

SOME THEOREMS CONCERNING BROWNIAN MOTION

BY

G. A. HUNT⁽¹⁾

Given a random time T and a Markoff process $X(\tau)$ with stationary transition probabilities, one can define a new process $X'(\tau) = X(\tau + T)$. It is plausible, if T depends only on the $X(\tau)$ for τ less than T , that $X'(\tau)$ is a Markoff process with the same transition probabilities as $X(\tau)$ and that, when the value of $X'(0) \equiv X(T)$ is fixed, the process $X'(\tau)$ is independent of the history of $X(\tau)$ before time T . Although mathematicians use this extended Markoff property, at least as a heuristic principle, I have nowhere found it discussed with rigor. We begin by proving a precise version for Brownian motion (Theorem 2.5, with an extension in §3.3). Our statement has the good points that the hypotheses are easy to verify, that the proof is thoroughly elementary (it even avoids conditional probabilities), and that it holds for all processes with stationary independent increments (see §3.1). I have not pushed the scope of the proof to the limit because it requires continuity of the sample functions on the right and it is probable that a version of the extended Markoff property holds for processes which are only separable.

In §4 and §5 we use the extended Markoff property to study the transition probability density $q(\tau, r, s)$ of a Brownian motion in R^n in which a particle is extinguished the moment it hits a closed set E . It turns out that $q(\tau, r, s)$, which can be defined without ambiguity for every r and s in R^n , is the fundamental solution of the heat equation on $R^n - E$ with boundary values 0 (at the regular points of E), and that $G(r, s) = \int_0^\infty q(\tau, r, s) d\tau$ is the Green's function for the Laplacian on $R^n - E$. The content of these sections must be considered more or less as common knowledge; however, I believe some of the details are new, especially those concerning irregular points and the probabilistic interpretation.

In writing §4 and §5 I profited from many discussions with Mark Kac and Daniel Ray. Had I known of Doob's fundamental paper [4] on the heat equation it is likely that these sections would have been cast in a different form.

Kac has made the following conjecture. Consider in the plane a closed bounded domain E and a point r outside E ; the probability that a Brownian motion starting from r does not meet E by time τ is for large τ asymptotic to $2\pi H(r)/\ln \tau$. Here $H(r)$ is the Green's function of $R^2 - E$ with pole at in-

Received by the editors July 20, 1955.

⁽¹⁾ This research was supported in part by the United States Air Force, through the Office of Scientific Research of the Air Research and Development Command.

finity. In §6 we prove a slightly more general statement: Let f be a bounded Borel measurable function defined on the compact set E of R^2 and let $F(r)$ be the solution of the Dirichlet problem on $R^2 - E$ with boundary value f . The function $F(r)$ is the limit, as τ becomes large, of the solution $F(\tau, r)$ of the heat equation $v_\tau = \Delta v/2$ on $R^2 - E$ with initial value 0 and boundary value f . Both $F(r)$ and $F(\tau, r)$ can be defined naturally for every point r in the plane. Then without exceptions the product $[F(r) - F(\tau, r)] \ln \tau$ approaches $2\pi H(r)F(\infty)$, where $F(\infty)$ is the limit of $F(r)$ as $|r| \rightarrow \infty$ and $H(r)$ has the meaning given above.

1. Preliminary. This section presents the definitions and the elementary statements which are used in §2. The phrases in *italics* are defined by the sentences in which they occur.

1.1. A *space* is a set Z together with a Borel field $\mathcal{B}(Z)$ of subsets of Z which has Z as a member. If Z is a topological space we shall take $\mathcal{B}(Z)$ to be the Borel field of the topology. If Z_1 and Z_2 are spaces we make $Z_1 \times Z_2$ a space by taking $\mathcal{B}(Z_1 \times Z_2)$ to be $\mathcal{B}(Z_1) \times \mathcal{B}(Z_2)$. A *measurable* function f from the space Z_1 to the space Z_2 is a function with the property that $f^{-1}(A) \in \mathcal{B}(Z_1)$ for every A in $\mathcal{B}(Z_2)$.

1.2. A triple (Ω, \mathcal{F}, P) is a *probability field* if the pair (Ω, \mathcal{F}) is a space and if P is a positive measure on \mathcal{F} such that $P(\Omega) = 1$. We write $d\omega$ instead of $P(d\omega)$ for the element of measure; thus the expectation of a random variable X is written $\int_{\Omega} X(\omega) d\omega$.

