1^{-1}k_2 x_{i_2}, & \cdots \end{pmatrix}.$$

If y is to belong to F_c the x's of this result must be the same as the ones that c moves, and this gives a condition on y such that y has the form

$$y = \begin{pmatrix} \cdots, & x_{-1}, & x_0, & x_1, & \cdots \\ \cdots, & k_{-1}x_{j-1}, & k_0 x_j, & k_1 x_{j+1}, & \cdots \end{pmatrix}.$$

The factors of y may now be obtained. A computation of $y^{-1}cy$ using the new form shows that

$$y^{-1}cy = \begin{pmatrix} \cdots, & x_{j+1}, & x_j, & x_{j+1}, & \cdots \\ \cdots, & k_{-1}^{-1}k_0 x_j, & k_0^{-1}k_1 x_{j+1}, & k_1^{-1}k_2 x_{j+2}, & \cdots \end{pmatrix}.$$

In order for this result to be c let k_0 be arbitrary in H and it follows that

$k_i = k_0$ for $i = 1, -1, 2, -2, \cdots$. The final form for y to belong to F_c is then given by

$$y = \begin{pmatrix} \cdots, & x_{-1}, & x_0, & x_1, & \cdots \\ \cdots, & kx_{j-1}, & kx_j, & kx_{j+1}, & \cdots \end{pmatrix}.$$

A computation shows that $y = \{k\}c^i = c^i\{k\}$ where $\{k\}$ is not a true scalar but is a multiplication with k as factor in the positions corresponding to x that c moves and e as factor elsewhere. This shows that F_c is isomorphic to the direct product $H \times Z$ where Z is the infinite cyclic group, and this is independent of c (up to an isomorphism).

The centralizer F_y of $y = \prod_\alpha c_\alpha$, where c_α are cycles of length d in the symmetry of degree corresponding to the number of x involved, is now determined. Let α run over a set of cardinal C where $1 \leq C \leq B$. Then the number of variables that appear in y is (dC). Any permutation of the c_α among themselves belongs to F_y. An element y_1 of F_y will have the form $y_1 = (\prod_\alpha f_\alpha)s$ where s is an element of the symmetric group $S(C, C^+)$ and f_α belongs to F_{c_α}, the centralizer of c_α in the symmetry on its variables. It is clear that all of the F_{c_α} are isomorphic since each is isomorphic to $H \times Z$. So F_y is isomorphic to the symmetry $\sum(F_{c_\alpha}; C, C^+, C^+)$ where F_{c_α} is the same for all α.

Consider $y = \prod_\alpha c_\alpha$ where the c_α are finite cycles of the same length n and which have the same determinant class. Let α run over a set of cardinal C where $1 \leq C \leq B$. In a manner similar to that used by Ore [1, p. 21], one finds that F_y is isomorphic to the symmetry $\Sigma(F_{c_\alpha}; C, C^+, C^+)$.

This proves:

THEOREM 3. *Let y be conjugate to y_1 written in the normal form $y_1 = \prod_\alpha c_\alpha$, $c_\alpha = \prod_{\beta(\alpha)} c^\alpha_{\beta(\alpha)}$, where for a fixed α the $c^\alpha_{\beta(\alpha)}$ are the normalized cycles of the same length L_α, and the same determinant class a_α if $L_\alpha < d$. Let $\beta(\alpha)$ run over a set of cardinal $C_{\beta(\alpha)}$ where $0 \leq C_{\beta(\alpha)} \leq B$. Then the centralizer F_y of y in $\Sigma(H; B, B^+, B^+)$ is isomorphic to the strong direct product of symmetries*

$$F_y \cong \prod_\alpha \left(\Sigma(F_{c_{\beta(\alpha)}}; C_{\beta(\alpha)}, C^+_{\beta(\alpha)}, C^+_{\beta(\alpha)}) \right)$$

where $F_{c_{\beta(\alpha)}}$ is the centralizer of a single cycle c^α_ϵ in $\Sigma(H; L_\beta C_{\beta(\alpha)}, (L_\beta C_{\beta(\alpha)})^+,$ $(L_\beta C_{\beta(\alpha)})^+)$. The group $F_{c_{\beta(\alpha)}}$ consists of all elements y_1 of the form $y_1 = \{k_\alpha\}(c_1^\alpha)^i$ where k belongs to the centralizer of a_α in H (k belongs to H if $C_{\beta(\alpha)}$ is a d cycle).

For elements of the group $\Sigma(H; B, B^+, C)$ where $d \leq C \leq B$ the result is the same. When y is written in its cyclic decomposition, the cycles are still of length n or d and all the previous argument is valid including a revised statement of the theorem with $\Sigma(H; B, B^+, B^+)$ replaced by $\Sigma(H; B, B^+, C)$.

The elements of certain subgroups of $\Sigma(H; B, B^+, B^+)$, which are discussed in detail later, have only finite cycles in their decomposition and the corresponding theorem, not stated here, is somewhat simpler.

Chapter II. Splitting of the symmetry

1. **The splitting of** $\Sigma(H; B, B^+, C)$. Let H be a given fixed group, B a fixed infinite cardinal, and let C be such that $d \leq C \leq B^+$. It has already been seen that $\Sigma(H; B, B^+, C) = V(B, B^+) \cup S(B, C)$, $V(B, B^+) \cap S(B, C) = E$, and hence that Σ splits over V. The problem of finding all groups T such $\Sigma(H; B, B^+, C) = V(B, B^+) \cup T$, $V(B, B^+) \cap T = E$ will now be considered.

If there exists such a T, it is isomorphic to $S(B, C)$. Denote by θ the natural isomorphism such that $s\theta = vs = t$.

The group $S(B, C)$ contains B elements of the form $s = (1, \alpha)$ where $\alpha = 2, 3, \cdots$. Thus T must contain B elements of the form

$$t_\alpha = (1, \alpha)\theta = \{ h_{1,\alpha}, h_{2,\alpha}, \cdots, h_{\epsilon,\alpha}, \cdots \}(1, \alpha).$$

In the same way as was used by Ore [1] one sees that there exists a multiplication v such that the group $T' = vTv^{-1}$ has elements t'_α whose first factors are e.

If $t'_\alpha = \{ e, h_{2,\alpha}, \cdots, h_{\epsilon,\alpha}, \cdots \}(1, \alpha)$, then, since $(t'_\alpha)^2 = E$, it follows that
(i) $h_{\alpha,\alpha} = e$,
(ii) $h_{\epsilon,\alpha}^2 = e$ for $\epsilon \neq 1$, $\epsilon \neq \alpha$.

If s belongs to $S(B, C)$ and moves x_1, then s can be written uniquely as $s = (1, \alpha)s_1$ where s_1 leaves x_1 fixed. The image of $(1, \alpha)$ under θ has already been partially described. To find the image of any element of $S(B, C)$ it is sufficient to discuss those elements that leave x_1 fixed before returning to elements of the form $(1, \alpha)$.

Denote by $S_1(B, C)$ that subgroup of $S(B, C)$ whose elements have the property that they do not move x_1. Let s_1 belong to $S_1(B, C)$ and be such that $x_\alpha s_1 = x_\alpha$ for some x_α, $\alpha \neq 1$. Then $s_1 = (1, \alpha)s = s(1, \alpha)$, where s sends x_1 into x_α, x_α into x_1. Let $s\theta = \{ k_1, k_2, \cdots, k_\epsilon, \cdots \}s$. Since

$$(1, \alpha)\theta = \{ e, h_{2,\alpha}, \cdots, e, \cdots, h_{\epsilon,\alpha}, \cdots \}(1, \alpha),$$

where e occurs as a factor in the first and αth positions,

$$s_1\theta = (1, \alpha)\theta s\theta = \begin{pmatrix} x_1, & \cdots, & x_\alpha, & \cdots \\ k_\alpha x_1, & \cdots, & k_1 x_\alpha, & \cdots \end{pmatrix},$$

$$s_1\theta = s\theta(1, \alpha)\theta = \begin{pmatrix} x_1, & \cdots, & x_\alpha, & \cdots \\ k_1 x_1, & \cdots, & k_\alpha x_\alpha, & \cdots \end{pmatrix}.$$

This shows that if s_1 belongs to $S_1(B, C)$ then the factors of v, where $s_1\theta = vs_1$, in the positions corresponding to those x which s_1 leaves fixed are equal to the first factor of v.

LEMMA. *Let s belong to $S(B, C)$ and have the following properties: s moves x_1, s sends at least one x into itself. Denote by x_β the x which s sends into x_1. Then s has the following form:*

$$s = \begin{pmatrix} x_1, & \cdots, & x_\beta, & \cdots, & x_\alpha, & \cdots \\ x_\delta, & \cdots, & x_1, & \cdots, & x_\alpha, & \cdots \end{pmatrix}$$

where $\delta \neq 1$. Let $s\theta = vs$ where v is some element of $V(B, B^+)$. Then the factors which occur in the first and βth positions of v are equal.

