

A CLASS OF AFFINELY CONNECTED SPACES

BY

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It is known that the existence of a group G of motions of sufficiently high dimension in an n -dimensional Riemannian space imposes strong restrictions on the space [10; 15; 16]. This restriction is related to the fact that the isotropic subgroup, being a "high" dimensional subgroup of the orthogonal group $O(n)$, is very restricted [12]. Some results of this nature for affine connections have been known [4; 5; 6; 7; 8; 11; 13]; in particular, if an affinely connected space \mathfrak{A}_n of dimension n admits a group G of collineations with $\dim G \geq n^2 - n + 5$, then \mathfrak{A}_n is projectively flat. We know however that, unlike the Riemannian case, there are numerous non-equivalent, simply-connected, homogeneous⁽¹⁾ affinely connected spaces with the same local structure. For example, over the Euclidean space (in the topological sense), many flat and homogeneous affine connections can be defined such that any two of them are different in the global sense⁽²⁾. This tells us that the above class of spaces \mathfrak{A}_n needs some further clarifications.

In this paper, we give a more detailed study of the symmetric affinely connected spaces \mathfrak{A}_n having the above property. All the possible curvature tensors of \mathfrak{A}_n as well as the isotropic subgroup of G are determined (Theorem 2, §9). In case G is transitive, we exhibit all the simply-connected \mathfrak{A}_n 's (Theorem 3, §10). They fall into three individual cases and two classes depending on a non-negative constant. Among them, there are three affinely flat spaces, two of which are homeomorphic with the euclidean space while the other one homeomorphic with the product of a line and an $(n-1)$ -sphere, $n > 2$. As for the nonflat cases, they are all homeomorphic with the euclidean space. For the transitive case, all the isotropic subgroups given in Theorem 2 are realized; some of them can moreover be realized over nonequivalent affinely connected spaces.

1. Reducibility of certain linear groups. Let P_n denote the special linear group of n real variables x^1, \dots, x^n . In this section and the next, we shall determine all the closed subgroups of P_n with dimension not less than $n^2 - 2n + 4$. This information plays an important rôle in our further discussions. Let us first establish the following

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⁽¹⁾ An affinely connected space is called homogeneous if it admits a transitive group of collineations.

⁽²⁾ Let \mathfrak{B}_n be the entire n -space with coordinates x^1, x^2, \dots, x^n and $\mathfrak{B}_n(m)$ the subset of \mathfrak{B}_n consisting of all the points whose first m coordinates are positive. The affinely connected spaces $\{\mathfrak{B}_n(m); \Gamma^i_{jk}(x) = 0\}$ ($m=0, 1, 2, \dots, n$) are homogeneous, flat, and homeomorphic with one another. But no two of them are equivalent.

LEMMA 1. Let G be a proper subgroup of P_n . If $\dim G \geq n^2 - 2n + 4$, then G is reducible.

Proof. We shall prove it by using the method of contradiction. Suppose G to be irreducible. Let \mathcal{G} denote the same group G when the range of the variables x^1, \dots, x^n is extended to the field of complex numbers. Thus \mathcal{G} is a transformation group of the complex vector space of n -dimensions. Of course,

$$\dim \mathcal{G} = \dim G \geq n^2 - 2n + 4.$$

If \mathcal{G} is reducible, then as a direct consequence of Cartan's arguments [1, p. 155] we would have $\dim \mathcal{G} \leq (n/2)^2$ which is impossible. Thus \mathcal{G} is irreducible, and thus the complex form Γ of \mathcal{G} is also irreducible. We know that [1, p. 151] a complex irreducible linear group is either semi-simple or the direct product of a semi-simple group and the homothetic group defined by $y^i = \rho x^i$ where ρ is an arbitrary nonvanishing complex number. From our assumption $G \subset P_n$, no homothetic transformation can appear in Γ . Hence Γ is semi-simple. Here we find it convenient to divide our discussion into the following two cases:

CASE 1. Γ is simple. Since the complex dimension of Γ which is the same as $\dim G$ is not less than $n^2 - 2n + 4$, we know that Γ cannot be isomorphic with a complex linear group in less than n variables. Such simple linear groups have been determined by Cartan. Taking account of the fact $\dim \Gamma \geq n^2 - 2n + 4$, a survey of Cartan's table [3, pp. 147-148] tells us at once that $\dim \Gamma = n^2 - 1$ where $\dim \Gamma$ means the complex dimension. Hence $\dim G = n^2 - 1$. This contradicts our assumption that G is a proper subgroup of P_n .

CASE 2. Γ is not simple. From representation theory, Γ can be written as a Kronecker product $\Gamma_1 \times \Gamma_2$ of two nontrivial irreducible linear groups Γ_1 and Γ_2 . Let r_i and n_i denote, respectively, the complex dimension and degree^(*) of Γ_i ($i=1, 2$). Then

$$n_1 n_2 = n, \quad r_1 + r_2 = \dim \Gamma \geq n^2 - 2n + 4, \quad n_1 \geq 2, \quad n_2 \geq 2.$$

Without loss of generality, we can assume $r_1 \geq (n^2 - 2n + 4)/2$. Since $n_1 \leq n/2$, it follows that $r_1 \geq n_1^2 - 1$. A contradiction is thus obtained.

The contradiction in both alternatives proves our lemma.

2. Subgroups of the affine groups. Let H_n be the general linear group in n real variables. Then each element of H_n can be regarded as a nonsingular real matrix (a_{ij}) , and the special linear group P_n becomes the totality of matrices (a_{ij}) whose determinants $\det (a_{ij})$ are equal to one. For simplicity, we shall use the following notations throughout:

$$H_n^+ = \{(a_{ij}) : \det (a_{ij}) > 0\},$$

$$K = \{(a_{ij}) : a_{ij} = \lambda \delta_{ij}, \lambda = \text{positive number}\}^{(4)},$$

(*) By the degree of a linear group, we mean the dimension of the vector space on which it acts.

(4) δ_{ij} , δ_j^i denote the Kronecker deltas.

$$\begin{aligned}
L &= \{(a_{ij}): a_{11} = 1, a_{\alpha 1} = 0, \det(a_{ij}) = 1; \alpha = 2, 3, \dots, n\}, \\
L' &= \{(a_{ij}): a_{11} = 1, a_{1\alpha} = 0, \det(a_{ij}) = 1; \alpha = 2, 3, \dots, n\}, \\
M &= \{(a_{ij}): a_{11} > 0, a_{\alpha 1} = 0, \det(a_{ij}) = 1; \alpha = 2, 3, \dots, n\}, \\
M' &= \{(a_{ij}): a_{11} > 0, a_{1\alpha} = 0, \det(a_{ij}) = 1; \alpha = 2, 3, \dots, n\},
\end{aligned}$$

where, and in the following, the indices $a, b, c, \dots, i, j, k, \dots$ take the values in the range $1, 2, 3, \dots, n$.

