

THE LATTICE OF SUBMODULES OF A MODULE OVER A NONCOMMUTATIVE RING

BY

EDMUND H. FELLER

Introduction and summary. Of fundamental importance to the theory of ideals in commutative rings satisfying the ascending chain condition for ideals has been the works of Noether [6] and Krull [4]. What follows, in the first two sections, is an attempt to extend more of the Krull-Noether theory of commutative rings to one-sided ideals in noncommutative rings. A decomposition theory for one-sided ideals in the noncommutative case was provided by Fitting [1]. Although Fitting was able to prove uniqueness theorems in certain cases, he did not prove uniqueness theorems for arbitrary noncommutative rings satisfying the ascending chain condition for right ideals. By methods based on the works of Noether and Fitting, we shall prove decomposition and uniqueness theorems for arbitrary noncommutative rings with identity that satisfy the ascending chain condition for right ideals. Actually, the theorems in this paper are proved for what we define to be A - R modules, which is a generalization of noncommutative rings.

In the third section, we discuss completely indecomposable A - R modules, which follow along the lines of a paper by Snapper [7]. A sufficient condition for two faithful representations of a noncommutative ring with identity to be equivalent follows from this discussion.

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1. A property of the A.C.C. for A - R modules. We begin by defining an A - R module

DEFINITION 1.1. An A - R module is a system consisting of an additive abelian group M , a ring A containing the identity element 1, a ring R containing the identity element e , and two functions defined on the product sets $(M \times A)$ and $(M \times R)$ having values in M , such that if xa and $y\alpha$ denote the elements in M determined by the elements x, y in M , a in A , and α in R , then

$$\text{I-1.} \quad x(a + b) = xa + xb,$$

$$\text{I-2.} \quad x(ab) = (xa)b,$$

$$\text{I-3.} \quad x(\alpha + \beta) = x\alpha + x\beta,$$

$$\text{I-4.} \quad x(\alpha\beta) = (x\alpha)\beta,$$

$$\text{I-5.} \quad (x + y)a = xa + ya,$$

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$$\text{I-6.} \quad (x + y)\alpha = x\alpha + y\alpha,$$

$$\text{I-7.} \quad x1 = xe = x,$$

$$\text{I-8.} \quad (xa)\alpha = (x\alpha)a$$

hold for any a, b in A and α, β in R .

DEFINITION 1.2. A subgroup N of M is said to be an R -submodule of M if $x\alpha \in M$ for every x in N and every α in R . A subgroup N of M is said to be an A submodule of M if $xa \in N$ for every x in N and every a in A . A subgroup N is an A - R submodule if it is both an A and R submodule.

EXAMPLES. Every ring with identity can be considered as an A - R module by taking the ring of right multiplications as R and the ring of left multiplications as A .

A module M over a ring R containing the identity element can be handled as an A - R module by taking as A the ring of endomorphisms that commute with the endomorphisms which are the multiplications of the elements in M by the elements of R .

CONVENTIONS. In the future, except where the contrary is stated, the letters N, N_1, N_2 , etc., will stand for R and A submodules; the letters a, b, c , for elements of A ; the letters α, β, γ , for elements of R ; the symbols m, n, p, q , will stand for ideals in R and A .

The proofs of the theorems, except for Theorems 1.2 and 2.2, in the first two sections follow closely the proofs of similar theorems for the commutative case given by Noether and Krull, and are therefore omitted.

For proof of the following theorem we refer to [2, p. 242].

THEOREM 1.1. Let G_1, G_2, \dots, G_n be normal N subgroups of the group G with operators N . If the A.C.C. holds for N subgroups of $G/G_i, i=1, 2, \dots, n$, then the A.C.C. holds for N subgroups of $G/G_1 \cap \dots \cap G_n$.

THEOREM 1.2. If M is an A - R module, which satisfies the A.C.C. for A submodules and R submodules, then the rings A/q and R/q' satisfy the A.C.C. for right ideals, where q and q' are the annihilating ideals of M in A and R respectively.

Proof. We shall show that the A.C.C. holds for right ideals in A/q . A similar discussion proves this statement for R/q' .

Consider A as a right module over its ring of right multiplications. For a fixed x_i in M the set $(0:x_i; A) = \{a \mid x_i a = 0, x_i \in M, a \in A\}$ is a right ideal of A , and hence an A submodule of A . The mapping ψ , which takes $a + (0:x_i; A)$ into $x_i a$, is an A -isomorphism of the A module $A/(0:x_i; A)$ upon $x_i A = \{x_i a \mid a \in A\}$, which is an A submodule of M . Since $x_i A$ is an A submodule of M , it satisfies the A.C.C. for A submodules, i.e., right ideals. We have also assumed that M satisfies the A.C.C. for R submodules. Hence there must exist a set of generators x_1, x_2, \dots, x_n , for M as an R module. Then,

from $q = (0: M; A) = (0: \{x_1, \dots, x_n\}; A) = (0: x_1; A) \cap \dots \cap (0: x_n; A)$ and Theorem 1.1, it follows that A/q satisfies the A.C.C. for right ideals.

2. Decomposition of submodules of an A - R module. Let M be an A - R module.

DEFINITION 2.1. If N is an R submodule of M , the set $V^*(N) = \{a \mid Na \subseteq N, a \in A\}$ is called the *centralizer* of N .

$V^*(N)$ is a subring of A which contains the identity element. We denote an arbitrary subring of $V^*(N)$ that contains the identity element by $V(N)$.

DEFINITION 2.2. If T is a set in M , and N is an R submodule of M , then $(N: T; V) = \{a \mid Ta \subseteq N, a \in V(N)\}$ is a right ideal of $V(N)$. If now $T = M$, this set is an ideal of $V(N)$ and will be called the V shadow of N . In general we shall denote the V shadow of N by \mathfrak{n} .

DEFINITION 2.3. If N is an R submodule of M and H is a set in $V(N)$, then $(N: H) = \{x \mid xh \in N, x \in M, \text{ all } h \in H\}$ is an R submodule.

DEFINITION 2.4. The V radical of an R submodule N of M is defined to be the set of all $a \in V(N)$ such that a^t is contained in the V shadow \mathfrak{n} of N for some positive integer t .

DEFINITION 2.5. An R submodule N is said to be V primary provided that $xa \in N$, for $x \notin N$ and $a \in V(N)$, implies $Ma^t \subseteq N$ for some positive integer t , i.e., a is contained in the V radical of N .