Proof. It is possible to write the s as described above in the following two ways:

$$s = (1, \beta)\begin{pmatrix} x_1, & \cdots, & x_\beta, & \cdots, & x_\alpha, & \cdots \\ x_1, & \cdots, & x_\delta, & \cdots, & x_\alpha, & \cdots \end{pmatrix} = (1, \beta)s_1,$$

$$s = s_1(1, \delta).$$

But s_1 is of the form discussed earlier. Therefore,

$$s_1\theta = \{ h_\alpha, \cdots, h_\beta, \cdots, h_\alpha, \cdots, h_\delta, \cdots \}s_1,$$

$$(1, \beta)\theta = \{ e, \cdots, e, \cdots, h_{\alpha,\beta}, \cdots, h_{\delta,\beta}, \cdots \}(1, \beta),$$

$$(1, \delta)\theta = \{ e, \cdots, h_{\beta,\delta}, \cdots, h_{\alpha,\delta}, \cdots, e, \cdots \}(1, \delta).$$

Using the decomposition of s indicated above and the fact that θ is an isomorphism, a computation shows that

$$s\theta = \begin{pmatrix} x_1, & \cdots, & x_\beta, & \cdots \\ h_\beta x_\delta, & \cdots, & h_\alpha x_1, & \cdots \end{pmatrix},$$

$$s\theta = \begin{pmatrix} x_1, & \cdots, & x_\beta, & \cdots \\ h_\alpha x_\delta, & \cdots, & h_\beta x_1, & \cdots \end{pmatrix}.$$

This establishes that $h_\alpha = h_\beta$.

It will now be shown that if s_1 is any element of $S_1(B, C)$ leaving x_α fixed, $\alpha \neq 1$, and $s_1\theta = vs_1$, then v is a scalar. It is sufficient, in view of what has already been established, to show that the factors occupying positions corresponding to x which s_1 moves are the same as the first factor of v. Let

$$s_1 = \begin{pmatrix} x_1, & \cdots, & x_\alpha, & \cdots, & x_\beta, & \cdots \\ x_1, & \cdots, & x_\alpha, & \cdots, & x_\delta, & \cdots \end{pmatrix}$$

$$= \begin{pmatrix} x_1, & \cdots, & x_\alpha, & \cdots, & x_\beta, & \cdots \\ x_\delta, & \cdots, & x_\alpha, & \cdots, & x_1, & \cdots \end{pmatrix}(1, \delta) = s(1, \delta),$$

where $\delta \neq \beta$, $\delta \neq 1$. By the lemma, $s\theta = \{ k_1, \cdots, k_\alpha, \cdots, k_1, \cdots \}s$. Furthermore, $(1, \delta)\theta = \{ e, \cdots, k_{\alpha,\delta}, \cdots, e, \cdots \}(1, \delta)$, where e occurs in the first and δth positions. Using the decomposition of s_1 and the fact that θ is an isomorphism,

$$s_1\theta = s\theta(1, \delta)\theta = \begin{pmatrix} x_1, & \cdots, & x_\beta, & \cdots \\ k_1 x_1, & \cdots, & k_1 x_\delta, & \cdots \end{pmatrix}.$$

This shows that if s_1 belongs to $S_1(B, C)$ and s_1 leaves some x_α fixed, $\alpha \neq 1$, then $s_1\theta = vs_1$, where v is a scalar.

If s_1 belongs to $S_1(B, C)$ but moves all other x, then

$$s_1 = \begin{pmatrix} x_1, & x_2, & \cdots, & x_\beta, & \cdots \\ x_1, & x_\beta, & \cdots, & x_\alpha, & \cdots \end{pmatrix}$$

$$= (2, \beta)\begin{pmatrix} x_1, & x_2, & \cdots, & x_\beta, & \cdots \\ x_1, & x_\alpha, & \cdots, & x_\beta, & \cdots \end{pmatrix} = \bar{s}_1 s_1',$$

where $\bar{s}_1$, s_1' belong to $S_1(B, C)$. Therefore, $s_1\theta = \bar{s}_1\theta s_1' \theta = \bar{v}_1\bar{s}_1 v_1' s_1' = \bar{v}_1 v_1' \bar{s}_1 s_1'$ $= vs_1$. It has already been established that $\bar{v}_1$, v_1' are scalars. Since scalars commute with all permutations, v is a scalar.

Define a homomorphism between $S_1(B, C)$ and a subgroup of H by $s_1\phi = h_{s_1}$, where $s_1\theta = \{h_{s_1}\} s_1$.

A computation shows that if

$$(1, \alpha)\theta = \{e, \cdots, h_{\beta,\alpha}, \cdots, e, \cdots\}(1, \alpha),$$

$$(1, \beta)\theta = \{e, \cdots, h_{\alpha,\beta}, \cdots, e, \cdots\}(1, \beta),$$

then

$$((1, \alpha)(1, \beta)(1, \alpha))\theta = (\alpha, \beta)\theta = \begin{pmatrix} x_1, & \cdots, & x_\alpha, & \cdots, & x_\beta, & \cdots \\ h_{\alpha,\beta}x_1, & \cdots, & h_{\beta,\alpha}x_\beta, & \cdots, & h_{\beta,\alpha}x_\alpha, & \cdots \end{pmatrix},$$

where $\alpha \neq 1$, $\beta \neq 1$, $\alpha \neq \beta$. On the other hand $(\alpha, \beta)\theta = \{g_{\alpha,\beta}\}(\alpha, \beta)$. Therefore, the βth factor of $(1, \alpha)\theta = v(1, \alpha)$ is simply the homomorphic image of (β, α) under ϕ.

This leads to the following theorem:

THEOREM 1. *The symmetry $\Sigma(H; B, B^+, C)$ splits over the basis group, $\Sigma(H; B, B^+, C) = V(B, B^+)\cup T$, $V(B, B^+)\cap T = E$. Any such group T is the conjugate of some group T' obtained by the following construction. Let G be a subgroup of H that is the homomorphic image of $S_1(B, C)$ where $d \leq C \leq B^+$. Let $s\phi = g_s$ indicate the homomorphism. In particular, $(\alpha, \beta)\phi = g_{\alpha,\beta}$. Then the elements of T' are obtained from the elements of $S(B, C)$ by the isomorphism defined as follows:*

$s* = \{g_s\}$ *for s belonging to $S_1(B, C)$,*

$$(1, \alpha)* = \{e, g_{2,\alpha}, \cdots, g_{\epsilon,\alpha}, \cdots, e, \cdots\}(1, \alpha)$$

where e occurs in the first and αth positions.

It has already been shown that any group T, after suitable transformation, must have the form indicated in the correspondence above. It remains to show that the set of substitutions defined by the correspondence above actually forms a group isomorphic to $S(B, C)$.

Any element s of $S(B, C)$ may be written uniquely in the form $s = (1, \alpha)s_1$ where s_1 belongs to $S_1(B, C)$ (α may be 1).

The correspondence of the theorem defines a unique substitution $s*$ corresponding to each s of $S(B, C)$ by

$$s* = (1, \alpha)*s_1*$$

$$= \{e, g_{2,\alpha}, \cdots, g_{\epsilon,\alpha}, \cdots, e, \cdots\}(1, \alpha)\{g_{s_1}\}s_1$$

$$= \{g_{s_1}, g_{2,\alpha}g_{s_1}, \cdots, g_{\epsilon,\alpha}g_{s_1}, \cdots, g_{s_1}, \cdots\}s.$$

Let $\bar{s} = (1, \beta)\bar{s}_1$ be another element of $S(B, C)$ that has been written in the normal form with $\bar{s}_1$ belonging to $S_1(B, C)$. To establish that the correspondence preserves multiplication it must be shown that

(1) $(s\bar{s})* = s*\bar{s}*.$

By the same argument used by Ore $[1, p.\ 25]$ it follows that it is sufficient to prove that

(2) $(s(1, \beta))* = s*(1, \beta)*.$

This will follow if it can be established that

(2.1) $((1, \alpha)(1, \beta))* = (1, \alpha)*(1, \beta)*$

and

(2.2) $(s_1(1, \beta))* = s_1*(1, \beta)*$ for any s_1 of $S_1(B, C)$.