We see at once that they are closed and connected subgroups of H_n^+ , and

$$\begin{aligned}
\dim H_n^+ &= n^2, & \dim K &= 1, \\
\dim L &= \dim L' = n^2 - n - 1, & \dim M &= \dim M' = n^2 - n.
\end{aligned}$$

LEMMA 2. *Let G be a closed and connected subgroup of M with $\dim G \geq n^2 - 2n + 4$. Then either $G = L$, or $G = M$.*

Proof. Let

$$P_{n-1} = \{(a_{ij}): a_{11} = 1, a_{1\alpha} = a_{\alpha 1} = 0, \det(a_{ij}) = 1; \alpha = 2, 3, \dots, n\}.$$

Since G, L , and P_{n-1} are all subgroups of M , we have

$$(2.1) \quad \dim(G \cap P_{n-1}) = \dim G + \dim P_{n-1} - \dim M \geq n^2 - 3n + 4,$$

$$(2.2) \quad \dim(L \cap P_{n-1}) = \dim L + \dim P_{n-1} - \dim M \geq n^2 - 3n + 3.$$

It is a well known result (due to S. Lie) that the projective group P_k has no proper subgroup with dimension higher than $k^2 - k$. Thus P_{n-1} cannot have proper subgroup with dimension higher than $n^2 - 3n + 2$. (2.1) then implies that $G \cap P_{n-1} = P_{n-1}$, or what is the same, $P_{n-1} \subset G$. Thus $P_{n-1} \subset G \cap L \subset L$. By using matrix multiplication we can easily verify that P_{n-1} is a maximal subgroup of L . It follows then that $G \cap L$ is either P_{n-1} or L . On account of (2.2), the first alternative cannot happen, and therefore, $G \cap L = L$, i.e., $L \subset G$. But we know that the difference between the dimensions of L and M is equal to one. It follows then that G is either L or M . Lemma 2 is proved.

THEOREM 1. *Let G be a closed and connected subgroup of P_n . If $\dim G \geq n^2 - 2n + 4$, then G is conjugate to one of the groups P_n, L, M, L', M' .*

Proof. If $G = P_n$, our theorem evidently holds. Now, assume $G \neq P_n$. Lemma 1 then tells us that G is reducible. In other words, G leaves invariant a linear subspace of m dimensions with $0 < m < n$. Thus $\dim G \leq n^2 - 1 - m(n - m)$. From the inequality $\dim G \geq n^2 - 2n + 4$, it follows that m is either 1 or $n - 1$. These two cases are dual to each other. Suppose that $m = 1$. Then G leaves invariant a line. Up to an inner automorphism of P_n , we can assume this invariant line to be

$$x^2 = x^3 = \dots = x^n = 0.$$

Thus G is subgroup of M . From Lemma 2, it follows that G is either L

or M . By a duality argument, we see that in case $m = n - 1$, G is either L' or M' . Theorem 1 is hereby proved.

COROLLARY. For each real number b , let us denote by $I(b)$ the totality of diagonal matrices of the form:

$$\begin{pmatrix} e^{(1+b)t} & 0 & 0 & 0 \dots 0 \\ 0 & e^{bt} & 0 & 0 \dots 0 \\ 0 & 0 & e^{bt} & 0 \dots 0 \\ \cdot & \cdot & \cdot & \cdot \dots \cdot \\ 0 & 0 & 0 & 0 \dots e^{bt} \end{pmatrix}$$

where t runs through all real numbers. Then each closed and connected subgroup G of H_n with $\dim G \geq n^2 - 2n + 5$ is conjugate to one of the groups: H_n^+ , P_n , $K \times M$, $K \times M'$, $K \times L$, $K \times L'$, $I(b) \times L$, $I(b) \times L'$, L , L' .

Proof. From the connectedness of G , we know $G \subset H_n^+$. Let $G^* = G \cap P_n$. Since $H_n^+ = K \times P_n$ and $\dim K = 1$, it follows that G^* is a normal subgroup of G and $\dim G^* \leq \dim G \leq \dim G^* + 1$. This tells us that $\dim G^* \geq n^2 - 2n + 4$. Then on account of Theorem 1, we can assume G^* to be one of the groups P_n , L , M , L' , M' . When $G^* = P_n$, G is evidently either P_n or H_n^+ . We shall only discuss the cases $G^* = L$ and $G^* = M$; the remaining two cases can be reduced to these two by a duality.

CASE 1. $G^* = M$. Since G^* is a normal subgroup of G , G is contained in the normalizer $K \times M$ of M . From the fact that $M \subset G \subset K \cap M$, $\dim K = 1$, we know that G can only be M or $K \times M$. But $M = I(-1/n) \times L$. Thus our corollary holds in this case.

CASE 2. $G^* = L$. The normalizer of L in H_n^+ is $K \times M$. Hence $L \subset G \subset K \times M$. Passing from these groups to their Lie algebras, we find by a short calculation that G is either L or $K \times L$ or $I(b) \times L$ for a certain b . This completes the proof of the corollary.

3. Groups of affine collineations. Let \mathfrak{A}_n be an n -dimensional space with symmetric affine connection $\Gamma_{jk}^i(x)$ covered by a system of coordinate neighborhoods (x^i) . Then the paths of this space are defined to be integral curves of the differential equations

$$(3.1) \quad \frac{d^2 x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} = 0,$$

s being the so-called affine parameter on each path. A point transformation

$$(3.2) \quad \bar{x}^i = f^i(x)$$

is called an affine collineation when it carries any path into a path of the space and preserves the affine character of the parameter s . A necessary and suffi-

cient condition that (3.2) be an affine collineation is that

$$(3.3) \quad \Gamma_{jk}^i(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^a} \left(\frac{\partial x^b}{\partial \bar{x}^j} \frac{\partial x^c}{\partial \bar{x}^k} \Gamma_{bc}^a(x) + \frac{\partial^2 x^a}{\partial \bar{x}^j \partial \bar{x}^k} \right).$$

If (3.2) is an infinitesimal transformation

$$(3.4) \quad \bar{x}^i = x^i + \xi^i(x)dt,$$

$\xi^i(x)$ being a vector field and dt an infinitesimal, then (3.3) gives

$$(3.5) \quad X\Gamma_{jk}^i \equiv \frac{\partial^2 \xi^i}{\partial x^j \partial x^k} + \xi^a \frac{\partial \Gamma_{jk}^i}{\partial x^a} - \frac{\partial \xi^i}{\partial x^a} \Gamma_{jk}^a + \frac{\partial \xi^a}{\partial x^j} \Gamma_{ak}^i + \frac{\partial \xi^a}{\partial x^k} \Gamma_{ja}^i = 0$$

where X is the operator of Lie derivation with respect to (3.4). Equation (3.5) may be written also in tensor form:

$$(3.6) \quad X\Gamma_{jk}^i \equiv \xi^i_{;jk} + R^i_{jkl}\xi^l = 0,$$

where the semi-colon followed by an index denotes the covariant differentiation with respect to Γ^i_{jk} and R^i_{jkl} the curvature tensor formed with Γ^i_{jk} .