One should note here that if an R module N is V primary, then it is necessarily V' primary, where V' is any subring of $V(N)$ containing the identity element.

DEFINITION 2.6. An ideal \mathfrak{p} of an arbitrary ring K will be called *completely prime* provided $ab \in \mathfrak{p}$, $a, b \in K$, implies at least one of these elements is contained in \mathfrak{p} .

DEFINITION 2.7. An ideal \mathfrak{q} of an arbitrary ring K is said to be *right primary* provided $ab \in \mathfrak{q}$, $a \notin \mathfrak{q}$, $a, b \in K$, implies $b^s \in \mathfrak{q}$ for some positive integer s .

One can easily show that the V shadow of a V primary R submodule is a right primary ideal of A .

DEFINITION 2.8. An R submodule is *irreducible* provided it is not the intersection of two R submodules that properly contain it.

THEOREM 2.1. If M is an A - R module satisfying the A.C.C. for R submodules, then every irreducible R submodule N is $V(N)$ primary.

THEOREM 2.2. The V radical of a V primary and irreducible R submodule is a completely prime two-sided ideal of $V(N)$.

Proof. The V radical \mathfrak{p} of N is closed under addition. For $a, b \in \mathfrak{p}$, let s and t be the least positive integers such that a^s and b^t are contained in the V shadow \mathfrak{n} of N . Suppose $s > 1$ and $t > 1$. Then, since $(Ma^{s-1})a \subseteq N$ and $Ma^{s-1} \notin N$, we have $N \subset (N:a)$, and similarly $N \subset (N:b)$. Suppose $a+b \notin \mathfrak{p}$.

Then, since $[(N:a) \cap (N:b)] \cdot (a+b) \subseteq N$ and N is V primary, it follows that $(N:a) \cap (N:b) \subseteq N$. However, this is impossible since this would imply that N is reducible, i.e., $N = (N:a) \cap (N:b)$. If $s=1$, then $(a+b)^t \in n$, and similarly, if $t=1$, then $(a+b)^s \in n$.

The radical \mathfrak{p} is closed under right and left multiplication by elements of $V(N)$. Again, for $a \in \mathfrak{p}$, let s be the least positive integer such that $a^s \in n$. Since n is an ideal of $V(N)$, $a^s c \in n$, for any $c \in V(N)$. If $s > 1$, then $a^{s-1}(ac) \in n$. Hence, since the V shadow n of N is right primary and $a^{s-1} \notin n$, it follows $ac \in \mathfrak{p}$. If $s=1$, the result is trivial. Since $(ca)^s = c(ac)^{s-1}a$, $ca \in \mathfrak{p}$.

We conclude by showing that \mathfrak{p} is completely prime in $V(N)$. Suppose $a_1 \cdot a_2 \in \mathfrak{p}$, $a_1, a_2 \in V(N)$. Then $(a_1 a_2)^t \in n$ for some positive integer t . From these elements let $a_{i_1}, a_{i_2}, \dots, a_{i_k}$ be the smallest subcollection whose product $a_{i_1} \cdot a_{i_2} \cdot \dots \cdot a_{i_k}$ is in n . Since $a_{i_1} \cdot \dots \cdot a_{i_{k-1}}$ is not in n and n is right primary, it follows $a_{i_k} \in \mathfrak{p}$.

Usual lattice theoretic methods give us the following theorem.

THEOREM 2.3. *If M is an A - R module satisfying the A.C.C. for R submodules, every R submodule can be expressed as the intersection of a finite number of irreducible R submodules.*

DEFINITION 2.9. The intersection $N = N_1 \cap \dots \cap N_t$ of R submodules is said to be *irredundant* if $N \subset N_1 \cap \dots \cap N_{i-1} \cap N_{i+1} \cap \dots \cap N_t$, $i=1, 2, \dots, t$.

THEOREM 2.4. *If $N = N_1 \cap \dots \cap N_s = N_1^\# \cap \dots \cap N_s^\#$, where $N_i, N_i^\#$ are irreducible, and if these intersections are irredundant, then:*

(i) *For any i , $1 \leq i \leq s$, there is a j , $1 \leq j \leq s^\#$, such that*

$$N = N_1 \cap \dots \cap N_{i-1} \cap N_j^\# \cap N_{i+1} \cap \dots \cap N_s.$$

(ii) $s = s^\#$.

For proof we refer to [2, p. 252].

DEFINITION 2.10. If N is an R submodule of M , then N' is a *component element* of N' if it appears in any irredundant representation of N as the intersection of irreducible R submodules.

DEFINITION 2.11. The ring $F^*(N) = \bigcap_\alpha V^*(N_\alpha)$, where N_α ranges over the entire set of component elements of N , is called the *subcentralizer* of N .

The subcentralizer $F^*(N)$ is contained in the centralizer of N and is not zero since it contains the identity element. We denote an arbitrary subring of $F^*(N)$ that contains the identity element by $F(N)$. The radicals, which we shall now consider, will be in just such a subring $F(N)$.

THEOREM 2.5. *Let $N = N_1 \cap \dots \cap N_s = N_1^\# \cap \dots \cap N_s^\#$ be irredundant intersections where the $N_i, N_i^\#$ are irreducible R submodules, $i=1, 2, \dots, s$. Then the set of distinct F radicals of N_1, \dots, N_s is identical with the set of distinct F radicals of $N_1^\#, \dots, N_s^\#$ in $F(N)$.*

THEOREM 2.6. *Let N_1, \dots, N_r be irreducible V' primary R modules, where V' is any subring of $V^*(N_1) \cap \dots \cap V^*(N_r)$ containing the identity element. If N_i has V' radical \mathfrak{p} for all i , then $N = N_1 \cap \dots \cap N_r$ is again V' primary with radical \mathfrak{p} . If the intersection $N = N_1 \cap \dots \cap N_r$ is irredundant and N has radical \mathfrak{p}_i , $i = 1, 2, \dots, r$, where $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$, for $i \neq j$, then N is not V' primary.*

From the preceding theorems, we can state a theorem quite similar to that given by Noether and Krull. However the radicals will not be ideals of the ring A , but will be ideals in a subring of A . In the example following we show that the same result cannot be stated unless the ring $F(N)$ is used.