The verification of (2.1) is immediate in the case $\alpha = \beta$ when the fact that ϕ is a homomorphism is used. If $\alpha \neq \beta$, let

$$(1, \alpha)* = \{e, \cdots, g_{\epsilon,\alpha}, \cdots, e, \cdots, g_{\beta,\alpha}, \cdots\}(1, \alpha),$$

$$(1, \beta)* = \{e, \cdots, g_{\epsilon,\beta}, \cdots, g_{\alpha,\beta}, \cdots, e, \cdots\}(1, \beta).$$

A computation shows that

$$(1, \alpha)*(1, \beta)* = \{g_{\alpha,\beta}, \cdots, g_{\epsilon,\alpha}g_{\epsilon,\beta}, \cdots, e, \cdots, g_{\beta,\alpha}, \cdots\}(1, \alpha)(1, \beta).$$

On the other hand, since $(1, \alpha)(1, \beta) = (1, \beta)(\alpha, \beta)$,

$$((1, \alpha)(1, \beta))* = (1, \beta)*\{g_{\alpha,\beta}\}(\alpha, \beta)$$

$$= \{g_{\alpha,\beta}, \cdots, g_{\epsilon,\beta}g_{\alpha,\beta}, \cdots, \overset{2}{g}_{\alpha,\beta}, \cdots, g_{\alpha,\beta}, \cdots\}(1, \alpha)(1, \beta).$$

By comparison of the two computations, one sees that the factors in the first and βth positions are the same. But $g_{\alpha,\beta}^2 = e$ since $(\alpha, \beta)^2 = I$, $g_{\epsilon,\alpha}g_{\epsilon,\beta} = g_{\epsilon,\beta}g_{\alpha,\beta}$ since $(\epsilon, \alpha)(\epsilon, \beta) = (\epsilon, \beta)(\alpha, \beta)$ and ϕ is a homomorphism.

The verification of (2.2) is also discussed in two cases, namely, when s_1 moves x_β and when s_1 does not move x_β. If x_{s_1} does not move x_β, $s_1*(1, \beta)*$ is computed and $(s_1(1, \beta))*$ is computed using the fact that $s_1(1, \beta) = (1, \beta)s_1$ in this case. Factors of the two computations are compared and, again using

the fact that ϕ is a homomorphism, it can be shown that they are the same. If s_1 does move x_β, then $(s_1(1, \beta))*$ cannot be computed directly. However,

$$s_1(1, \beta) = \begin{pmatrix} x_1, & \cdots, & x_\beta, & \cdots, & x_\delta, & \cdots \\ x_1, & \cdots, & x_\alpha, & \cdots, & x_\beta, & \cdots \end{pmatrix}(1, \beta) = (1, \delta)s_1$$

and $(1, \delta)s_1$ may be computed. If

$$s_1* = \{g_{s_1}\}s_1,$$

$$(1, \beta)* = \{e, \cdots, e, \cdots, g_{\delta,\beta}, \cdots, g_{\epsilon,\beta}, \cdots\}(1, \beta),$$

$$(1, \delta)* = \{e, \cdots, g_{\beta,\delta}, \cdots, e, \cdots, g_{\epsilon,\delta}, \cdots\}(1, \delta),$$

then

$$s_1*(1, \beta)* = \begin{pmatrix} x_1, & \cdots, & x_\beta, & \cdots, & x_\delta, & \cdots, & x_\epsilon, & \cdots \\ g_{s_1}x_\beta, & \cdots, & g_{s_1}g_{\alpha,\beta}x_\alpha, & \cdots, & g_{s_1}x_1, & \cdots, & g_{s_1}g_{i_\epsilon,\beta}x_{i_\epsilon}, & \cdots \end{pmatrix},$$

$$(s_1, (1, \beta))* = ((1, \delta)s_1)* = (1, \delta)*s_1*$$

$$= \begin{pmatrix} x_1, & \cdots, & x_\beta, & \cdots, & x_\delta, & \cdots, & x_\epsilon, & \cdots \\ g_{s_1}x_\beta, & \cdots, & g_{\beta,\delta}g_{s_1}x_\alpha, & \cdots, & g_{s_1}x_1, & \cdots, & g_{\epsilon,\delta}g_{s_1}x_{i_\epsilon}, & \cdots \end{pmatrix}.$$

The proof that factors of the two computations are the same again follows from the fact that ϕ is a homomorphism.

This concludes showing that the correspondence defined in the theorem preserves multiplication. The images of the elements of $S(B, C)$ form a group isomorphic to $S(B, C)$ which we shall now call T. It is also clear that $V(B, B^+) \cap T = E$. Furthermore, $V(B, B^+) \cup T = \Sigma(H; B, B^+, C)$, since all $y = vs$ belonging to Σ may be written $y = vv_1^{-1}v_1s = v_2t$, where $s* = v_1s = t$.

THEOREM 2. *A necessary and sufficient condition for* $\Sigma(H; B, B^+, C)$, *where* $d^+ \le C \le B^+$, *to split regularly over the basis group is that H contain no subgroup isomorphic to $S(B, C)$.*

Proof. If Σ splits regularly over the basis group, T' can be transformed by an element y into S. This element may be assumed to be a multiplication; $y = \{k_1, \cdots\}$. Consider the element $t = \{g_{s_1}\}s_1$ of T'. When yty^{-1} is computed, since s_1 leaves x_1 fixed, $k_1g_sk_1^{-1}$ must be e. Therefore, $g_s = e$ for all g of G. This means that H contains no proper subgroup homomorphic to $S_1(B, C)$, and, since for $C > d$ the group $S_1(B, C)$ is isomorphic to $S(B, C)$ and has no proper normal subgroups, H contains no subgroup isomorphic to $S(B, C)$.

Conversely, assume H contains no subgroup isomorphic to $S(B, C)$ and that Σ does not split regularly. Then H contains a group G which is the homomorphic image of $S_1(B, C)$. Scott[3] has shown that this implies that G contains a subgroup isomorphic to $S(B, C)$, contradicting the hypothesis.

---

[3] Scott, W. R., oral communication.

Therefore Σ splits regularly.

THEOREM 3. *A necessary and sufficient condition for $\Sigma(H; B, B^+, d)$ to split regularly over its basis group is that H contain no element of order 2.*

Proof. If Σ splits regularly, then by the method used in the proof of Theorem 2 it can be shown that H contains no subgroup homomorphic to $S_1(B, d)$. Baer [2, p. 16] has shown that the only proper normal subgroup of $S(B, d)$ which is isomorphic to $S_1(B, d)$ is $A(B, d)$. H contains no element of order 2, since the quotient group $S(B, d)/A(B, d)$ is of order 2.

Conversely, if H contains no element of order 2, it does not contain a cyclic group of order 2 or a subgroup isomorphic to $S_1(B, d)$. Therefore, H contains no subgroup which is homomorphic to $S_1(B, d)$. The group T' constructed in Theorem 1 is simply $S(B, d)$ and Σ splits regularly.

COROLLARY. *For every group H there exists a group $\Sigma(H; B, B^+, B^+)$ such that the monomial group splits regularly over the basis group.*

Proof. This follows from Theorem 2 if the cardinal B is chosen such that $o(S(B, B^+)) > o(H)$.

2. The splitting of $\Sigma_{n,A}(H)$. The problem of finding all groups T such that $\Sigma_{n,A}(H) = V_n \cup T$, $V_n \cap T = E$ will now be considered.

If there exists such a group T, T is isomorphic to A_n. The natural isomorphism, denoted by θ, may be taken such that $s\theta = vs = t$.

The elements $s_i = (1, i, 2)$, $i = 3, \cdots, n$, generate the group A_n. T must contain elements t_i, $i = 3, \cdots, n$, such that $s_i\theta = t_i$. Let $t_i = \{h_{1i}, h_{2i}, \cdots, h_{ji}, \cdots, h_{ni}\}(1, i, 2)$, for $i = 3, \cdots, n$. If T is transformed by $v = \{k_1, \cdots, k_n\}$, then $T' = vTv^{-1}$ contains elements of the form

$$t_i = vt_iv^{-1} = \{k_1h_{1i}k_i^{-1}, k_2h_{2i}k_1^{-1} \cdots \cdots \}(1, i, 2).$$

Let k_1 be an arbitrary fixed element of H and choose $k_i = k_1h_{1i}$ for $i = 3, \cdots, n$. Choose $k_2 = k_1h_{23}^{-1}$. Then T' contains

$$t_3' = \{e, e, g_{33}, \cdots, g_{n3}\}(1, 3, 2),$$
$$t_i' = \{e, g_{2i}, \cdots, g_{ni}\}(1, i, 2)$$

for $i = 4, \cdots, n$. The equations $s_i^3 = (s_is_j)^2 = I$ for $i \neq j$ imply

(1) $(t_i')^3 = (t_i't_j')^2 = E,$ $i \neq j, i, j = 3, 4, \cdots, n.$

The first position factors of $(t_i')^3$ and $(t_it_j)^2$ are

(2) $g_{1i}g_{ii}g_{2i} = e,$

(3) $g_{1i}g_{ij}g_{ii}g_{2j} = e,$

where i, j are distinct integers between 3 and n. Noting that $g_{1i} = g_{1j} = g_{23} = e$ and writing g_i for g_{2i} we have from (2) and (3)

(4) $$g_i = g_{2i} = \overset{-1}{g_{ii}}, \; g_{33} = g_3 = e,$$

(5) $$g_{ij} = (g_{ii}g_{2j})^{-1} = \overset{-1}{g_j}g_i = \overset{-1}{g_{ji}}, \; g_{i3} = g_i = \overset{-1}{g_{3i}}.$$

Thus t_i' has the form

(6) $$t_i' = \{e, \, g_i, \, \overset{-1}{g_i}, \, \overset{-1}{g_i}, g_4, \cdots, \overset{-1}{g_i} \, g_n\}, \qquad \text{where } g_3 = e.$$

If $k > 2$ and $k \neq i$, $k \neq j$ the kth position factors of $(t_i')^3$ and $(t_i't_j')^2$ are

(7) $$\overset{3}{g_{ki}} = (\overset{-1}{g_i}g_k)^3 = e, \qquad (g_{ki}g_{kj})^2 = e.$$

Hence for $k = 3$, we find

(8) $$\overset{3}{g_i} = (g_ig_j)^2 = e \qquad\qquad 4 \leq i, j \leq n, \; i \neq j.$$

Therefore, the g_i, $i = 3, \cdots, n$ generate a group homomorphic to A_{n-1}.