1.3. A *random point* of the space Z over (Ω, \mathcal{F}, P) is a measurable function Z from the space Ω (with Borel field \mathcal{F}) to the space Z . We shall ordinarily omit the phrase "over (Ω, \mathcal{F}, P) ." The measure $\mu(A) = P\{Z^{-1}(A)\}$ on $\mathcal{B}(Z)$ is the *distribution* of the random point Z .

1.4. If Z_1 and Z_2 are spaces, Z_1 a random point of Z_1 , and $f: Z_1 \rightarrow Z_2$ a measurable function, then the composition $Z_2 = f \circ Z_1$ is a random point of Z_2 . If Z_i is a random point of Z_i (for $1 \leq i \leq k$) then $Z = (Z_1, \dots, Z_k)$ is a random point of the product space $Z_1 \times \dots \times Z_k$. The family $(Z_i)_{1 \leq i \leq k}$ is *independent* if

$$P\{Z^{-1}(A_1 \times \dots \times A_k)\} = \prod_{1 \leq i \leq k} P\{Z_i^{-1}(A_i)\}$$

whenever the A_i belong to $\mathcal{B}(Z_i)$. In proving independence we shall use the fact that it suffices to verify this equality for A_i restricted to a ring of subsets of Z_i which generates $\mathcal{B}(Z_i)$ and which has Z_i as an element.

1.5. We shall be concerned with spaces of curves in number space R^n of dimension n . If $r = (r_1, \dots, r_n)$ and $s = (s_1, \dots, s_n)$ are two points of R^n , set $r \pm s = (r_1 \pm s_1, \dots, r_n \pm s_n)$ and $|r| = (\sum r_i^2)^{1/2}$. Let I_α be the interval $0 \leq \tau \leq \alpha$ (the interval $0 \leq \tau < \infty$ if α is infinite) and \mathcal{X}_α^n the set of continuous functions from I_α to R^n . We take $\mathcal{B}(\mathcal{X}_\alpha^n)$ to be the least Borel field containing all sets $\{x | \beta < x_i(\tau) < \gamma\}$, where β and γ are real, $1 \leq i \leq n$, and $\tau \in I_\alpha$. Let \mathcal{W}_α^n be the

subspace of \mathcal{X}_α^n comprising those curves which begin at the origin of R^n .

1.6. The Borel field $\mathcal{B}(\mathcal{X}_\alpha^n)$ is generated by the sets

$$(1) \quad \{x \mid \beta_{ij} < x_i(\tau_i) \leq \gamma_{ij} \text{ for } 1 \leq i \leq k, 1 \leq j \leq n\}$$

where k is arbitrary, the β_{ij} and γ_{ij} are reals or $\pm \infty$, and the τ_i are taken from I_α . Finite disjoint unions of such sets form a ring of sets having \mathcal{X}_α^n as an element; consequently a measure on $\mathcal{B}(\mathcal{X}_\alpha^n)$ is determined by its values on the sets (1).

1.7. For x in \mathcal{X}_∞^n and β in I_∞ define the element x_β of \mathcal{X}_∞^n by $x_\beta(\tau) = x(\tau + \beta)$. The mapping $\phi(x, \beta) = x_\beta$ of $\mathcal{X}_\infty^n \times I_\infty$ into \mathcal{X}_∞^n is measurable (see the proof in §3.1). Also define x'_β in \mathcal{X}_β^n by $x'_\beta(\tau) = x(\tau)$ for $0 \leq \tau \leq \beta$; it is clear that $x \rightarrow (x'_\beta, x_\beta - x(\beta))$ is a one-one measurable mapping of \mathcal{X}_∞^n onto $\mathcal{X}_\beta^n \times \mathcal{W}_\infty^n$.

1.8. A random point X of \mathcal{X}_α^n may be considered a measurable function from $I_\alpha \times \Omega$ to R^n . We write $X(\tau, \omega)$ to denote the value of this function at the point (τ, ω) and $X(\tau)$ to denote the random point $X(\tau, \cdot)$ of R^n . Of course $X(\omega)$ denotes the curve assigned to ω by X , but there is little danger of confusing $X(\tau)$ with $X(\omega)$. We also write $X = (X_1, \dots, X_n)$ to display the coordinates of X .