THEOREM 2.7. *Let M be an A - R module satisfying the A.C.C. for R submodules. Let N be an R submodule and $F(N)$ a subring of the subcentralizer of N containing the identity element. Then there exists F primary R submodules N_1, \dots, N_s , with distinct F radicals $\mathfrak{p}_1, \dots, \mathfrak{p}_s$, such that $N = N_1 \cap \dots \cap N_s$, where this intersection is irredundant. If $N = N_1^\# \cap \dots \cap N_r^\#$ is another such reduction of N , then $r = s$, and for a suitable rearrangement of the subscripts the corresponding prime F radicals are equal.*

We shall call such a reduction of N an F -short representation of N , and the prime radicals the F associated prime ideals of N .

If the A - R module M satisfies the A.C.C. for R submodules, then for any R submodule N and any ideal q of $V(N)$, the R submodules $(N:q)$ are fixed for large r . We denote this limit by $N(q)$.

LEMMA 1. *If M satisfies the A.C.C. for R submodules and $N = N_1 \cap N_2$, where N_1 and N_2 are R submodules, and q is an ideal of V' , which is any subring of $V^*(N_1) \cap V^*(N_2)$ containing 1, then $(N_1 \cap N_2)(q) = N_1(q) \cap N_2(q)$.*

LEMMA 2. *If N is a V primary R submodule of M , with V radical ideal \mathfrak{p} , such that $\mathfrak{p}^r \subseteq n$, the V shadow of N , then*

- (i) $N(q) = N$, if $q \not\subseteq \mathfrak{p}$,
- (ii) $N(q) = M$, if $q \subseteq \mathfrak{p}$,

for an ideal q contained in $V(N)$.

From the preceding lemmas the following theorem is immediate.

THEOREM 2.8. *If M satisfies the A.C.C. for R submodules and $N = N_1 \cap \dots \cap N_s$ is an F -short representation of N with distinct F radicals $\{\mathfrak{p}_i\}$, $i = 1, 2, \dots, s$, such that $\mathfrak{p}_i^{r_i} \subseteq n$ for some positive integers r_i , $i = 1, 2, \dots, s$, and q is an ideal of $F(N)$ contained in $\mathfrak{p}_{m+1}, \dots, \mathfrak{p}_s$, but not in $\mathfrak{p}_1, \dots, \mathfrak{p}_m$, then $N(q) = N_1 \cap \dots \cap N_m$.*

DEFINITION 2.12. If $N = N_1 \cap \dots \cap N_s$ is an F -short representation of N , we call $N' = N_{i_1} \cap \dots \cap N_{i_r}$ an F isolated component of N , if no one of the prime ideals $\mathfrak{p}_{i_1}, \dots, \mathfrak{p}_{i_r}$ contains an F associated prime ideal of N which is not an ideal of this set.

THEOREM 2.9. *If M is an A - R module satisfying the A.C.C. for R submodules and $N = N_1 \cap \dots \cap N_r = N_1^\# \cap \dots \cap N_s^\#$ are two F -short representations of N with distinct F radicals $\mathfrak{p}_1, \dots, \mathfrak{p}_s$, such that $\mathfrak{p}_i^{r_i} \subseteq \mathfrak{n}_i$ and $\mathfrak{p}_i^{t_i} \subseteq \mathfrak{n}_i^\#$ for positive integers r_i and t_i , where \mathfrak{n}_i and $\mathfrak{n}_i^\#$ are the corresponding F shadows, $i = 1, 2, \dots, s$, then the corresponding isolated components are equal.*

One may ask in what cases, as in Theorem 2.9, would $\mathfrak{p}_i^{r_i} \subseteq \mathfrak{n}$. If M satisfies the A.C.C. for R and $F(N)$ submodules, then this is true. For it follows from Theorem 1.2 that $F(N)/\mathfrak{q}$, where \mathfrak{q} is the annihilating ideal of M in $F(N)$, satisfies the A.C.C. for right ideals. Therefore $F(N)/\mathfrak{n}_i$ satisfies the A.C.C. for right ideals. Since \mathfrak{p}_i is the set of all nilpotent elements in $F(N)/\mathfrak{n}_i$, it follows $\mathfrak{p}_i^{r_i} \subseteq \mathfrak{n}$ (see [5]). In addition, if $F(N)$ satisfies the D.C.C. for right ideals, then $\mathfrak{p}_i^{t_i} \subseteq \mathfrak{n}$. In fact, in this case, $F(N)/\mathfrak{p}_i$ is a division ring, and the \mathfrak{p}_i are maximal ideals of $F(N)$. Therefore each N_i would be an isolated component. Hence, in this case, the representation would be unique.

2'. Example. Let D be a Noetherian ring with identity 1. Consider D_2 , the ring of 2×2 matrices with elements in D . Then by taking as R the ring of right multiplications and as A the ring of left multiplications, we have an A - R module.

Denote by

$$\begin{pmatrix} D \\ I \end{pmatrix}$$

the set of all matrices (a_{ij}) in D with $a_{11}, a_{12} \in D$; $a_{21}, a_{22} \in I$, where I is an ideal in D . Then

$$\begin{pmatrix} D \\ I \end{pmatrix}$$

is an R submodule.

The proof of the following lemma we leave as an exercise.

LEMMA 1.

$$\begin{pmatrix} D \\ I \end{pmatrix} = \begin{pmatrix} D \\ I_1 \end{pmatrix} \cap \begin{pmatrix} D \\ I_2 \end{pmatrix}$$

if and only if $I = I_1 \cap I_2$.

Denote by $(I_1, I_2)_l$ the set of all matrices $(a_{ij})_l$ in A with $a_{11}, a_{12} \in D$, $a_{21} \in I_1$, $a_{22} \in I_2$, where I_1 and I_2 are ideals in D .

THEOREM 2'.1. *If*

$$N = \begin{pmatrix} D \\ I \end{pmatrix},$$

then $V^(N)$ is $(I, D)_l$.*

Proof. Certainly this set is contained in the centralizer of N . Suppose $(a_{ij})_i \notin (I, D)_i$, i.e., $a_{21} \notin I$. Then for

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in N$$

we have

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix} \in N.$$

THEOREM 2'.2. *If*

$$N = \begin{pmatrix} D \\ I \end{pmatrix},$$

then the $V^(N)$ shadow of N will be $(I, I)_i$, and the $V^*(N)$ radical of N will be $(I, p)_i$, where p is the radical of I in D .*

Proof. Certainly $(I, I)_i$ is contained in the $V^*(N)$ shadow of N . If now $(a_{ij})_i \in V^*(N)$ but not contained in $(I, I)_i$, i.e., $a_{22} \notin I$. Then for

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M$$

we have

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix} \in N.$$

If $(a_{ij})_i \in V^*(N)$, then $(a_{ij})_i^t = (b_{ij})_i$, where $b_{11}, b_{12} \in D$, $b_{21} \in I$, $b_{22} = c + a_{22}^t$ for $c \in I$. Hence, $(a_{ij})_i$ is contained in the $V^*(N)$ shadow if and only if $a_{22} \in p$.