This leads to the following theorem:

THEOREM 1. *The group $\Sigma_{n,A}(H)$ splits over the basis group, $\Sigma_{n,A}(H) = V_n \cup T$, $V_n \cap T = E$. The group T is conjugate to some group T' obtained as follows. Let G be a subgroup of H which is the homomorphic image of A_{n-1}. Let $g_4, \cdots, g_n$ be generators of G, satisfying the following relations:*
  (α) $g_i^3 = e$, $i = 4, \cdots, n$,
  (β) $(g_ig_j)^2 = e$ where $i \neq j$.
Let $s_i = (1, i, 2)$ for $i = 3, \cdots, n$ generate the group A_n. Then the elements of T' are obtained from the elements of A_n by the isomorphism θ defined by

$$s_3\theta = t_3' = \{e, e, e, g_4, \cdots, g_n\}(1, 3, 2),$$
$$s_i\theta = t_i' = \{e, g_i, \overset{2}{g_i}, \overset{2}{g_i}g_4, \cdots, \overset{2}{g_i}g_{i-1}, \overset{2}{g_i}, \overset{2}{g_i}g_{i+1}, \cdots, \overset{2}{g_i}g_n\}(1, i, 2)$$

for $i = 4, \cdots, n$.

It has already been shown that any group T after proper transformation must have the form indicated by the theorem.

It will now be shown that $(t_i')^3 = (t_it_j)^2 = E$, $i \neq j$, $i, j = 3, 4, \cdots, n$. The 1st, 2nd, and ith factors of $(t_i')^3$ and $(t_i't_j')^2$ and the jth factor of the latter are

(9) $$\overset{3}{g_i}, \; \overset{3}{g_i}, \; \overset{3}{g_i}$$

(10) $$\overset{2}{g_j}g_i\overset{2}{g_i}g_j, \; g_i\overset{2}{g_i}\overset{2}{g_j}g_j, \; \overset{2}{g_i}\overset{2}{g_j}g_jg_i, \; g_ig_j\overset{2}{g_j}\overset{2}{g_i}.$$

These factors are e by α. If $k > 2$, $k \neq i$, $k \neq j$ the kth position factors of $(t_i')^3$ and $(t_i't_j')^2$ are

(11) $$(\overset{2}{g_i}g_k)^3, \qquad (\overset{2}{g_i}g_k\overset{2}{g_j}g_k)^2$$

and these are seen to be e by α and β.

Thus we have $n - 2$ elements which generate a group that is the homo-

morphic image of A_n. Since the permutation part of the elements of T' run through A_n, the group T' is actually isomorphic to A_n.

Since T is isomorphic to A_n by $s\theta = vs = t$, $V \cap T' = E$. Furthermore, if $y = vs$ is any element of $\Sigma_{n,A}(H)$, then $y = vv_1^{-1}v_1 s = v_2 t$, where $s\theta = v_1 s$. This shows that $\Sigma_{n,A}(H) = V_n \cup T$.

COROLLARY. *The group $\Sigma_{3,A}(H)$ splits regularly over its basis group.*

Proof. Since $s_3 = (1, 3, 2)$ generates A_3 and $t_3' = \{e, e, e\}$ $(1, 3, 2) = s_3$ generates T', it follows that T' is A_3.

Theorem 1 describes the splitting of the group $\Sigma_{4,A}(H)$; however, it is interesting to note that another and perhaps more pleasing construction for the group T' can be given by a slightly different approach.

If, instead of discussing the images of $(1, i, 2)$, the images of $(1, 2)(3, 4)$, $(1, 3)(2, 4)$, and $(1, 4)(2, 3)$ are first considered, the following theorem may be proved in a similar fashion.

THEOREM 2. *The group $\Sigma_{4,A}(H)$ splits over the basis group, $\Sigma_{4,n}(H) = V_4 \cup T$, $V_4 \cap T = E$. Any group T in this decomposition is conjugate to some group T' obtained as follows. Let G be a subgroup of H which is the proper homomorphic image of A_4. Denote this homomorphism by $s\phi = g_s$ for s in A_4. The elements of T' are obtained from those of A_4 by the isomorphism*

$$s\theta = \{g_s\}s$$

for s belonging to A_4.

The general group $\Sigma_{n,A}(H)$ will now again be discussed.

THEOREM 3. *A necessary and sufficient condition for the group $\Sigma_{n,A}(H)$ to split regularly over the basis group is that H contain no subgroup, except e, which is the homomorphic image of A_{n-1}.*

Proof. If H contain no·subgroup which is the homomorphic image of A_{n-1}, then the group T' constructed in Theorem 1 is simply A_n.

Conversely, if $\Sigma_{n,A}(H)$ splits regularly, then T' can be transformed into A_n. Such a transformation need only be by a multiplication $v = \{k_1, k_2, \cdots, k_n\}$. Consider

$$vt_3'v^{-1} = \{k_1k_3^{-1}, k_2k_1^{-1}, k_3k_2^{-1}, k_4g_4k_4^{-1}, \cdots, k_ng_nk_n^{-1}\}(1, 3, 2).$$

If this is a permutation then $k_ig_ik_i^{-1} = e$ for $i = 4, \cdots, n$. Therefore, $g_i = e$ for $i = 4, \cdots, n$, and $G = e$.

COROLLARY 1. *A necessary and sufficient condition for $\Sigma_{n,A}(H)$, for $n = 4, 5$, to split regularly over the basis group is that H contain no element of order 3.*

COROLLARY 2. *A necessary and sufficient condition for $\Sigma_{n,A}(H)$, for $n \geq 6$,*

to split regularly over the basis group is that H contain no subgroup isomorphic to A_{n-1}.

3. The splitting of $\Sigma_A(H; B, B^+, d)$. The group $\Sigma_A(H; B, B^+, d)$ splits over the basis group. Any group T such that $\Sigma_A(H; B, B^+, d) = V(B, B^+) \cup T$, $V(B, B^+) \cap T = E$ is isomorphic to $A(B, d)$. The natural isomorphism θ may be taken such that $s\theta = vs = t$. The elements $s_\alpha = (1, \alpha, 2)$, $\alpha = 3, 4, \cdots$, generate $A(B, d)$.

By a process almost identical with the one used in §2, Chapter III, the following theorem may be established.

THEOREM 1. *The group $\Sigma_A(H; B, B^+, d)$ splits over the basis group, $\Sigma_A(H; B, B^+, d) = V(B, B^+) \cup T$, $V(B, B^+) \cap T = E$. The group T is conjugate to some group T' obtained as follows. Let G be a subgroup of H which is the homomorphic image of $A(B, d)$. Let $g_4, \cdots, g_\epsilon, \cdots$ be generators of G, satisfying the relations,* (i) $(g_\epsilon)^3 = e$ *and* (ii) $(g_\epsilon g_\delta)^2 = e$ *when $\epsilon \neq \delta$. Let $s_\alpha = (1, \alpha, 2)$, $\alpha = 3, \cdots$, denote the generators of the group $A(B, d)$. Then the elements of T' are obtained from the elements of $A(B, d)$ by the isomorphism θ defined by*

$$s_3\theta = t_3' = \{e, e, e, g_4, \cdots, g_\epsilon, \cdots\}(1, 3, 2),$$
$$s_\alpha\theta = t_\alpha' = \{e, g_\alpha, g_\alpha^2, g_\alpha g_4, \cdots, g^2 \cdots, g_\alpha g_\epsilon, \cdots\}(1, \alpha, 2).$$

THEOREM 2. *A necessary and sufficient condition for $\Sigma_A(H; B, B^+, d)$ to split regularly over the basis group is that H contain no subgroup isomorphic to $A(B, d)$.*

Proof. Since Baer [2, p. 16] has shown $A(B, d)$ is simple, if H contains no subgroup isomorphic to $A(B, d)$, then H contains no subgroup, except e, which is the homomorphic image of $A(B, d)$. Therefore, the group T constructed in Theorem 1 is simply $A(B, d)$.