The integrability conditions of system (3.6) of partial differential equations, or those of

$$(3.7) \quad \xi^i_{;jk} = -R^i_{jkl}\xi^l$$

are given by a sequence of equations

$$(3.8)_1 \quad XR^i_{jkl} \equiv \xi^a R^i_{jkl;a} - \xi^i_{;a} R^a_{jkl} + \xi^a_{;j} R^i_{akl} + \xi^a_{;k} R^i_{jal} + \xi^a_{;l} R^i_{jka} = 0,$$

$$(3.8)_2 \quad \begin{aligned} XR^i_{jkl;m} &\equiv \xi^a R^i_{jkl;m;a} - \xi^i_{;a} R^a_{jkl;m} + \xi^a_{;j} R^i_{akl;m} + \xi^a_{;k} R^i_{jal;m} \\ &\quad + \xi^a_{;l} R^i_{jka;m} + \xi^a_{;m} R^i_{jkl;a} = 0, \end{aligned}$$

.

Thus, in order that system (3.7) of partial differential equations be integrable, it is necessary and sufficient that there exist a positive integer N such that equations (3.8)₁, (3.8)₂, . . . , (3.8)_{N+1} are automatically satisfied by ξ^i and $\xi^i_{;j}$ satisfying equations (3.8)₁, (3.8)₂, . . . , (3.8)_N.

In this case, if N equations (3.8)₁, (3.8)₂, . . . , (3.8)_N give just r linearly independent equations with respect to ξ^i and $\xi^i_{;j}$, then the space admits a group of affine collineations with dimension $n^2 + n - r$.

Conversely, if the space admits a group of affine collineations with dimension $n^2 + n - s$, then there exist s linearly independent relations between ξ^i and $\xi^i_{;j}$ and the integrability conditions (3.8)₁, (3.8)₂, . . . should be automatically satisfied by ξ^i and $\xi^i_{;j}$ satisfying these s relations [9; 15].

Now, when the space admits a group G of affine collineations with dimension r , if we take a point P in the space and consider all the transformations of the group which fix this point P , then such transformations form a sub-

group G_P , called the isotropic subgroup at P . This subgroup G_P consists of transformations

$$T: \bar{x}^i = f^i(x; \alpha)$$

such that

$$x_0^i = f^i(x_0; \alpha)$$

where x_0^i are the coordinates of the points P , and α denotes the parameters.

To each transformation T_α in G_P , there corresponds a linear transformation

$$\tau(T_\alpha): d\bar{x}^i = \frac{\partial f^i(x_0; \alpha)}{\partial x^j} dx^j$$

of the tangent space \mathcal{L}_P at the point P . It can be easily proved that this linear representation τ of G_P is an isomorphism in the sense of topological groups [16].

Now consider the matrix (ξ_a^i) ($a=1, 2, \dots, r$) of the components of a basis of the infinitesimal group of G , and denote by q the generic rank of this matrix. A point is called an ordinary point if, at this point, the matrix assumes the maximum rank q , and is called a singular point if otherwise.

Let \mathfrak{A}_n be an n -dimensional space with a symmetric affine connection admitting a group G of affine collineations of dimension greater than or equal to $n^2 - n + 5$. We confine ourselves to an open domain containing only ordinary points. Let G_P denote the isotropic subgroup at P . Then evidently $\dim G_P = \dim \tau(G_P) \geq n^2 - 2n + 5$. Thus, by corollary to Theorem 1, the identity component A_P of $\tau(G_P)$ should be conjugate to one of the groups H_n^+ , P , $K \times M$, $K \times M'$, $K \times L$, $K \times L'$, $I(b) \times L$, $I(b) \times L'$, L , L' .

4. The case by which A_P is conjugate to H_n^+ or P_n . In these two cases, the group G is transitive. Because if G is not transitive, there would be an invariant subvariety passing through P , and consequently, A_P would leave invariant a proper linear subspace of the tangent space \mathcal{L}_P at the point P which is impossible.

1. CASE $A_P = H_n^+$. In this case, G is of dimension $n^2 + n$. Thus the integrability conditions (3.8)₁ should be satisfied identically by any ξ^i and $\xi^i_{;j}$. Thus by writing (3.8)₁ in the form

$$(4.1) \quad \xi^a R^i_{jkl;a} - \xi^b_{;a} (\delta^i_b R^a_{jkl} - \delta^a_j R^i_{bkl} - \delta^a_k R^i_{jbl} - \delta^a_l R^i_{jkb}) = 0^{(4)},$$

we obtain

$$(4.2) \quad R^i_{jkl;a} = 0$$

and

$$(4.3) \quad \delta^i_b R^a_{jkl} - \delta^a_j R^i_{bkl} - \delta^a_k R^i_{jbl} - \delta^a_l R^i_{jkb} = 0.$$

From (4.3), we get, by contraction with respect to a and b ,

$$(4.4) \quad R^i_{jkl} = 0.$$

It is evident that if $R^i_{jkl}=0$, then conditions (3.8)₁, (3.8)₂, \dots are automatically satisfied. Thus, in order that $A_P = H_n^+$, it is necessary and sufficient that the space is flat.

2. CASE $A_P = P_n$. In this case, the group A_P being of dimension $n^2 - 1$ and the group G being transitive, we know that $\dim G = n^2 + n - 1$. Since $A_P = P_n$, $\xi^i_{;j}$ should satisfy

$$(4.5) \quad \xi^a_{;a} = 0$$

and the integrability conditions (3.8) should be satisfied identically by any ξ^i and $\xi^i_{;j}$ satisfying (4.5).

Thus comparing (3.8)₁ and (4.5), we see that there should exist functions F^i_{jkl} such that

$$\xi^a R^i_{jkl;a} - \xi^i_{;a} R^a_{jkl} + \xi^a_{;j} R^i_{akl} + \xi^a_{;k} R^i_{jal} + \xi^a_{;l} R^i_{jka} = -\xi^a_{;a} F^i_{jkl}$$

become identities in ξ^i and $\xi^i_{;j}$. Thus we must have

$$(4.6) \quad R^i_{jkl;a} = 0$$

and

$$(4.7) \quad \delta^i_b R^a_{jkl} - \delta^a_j R^i_{bkl} - \delta^a_k R^i_{jbl} - \delta^a_l R^i_{jkb} = \delta^a_b F^i_{jkl}.$$

By contraction with respect to a and b , we find from (4.7)

$$F^i_{jkl} = -(2/n) R^i_{jkl},$$

and by contraction with respect to i and b , we get

$$(4.8) \quad n R^a_{jkl} - \delta^a_l R_{jk} + \delta^a_k R_{jl} + \delta^a_j (R_{kl} - R_{lk}) = -(2/n) R^a_{jkl},$$

where $R_{jk} = R^a_{jka}$. Contracting again with respect to a and l , we find $R_{jk} = 0$ for $n > 2$. Thus we have, from (4.8), $R^i_{jkl} = 0$.

5. **The case in which A_P is conjugate to $K \times M$, $K \times M'$, $K \times L$ or $K \times L'$.** In these cases, the group G is transitive. We shall prove this by method of contradiction.

We first suppose that $A_P = K \times M$ or $K \times L$ and that the group G is intransitive. Then the invariant subvariety passing through P should be one-dimensional, because the linear manifold tangent to this subvariety at P is left invariant by $K \times M$ or $K \times L$ which fixes one and only one direction. Thus the rank of the matrix (ξ^i_a) is equal to 1 at P and, consequently, is equal to 1 at every point of the domain under consideration. It follows that through every point of this domain there passes one and only one invariant curve.