THEOREM 2'.3. *If*

$$N = \begin{pmatrix} D \\ I \end{pmatrix},$$

where I is a primary ideal in D , then N is $V^(N)$ primary.*

Proof. Suppose (a_{ij}) is contained in M and is not contained in N , say $a_{21} \notin I$. Let $(b_{ij})_i \in V^*(N) = (I, D)_i$, i.e., $b_{21} \in I$. Then if $(a_{ij})(b_{ij})_i = (b_{ij})(a_{ij}) = (c_{ij}) \in N$, we have $c_{21} = d + a_{21}b_{22}$, with $d \in I$. Since $(c_{ij}) \in N$, $c_{21} = d + a_{21}b_{22} \in I$. Inasmuch as I is primary in D and $a_{21} \notin I$, then $b_{22} \in p$, the radical of I in D . Hence $(b_{ij})_i \in (I, p)_i$, the $V^*(N)$ radical of N .

A similar discussion holds if $a_{22} \notin I$.

Let $D = F[1, x, y]$, the ring of polynomials in x and y over the field F . Let $I = (x^2, xy)$, $I_1 = (x)$, and $I_\alpha = (x^2, y + \alpha x)$, where $\alpha \in F$ and $I_\alpha \neq I_\beta$ if $\alpha \neq \beta$. Then in D , $I = I_1 \cap I_\alpha$, where the radical $p = (x, y)$ of I_α is the same for each

decomposition of I in primary ideals.

In D_2 , let

$$N = \begin{pmatrix} D \\ I \end{pmatrix}, \quad N_1 = \begin{pmatrix} D \\ I_1 \end{pmatrix}, \quad \text{and} \quad N_\alpha = \begin{pmatrix} D \\ I_\alpha \end{pmatrix}.$$

Then, by Lemma 1,

$$N = N_1 \cap N_\alpha.$$

However, here the $V^*(N_\alpha)$ radicals (N_α, \mathfrak{p}) are different for each decomposition. If we let $F = \bigcap I_\alpha \cap I_1$, then in $(F, D) = \bigcap_\alpha V^*(N_\alpha) \cap V^*(N_1)$ the radical $(F, \mathfrak{p})_1$ is the same for each decomposition.

Thus although the corresponding V^* radicals may be distinct for two decompositions of an R submodule, the corresponding F^* radicals are the same.

3. Completely indecomposable A - R modules.

DEFINITION 3.1. An R submodule of an A - R module M is said to be *indecomposable* if it can not be expressed as the direct sum of R submodules. An A submodule of M is said to be *indecomposable* if it can not be expressed as the direct sum of A submodules.

DEFINITION 3.2. A *completely indecomposable A - R module* is an A - R module where the following four conditions are satisfied.

II-1. Every R submodule of M is indecomposable.

II-2. Every A submodule of M is indecomposable.

II-3. M satisfies the A.C.C. for A submodules and R submodules.

II-4. M satisfies the D.C.C. for A submodules and R submodules.

DEFINITION 3.3. A *simple* or *minimal* R submodule is an R submodule N such that if $0 \subseteq N' \subseteq N$, where N' is an R submodule, then either $0 = N'$ or $N = N'$. A *simple* or *minimal* A submodule is defined in the same manner.

From II-3 and Theorem 1.2 we conclude that R/\mathfrak{q}' and R/\mathfrak{q} both satisfy the A.C.C. for right ideals, where \mathfrak{q}' and \mathfrak{q} are the annihilating ideals of M in R and A respectively. By II-1 the 0 module is irreducible as an R submodule, for if N_1 and N_2 are R submodules such that $0 = N_1 \cap N_2$, then $N_1 + N_2 = N_1 \oplus N_2$. Hence, by Theorem 2.1, the 0 submodule is $A = V^*(0)$ primary, and, by Theorem 2.2, the radical \mathfrak{p} of 0 in A is a completely prime ideal of A . Furthermore, since A/\mathfrak{q} satisfies the A.C.C. for right ideals and \mathfrak{p} is a nil ideal in A/\mathfrak{q} , then from a theorem proved by Levitski [5], it follows that $\mathfrak{p}^s \subseteq \mathfrak{q}$ for some positive integer s . Similarly by II-2, the 0 module is an irreducible A submodule. Hence 0 is R primary and its radical \mathfrak{p}' in R is a completely prime ideal in R , where $\mathfrak{p}'^r \subseteq \mathfrak{q}'$ for some positive integer r .

For a set H in M and a set \mathfrak{m} in A denote by $H\mathfrak{m}$ the set of all sums $\sum x_i a_i$ for $x_i \in H$ and $a_i \in \mathfrak{m}$. For a set $\mathfrak{m}' \subseteq R$ we denote by $H\mathfrak{m}'$ the set of all sums $\sum x_i \alpha_i$ for $x_i \in H$ and $\alpha_i \in \mathfrak{m}'$.

LEMMA 1. *Every completely indecomposable A - R module M contains a unique minimal A - R submodule.*

Proof. By II-4, M contains a minimal R submodule N_0 , which is necessarily a unique minimal R submodule. For if N_1 is another such minimal R submodule, then $0 = N_0 \cap N_1$. Similarly M contains a unique minimal A submodule N' .

We shall now show that N_0 is an A submodule. Let $H(a)$ be the homomorphism $x \rightarrow xa$ of M for $a \in A$. Then $H(a)$ will be an R homomorphism of N_0 with kernel N_0 or 0. If the kernel is N_0 , then $N_0a = 0$. If the kernel is 0, then N_0a is an isomorphic image of N_0 and therefore a minimal R submodule. Since N_0 is the unique minimal R submodule, then $N_0a = N_0$. In both cases $N_0a \subseteq N_0$, and therefore N_0 is an A submodule. Similarly N' is an R submodule. Hence it follows that $N_0 \subseteq N'$ and $N' \subseteq N_0$, and therefore $N_0 = N'$.