1.9. We define a *Brownian motion in R^n* to be a random point $X = (X_1, \dots, X_n)$ of \mathcal{X}_α^n with the properties

(i) for $0 < \tau_1 < \dots < \tau_k < \alpha$ the random points

$$X(0), X(\tau_1) - X(0), \dots, X(\tau_k) - X(\tau_{k-1})$$

of R^n are independent;

(ii) for $0 \leq \sigma < \tau < \alpha$ the random variables

$$X_1(\tau) - X_1(\sigma), \dots, X_n(\tau) - X_n(\sigma)$$

are independent and each is Gaussian with mean 0 and variance $\tau - \sigma$.

The usual definition of a (separable) Brownian motion requires continuity of the sample paths only with probability 1. Such a process, after being modified on a set of probability 0, becomes a Brownian motion in our sense and it will be clear that the results we obtain apply to all separable Brownian motions.

It follows from §1.6 that conditions (i) and (ii) and the distribution of $X(0)$ on R^n determine $\mathcal{P}\{X^{-1}(A)\}$ for every A in $\mathcal{B}(\mathcal{X}_\alpha^n)$.

A *Wiener process in R^n* is a Brownian motion whose range is in \mathcal{W}_∞^n , that is to say, a Brownian motion starting at the origin of R^n and defined for all positive time.

2. The extended Markoff property. The first four numbers of this section are the hypotheses and definitions of Theorem 2.5. The space \mathfrak{V} , function g , and random point V are introduced only to make precise a certain independence. It is f and T that are important. Indeed, the theorem should be thought of as a statement concerning only the random time T and the

Brownian motion X . In §2.10 we restate a part of Theorem 2.5 from this point of view in the language used in [2].

The conditions in the first number, which are a little too stringent for most applications, will be weakened in §3.3.

2.1. Let \mathcal{U} and \mathfrak{V} be spaces. Let $f: \mathcal{X}_\infty^n \times \mathcal{U} \rightarrow I_\infty \cup \{\infty\}$ and $g: \mathcal{X}_\infty^n \times \mathcal{U} \rightarrow \mathfrak{V}$ be measurable functions with the property: If $f(x, u) = \alpha$ and $x'(\tau) = x(\tau)$ for $0 \leq \tau \leq \alpha$ then $f(x', u) = f(x, u) = \alpha$ and $g(x', u) = g(x, u)$.

2.2. Let $X: \Omega \rightarrow \mathcal{X}_\infty^n$ be a Brownian motion, U a random point of \mathcal{U} , and (X, U) an independent pair.

2.3. Let $\Omega' = \{\omega \mid f(X(\omega), U(\omega)) < \infty\}$. We assume that $\mathcal{P}\{\Omega'\} > 0$ (note that $\Omega' \in \mathcal{F}$) and define a new probability field $(\Omega', \mathcal{F}', \mathcal{P}')$ by taking \mathcal{F}' to comprise the sets $A = B \cap \Omega'$ with B in \mathcal{F} and setting $\mathcal{P}'(A) = \mathcal{P}(A)/\mathcal{P}(\Omega')$.

2.4. Define functions $T: \Omega' \rightarrow I_\infty$, $V: \Omega' \rightarrow \mathfrak{V}$, $W: \Omega' \rightarrow \mathcal{W}_\infty^n$ by

$$T(\omega) = f(X(\omega), U(\omega)),$$

$$V(\omega) = g(X(\omega), U(\omega)),$$

$$W(\tau, \omega) = X(\tau + T(\omega), \omega) - X(T(\omega), \omega).$$

2.5. THEOREM. T, V, W are random points of $I_\infty, \mathfrak{V}, \mathcal{W}_\infty^n$ over $(\Omega', \mathcal{F}', \mathcal{P}')$; the pair (V, W) is independent; and W is a Wiener process.