Now take an invariant curve passing through a point Q which is not on the invariant curve passing through P and which is in the domain under

consideration, and consider all the paths joining P to the points on the invariant curve passing through Q . These paths constitute a two-dimensional surface. This surface is left invariant by the isotropic subgroup G_P . Consequently, the corresponding linear group A_P must fix the two-dimensional plane tangent to this surface at P which contradicts our assumption.

We next suppose that $A_P = K \times M'$ or $K \times L'$ and that the group G is intransitive. The invariant subvariety passing through P should be $(n-1)$ -dimensional, because the linear manifold tangent to this subvariety at P is left invariant by $K \times M'$ or $K \times L'$ which fixes one and only one hyperplane. Thus the rank of the matrix (ξ_a^i) is equal to $n-1$ at P and, consequently, is equal to $n-1$ at every point of the domain under consideration. It follows that through every point of the domain there passes one and only one invariant hypersurface.

Now, consider a path through P which intersects these invariant hypersurfaces; then the points of intersections can be transformed by K into one another (except the point P , of course), which is a contradiction.

Thus, in these cases, the group G is transitive, and consequently two isotropic groups at any two ordinary points in the domain under consideration are conjugate to each other.

The groups $K \times M$, $K \times M'$, $K \times L$, $K \times L'$ being respectively with dimension $n^2 - n + 1$, $n^2 - n + 1$, $n^2 - n$, $n^2 - n$, and the group G being transitive, the group G is respectively with dimension $n^2 + 1$, $n^2 + 1$, n^2 , n^2 .

Now, at the point P of the domain, we choose the normal coordinates x^i whose origin is P , then the space admits a one-parameter group of affine collineations

$$(5.1) \quad \bar{x}^i = e^t x^i.$$

In this coordinate system, the vector ξ^i defining the infinitesimal transformation of this one-parameter group is given by

$$(5.2) \quad \xi^i = x^i.$$

Thus, the integrability condition $(3.8)_1$ becomes

$$x^a \frac{\partial R^i_{jkl}}{\partial x^a} + 2R^i_{jkl} = 0$$

which shows that R^i_{jkl} are homogeneous functions of degree -2 of x^i .

But we know that the components R^i_{jkl} of the curvature tensor are well defined at the origin of the normal coordinates system. Thus the components R^i_{jkl} must vanish at P and consequently at any point of the domain. Thus, in these cases, the space is affinely flat.

6. The case in which A_P is conjugate to $I(b) \times L$ or L . In these cases, the group G is transitive. This can be proved by the same argument as that used at the beginning of §5.

The group G being transitive, the isotropic groups at any two points of the domain under consideration are conjugate to each other. On the other hand, the isotropic group G_Q at an ordinary point Q fixes one and only one direction which we denote by u_Q . Thus, at every point Q of the domain under consideration, there is associated a direction u_Q .

Consider a path which passes through a point Q and tangent to u_Q ; then the isotropic group G_Q , being an affine collineation, fixes this path. We take a point R different from Q on this path and consider the transformations of G_Q which fix this point R . These transformations form the group L .

Now, we consider an affine frame at Q whose first axis is in the direction u_Q and transport it parallelly along the path to the point R . Then we have at R an affine frame whose first axis is tangent to the path. The parallelism of vectors along a curve being preserved by an affine collineation, the transformation of G_Q fixing the point R gives the same effect on the affine frame at R as on that at Q . This shows that the subgroup of G_Q fixing R coincides with the subgroup of G_R fixing Q . The subgroup of G_R fixing Q fixes the tangent to the path and u_R , and consequently the tangent must coincide with u_R , which shows that the path is the trajectory of the field of directions u .

Now, the isotropic groups $I(b) \times L$ and L being respectively with dimension $n^2 - n$ and $n^2 - n - 1$, and the group G being transitive, the group G is respectively with dimension n^2 and $n^2 - 1$.

Now, the group G of affine collineations being transitive, we denote by T a transformation of G which carries a point Q into a point R . Then, by the same method as in [16], we can prove that

$$Tu_Q = u_R$$

and that u_Q is a parallel vector field.

If we denote this vector field by $u^i(x)$, then we have

$$(6.1) \quad Xu^i = \alpha u^i,$$

$$(6.2) \quad u^i{}_{;k} = u^i \lambda_k,$$

where α is a certain scalar and λ_k a certain covariant vector field. From (6.2), we find

$$(6.3) \quad u^i R^i{}_{jkl} = u^i \lambda_{kl}$$

where

$$(6.4) \quad \lambda_{kl} = \lambda_{k;i} - \lambda_{i;k}.$$

We first suppose that $A_P = I(b) \times L$. Then equations (3.8)₁ should be satisfied by any ξ^i and $\xi^i{}_{;j}$ satisfying

$$(6.5) \quad (1 + nb)Xu^i = (1 + b)\xi^a{}_{;a}u^i.$$

We see that conditions

$$(6.6) \quad Xu^i \equiv \xi^a u^i_{;a} - \xi^i_{;a} u^a = 0$$

and

$$(6.7) \quad \xi^a_{;a} = 0$$

put together are stronger than (6.5). Hence any ξ^i and $\xi^i_{;j}$ satisfying (6.6) and (6.7) must satisfy (6.5) and hence satisfy (3.8)₁.

The group being that of affine collineations, the covariant differentiation and the Lie derivation are commutative and consequently, from (6.2) and (6.6), we find $X\lambda_k = 0$. But the group G_P does not fix a hyperplane and consequently we should have $\lambda_k = 0$. Consequently we have

$$(6.8) \quad u^i_{;k} = 0 \quad \text{and} \quad u^i R^i_{jkl} = 0.$$

Thus the integrability conditions (3.8)₁ should be satisfied by any ξ^i and $\xi^i_{;j}$ satisfying

$$(6.9) \quad \xi^i_{;a} u^a = 0 \quad \text{and} \quad \xi^a_{;a} = 0$$

and consequently there must exist functions F^i_{jkl} and G^i_{jklb} such that

$$(6.10) \quad R^i_{jkl;a} = 0$$

and

$$(6.11) \quad \delta_b^i R^a_{jkl} - \delta_j^a R^i_{bkl} - \delta_k^a R^i_{jbl} - \delta_l^a R^i_{jkb} = \delta_b^a F^i_{jkl} + u^a G^i_{jklb}.$$

From (6.11), after some calculation, we can deduce $R^i_{jkl} = 0$.

The case $A_P = L$ is characterized by (6.6) and (6.7) and consequently the above discussion shows that when $A_P = L$ the space is also affinely flat.

7. The case in which A_P is conjugate to $I(b) \times L'$ or L' and G is transitive. The group G being transitive, two isotropic groups at any two ordinary points in the domain under consideration are conjugate to one another.

On the other hand, the isotropic group G_Q at an ordinary point Q fixes one and only one hyperplane which we denote by v_Q . Thus with every point Q of the domain under consideration, there is associated a hyperplane v_Q .

The isotropic groups $I(b) \times L'$ and L' being respectively with dimension $n^2 - n$ and $n^2 - n - 1$ and the group G being transitive, the group G is respectively with dimension n^2 and $n^2 - 1$.

By exactly the same method as in [16], we can prove that

$$Tv_Q = v_R,$$

where T is an arbitrary transformation carrying a point Q into a point R .