Conversely, if Σ splits regularly over the basis group, T may be transformed into $A(B, d)$. By the method used to prove Theorem 3 of §2 of this chapter it can be shown that $G = e$.

COROLLARY. *For every group H there exists a group $\Sigma_A(H; B, B^+, d)$ such that the monomial group splits regularly over the basis group.*

Proof. This follows from Theorem 2 if the cardinal B is chosen such that $o(A(B, d)) > o(H)$.

CHAPTER III. NORMAL SUBGROUPS OF THE SYMMETRY

In this chapter all of the normal subgroups of the groups $\Sigma(H; B, d, d)$, $\Sigma_A(H; B, d, d)$, and $\Sigma_{n,A}(H)$ for $n \geq 5$ are found. The method of investigation is that employed by Ore [1] for $\Sigma_n(H)$.

1. Normal subgroups of $\Sigma(H; B, d, d)$. Before the normal subgroups of the

symmetry can be determined we must first solve a preliminary problem.

A subgroup of the symmetry Σ is called a *permutation invariant subgroup* if it is transformed into itself by all permutations s of the symmetric group.

The first problem to be solved is that of finding all permutation invariant subgroups of $\Sigma(H; B, d, d)$ contained in the basis group.

1.1 Permutation invariant subgroups of $\Sigma(H; B, d, d)$ contained in the basis group. Let N be a fixed permutation invariant subgroup of Σ contained in the basis group. All elements g_α of H which occur in the αth position of multiplications in N will form a subgroup of H. Since N is permutation invariant, this subgroup G will be the same for all indices α.

The set S_1 of all the multiplications in N that have every factor $h_\alpha = e$ for $\alpha > 1$ forms a normal subgroup of N.

The factors that occur in the first position of multiplications of S_1 form a normal subgroup G_1 of G.

If g_1 belongs to G_1, then $v = \{g_1, e, \cdots, e, \cdots\}$ belongs to S_1. Since N is permutation invariant, N must contain $v_\alpha = \{e, \cdots, e, g_1, e, \cdots\}$.

The preceding shows that if v is an arbitrary element of N, then any of the nonidentity factors of v can be multiplied by an arbitrary g_1 of G_1, and the multiplication v' so obtained will again be in N. Thus the relations between the factors of v can only be determined modulo G_1. It is, therefore, no limitation if we consider the quotient group G/G_1 in the following and assume $G_1 = e$.

The set of elements S_2 of N which have every factor $h_\alpha = e$ for $\alpha > 2$ forms a normal subgroup of N. The first factors of elements of S_2 run through a normal subgroup G_2 of G. Since N is permutation invariant, the second factors of elements of S_2 also run through G_2.

The result of Ore [1, Theorem 1, p. 29] in regard to elements S_2 may be applied, and v_2 belongs to S_2 implies

$$v_2 = \{g_2, g_2\theta, e, \cdots, e, \cdots\}$$

where θ is some automorphism of order two of the group G_2.

Since N is permutation invariant, when N contains $v_1 = \{g_2, g_2\theta, e, \cdots, e, \cdots\}$, it must also contain $v_2 = \{g_2, e, g_2\theta, e, \cdots, e, \cdots\}$. The element $v_1 v_2^{-1} = \{e, g_2\theta, (g_2\theta)^{-1}, e, \cdots, e, \cdots\}$ also belongs to N. Again using the fact that N is permutation invariant, N must contain $v_3 = \{g_2\theta, (g_2\theta)^{-1}, e, \cdots, e, \cdots\}$. This shows that $g_2\theta = g_2^{-1}$. A group that has an automorphism changing every element into its inverse is Abelian. This establishes the following:

THEOREM 1. *Let N be a fixed permutation invariant subgroup of $\Sigma(H; B, d, d)$ contained in the basis group. Then the set G of H consisting of all of the factors that occur in a fixed αth position of all the multiplications in N forms a subgroup of H. This group is the same for all α. The set S_1 of all multiplications of N which have $h_\alpha = e$ for $\alpha > 1$ forms a normal subgroup of N. The set G_1, con-*

sisting of first factors of multiplications of S_1, forms a normal subgroup of G.
Assume $G_1 = e$. The set of elements S_2 of N that have $h_\alpha = e$ for $\alpha > 2$ forms a
normal subgroup of N. The set G_2 of first factors of elements of S_2 forms a normal,
Abelian subgroup of G. The elements of S_2 are of the form

$$v_2 = \{g,\ g_2^{-1},\ e,\ \cdots,\ e,\ \cdots\}$$

where g_2 runs through G_2.

When the factors g_2, g_2^{-1} in v_2 are permuted into all possible positions, the corresponding substitutions generate a normal subgroup R of N which is also permutation invariant. It follows that if v is an element of R, since all but a finite number of factors of any element of Σ are e, then the nonidentity factors $r_{i_1}, \cdots, r_{i_n}$ satisfy $r_{i_1} \cdots r_{i_n} = e$.

In a manner similar to that used by Ore [1, p. 30] for $\Sigma_n(H)$ the following theorem may be proved.

THEOREM 2. *The group R, generated by the substitutions obtained by all possible permutations of elements of S_2, consists of elements of the form*

$$v = \{e,\ \cdots,\ e,\ r_{i_1},\ e,\ \cdots,\ e,\ r_{i_{n-1}},\ e,\ \cdots,\ e,\ (r_{i_1} \cdots r_{i_{n-1}})^{-1}, e,\ \cdots\}$$

where the r_{i_j} run through the Abelian group G_2 independently.

The final step in the determination of the permutation invariant subgroups of Σ that are contained in the basis group is now reached. Let v be an arbitrary element of N and let the nonidentity factors occur in the $i_1, \cdots, i_n$ positions. Since N is permutation invariant, it must contain v transformed by (i_1, α) where α is different from each of the indices $i_1, \cdots, i_n$. Let $v_1 = (i_1, \alpha)v(i_1, \alpha)$. Then v_1 differs from v only by having the factor g_{i_1} in different positions. Now

$$vv_1^{-1} = \{e,\ \cdots,\ e,\ g_{i_1},\ e,\ \cdots,\ e,\ g_{i_1}^{-1},\ e,\ \cdots\}$$

where g_{i_1} is in the i_1th position and $g_{i_1}^{-1}$ is in the αth position. This shows that g_{i_1} belongs to G_2. A similar procedure shows that all the factors of v are in G_2 and this means that N is simply R.

It has now been shown that if N is permutation invariant and contained in V, with $G_1 = e$, then N must be of the form of the group R of Theorem 2.

Conversely, any group R whose elements are of the form

$$v = \{e,\ \cdots,\ e,\ r_{i_1},\ e,\ \cdots,\ e,\ r_{i_{n-1}},\ e,\ \cdots,\ e,\ (r_{i_1} \cdots r_{i_{n-1}})^{-1},\ e,\ \cdots\}$$

where r_{i_j} runs through an Abelian subgroup G_2 of H is permutation invariant.

This establishes the following result:

THEOREM 3. *Let an Abelian subgroup G be chosen in H. The group N consisting of all elements of the form*

$$v = \left\{ e, \cdots, e, g_{i_1}, e, \cdots, e, g_{i_{n-1}}, e, \cdots, e, (g_{i_1} \cdots g_{i_{n-1}})^{-1}, e, \cdots \right\},$$

where the g_{i_j} run through G independently, is a permutation invariant subgroup of $\Sigma(H; B, d, d)$.

One recalls that for convenience the subgroup G_1 was assumed to be e. The results obtained may, therefore, be generalized by working with G_1 a normal subgroup of $G = G_2$. Such a consideration leads to a determination of all permutation invariant subgroups of $\Sigma(H; B, d, d)$ contained in the basis group. This result is stated in the following theorem.

THEOREM 3'. *All permutation invariant subgroups N of $\Sigma(H; B, d, d)$ that are contained in the basis group may be obtained by the following construction. A subgroup G of H is chosen. In G a normal subgroup G_1 is chosen such that the quotient group G/G_1 is Abelian. Then let N consist of elements of the form*

$$v = \left\{ e, \cdots, e, g_{i_1}, e, \cdots, g_{i_n}, e, \cdots \right\}$$

where the nonidentity factors run through G subject to the condition that $g_{i_1} \cdots g_{i_n}$ belongs to G_1. Conversely, any such N is permutation invariant.

1.2 Normal subgroups of $\Sigma(H; B, d, d)$ contained in the basis group. A necessary and sufficient condition for a subgroup N of the basis group to be normal in Σ is that N be permutation invariant and normal in V.