LEMMA 2. $(0; N_0; A)$ and $(0; N_0; R)$ are maximal right ideals in A and R respectively.

Proof. Let $xR = xA = N_0$, which is necessarily cyclic. Let ψ be the homomorphism that takes a into xa , for all $a \in A$. If we treat A as a right module over its set of right multiplications, then ψ is an A homomorphism of A upon N_0 with kernel $(0; x; A)$. Hence $A/(0; x; A)$ is A isomorphic to xA . Since xA is a simple A module, it follows that $(0; x; A)$ is a maximal right ideal of A . From the relation

$$N_0(0; x; A) = xR(0; x; A) = x(0; x; A)R = 0$$

it follows that $(0; x; A) = (0; N_0; A)$.

Similarly $(0; N_0; R)$ is a maximal right ideal in R .

Inasmuch as the 0 module of M is A primary, we have $(0; N_0; A) \subseteq \mathfrak{p}$. Consequently \mathfrak{p} is a maximal right ideal in A , and since \mathfrak{p} is also a completely prime ideal in A , it follows that A/\mathfrak{p} is a division ring. Similarly R/\mathfrak{p}' is a division ring.

LEMMA 3. The unique minimal A - R submodule N_0 is equal to $(0; \mathfrak{p})$ and $(0; \mathfrak{p}')$.

Proof. We have $(0; \mathfrak{p})$ is a vector space over A/\mathfrak{p} . If $N_0 \subset (0; \mathfrak{p})$, then $(0; \mathfrak{p}) = N_0 \oplus N'$, where N' as an A/\mathfrak{p} subspace. Since $N' \subseteq (0; \mathfrak{p})$, it follows that N' is an R submodule. This contradicts II-1. Hence $N_0 = (0; \mathfrak{p})$. Similarly $N_0 = (0; \mathfrak{p}')$.

Lemmas 5 and 6 are immediate applications of the following known lemma, which holds true for arbitrary modules over a noncommutative ring.

LEMMA 4. Let M be a unitary right module over a ring R , where R contains an ideal \mathfrak{p} such that $M\mathfrak{p}^n = 0$ for some integer n . If N is a simple R submodule of M , then $N\mathfrak{p} = 0$.

LEMMA 5. If M is a completely indecomposable A - R module and N_1 is a maximal R submodule of the R submodule N_2 , then $N_2\mathfrak{p}' \subseteq N_1$, where \mathfrak{p}' is the radical of 0 in R .

If we consider A/q as a right module over its set of right multiplications, we have the following lemma.

LEMMA 6. *Let M be a completely indecomposable A - R module. If n_1 and n_2 are right ideals in A containing q , where n_1 is maximal in the set of right ideals contained in n_2 , then $n_2p \subseteq n_1$ where p is the radical of 0 in A .*

THEOREM 3.1. *If M is a completely indecomposable A - R module, then A/q satisfies the D.C.C. for right ideals, where q is the annihilating ideal of M in A . Then if*

$$(1) \quad q \subset n_0 \subset n_1 \cdots \subset n_\mu = A$$

is a composition series of right ideals from q to A and

$$(2) \quad 0 \subset N_0 \subset N_1 \subset \cdots \subset N_\lambda = M$$

is a composition series of R submodules for M , then

$$(3) \quad 0 = (0:n_\mu) \subset \cdots \subset (0:n_0) \subset M$$

and

$$(4) \quad q = (0:N_\lambda; A) \subset \cdots \subset (0:N_0; A) \subset A$$

are composition series of R submodules and right ideals of M and A/q respectively.

Proof. Consider A as a right module over its set of right multiplications.

Since M satisfies both chain conditions for R submodules, there exists a chain (2). From this chain we shall show that the factors of (4) are either 0 or isomorphic to a simple A module, i.e., the chain is a normal chain which can be refined to a composition series of right ideals in A/q . From this we can conclude that A/q satisfies the D.C.C. for right ideals, and the length μ of A/q in right ideals is less than or equal to the length λ of M in R submodules.

In the second part of this proof we shall show in a similar manner that the chain (3) is a normal chain that can be refined to a composition series of R submodules in M . From this we can conclude that $\mu \geq \lambda$, which completes the proof of the theorem.

Since (2) is a composition series of M in R submodules, we have, by Lemma 5, $N_j p' \subseteq N_{j-1}$, where p' is the radical of 0 in R . Hence $N_j p' (0:N_{j-1}; A) = N_j (0:N_{j-1}; A) p' = 0$. From Lemma 3, we have $(0:p') = N_0$, hence $N_j (0:N_{j-1}; A) \subseteq N_0$. Since N_0 is a simple R submodule, it follows that either the R submodule $N_j (0:N_{j-1}; A)$ equals 0 or N_0 . If this R submodule is zero, we have $(0:N_{j-1}; A) / (0:N_j; A) = 0$. Suppose however that this R submodule is equal to N_0 . Then there must exist an $x \in N_j$, $x \notin N_{j-1}$, and an $a \in (0:N_{j-1}; A)$ such that $xa \neq 0$. Since $x \in N_j$, $x \notin N_{j-1}$, then $N_{j-1} + xR = N_j$. It follows, since $xa \neq 0$, that the A submodule $x(0:N_{j-1}; A) = \{xa \mid a \in (0:N_{j-1}; A)\} = N_0$ since N_0 is also a minimal A submodule.

If we treat A as a right module over its set of right multiplications, the correspondence $\psi: a \rightarrow xa$ for $a \in (0: N_{j-1}; A)$ is an A homomorphism of $(0: N_{j-1}; A)$ onto N_0 with kernel $(0: N_{j-1}; A) \cap (0: x; A)$, which equals $(0: N_j; A)$. Hence (4) can be refined to a composition series of A/q , and $\lambda \geq \mu$.