Furthermore, if we represent this hyperplane by a covariant vector $v_j(x)$, then we can prove that

$$(7.1) \quad Xv_j = \beta v_j,$$

$$(7.2) \quad v_{j;k} = \alpha v_j v_k,$$

where α is a certain scalar. From (7.2), we find

$$(7.3) \quad -v_i R^i_{jkl} = v_j \alpha_{kl},$$

where

$$(7.4) \quad \alpha_{kl} = v_k \alpha_{;l} - v_l \alpha_{;k}.$$

We first suppose that $A_P = I(b) \times L'$. Then equations (3.8)₁ should be satisfied by any ξ^i and $\xi^i_{;j}$ satisfying

$$(7.5) \quad (1 + nb)Xv_j = (1 + b)\xi^a_{;a}v_j.$$

We see that conditions

$$(7.6) \quad Xv_j = \xi^a \alpha v_{;j} - \xi^a_{;j} v_a = 0,$$

$$(7.7) \quad \xi^a_{;a} = 0$$

put together are stronger than (7.5). Hence any ξ^i and $\xi^i_{;j}$ satisfying (7.6) and (7.7) must satisfy (7.5) and hence satisfy (3.8)₁.

The group being that of affine collineations, the covariant differentiation and the Lie derivation are commutative, and consequently, from (7.2) and (7.6) we find $X\alpha = 0$, which shows, the group G being transitive, that α is a constant.

Thus the integrability conditions (3.8)₁ should be satisfied by any ξ^i and $\xi^i_{;j}$ satisfying (7.6) and (7.7) and consequently there must exist functions F^i_{jkl} and $G^i_{jkl}{}^a$ such that

$$(7.8) \quad R^i_{jkl;a} = -\alpha G^i_{jkl}{}^a v_a$$

and

$$(7.9) \quad \delta^i_b R^a_{jkl} - \delta^a_j R^i_{bkl} - \delta^a_k R^i_{jbl} - \delta^a_l R^i_{jkb} = \delta^a_b F^i_{jkl} + v_b G^i_{jkl}{}^a.$$

From (7.9), after some calculation, we can conclude that

$$(7.10) \quad R^i_{jkl} = kv_j(v_k \delta^i_l - v_l \delta^i_k),$$

where k is a constant. Thus equations (3.8)₁ become

$$XR^i_{jkl} = \beta R^i_{jkl},$$

where β is given by $Xv_j = \beta v_j$.

When $1+b \neq 0$ there exists X such that $\beta \neq 0$ and thus we have $R^i_{jkl} = 0$. When $1+b = 0$ then $Xv_j = 0$ and thus (3.8)₁ is really satisfied by all the infinitesimal transformations X of the group G .

8. The case in which A_P is conjugate to $I(b) \times L'$ or L' and G is intransitive. Let us consider the invariant variety through P . All the points on this invariant variety being equivalent under the group G , isotropic groups at points of this invariant variety are conjugate to each other. Thus the invariant variety should be $(n-1)$ -dimensional, because the hyperplane tan-

gent to this invariant variety at a point should be left invariant by the isotropic group $I(b) \times L'$ or L' at this point which fixes one and only one hyperplane.

Take a point Q not on this invariant variety. If the isotropic group at Q is one of the groups hitherto examined except $I(b) \times L'$ and L' , then the group G should be transitive. Thus the isotropic group at Q should be also $I(b) \times L'$ or L' .

Consequently, passing through every ordinary point on the domain under consideration, there exists an $(n-1)$ -dimensional invariant variety whose tangent hyperplane is fixed by the isotropic group at the point of contact. We denote this hyperplane at Q by v_Q .

The isotropic group $I(b) \times L'$ and L' being respectively with dimension n^2-n and n^2-n-1 , and the invariant varieties being $(n-1)$ -dimensional, the group G is respectively with dimension n^2-1 and n^2-2 .

Thus if we denote by

$$(8.1) \quad f(x) = \text{constant}$$

the family of invariant varieties and put

$$(8.2) \quad \lambda v_j = \partial f / \partial x^j,$$

then, using the so-called adopted frames, we can prove that

$$(8.3) \quad (\lambda v_j)_{;k} = (\lambda v_j) p_k + (\lambda v_k) p_j,$$

p_k being a certain covariant vector.

On the other hand, we know that

$$Xf = 0, \quad X(\lambda v_j) = 0, \quad X(\lambda v_j)_{;k} = 0$$

and consequently, from (8.3), we find

$$Xp_k = 0.$$

But the hyperplane represented by v_j is the only one hyperplane fixed by the isotropic group and consequently, we should have

$$p_k = (1/2)\alpha v_k$$

where α is a certain function of f .

Thus substituting this into (8.3), we get

$$(8.4) \quad (\lambda v_j)_{;k} = \alpha (\lambda v_j) (\lambda v_k)$$

from which

$$(8.5) \quad v_i R^i_{jkl} = 0,$$

We first suppose that $A_P = I(b) \times L'$. Then equations (3.8)₁ should be satisfied by any ξ^i and $\xi^i_{;j}$ satisfying

$$(8.6) \quad (1 + nb)Xv_j = (1 + b)\xi^a_{;a}v_j.$$

We see that conditions

$$(8.7) \quad Xf \equiv \lambda v_a \xi^a = 0,$$

$$(8.8) \quad Xv_j \equiv \xi^a v_{j;a} + \xi^a_{;j} v_a = 0,$$

$$(8.9) \quad \xi^a_{;a} = 0$$

put together are stronger than (8.6). Hence any ξ^i and $\xi^i_{;j}$ satisfying (8.7), (8.8), and (8.9) must satisfy (8.6) and hence satisfy (3.8)₁.

Equations $X(\lambda v_j) = 0$ and $Xv_j = 0$ show that $X\lambda = 0$ and consequently that λ is a function of f . Thus, from (8.2), we can see that we can suppose $\lambda = 1$. Thus equation (8.8) can be written as

$$(8.10) \quad Xv_j \equiv \xi^a_{;j} v_a = 0$$

by virtue of (8.4) and (8.7).

Thus the integrability conditions (3.8)₁ should be satisfied by any ξ^i and $\xi^i_{;j}$ satisfying (8.7), (8.9), and (8.10) and consequently there must exist functions E^i_{jkl} , F^i_{jkl} and G^i_{jkl} such that

$$(8.11) \quad R^i_{jkl;a} = E^i_{jkl} v_a$$

$$(8.12) \quad \delta^i_b R^a_{jkl} - \delta^a_j R^i_{bkl} - \delta^a_k R^i_{jbl} - \delta^a_l R^i_{jkb} = \delta^a_b F^i_{jkl} + G^i_{jkl} v_b.$$

From (8.12) we can conclude that the curvature tensor R^i_{jkl} should be of the form

$$(8.13) \quad R^i_{jkl} = kv_j(v_k \delta^i_l - v_l \delta^i_k).$$

But since we have $XR^i_{jkl} = 0$, $Xv_j = 0$, we find from this $Xk = 0$, which shows that k is a certain function of f .

Thus equations (3.8)₁ become

$$XR^i_{jkl} = \beta R^i_{jkl},$$

where β is given by $Xv_j = \beta v_j$. When $1 + b \neq 0$, there exists X such that $\beta \neq 0$ and thus $R^i_{jkl} = 0$. When $1 + b = 0$, then $Xv_j = 0$ and thus (3.8)₁ is really satisfied by all the infinitesimal transformations X of the group G .