2.6. The restrictions X', U' of X, U to Ω' are random points over $(\Omega', \mathcal{F}', \mathcal{P}')$; thus §1.4 implies that T and V are random points. So (X', T) is a random point of $\mathcal{X}_\infty^n \times I_\infty$; it follows from §1.4 and §1.7 that $X_{T'}$ is a random point of \mathcal{X}_∞^n if we define $X_{T'}(\tau, \omega) = X'(\tau + T(\omega), \omega)$ for ω in Ω' . Consequently $X_{T'}(0)$ is a random point of R^n , and finally $W = X_{T'} - X_{T'}(0)$ is a random point of \mathcal{W}_∞^n over $(\Omega', \mathcal{F}', \mathcal{P}')$.

The rest of the proof is broken into three steps; only the second requires some ingenuity. We assume without loss of generality that \mathfrak{V} has at least two points, for a second point may be adjoined to \mathfrak{V} if necessary.

2.7. f is constant. The value α of f must be finite, so that $\Omega' = \Omega$. The random point W of \mathcal{W}_∞^n is $W(\tau, \omega) = X(\tau + \alpha, \omega) - X(\alpha, \omega)$. Define also the random point X'_α of \mathcal{X}_∞^n by $X'_\alpha(\tau, \omega) = X(\tau, \omega)$ for $0 \leq \tau \leq \alpha$. It follows at once from the definition of Brownian motion that the pair (X'_α, W) is independent; since (U, X) is an independent pair and both X'_α and W are functions of X , the triple (U, X'_α, W) is also independent. There is clearly a function $g': \mathcal{X}_\infty^n \times \mathcal{U} \rightarrow \mathfrak{V}$ such that $g(x, u) = g'(x'_\alpha, u)$, where x'_α is $x'_\alpha(\tau) = x(\tau)$ for $0 \leq \tau \leq \alpha$. So $V(\omega) = g'(X'_\alpha(\omega), U(\omega))$, and consequently the pair (V, W) is independent. It is also clear that W is a Wiener process.

2.8. f takes on countably many values. Let $A \in \mathcal{B}(\mathcal{W}_\infty^n)$ and let $B \in \mathcal{B}(\mathfrak{V})$ with B a proper subset of \mathfrak{V} . We shall verify

$$(1) \quad \mathcal{P}\{W^{-1}(A) \cap V^{-1}(B)\} = Q\mathcal{P}\{V^{-1}(B)\},$$

where Q is the probability that a Wiener process belongs to A . This is enough

to prove the theorem. For equation (1), written both for B and $\mathfrak{V} - B$, implies $\mathcal{P}'\{W(A)\} = Q$, so that W must be a Wiener process. Then (1) shows that V and W are independent.

Let $\alpha_1, \alpha_2, \dots$ be the finite values of f and let v be a point of \mathfrak{V} not in B . For each i define the random point V_i of \mathfrak{V} over $(\Omega, \mathcal{F}, \mathcal{P})$ by setting

$$V_i(\omega) = \begin{cases} V(\omega) & \text{if } \omega \in \Omega' \text{ and } T(\omega) = \alpha_i, \\ v & \text{otherwise.} \end{cases}$$

Let W_i be the Wiener process $W_i(\tau, \omega) = X(\tau + \alpha_i, \omega) - X(\alpha_i, \omega)$.

The justification of the following computation becomes clear if we define the V_i in another way. Let $f_i: \mathcal{X}_\infty^n \times \mathcal{U} \rightarrow I_\infty$ be the constant function with value α_i and let $g_i: \mathcal{X}_\infty^n \times \mathcal{U} \rightarrow \mathfrak{V}$ be the function

$$g_i(x, u) = \begin{cases} g(x, u) & \text{if } f(x, u) = \alpha_i, \\ v & \text{if } f(x, u) \neq \alpha_i. \end{cases}$$

It is easy to verify that the pair (f_i, g_i) satisfies the conditions of §2.1 and that V_i, W_i are defined by f_i, g_i precisely as V, W are defined by f, g . According to the preceding number the pair (V_i, W_i) is independent and W_i is a Wiener process. Note also that the condition $\{W_i(\omega) \in A, V_i(\omega) \in B, \omega \in \Omega', T(\omega) = \alpha_i\}$ on ω can be simplified to $\{W_i(\omega) \in A, V_i(\omega) \in B\}$ since $V_i(\omega) \in B$ implies both $\omega \in \Omega'$ and $T(\omega) = \alpha_i$.