It is now necessary to find those normal subgroups of the basis group which are permutation invariant. The same notation as in the previous section will be used. Let N be a normal subgroup of Σ contained in V. Now the group $G = G_2$ must be a normal subgroup of H, and the group G_1 is also normal in H. Since N is permutation invariant, an element v of N must have the form described in Theorem 3'. For convenience, for the moment again assume that $G_1 = e$. Then any nonidentity factor of an element of N is uniquely determined by the other nonidentity factors. Let $v_1 = \left\{ e, \cdots, e, h, e, \cdots \right\}$ be an element of $V(B, d)$ that has h as its only nonidentity factor. Furthermore, let h occupy the position occupied by g_{i_1} of v and let h be arbitrary in H. The element

$$v_1 v v_1^{-1} = \left\{ e, \cdots, e, h g_{i_1} h^{-1}, e, \cdots, e, (g_{i_1} \cdots g_{i_{n-1}})^{-1}, e, \cdots \right\}$$

must be in N since N is normal in $V(B, d)$. This means that $h g_{i_1} h^{-1} = g_{i_1}$, and, since this is true for all h of H and all g of G, G belongs to the center of H.

Conversely, if $G_1 = e$, $G_2 = G$ belongs to the center of H and N is permutation invariant, then N is normal in $\Sigma(H; B, d, d)$.

This establishes the following theorem:

THEOREM 4. *If N is as given by Theorem 3 and the additional requirement that G belongs to the center of H is met, then N is normal in $\Sigma(H; B, d, d)$.*

The generalization of this theorem gives all of the normal subgroups of

$\Sigma(H; B, d, d)$ contained in the basis group. This result is stated in

THEOREM 4'. *The normal subgroups of $\Sigma(H; B, d, d)$ that are contained in the basis group are obtained from the construction of Theorem 3' with the additional conditions that $G_1 \subset G$ are normal in H and G/G_1 belongs to the center of H/G_1.*

1.3 Other normal subgroups of $\Sigma(H; B, d, d)$. The problem of finding all of the normal subgroups of Σ will now be solved by finding those normal subgroups M of Σ not contained in V. Let M be such a normal subgroup of Σ. Then $N = M \cap V$ is normal in Σ and of the form described in Theorem 4'.

Let y be a substitution of M and let c be a cycle of y,

$$c = \begin{pmatrix} x_{i_1}, & \cdots, & x_{i_n} \\ h_1 x_{i_2}, & \cdots, & h_n x_{i_1} \end{pmatrix}, \qquad \text{where } n \geq 2.$$

Let $v = \{k_1, k_2, \cdots, k_\epsilon, \cdots\}$ be an element of the basis group. Since M is normal in Σ, the multiplication defined by the commutator $y^{-1}v^{-1}yv$ is in M. The factor that appears in the position occupied by x_{i_1} is $h_n^{-1}k_{i_n}^{-1}h_n k_{i_1}$. The element v is arbitrary so it may now be chosen such that it has factors in the positions corresponding to x_{i_1} and x_{i_n} such that the factor above is any element of H. But $y^{-1}v^{-1}yv$ is in M and is a multiplication, so it is also in N. This means that the group G is H for $N = M \cap V$.

This establishes the following:

THEOREM 5. *Let M be a normal subgroup of $\Sigma(H; B, d, d)$ not contained in the basis group. The multiplications contained in M form a normal subgroup N of $\Sigma(H; B, d, d)$ in which $G = H$, i.e., the factors in any fixed position run through the whole group and H/G_1 for N is an Abelian group.*

The group $P = M \cap S$ is a normal subgroup of Σ, hence a normal subgroup of S. By the result of Baer [2, p. 16], P is e, $A(B, d)$ or $S(B, d)$.

Since M is not contained in V, M contains an element y with a cycle c in its decomposition of length $n \geq 2$. Then y is conjugate to y' which contains a cycle

$$c' = \begin{pmatrix} x_1, & \cdots, & x_n \\ x_2, & \cdots, & ax_1 \end{pmatrix}.$$

Since M is normal in Σ, M must contain y'. Every element of Σ maps an infinite number of x into themselves with factors of e. Let x_α, x_β be two of these, where α, β are each different from each of $1, \cdots, n$. Let $s = (1, \alpha, \beta)$. Then M must contain the commutator $(y')^{-1}sy's^{-1} = (1, \beta, 2)$.

This establishes:

THEOREM 6. *Every normal subgroup M of $\Sigma(H; B, d, d)$ not contained in the basis group contains permutations.*

The normal subgroups of $\Sigma(H; B, d, d)$ will now be constructed. The quotient group Σ/V is isomorphic to S. Since $M \cup V$ is the union of two normal subgroups of Σ, it is normal in Σ and $(M \cup V)/V \cong M/N$. This means that M/N is isomorphic to $S(B, d)$ or $A(B, d)$. P is also one of these groups.

The case where $P \cong M/N$, and hence $M = N \cup P$, will be discussed first.

THEOREM 7. *Let P be a normal subgroup of $S(B, d)$ and let N be a normal subgroup of the type discussed in Theorem 5. Then $M = N \cup P$ is a normal subgroup of $\Sigma(H; B, d, d)$.*

Proof. The group N is normal in Σ by assumption, and hence it is sufficient to show that an element in P is transformed into an element in M. Since P is normal in S, it is sufficient to show that a permutation s in P is transformed into an element of M by a multiplication. The commutator $s^{-1}v^{-1}sv$ is a multiplication. Although it cannot be said what factors occur in which positions, it can be said that the product of the factors is in G_1 since H/G_1 is Abelian. Therefore, the commutator is in $N \subset M$ which implies that $v^{-1}sv$ is in M, and M is normal in Σ.

It remains to discuss the case where $(M \cup V)/V \cong M/N \cong S(B, d)$ and $P = A(B, d)$. Since $P \subset M$ it follows that $(V \cup A) \subset (M \cup V)$. It follows readily that M must contain an element y of the form $y = vs$ where v belongs to V, s belongs to S, and s does not belong to A. Now s leaves an infinite number of x fixed. Let x_1 and x_2 be two of these. The permutation $s^{-1}(1, 2)$ belongs to $A = P \subset M$. So $ys^{-1}(1, 2) = vss^{-1}(1, 2) = v(1, 2)$ must belong to M by the fact that M is closed under multiplication. It has been shown that M must contain an element y_1, which has the cycle

$$c = \begin{pmatrix} x_1, & x_2 \\ a_1 x_2, & a_2 x_1 \end{pmatrix}$$

in its cyclic decomposition and which maps all other x into themselves with only a finite number of factors different from e. According to Theorem 5 the factors of any element of N may be taken arbitrarily in H except for one of them, so an element v_1 of M from N is chosen such that all the factors of $v(1, 2)v_1$ are e except those of x_1 and x_2. The element $v(1, 2)v_1$ may be transformed so that M must contain an element of the form

$$y = \begin{pmatrix} x_1, & x_2 \\ x_2, & a x_1 \end{pmatrix}.$$

Since $y^2 \in N$, a^2 belongs to G_1. The following can now be proved:

THEOREM 8. *Let $M = N \cup A(B, d)$ be a normal subgroup of $\Sigma(H; B, d, d)$ defined by the procedure of Theorem 7 and let L be the cyclic subgroup generated by*

$$y = \begin{pmatrix} x_1, & x_2 \\ x_2, & ax_1 \end{pmatrix}$$

where a^2 belongs to G_1. Then $M_1 = L \cup M$ is a normal subgroup of $\Sigma(H; B, d, d)$.

Proof. Since M is normal in Σ, it is sufficient to show that $(vs)y(vs)^{-1}$ belongs to M_1 for all vs of Σ because y^2 belongs to N. It is, therefore, sufficient to show that sys^{-1} and vyv^{-1} belong to M_1.

It will now be shown that, if s is any element of $S(B, d)$ and

$$y = \begin{pmatrix} x_1, & x_2 \\ x_2, & ax_1 \end{pmatrix},$$

then sys^{-1} belongs to M_1. One can write

$$s = \begin{pmatrix} x_i, & x_j, & \cdots \\ x_1, & x_2, & \cdots \end{pmatrix} \qquad\qquad \text{where } i \neq j.$$

Then

$$sys^{-1} = \begin{pmatrix} x_i, & x_j \\ x_j, & ax_i \end{pmatrix}.$$

The group $A(B, d)$ must contain an element of the form

$$s_1 = \begin{pmatrix} x_i, & x_j, & \cdots \\ x_1, & x_2, & \cdots \end{pmatrix}$$

since the mapping of the x's not shown may always be chosen in such a way that s_1 belongs to $A(B, d)$. Now consider

$$s_1 y s_1^{-1} = \begin{pmatrix} x_i, & x_j \\ x_j, & ax_i \end{pmatrix}.$$

The element $s_1 y s_1^{-1} = sys^{-1}$ is in $A \cup L \subset M_1$ for all s belonging to $S(B, d)$. This implies that sys^{-1} belongs to M_1.