We shall now show that $\mu \geq \lambda$. Since (1) is a composition series we have by Lemma 6 that $n_j p \subseteq n_{j-1}$. Hence $(0: n_{j-1}) n_j p = 0$, and since $(0: p) = N_0$, it follows that the A submodule $(0: n_{j-1}) n_j$ equals 0 or N_0 since N_0 is a minimal A submodule. If this A module is 0, we have $(0: n_{j-1}) / (0: n_j) = 0$. However, if this A submodule is N_0 , there must exist an $a \in n_j$, $a \notin n_{j-1}$ and an $x \in (0: n_{j-1})$ such that $xa \neq 0$. Since $a \in n_j$, $a \notin n_{j-1}$ it follows that $n_{j-1} + aA = n_j$. Since $xa \neq 0$, we have $(0: n_{j-1})a = \{xa \mid x \in (0: n_{j-1})\}$ is an R submodule equal to N_0 , since N_0 is a minimal R submodule. The correspondence $y \rightarrow ya$ for $y \in (0: n_{j-1})$ is an R homomorphism of $(0: n_{j-1})$ onto N_0 with kernel $(0: n_{j-1}) \cap (0: a) = (0: n_j)$. Therefore (3) except for repetition is a composition and $\mu \geq \lambda$.

The following theorem is proved in the same manner as Theorem 3.1.

THEOREM 3.2. *If M is a completely indecomposable A - R module, then R/q' satisfies the D.C.C. for right ideals, where q' is the annihilating ideal of M in R . In addition, the length of R/q' in right ideals is equal to the length of M in A submodules.*

An immediate application of these theorems gives us the following corollary.

COROLLARY 1. *Let R be a ring with identity that satisfies the A.C.C. and the D.C.C. for both right and left ideals. If no right ideal and no left ideal can be expressed as the direct sum of right ideals and left ideals respectively, then the length of R in left ideals and right ideals is the same.*

LEMMA 7. *If M is a completely indecomposable A - R module, then $[0: (0: n); A] = n$ and $[0: (0: N; A)] = N$, where n is a right ideal in A and N is an R submodule.*

Proof. Let $N = N_j$ in the composition series

$$0 \subset N_0 \subset N_1 \subset \cdots \subset N_\mu = M$$

of R submodules for M . From Theorem 3.1 it follows that

$$0 \subset [0: (0: N_0; A)] \subset \cdots \subset [0: (0: N_\mu; A)] = M$$

is a composition series of R submodules for M . Clearly $[0: (0: N_i; A)] \subseteq N_i$, for all i . Since N_0 is a minimal R submodule, it follows that $N_0 = [0: (0: N_0; A)]$, and this implies that $N_i = [0: (0: N_i; A)]$ for all i . The other part of this lemma is proved in like manner.

A similar lemma can be stated for a right ideal n in R and an A submodule

N , by interchanging the letters R and A in the preceding lemma. The following lemma follows immediately.

LEMMA 8. *In a completely indecomposable A - R module M , an R submodule N is completely determined by its annihilating right ideal in A , and an A submodule is completely determined by its annihilating right ideal in R .*

THEOREM 3.3. *If M is a completely indecomposable A - R module, then the set \mathfrak{E} of A endomorphisms of M consists of the multiplications of the elements of M with the elements of R . Hence, if q' is the annihilating ideal of M in R , then \mathfrak{E} is ring isomorphic with the factor ring R/q' .*

Proof. We shall consider \mathfrak{E} not as a ring, but as a module with R as right operator domain, according to the following definitions.

$$\begin{aligned}x(H_1 + H_2) &= xH_1 + xH_2, \\x(H\alpha) &= (xH)\alpha,\end{aligned}$$

for all $x \in M$, $H_1, H_2, H \in \mathfrak{E}$, $\alpha \in R$.

To any $\alpha \in R$ there corresponds the mapping $H(\alpha)$ in \mathfrak{E} defined by $xH(\alpha) = x\alpha$ for all $x \in M$. If R is considered as a right module over its set of right multiplications, then the correspondence $\alpha \rightarrow H(\alpha)$ is clearly an R endomorphism from R into \mathfrak{E} with kernel q' . Hence \mathfrak{E} contains an R submodule E' which is R isomorphic to R/q' . We have but to show that $\mathfrak{E} = E'$. Since E' has finite length, namely the length μ of R/q' in right ideals, we shall show that \mathfrak{E} as an R module has length μ . This would imply that $\mathfrak{E} = E'$.

By Theorem 3.2, since R/q' has length μ for R submodules (right ideals), the length of M for A submodules will be μ . Let

$$(1) \quad 0 \subset N_0 \subset N_1 \subset \cdots \subset N_\mu = M$$

be a composition series of M in A submodules. Let E_i , for $i=0, 1, \cdots, \mu$, be the R submodules of \mathfrak{E} which annihilate N_i , i.e., $H \in E_i$ if $N_i H = 0$. We shall show that the chain

$$(2) \quad \mathfrak{E} \supseteq E_0 \supseteq E_1 \supseteq \cdots \supseteq E_\mu = 0',$$

where $0'$ denotes the zero-endomorphism, will be a composition series of R submodules for \mathfrak{E} . This we shall do by showing that E_{i-1}/E_i is R isomorphic to the division ring R/\mathfrak{p}' .

If $H \in E_{i-1}$, then $N_i H$ is an A submodule which is homomorphic with N_i . Since $N_{i-1} H = 0$, the length of $N_i H$ is at most 1; hence $N_i H \subseteq N_0 = (0: \mathfrak{p}')$. Let x be a fixed nonzero element of N_0 and x_i be a fixed element of N_i not in N_{i-1} . Then $xR = N_0$ and $N_{i-1} + x_i A = N_i$. Since $N_i H \subseteq N_0 = xR$, there exists an $\alpha \in R$ such that $x_i H = x\alpha$. Let $\bar{\alpha}$ denote the coset of α in R/\mathfrak{p}' . Since \mathfrak{p}' is the annihilating ideal of N_0 in R , the correspondence $H \rightarrow \bar{\alpha}$ is an R homomorphism of E_{i-1} into R/\mathfrak{p}' with kernel E_i . Hence E_{i-1}/E_i is R isomorphic with an R submodule of R/\mathfrak{p}' . However, since the only submodules of R/\mathfrak{p}' are 0 and

R/\mathfrak{p}' itself, all that remains to show is that $E_{i-1}/E_i \neq 0'$.

Let $E_i^\#$ be the set of all elements α in R such that $N_i\alpha = 0$, $i = 0, 1, \dots, \mu$. Then by Theorem 3.2, $E_{i-1}^\# \supset E_i^\#$, and consequently there exists an element $\alpha \in R$ such that $N_i\alpha \neq 0$ and $N_{i-1}\alpha = 0$. From this it follows that $H(\alpha)$ is contained in E_{i-1} and not in E_i . Hence $E_{i-1}/E_i \neq 0'$, and therefore (2) is a composition series. Thus $E' = \mathcal{C}$.