The case $A_P = L'$ is characterized by (8.7), (8.8), and (8.9) also and consequently the above discussion shows that when $A_P = L'$, the space has also the curvature tensor of the form (8.13).

9. Theorems. Gathering all the results in §§3–8, we have

THEOREM 2. *If an n -dimensional space with a symmetric affine connection admits a group of affine collineations with dimension greater than $n^2 - n + 5$, then the isotropic group G_P at a point P , the dimension of G_P , the groups of affine collineations G , the dimension of G , and the structure of the space should be one of those on the opposite page:*

isotropic group G_P	dimension of G_P	group of affine col- lineations G	dimension of G	structure of the space
H_n	n^2	transitive	n^2+n	affinely flat
H_n^+	n^2	"	n^2+n	"
P_n	n^2-1	"	n^2+n-1	"
$K \times M$	n^2-n+1	"	n^2+1	"
$K \times M'$	n^2-n+1	"	n^2+1	"
$K \times L$	n^2-n	"	n^2	"
$K \times L'$	n^2-n	"	n^2	"
$I(b) \times L$	n^2-n	"	n^2	"
L	n^2-n-1	"	n^2-1	"
$I(b) \times L'$	n^2-n	"	n^2	(i) $1+b \neq 0$, $R^i_{jkl} = 0$. (ii) $1+b = 0$, $v_{j;ik} = \alpha v_j v_k$, $R^i_{jkl} = k v_j (v_k \delta_l^i - v_l \delta_k^i)$, α, k : constants.
		intransitive	n^2-1	(i) $1+b \neq 0$, $R^i_{jkl} = 0$. (ii) $1+b = 0$, $v_{j;ik} = \alpha v_j v_k$, $\left(v_j = \frac{\partial f}{\partial x^j} \right)$, $R^i_{jkl} = k v_j (v_k \delta_l^i - v_l \delta_k^i)$, α, k : functions of f .
L'	n^2-n-1	transitive	n^2-1	$v_{j;ik} = \alpha v_j v_k$, $R^i_{jkl} = k v_j (v_k \delta_l^i - v_l \delta_k^i)$, α, k : constants.
		intransitive	n^2-2	$v_{j;ik} = \alpha v_j v_k$, $\left(v_j = \frac{\partial f}{\partial x^j} \right)$, $R^i_{jkl} = k v_j (v_k \delta_l^i - v_l \delta_k^i)$, α, k : functions of f .

10. Determination of \mathfrak{A}_n when G is transitive.

THEOREM 3. *Let \mathfrak{A}_n be a simply-connected, n -dimensional manifold with a symmetric affine connection. Suppose that \mathfrak{A}_n admits a group G of affine collineations with $\dim G \geq n^2 - n + 5$. Let \mathfrak{B}_n denote the entire coordinate space with coordinates x^1, x^2, \dots, x^n , and let i, j, k, \dots be indices running from 1 to n while α, β indices running from 2 to n . Then \mathfrak{A}_n is equivalent to one of those on the following page. Moreover, the affinely connected spaces listed above are nonequivalent.*

Before proving the theorem, we shall firstly give three remarks and establish a lemma.

REMARK 1. Let G^* be the group of all affine collineations of \mathfrak{A}_n . Then with respect to a suitable topology, G^* forms a Lie group^(*). Thus the closure of any subgroup of G^* is a Lie group. For this reason, we can always assume the group G in Theorem 3 to be a connected Lie group, for otherwise, we can take the identity component of the closure of G instead of G . This does not effect the transitivity and the dimension restriction.

REMARK 2. Let G_p be the isotropic subgroup of G at a point p , and \mathbf{G}, \mathbf{G}_p be the Lie algebras of G, G_p respectively. Since \mathfrak{A}_n is simply-connected, the space \mathfrak{A}_n as well as the action of G on \mathfrak{A}_n is uniquely determined by the pair $(\mathbf{G}, \mathbf{G}_p)$.

REMARK 3. Let \mathbf{E} be the quotient \mathbf{G}/\mathbf{G}_p in the sense of linear space, and $\phi: \mathbf{G} \rightarrow \mathbf{E}$ the natural linear mapping. Denoting respectively by "Ad" and "ad" the linear adjoint representations of G and \mathbf{G} over \mathbf{G} , we have

$$\text{Ad}(G_p)(G_p) \subset G_p, \quad \text{ad}(G_p)(G_p) \subset G_p.$$

Thus Ad and ad induce linear representations Ψ and ψ of G_p and \mathbf{G}_p over \mathbf{E} respectively. In fact,

$$\Psi(u)(b) = \phi \text{Ad}(u)\phi^{-1}(b), \quad \psi(x)(b) = \phi \text{ad}(x)\phi^{-1}(b),$$

$$u \in G_p, \quad x \in \mathbf{G}_p, \quad b \in \mathbf{E}.$$

On the other hand, there is a natural 1-1 linear correspondence between \mathbf{E} and the tangent space \mathcal{L}_p of \mathfrak{A}_n at p . Up to this correspondence,

$$\Psi(G_p) = \tau(G_p).$$

11. A lemma.

LEMMA. *Let P_t denote the Lie algebra of the real special linear group of degree t , and S a semi-simple Lie algebra containing P_r as a subalgebra. If $\dim S \leq r^2 + 2r + 2$, $r \geq 4$, then the least ideal S_m of S such that $P_r \subset S_m$ is either P_r itself or a P_{r+1} .*

(*) Cf. S. Kobayashi: *Groupe de transformations qui laissent invariante une connexion infinitésimale*, C. R. Acad. Sci. Paris vol. 238 (1954) pp. 644-645.

Types	1	2	3
Spaces	\mathfrak{B}_n	\mathfrak{B}_n -origin	\mathfrak{B}_n
Connections	$\Gamma_{jk}^i(x) = 0$	$\Gamma_{jk}^i(x) = 0$	$\Gamma_{11}^1(x) = 2, \Gamma_{1\alpha}^\beta(x) = \delta_\alpha^\beta,$ others = 0
Maximal group G^* of collineations	$y^i = C_0^i + C_k^i x^k,$ $ C_k^i \neq 0$	$y^i = C_k^i x^k,$ $ C_k^i \neq 0$	$y^1 = x^1 + C_0^1, C_\alpha^\beta \neq 0,$ $y^\alpha = C_0^\alpha + C_1^\alpha x^1 + C_\beta^\alpha x^\beta$
dim G^*	$n^2 + n$	n^2	n^2
R_{jkl}^i	0	0	$\delta_j^1(\delta_l^i \delta_k^1 - \delta_k^i \delta_l^1)$
Completeness	complete	not	not