It remains to show that vyv^{-1} belongs to M_1 for any v of V. Let $v = \{h_1, h_2, \cdots, h_\epsilon, \cdots \}$, where all but a finite number of the h are e. The commutator

$$y^{-1}vyv^{-1} = \{a^{-1}h_2ah_1^{-1}, h_1h_2^{-1}, e, \cdots, e, \cdots \}$$

belongs to $N \subset M \subset M_1$ if $a^{-1}h_2ah_1^{-1}h_1h_2^{-1} = a^{-1}h_2ah_2^{-1}$ belongs to G_1. But since H/G_1 is Abelian the desired result follows.

This section establishes:

THEOREM 9. *The normal subgroups of $\Sigma(H; B, d, d)$ are given by Theorems 4', 7, and 8.*

2. Normal subgroups of $\Sigma_A(H; B, d, d)$.

THEOREM 1. *The normal subgroups of $\Sigma_A(H; B, d, d)$ are precisely those normal subgroups of $\Sigma(H; B, d, d)$ which are contained in $\Sigma_A(H; B, d, d)$.*

Proof. Let N be a normal subgroup of $\Sigma(H; B, d, d)$ contained in $\Sigma_A(H; B, d, d)$. Then N is certainly normal in $\Sigma_A(H; B, d, d)$.

Conversely, let N be a normal subgroup of $\Sigma_A(H; B, d, d)$. If N is not normal in $\Sigma(H; B, d, d)$, then there exists an element y of N such that $(1, 2)y(1, 2)$ does not belong to N. The element $(1, 2)y(1, 2)$, like all elements of $\Sigma(H; B, d, d)$, maps an infinite number of x into themselves with e as a factor. Let x_α, x_β be two of these with the additional property that $1, 2, \alpha, \beta$ are distinct. Then the element

$$(\alpha, \beta)(1, 2)y(1, 2)(\alpha, \beta) = (1, 2)y(1, 2)$$

will not belong to N. But this contradicts the normality of N in $\Sigma_A(H; B, d, d)$ since $(\alpha, \beta)(1, 2)$ belongs to $A(B, d)$.

This establishes:

THEOREM 2. *The normal subgroups of $\Sigma_A(H; B, d, d)$ are* (i) *those described by Theorem 4' of §1, and* (ii) *the union of a group N, as described by Theorem 5 of §1, and $A(B, d)$.*

3. Normal subgroups of $\Sigma_{n,A}(H)$ for $n \geq 5$.

All of the normal subgroups of $\Sigma_{n,A}(H)$ will be found using the methods of §1.

3.1 Permutation invariant subgroups of $\Sigma_{n,A}(H)$ contained in the basis group. In almost the identical way that was used in §1 the following theorems can be established:

THEOREM 1. *Let N be a fixed permutation invariant subgroup of $\Sigma_{n,A}(H)$, for $n \geq 5$, contained in the basis group. Then the set G of H, consisting of all the factors that occur in a fixed ith position of all multiplications of N, forms a subgroup of H. This group is the same for all i. The set S_1 of all multiplications of N which have $h_i = e$, for $i > 1$, forms a normal subgroup of N. The set G_1, consisting of all first factors of multiplications of S_1, forms a normal subgroup of G. Assume $G_1 = e$. The set S_2 of elements of N that have $h_i = e$, for $i > 2$, forms a normal subgroup of N. The set G_2 of first factors of elements of S_2 forms a normal Abelian subgroup of G. The elements of S_2 are of the form*

$$v = \left\{ g_2, g_2^{-1}, e, \cdots, e \right\}$$

where g_2 runs through G_2.

THEOREM 2. *The group R, generated by the substitutions obtained by all possible permutations of elements of S_2, consists of elements of the form*

$$v = \left\{ r_1, r_2, \cdots, (r_1 \cdots r_{n-1})^{-1} \right\}$$

where the r_i run through the Abelian group G_2 independently.

The nature of the permutation invariant subgroups of $\Sigma_{n,A}(H)$ contained in the basis group is the same as that of those of $\Sigma_n(H)$ found by Ore [1].

Let $v = \{h_1, \cdots, h_n\}$ be an arbitrary element of N. Let $s_i = (1, i, j)$ where i runs over the set $2, \cdots, n$ and $j \neq 1, j \neq i$. The element

$$v_i = s_i v s_i^{-1} = \{h_i, \cdots, h_{i-1}, h_j, h_{i+1}, \cdots, h_{j-1}, h_1, h_{j+1}, \cdots, h_n\}$$

must belong to N. The element

$$v_2 = v_1 v_i^{-1} = \{h_1 h_i^{-1}, \cdots, e, h_i h_j^{-1}, e, \cdots, e, h_j h_1^{-1}, e, \cdots, e\}$$

must also belong to N. Since $n \geq 5$ there exist natural numbers k, m such that each is different from $1, i, j$. Then N must contain

$$v_3 = (1, k, m) v_2 (1, m, k)$$

$$= \{e, \cdots, e, h_1 h_i^{-1}, e, \cdots, e, h_i h_j^{-1}, e, \cdots, e, h_j h_1^{-1}, e, \cdots\}.$$

The element $v_2 v_3^{-1} = \{h_1 h_i^{-1}, e, \cdots, e, (h_1 h_i^{-1})^{-1}, e, \cdots\}$ also belongs to N. This shows $h_1 h_i^{-1}$ belongs to G_2, for $i = 2, \cdots, n$. Therefore, the elements of N are of the form

$$v = \{r_1 g, r_2 g, \cdots, r_n g\}$$

where the r_i are elements of G_2 and g runs through G. In exactly the same way as that used by Ore [1, p. 31] the following theorem can be proved.

THEOREM 3. *All permutation invariant subgroups N of $\Sigma_{n,A}(H)$, for $n \geq 5$, that are contained in the basis group may be obtained by the following construction. A subgroup G of H is chosen. In G two normal subgroups $G_1 \subset G_2$ are selected such that the quotient group G_2/G_1 is Abelian. Then let N consist of elements of the form*

$$v = \{k_1, k_2, \cdots, k_n\}$$

where the k_i runs through G subject to the conditions

(1) $k_i \equiv r_i k, \; i = 1, \cdots, n-1,$

(2) $k_n \equiv (r_1 \cdots r_{n-1})^{-1}(k\theta),$ $\mod G_1,$

where the r_i are arbitrary elements of G_2. Furthermore, θ is an endomorphism of G/G_1 multiplying each element of G/G_1 by an element of G_2/G_1. In particular, $(g)\theta \equiv g^{-(n-1)} \mod G_1$ for any element of G_2.

3.2 Normal subgroups of $\Sigma_{n,A}(H)$ contained in the basis group. It is now necessary to find those normal subgroups of the basis group which are permutation invariant. The same notation as in the previous section will be used. Let N be a normal subgroup of Σ contained in V_n. The groups $G_1 \subset G_2 \subset G$ are normal in H. Since N is permutation invariant, an element v of N must

have the form described in Theorem 3. For convenience assume $G_1 = e$. Then any factor of an element of N is uniquely determined by the other factors. Let $v_1 = \{h, e, \cdots, e\}$ be an element of the basis group that has only one nonidentity factor occurring in the first position. The element h is arbitrary in H. Let $v = \{r_1g, r_2g, \cdots, (r_1 \cdots r_{n-1})^{-1}(g\theta)\}$ be any element of N. Then

$$v_1 v v_1^{-1} = \{hr_1gh^{-1}, r_2g, \cdots, (r_1 \cdots r_{n-1})^{-1}(g\theta)\}.$$

This shows that $hr_1gh^{-1} = r_1g$. Therefore, r_1g belongs to the center of H. The following has been shown:

THEOREM 4. *Let N be a normal subgroup of $\Sigma_{n,A}(H)$, for $n \geqq 5$, contained in the basis group. Then N is permutation invariant and must meet the requirements of Theorem 3. Assume $G_1 = e$. The groups $G_2 \subset G$ are normal in H and G belongs to the center of H. Conversely, if N is as given by Theorem 3 and the additional requirements that $G_1 = e$, $G_2 \subset G$ are normal in H and G belongs to the center of H are met, then N is normal in $\Sigma_{n,A}(H)$, for $n \geqq 5$.*

THEOREM 4'. *For $n \geqq 5$ the normal subgroups of $\Sigma_{n,A}(H)$ are obtained by the construction of Theorem 3 with the additional conditions*
  (1) $G_1 \subset G_2 \subset G$ *are normal in H,*
  (2) G/G_1 *belongs to the center of H/G_1.*

3.3 **Other normal subgroups of $\Sigma_{n,A}(H)$.** By the method used to prove Theorem 5 of §1.3 the following can be proved:

THEOREM 5. *Let M be a normal subgroup of $\Sigma_{n,A}(H)$ not contained in the basis group. The multiplications $N = M \cap V$ form a normal subgroup of $\Sigma_{n,A}(H)$ in which $H = G$; i.e., the factors in any fixed position run through the whole group H and the quotient group H/G_1 for N is an Abelian group.*

Let P be the subgroup of M consisting of permutations only; $P = M \cap A$. Since M is normal in Σ, it follows that P is normal in A. Hence, $P = A_n$ or P is the identity. That $P = A_n$ is now proved.