COROLLARY 1. *Let M be a completely indecomposable A - R module and let N be an A - R submodule. Then every A endomorphism H^* of N can be extended to an A endomorphism H of M . If H^* is an automorphism of $N \neq 0$, then H is an automorphism of M .*

Proof. If N is an A - R submodule of M , then N is a completely indecomposable A - R module. Hence, it follows from Theorem 3.3 that if H^* is an A endomorphism, then there exists an element $\alpha \in R$ such that $xH^* = x\alpha$ for all $x \in N$. The endomorphism $xH = x\alpha$ for all $x \in M$ is clearly an extension of H^* to an A endomorphism H of M . If H^* is an automorphism of $N \neq 0$, then $\alpha \notin \mathfrak{p}^\#$, where $\mathfrak{p}^\#$ is the radical of the zero-module of N in R . If \mathfrak{p}' is the radical of the zero-module of M in R , clearly $\mathfrak{p}' \subseteq \mathfrak{p}^\#$, and since \mathfrak{p}' is a maximal ideal, then $\mathfrak{p}' = \mathfrak{p}^\#$. Hence $\alpha \notin \mathfrak{p}'$, which implies H is an automorphism of N , since α is a unit of $R^{(1)}$.

LEMMA 9. *If N_1 and N_2 are two R submodules of M with respectively the annihilating right ideals \mathfrak{n}_1 and \mathfrak{n}_2 , then the annihilating right ideal of $N_1 \cap N_2$ is $\mathfrak{n}_1 + \mathfrak{n}_2$.*

Proof. Let $N_3 = N_1 \cap N_2$. Then if \mathfrak{n}_3 is the annihilating right ideal of N_3 in A , we have, by Lemma 8, $\mathfrak{n}_3 = (0 : N_3; R)$. Since $(0 : \mathfrak{n}_1 + \mathfrak{n}_2) = (0 : \mathfrak{n}_1) \cap (0 : \mathfrak{n}_2) = N_1 \cap N_2 = N_3$, then, by Lemma 8, $0 : (0 : \mathfrak{n}_1 + \mathfrak{n}_2); R = (0 : N_3; R) = \mathfrak{n}_3$.

The following lemma which holds for arbitrary modules over a noncommutative ring is proved by Snapper [7, p. 131].

LEMMA 10. *Let M and W be two additive groups with the same ring R as operator domain. Let N_1 and N_2 be two R submodules of M and let W_1 and W_2 be two R submodules of W . Let I_1 be an R isomorphism between N_1 and W_1 , and let I_2 be an R isomorphism between N_2 and W_2 . Let I_1 map $N_3 = N_1 \cap N_2$ isomorphically onto $W_3 = W_1 \cap W_2$, i.e., I_1 induces an R isomorphism I_1^* between W_3 and N_3 . In the same way, let I_2 map N_3 isomorphically onto W_3 . Finally, let $I_1^* = I_2^* = I_3$. Then there exists an R isomorphism between $N_1 + N_2$ and $W_1 + W_2$ which is simultaneously an extension of I_1 , I_2 and I_3 .*

DEFINITION 3.4. A completely indecomposable A - R module M is said to be *duo*, provided every R submodule is an A submodule and every A submodule is an R submodule.

⁽¹⁾ It is shown in Jacobson [3, p. 57] that if R/\mathfrak{p} is a division ring, then the elements not in \mathfrak{p} are units of R , if \mathfrak{p} is a nil-ideal.

Since the set of annihilators of an A - R submodule in A or R is a two-sided ideal, we have by Theorem 3.2 and Lemma 7

THEOREM 3.4. *If M is a duo completely indecomposable A - R module, then the length of M in A - R submodules, the length of A/q in two-sided ideals, and the length of R/q' in two-sided ideals, where q and q' are the annihilating ideals of M in A and R respectively, are the same. In addition, if n is a two-sided ideal of A and N is an A - R submodule of M , then*

$$0:(0:n); A = n \quad \text{and} \quad 0:(0:N; A) = N^{(2)}.$$

LEMMA 11. *Let M be a duo completely indecomposable A - R module. If M has a unique maximal A - R submodule, then M is R isomorphic to R/q' and A isomorphic to A/q , where q' and q are the annihilating ideals of M in R and A respectively.*

Proof. If N is the unique maximal A - R submodule of M , then for $x \in M$, $x \notin N$, it follows that $xR = xA = M$. The mapping $a \rightarrow xa$, all $a \in A$, is an A homomorphism of A onto M with kernel q , for suppose $xa = 0$, then $xaR = 0$, which implies that $xRa = 0$, i.e., $Ma = 0$.

The following is an extension of a theorem proved by Snapper [7, p. 133].

THEOREM 3.5. *Let M be a duo completely indecomposable A - R module and W a duo completely indecomposable A - K module, then M is A isomorphic to W if and only if they have the same annihilating ideals in A .*

Proof. It is obvious that if M and W are A isomorphic, they have the same annihilating ideals in A . Hence we assume that M and W have the same annihilating ideal q in A and prove they are A isomorphic.

Since the length of M and W in A - R submodules are both equal to the length of the difference module A/q in A submodules (two sided ideals), let μ be the common length of W , M and A/q . If $\mu = 1$, both M and W have their zero submodules as unique maximal A - R submodules, and hence, by Lemma 11, are A isomorphic to A/q , and hence isomorphic to each other. Consequently, we assume $\mu > 1$ and make our induction hypothesis that Theorem 3.5 has been proved for $\mu = 1, 2, \dots, \mu - 1$. There are two cases to be considered.

CASE 1. *One of the two modules, say M , has a unique maximal A - R submodule.* Since M has a unique maximal submodule, it follows from Lemma 11 that M is A isomorphic to A/q . This implies that A/q contains a unique minimal two sided ideal. Consequently, there exists only one two-sided ideal q_1 where the length of A/q_1 is $\mu - 1$ in two-sided ideals. If $0'$ is the zero element of W , the length of $W_1 = (0':q_1)$ is $\mu - 1$ and hence W_1 is a maximal A - R submodule of W . We assert that W_1 is the only maximal A - R submodule of W .

(2) Notice that this implies that every right ideal of A or R is a two-sided ideal.