Types	4	5
Spaces	\mathfrak{B}_n	\mathfrak{B}_n
Connections	$\Gamma_{11}^1 = 2k, \Gamma_{1\beta}^\alpha = k\delta_\beta^\alpha, \Gamma_{11}^\alpha = -x^\alpha$ others = 0, $k = \text{constant} \geq 0$	$\Gamma_{11}^1 = 2k, \Gamma_{1\beta}^\alpha = k\delta_\beta^\alpha, \Gamma_{11}^\alpha = x^\alpha$ others = 0, $k = \text{constant} \geq 0$
Maximal group G^* of collineations	$y^1 = x^1 + C_0^1, C_\beta^\alpha \neq 0$ $y^\alpha = C_0^\alpha \cosh x^1 + C_1^\alpha \sinh x^1$ $+ C_\beta^\alpha x^\beta$	$y^1 = x^1 + C_0^1, C_\beta^\alpha \neq 0$ $y^\alpha = C_0^\alpha \cos x^1 + C_1^\alpha \sin x^1$ $+ C_\beta^\alpha x^\beta$
dim G^*	n^2	n^2
R_{jkl}^i	$(k^2 - 1)\delta_j^1(\delta_l^i \delta_k^1 - \delta_k^i \delta_l^1)$	$(k^2 + 1)\delta_j^1(\delta_l^i \delta_k^1 - \delta_k^i \delta_l^1)$
Completeness	complete when $k = 0$	complete when $k = 0$

Proof. Evidently S_m is semi-simple and $\dim S_m \leq r^2 + 2r + 2$. Let S_m^* be the complex form of S_m , and P_r^* the subalgebra of S_m^* which corresponds to P_r . We know that (1) S_m^* is semi-simple, (2) P_r^* is the complex simple Lie algebra of class A and rank $r-1$, (3) no proper ideal of S_m^* can contain P_r^* . Suppose that S_m^* is not simple. Then we can write it as a direct sum

$$S_m^* = S_1^* \oplus S_2^*$$

of two semi-simple nontrivial ideals. The intersection $S_1^* \cap P_r^*$ must be zero, for otherwise, we obtain from the simplicity of P_r^* that $P_r^* \subset S_1^*$ which is impossible. Similarly, $S_2^* \cap P_r^* = 0$. It follows then that $\pi_1(P_r^*) \approx P_r^*$, $\pi_2(P_r^*) \approx P_r^*$, where $\pi_1: S_m^* \rightarrow S_1^*$, $\pi_2: S_m^* \rightarrow S_2^*$ are the projections. Thus

$$\dim S_1^* \geq \dim P_r^* = r^2 - 1, \quad \dim S_2^* \geq \dim P_r^* = r^2 - 1,$$

whence

$$2r^2 - 2 \leq \dim S_1^* + \dim S_2^* = \dim S_m^* \leq r^2 + 2r + 2.$$

But this contradicts our hypothesis that $r \geq 4$. Hence S_m^* must be simple.

Since $P_r^* \subset S_m^*$, we know that

$$r^2 - 1 \leq \dim S_m^* \leq r^2 + 2r + 2, \quad \text{rank}(S_m^*) \geq r - 1, \quad r \geq 4.$$

A survey of the list of complex simple Lie algebras tells us that S_m^* has only three possibilities: (i) S_m^* is of class A and rank $r-1$; (ii) S_m^* is of class A and rank r ; or (iii) S_m^* is of class B or C and of rank 3 and $r=4$. But it is well known that the complex simple Lie algebras of class B and C of rank 3 cannot contain P_r^* . Therefore, only cases (i) and (ii) can happen.

Now we return to the real Lie algebras P_r and S_m . In case (i),

$$\dim S_m = \dim S_m^* = r^2 - 1 = \dim P_r$$

whence $S_m = P_r$. In case (ii), S_m is one of the real forms of the complex simple Lie algebras of class A and rank r . These real forms have been completely determined by Cartan [2]. We can see immediately from the list that P_{r+1} is the only such real form which can contain P_r . The lemma is thus proved.

12. Proof of Theorem 3. Firstly, we observe that the maximal group G^* of collineations of the five types of spaces listed in Theorem 3 are not isomorphic. The G^* corresponding to the first two types can be easily distinguished from the other. As for the G^* corresponding to the spaces of types 3, 4, 5, we can distinguish them by comparing the radicals of their Lie algebras. Thus two spaces belonging to different types in Theorem 3 are not equivalent. However, in each of the later two types, there are infinitely many spaces depending on k . We find, by a direct calculation,

$$R^i_{jkl;m} = -4k\delta_m^i R^i_{jkl}.$$

From these equalities and the expression for R^i_{jk} , it follows that the absolute value of k is a local scalar invariant. Thus two spaces of the type 4 or 5 with different k ($k \geq 0$) are not equivalent. The last sentence in Theorem 3 is proved.

Now we shall show that each \mathfrak{A}_n satisfying the hypothesis of Theorem 3 is equivalent to one of the listed spaces. Let p be a point of \mathfrak{A}_n , and let $G, G_p, \mathbf{G}, \mathbf{G}_p, \mathbf{E}, \tau, \Psi, \psi$ have the same meaning as in §10. Since \mathfrak{A}_n is simply-connected and G connected, G_p must be connected. Therefore, $\tau(G_p) = A_p$. We have determined all the possible linear groups A_p . The discussions for the various cases are similar. We select only three cases to study as patterns.

I. $\tau(G_p) = A_p = H_n^+$ or P_n . In this case, $\tau(G_p)$ is transitive over the nonzero tangent vectors at p , and thus G is transitive over the nonzero 1-elements of \mathfrak{A}_n . It follows that \mathfrak{A}_n is complete in the sense of affine connection. From Theorem 2, \mathfrak{A}_n is affinely flat. We know that a complete, simply-connected, homogeneous affinely connected space is *uniquely* determined by its local properties [9, p. 50]. Hence \mathfrak{A}_n must be equivalent to the space \mathfrak{B}_n with $\Gamma^i_{jk}(x) = 0$.

II. $\tau(G_p) = A_p = L$. In this case, $\dim G = n^2 - 1$. The hypothesis $\dim G \geq n^2 - n + 5$ then implies $n \geq 6$. Since τ is a faithful representation, G_p has a Levi-decomposition of the form

$$G_p = P_{n-1} + R_1, \quad R_1 = \text{radical of } G_p, \quad \dim R_1 = n - 1$$

where P_{n-1} is the Lie algebra of the real special linear group of degree $n - 1$. Choose a maximal semi-simple subalgebra S of G such that $P_{n-1} \subset S$. Let S_m be the minimal ideal, of S , which contains P_{n-1} . By the Lemma in §11, either $S_m = P_n$ or $S_m = P_{n-1}$. We shall discuss these two cases separately.

CASE II₁. $S_m = P_n$. Then $\dim S_m = n^2 - 1 = \dim G$ whence $S_m = P_n = G$. Passing the results in Theorem 1 to Lie algebras, we know that, up to an automorphism of P_n , P_n has only two subalgebras of dimension $n^2 - n - 1$, i.e., the Lie algebras L, L' of L and L' respectively. If $G_p = L'$, then it is easy to see that $\psi(G_p)$ and hence $\Psi(G_p)$ does not have any invariant vector. This contradicts the fact that $\Psi(G_p) = \tau(G_p)$ has an invariant vector. Therefore, $G_p = L$.