THEOREM 6. *Every normal subgroup M of $\Sigma_{n,A}(H)$ not contained in the basis group contains permutations.*

Proof. Since M is not contained in the basis group, there exists an element $y = vs$ where $s \neq I$. It is convenient to consider several cases.

CASE 1. If y contains a cycle c in its cyclic decomposition of length $n \geqq 4$, it is seen in Chapter I that y is conjugate to an element y' containing a cycle

$$c' = \begin{pmatrix} x_1, & x_2, & x_3, & \cdots, & x_n \\ x_2, & x_3, & x_4, & \cdots, & ax_1 \end{pmatrix}.$$

Since M is normal in Σ, M must contain y'. Let $s = (1, 2, 3)$ and M must contain $(y')^{-1}sy's^{-1} = (1, 3, 4)$.

CASE 2. If y contains a cycle c of length 3 and some other cycle of length greater than 1, then M contains an element y' conjugate to y of the form

$$y' = \begin{pmatrix} x_1, & x_2, & x_3 \\ x_2, & x_3, & ax_1 \end{pmatrix} \begin{pmatrix} x_4, & \cdots \\ x_5, & \cdots \end{pmatrix} \cdots .$$

Let $s = (1, 4, 2)$ and M must contain $(y')^{-1}sy's^{-1} = (1, 2, 5, 3, \cdots)$ which reduces to Case 1.

It may happen that y contains one 3-cycle and the remainder are 1-cycles. This case will be discussed later.

CASE 3. If y contains only 1 or 2 cycles in its cyclic decomposition and at least four 2-cycles, then M must contain an element y' conjugate to y of the form

$$y' = \begin{pmatrix} x_1, & x_2 \\ x_2, & ax_1 \end{pmatrix} \begin{pmatrix} x_3, & x_4 \\ x_4, & bx_3 \end{pmatrix} \begin{pmatrix} x_5, & x_6 \\ x_6, & cx_5 \end{pmatrix} \begin{pmatrix} x_7, & x_8 \\ x_8, & dx_7 \end{pmatrix} \cdots .$$

Let $s = (1, 3)(5, 7)$. Then M must contain

$$(y')^{-1}s^{-1}y's = (1, 3)(2, 4)(5, 7)(6, 8).$$

CASE 4. It may now be assumed that y contains at the most two 2-cycles in its cyclic decomposition. M contains an element y' conjugate to y of the form

$$y' = \begin{pmatrix} x_1, & x_2 \\ x_2, & ax_1 \end{pmatrix} \begin{pmatrix} x_3, & x_4 \\ x_4, & bx_3 \end{pmatrix} \begin{pmatrix} x_5, & \cdots \\ cx_5, & \cdots \end{pmatrix} .$$

Let $s = (1, 3, 5)$. Then M must contain

$$(y')^{-1}sy's^{-1} = \begin{pmatrix} x_1, & x_2, & x_3, & x_4, & x_5 \\ x_5, & x_4, & x_1, & cx_3, & c^{-1}x_2 \end{pmatrix}$$

which reduces to Case 1.

CASE 5. If y has one 3-cycle and the remainder 1-cycles, then M must contain an element y' conjugate to y of the form

$$y' = \begin{pmatrix} x_1, & x_2, & x_3 \\ x_2, & x_3, & ax_1 \end{pmatrix} \begin{pmatrix} x_4, & x_5, & \cdots \\ bx_4, & cx_5, & \cdots \end{pmatrix} .$$

Let $s = (1, 2, 4)$. Then M must contain

$$(y')^{-1}sy's^{-1} = \begin{pmatrix} x_1, & x_2, & x_3, & x_4 \\ x_4, & x_3, & bx_2, & b^{-1}x_1 \end{pmatrix}$$

which reduces to Case 4.

This concludes the proof of Theorem 6.

THEOREM 7. *The elements of the group $N = M \cap V$, where M is a normal sub-*

group of $\Sigma_{n,A}(H)$ and is not contained in the basis group, are of the form

$$v = \{h_1, \cdots, h_n\}$$

where the h_i runs through H subject to the condition $h_1 \cdots h_n$ belongs to G_1. Here G_1 is a normal subgroup of H such that H/G_1 is Abelian.

Proof. By Theorem 6, M contains $s = (1, 2, 3)$. Let $v = \{h_1, \cdots, h_n\}$ be an arbitrary element of the basis group. Then M must contain the commutator $s^{-1}v^{-1}sv = \{h_3^{-1}h_1, h_1^{-1}h_2, h_2^{-1}h_3, e, \cdots, e\}$. Choose $h_2 = h_3$ and $h_1 = h_3 h$, where h is arbitrary in H. Then M contains $\{h, h^{-1}, e, \cdots, e\}$. This shows $G_2 = H$.

That $h_1 \cdots h_n$ belongs to G_1 follows from the nature of N is given by Theorem 4'.

The normal subgroups of $\Sigma_{n,A}(H)$ will now be constructed. The quotient group Σ/V is isomorphic to A. Furthermore, M/N is isomorphic to A.

THEOREM 8. *Let N be a normal subgroup of $\Sigma_{n,A}(H)$ contained in the basis group of the type described in Theorem 7. Then $M = N \cup A_n$ is a normal subgroup of $\Sigma_{n,A}(H)$. Conversely, if M is a normal subgroup of $\Sigma_{n,A}(H)$ not contained in the basis group, then $M = N \cup A_n$, where N is of the type described in Theorem 7.*

Proof. N is normal in Σ by assumption, and hence it is only necessary to show an element of A_n is transformed into an element of M. Furthermore, it is only necessary to show an element s of A_n is transformed into an element of M by a multiplication. Let $v = \{h_1, \cdots, h_n\}$ be any element of V_n. The commutator $vsv^{-1}s^{-1} = \{h_1 h_{i_1}^{-1}, \cdots, h_n h_{i_n}^{-1}\}$ belongs to V_n. Since H/G_1 is Abelian the product of the factors is in G_1, so the commutator belongs to $N \subset M$. Therefore, M is normal in Σ.

Conversely, if M is normal in Σ and is not contained in the basis group, then M contains A_n by Theorem 6. Now let $y = vs$ be any element of M. M contains $ys^{-1} = v$. This shows the multiplication part of any element of M is in N as given by Theorem 7. It follows that $M = N \cup A_n$.

CHAPTER IV. THE BASIS GROUP AS CHARACTERISTIC SUBGROUP

As has been the case in all previous discussions, no assumptions about the order of the group H are made.

THEOREM 1. *The basis group of $\Sigma(H; B, d, d)$ is a characteristic subgroup of the symmetry.*

Proof. Deny the theorem. Then there exists some automorphism θ such that $V\theta \not\subseteq V$. There also exists some normal subgroup M of Σ such that $M\theta = V$. Therefore, $V \not\subseteq V\theta^{-1} = M$.

The quotient group Σ/V is isomorphic to S. Furthermore, $\Sigma/M \cong S \cong \Sigma/V$.

Consider the two normal groups K and N of Σ given by $K = V \cup M$,

$N = V \cap M$. The quotient group K/M is a normal subgroup of Σ/M. Since $V \nsubseteq M$, the group K/M is not the identity. Since K/M is isomorphic to a normal subgroup of S, by the result of Baer [2], previously referred to, K/M is isomorphic to S or A. Therefore, in any case, K/M is non-Abelian. On the other hand, from the second isomorphism law, it follows that $K/M \cong V/N$. By Theorem 5 of §1, Chapter III, V/N is Abelian, a contradiction.

THEOREM 2. *The basis group of $\Sigma_A(H; B, d, d)$ is a characteristic subgroup of the symmetry.*

The proof of Theorem 2 is similar to that of Theorem 1.

THEOREM 3. *The basis group of $\Sigma_{n,A}(H)$, for $n \geq 5$, is a characteristic subgroup of the symmetry.*

The proof of Theorem 3 is similar to that of Theorem 1.

BIBLIOGRAPHY

1. O. Ore, *Theory of monomial groups*, Trans. Amer. Math. Soc. vol. 51 (1942) pp. 15–64.
2. R. Baer, *Die Kompositionsreihe der Gruppe aller Eineindeutigen Abbildungen einer Unendlichen Menge auf sich*, Studia Mathematica vol. 5 (1934) pp. 15–17.

UNIVERSITY OF KANSAS,
    LAWRENCE, KAN.