Let W_2 be any maximal A - R submodule of W with q_2 as annihilating ideal in A . Then, by Theorem 3.4, the length of W_2 is $\mu - 1$, and the length of A/q_2 is $\mu - 1$. Consequently, $q_1 = q_2$ and $W_2 = (0' : q_2) = (0 : q_1) = W_1$. Lemma 11 then implies that M and W are A isomorphic to A/q and hence W and M are A isomorphic.

CASE 2. *Both M and W have at least two different maximal A submodules.* Let N_1 and N_2 be two different maximal A - R submodules of M , which implies that $M = N_1 + N_2$. The length of N_1 and N_2 is $\mu - 1$. If q_1 and q_2 are the annihilating ideals in A of N_1 and N_2 , then $q_1 = (0 : N_1; A)$ and $q_2 = (0 : N_2; A)$. Furthermore, by Theorem 3.4, $q_1 \neq q_2$ and the length of both the difference modules A/q_1 and A/q_2 in A submodules (two-sided ideals) is $\mu - 1$. Consider the A - R submodules $W_1 = (0' : q_1)$ and $W_2 = (0' : q_2)$ of W . Then, by Theorem 3.4, $W_1 \neq W_2$, and the length of W_1 and W_2 is $\mu - 1$, i.e., W_1 and W_2 are distinct maximal A - R submodules of W , which implies that $W_1 + W_2 = W$. Since N_1 is a duo completely indecomposable A - R module and W_2 is a duo completely indecomposable A - K module with the same annihilating ideal q_1 , it follows by induction that N_1 and W_1 are A isomorphic. In the same way N_2 and W_2 are A isomorphic. Hence, let I_1 be the A isomorphism which maps N_1 onto W_1 and I_2 the A isomorphism that maps N_2 onto W_2 . We claim that I_1 maps $N_3 = N_1 \cap N_2$ A isomorphically onto $W_3 = W_1 \cap W_2$ and that I_2 maps N_3 A isomorphically onto W_3 . In the first place, the annihilating ideal of N_3 and W_3 in A is $q_1 + q_2$ according to Lemma 9. In the second place, both $N_3 I_1$ and $N_3 I_2$ are A - R submodules of W which are A isomorphic with N_3 . Hence these modules have $q_1 + q_2$ as annihilating ideals in A , which implies, according to Theorem 3.4, that $N_3 I_1 = N_3 I_2 = W_3$. Let I_1^* and I_2^* be the A isomorphism which map N_3 onto W_3 and which are induced respectively by I_1 and I_2 . However, we cannot as yet apply Lemma 10 since it may be that $I_1^* \neq I_2^*$. However, we shall change I_1 into a new A isomorphism J_1 such that $J_1^* = I_2^*$. Let $(I_1^*)^{-1}$ be the inverse of I_1^* ; hence $(I_1^*)^{-1}$ maps $W_3 A$ isomorphically onto N_3 . Then $(I_1^*)^{-1} I_2^*$ is clearly an A isomorphism, say H^* , of W_3 onto itself. Since $W_3 \subseteq W_1$, it follows from Theorem 3.3, Corollary 1, that there exists an A isomorphism H of W_1 onto itself, where H is an extension of H^* . For J_1 we then take the A isomorphism $J_1 = I_1 H$ which clearly maps N_1 onto W_1 . Furthermore, if J_1^* is the A isomorphism between N_3 and W_3 , induced by J_1 , then $J_1^* = I_2^*$ because, if $x \in N_3$, then $x J_1^* = x I_1 H = x I_1^* H$ and, since $x I_1^* \in W_3$, then $x I_1^* H = x I_1^* H^* = x I_1^* (I_1^*)^{-1} I_2^* = x I_2^* = x I_2$. We can conclude from Lemma 10 that there exists an A isomorphism between $N_1 + N_2 = M$ and $W_1 + W_2 = W$ which proves our theorem.

In terms of representations this theorem has the following meaning. Let M and W be faithful representation spaces for a ring A with identity, where \mathfrak{C}_1 and \mathfrak{C}_2 are the ring of R endomorphisms of M and W respectively. Consider M and W as representation spaces for \mathfrak{C}_1 and \mathfrak{C}_2 . Then, if every A subspace of M is a \mathfrak{C}_1 subspace and if every \mathfrak{C}_1 subspace of M is an A subspace, and if

every A subspace of W is a \mathfrak{C}_2 subspace and every \mathfrak{C}_2 subspace of W is an A subspace, and if, in addition, M and W have composition series of A subspaces and unique minimal A subspaces, these representations are equivalent.

3'. Examples. Examples of completely indecomposable A - R modules where A and R are commutative are given by Snapper in [7].

EXAMPLE 3'.1. Every Grassmann⁽³⁾ Algebra with finite basis can be considered as a duo completely indecomposable A - R module by taking as R the ring of right multiplications and as A the ring of left multiplications. Suppose the basis elements are e_1, e_2, \dots, e_n . Then this A - R module is duo since $e_i e_j = -e_j e_i$, and satisfies II-1 and II-2 since $xR = xA$, where $x = e_1 \cdot e_2 \cdot \dots \cdot e_n$, is a unique minimal A - R submodule (two-sided ideal).

EXAMPLE 3'.2. Let F be a field and x_1, x_2 and x_3 be indeterminants. Consider $H = F[1, x_1, x_2, x_3]$, where the following conditions are satisfied:

- (1) Associate law of multiplication.
- (2) Two sided distributive law.
- (3) $a1 = 1a = a$, $ax_i = x_i a$, $i = 1, 2, 3$, for all $a \in F$.
- (4) $x_i x_j x_k = x_1$, $i, j, k = 2, 3$.
- (5) $x_i x_j x_k x_l = 0$, $i, j, k, l = 1, 2, 3$.

H is an algebra with finite bases over F , for if $x \in H$, then $x = a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_2^2 + a_5 x_3^2 + a_6 x_2 x_3 + a_7 x_3 x_2$. Properties (4) and (5) imply $0 = x_1^2 = x_1 x_i$, $i = 2, 3$. If we let R be the ring of right multiplications and A be the ring of left multiplications, then H is a completely indecomposable A - R module, for $x_1 R = x_1 H = H x_1 = x_1 A$ is a unique minimal A submodule and a unique minimal R submodule. H is not duo since $x_2 H = x_2 R$ is not an A submodule because it does not contain $x_3 x_2$.

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UNIVERSITY OF WISCONSIN,
MADISON, WIS.

⁽³⁾ For a discussion of Grassmann Algebras read *Theory of Lie groups* by C. C. Chevalley.