The real special linear group P_n acts transitively on the space $\mathfrak{M}_n = \mathfrak{B}_n - \text{origin}$. Its isotropic subgroup at the point $(1, 0, \dots, 0)$ is the subgroup L . Since $n > 2$, \mathfrak{M}_n is simply connected. By Remark 2 in §10, we can regard \mathfrak{A}_n to be \mathfrak{M}_n and the group G to be P_n . Using the coordinates x^1, \dots, x^n inherited from \mathfrak{B}_n , we find, by a direct calculation, that the only affine connection over \mathfrak{M}_n invariant under P_n is given by $\Gamma^i_{jk}(x) = 0$. This gives us the space of type 2 in our theorem.

CASE II₂. $S_m = P_{n-1}$. Here we shall be a little brief and omit the tedious Lie algebra arguments. Taking account of the fact $\Psi(G_p) = \tau(G_p) = L$, we first show that $S = P_{n-1}$, R_1 belongs to the radical R of G and that G_p is reduc-

tive⁽⁶⁾ in G . Then, by rather elaborate Lie algebra arguments, we can prove that G has a basis

$$e_2^2 - e_n^n, e_3^3 - e_n^n, \dots, e_{n-1}^{n-1}, e_\beta^\alpha, e_\alpha^1, e_0^1, e_0^\alpha \quad (\alpha \neq \beta; \alpha, \beta = 2, 3, \dots, n)$$

with the multiplication rule

$$[e_s^t, e_v^u] = \delta_s^u e_v^t - \delta_v^t e_s^u \quad (s, t, u, v = 0, 1, 2, \dots, n)$$

such that G_p is spanned by

$$e_2^2 - e_n^n, \dots, e_{n-1}^{n-1}, e_\beta^\alpha, e_\alpha^1 \quad (\alpha \neq \beta).$$

Now let us consider the group G' of all transformations of \mathfrak{B}_n of the form

$$y^1 = x^1 + C_\alpha^1 x^\alpha + C_0^1, \quad y^\alpha = C_\beta^\alpha x^\beta + C_0^\alpha, \quad |C_\beta^\alpha| = 1.$$

This group G' is transitive over \mathfrak{B}_n whose isotropic subgroup G'_0 at the origin consists of transformations:

$$y^1 = x^1 + C_\alpha^1 x^\alpha, \quad y^\alpha = C_\beta^\alpha x^\beta.$$

We see at once that, up to an isomorphism, $(G, G_p) = (G', G'_0)$. By Remark 2 in §11, we can regard $\mathfrak{A}_n = \mathfrak{B}_n$, $G = G'$. A direct calculation shows that the only affine connection over \mathfrak{B}_n invariant under G' is given by $\Gamma_{jk}^i(x) = 0$. This is the space of type 1.

III. $\tau(G_p) = A_p = L'$. In this case, we first show that P_{n-1} is at the same time a maximal semi-simple subalgebra of G , and then we can show that the pair (G, G_p) has only the following three possibilities:

CASE III₁. G has a basis

$$e_2^2 - e_n^n, \dots, e_{n-1}^{n-1} - e_n^n, e_\beta^\alpha, e_0^\alpha, e_1^\alpha, \lambda e_\alpha^\alpha - e_0^1, \quad \alpha \neq \beta; \alpha, \beta = 2, 3, \dots, n; \lambda \neq 0$$

such that G_p is spanned by

$$e_2^2 - e_n^n, \dots, e_{n-1}^{n-1} - e_n^n, e_\beta^\alpha, e_1^\alpha \quad (\alpha \neq \beta).$$

In this case, we can regard $\mathfrak{A}_n = \mathfrak{B}_n$ and regard G to be the group of all transformations of the form:

$$y^1 = x^1 - t, \quad y^\alpha = C_0^\alpha + C_1^\alpha x^1 + C_\beta^\alpha x^\beta, \quad |C_\beta^\alpha| = \exp(n-1)\lambda t.$$

It follows then that the affine connections over \mathfrak{B}_n invariant under G must be of the form:

⁽⁶⁾ A subalgebra L of a Lie algebra G is called reductive if there exists a linear subspace R of G such that $G = L + R$, $L \cap R = 0$, $[L, R] \subset R$.

$$\Gamma_{jm}^i = k(\delta_j^i \delta_m^1 + \delta_m^i \delta_j^1), \quad k = \text{constant.}$$

When $k=0$, the space is of type 1. When $k \neq 0$, we find that two connections corresponding to different k 's are affinely equivalent (in the global sense). Thus we can assume $k=1$, and obtain the space of type 3.

CASE III₂. G and G_p are spanned, respectively, by

$$e_2^2 - e_n^n, \dots, e_{n-1}^{n-1} - e_n^n, e_\beta^\alpha, e_1^\alpha, e_0^\alpha, \quad \lambda(e_0^0 + e_1^1) + e_0^1 + e_1^0, \quad \alpha \neq \beta$$

and

$$e_2^2 - e_n^n, \dots, e_{n-1}^{n-1} - e_n^n, e_\beta^\alpha, e_1^\alpha.$$

We can regard \mathfrak{A}_n to be \mathfrak{B}_n , and G to be the group of all transformations of the form

$$y^1 = x^1 - t, \quad y^\alpha = C_0^\alpha \cosh x^1 + C_1^\alpha \sinh x^1 + C_\beta^\alpha x^\beta, \quad |C_\beta^\alpha| = \exp(n-1)\lambda t.$$

The invariant affine connections are given by

$$\Gamma_{11}^1(x) = 2k, \quad \Gamma_{1\beta}^\alpha(x) = k\delta_\beta^\alpha, \quad \Gamma_{11}^\alpha(x) = -x^\alpha, \quad \text{other } \Gamma = 0, \\ k = \text{constant.}$$

But the affine connections corresponding to k and $-k$ are equivalent. Thus we can assume $k \geq 0$, and get the spaces of type 4.

CASE III₃. G and G_p are spanned, respectively, by

$$e_2^2 - e_n^n, \dots, e_{n-1}^{n-1} - e_n^n, e_\beta^\alpha, e_1^\alpha, e_0^\alpha, \quad \lambda(e_0^0 + e_1^1) - e_0^1 + e_1^0, \quad \alpha \neq \beta$$

and

$$e_2^2 - e_n^n, \dots, e_{n-1}^{n-1} - e_n^n, e_\beta^\alpha, e_1^\alpha.$$

We can regard \mathfrak{A}_n to be \mathfrak{B}_n and G to be the group of transformations of the form

$$y^1 = x^1 - t, \quad y^\alpha = C_0^\alpha \cos x^1 + C_1^\alpha \sin x^1 + C_\beta^\alpha x^\beta, \quad |C_\beta^\alpha| = \exp(n-1)\lambda t.$$

The invariant connections are given by

$$\Gamma_{11}^1(x) = 2k, \quad \Gamma_{1\beta}^\alpha(x) = k\delta_\beta^\alpha, \quad \Gamma_{11}^\alpha(x) = x^\alpha, \quad \text{other } \Gamma = 0, \quad k = \text{constant.}$$

Just as in the above case, the connections corresponding to k and $-k$ are equivalent. Thus we can assume $k \geq 0$, and obtain the spaces of type 5.

Thus we know that each \mathfrak{A}_n satisfying the restrictions in Theorem 3 is equivalent to one of the five types. The completeness, curvature tensor and the maximal group of affine collineations of these five types of spaces can be obtained by a direct computation.

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