Matters being so, we have

$$\begin{aligned} \mathcal{P}\{\Omega'\} \mathcal{P}'\{W^{-1}(A) \cap V^{-1}(B)\} &= \mathcal{P}\{W(\omega) \in A, V(\omega) \in B, \omega \in \Omega'\} \\ &= \sum_i \mathcal{P}\{W(\omega) \in A, V(\omega) \in B, \omega \in \Omega', T(\omega) = \alpha_i\} \\ &= \sum_i \mathcal{P}\{W_i(\omega) \in A, V_i(\omega) \in B\} \\ &= \sum_i \mathcal{P}\{W_i(\omega) \in A\} \mathcal{P}\{V_i(\omega) \in B\} \\ &= Q \sum_i \mathcal{P}\{V_i(\omega) \in B\} \\ &= Q \mathcal{P}\{V_i(\omega) \in B \text{ for some } i\} \\ &= Q \mathcal{P}\{V(\omega) \in B, \omega \in \Omega'\} \end{aligned}$$

and this equation becomes (1) upon dividing by $\mathcal{P}(\Omega')$.

2.9. f is arbitrary. Let A be a set in \mathcal{W}_∞^n defined by inequalities

$$\beta_{ij} \leq w_j(\tau_i) \leq \gamma_{ij}, \quad 1 \leq i \leq k, 1 \leq j \leq n, \tau_i > 0$$

and let A° be the interior of A , that is, the set defined by similar relations with strict inequalities. Assuming that $\mathcal{P}'\{W(\omega) \in A\} = \mathcal{P}'\{W(\omega) \in A^\circ\}$ we verify equation (1) for A and an arbitrary B in $\mathcal{B}(\mathfrak{V})$.

Define f_m by

$$f_m(x, u) = \begin{cases} \infty & \text{if } f(x, u) = \infty, \\ \frac{l+1}{m} & \text{if } \frac{l}{m} \leq f(x, u) < \frac{l+1}{m}. \end{cases}$$

Since $f_m \geq f$ the pair (f_m, g) satisfies the conditions of §2.1. If W_m is defined in terms of f_m as W in terms of f , the preceding number shows that the pair (V, W_m) is independent and that W_m is a Wiener process over $(\Omega', \mathcal{F}', \mathcal{P}')$. For every ω in Ω' and every τ , moreover, $W_m(\tau, \omega) \rightarrow W(\tau, \omega)$ as $m \rightarrow \infty$. So $W(\omega) \in A$ if $W_m(\omega) \in A$ for all large m , and $W_m(\omega) \in A^\circ$ for sufficiently large m if $W(\omega) \in A^\circ$. Denoting by Q the probability that a Wiener process belongs to A or to A° (the two events have the same probability), we have

$$\begin{aligned} \mathcal{P}'\{W(\omega) \in A^\circ, V(\omega) \in B\} &\leq \lim_{m \rightarrow \infty} \mathcal{P}'\{W_m(\omega) \in A^\circ, V(\omega) \in B\} \\ &= Q\mathcal{P}'\{V(\omega) \in B\} \\ &= \lim_{m \rightarrow \infty} \mathcal{P}'\{W_m(\omega) \in A, V(\omega) \in B\} \\ &\leq \mathcal{P}'\{W(\omega) \in A, V(\omega) \in B\}. \end{aligned}$$

Our assumption implies the equality of the two extreme members. Thus (1) is established.

For a fixed choice of the τ_i the equation $\mathcal{P}'\{W(\omega) \in A\} = \mathcal{P}'\{W(\omega) \in A^\circ\}$ holds for all but countably many choices of the β_{ij} and γ_{ij} . This implies by an easy passage to the limit that (1) is true whenever A is a set of the type mentioned in §1.6. The remark of that number then shows that (1) must be true for every A in $\mathcal{B}(\mathcal{W}_\infty^n)$.

The proof is completed by a repetition of the beginning of §2.8. Note that the proof has used only the facts that the theorem is true for each pair (f_k, g) , that f_k and f are infinite together, and that f_k tends to f .

2.10. The following is the most useful version of Theorem 2.5, rephrased in the language used in [2].

Let T be a non-negative random variable and let $\{X(\tau); \tau \in I_\infty\}$ be a separable Brownian motion with initial position independent of the later motion. Define $X^(\tau)$ to be $X(\tau)$ if $\tau < T$ and to be $X(T)$ if $\tau \geq T$. If T is measurable on the sample space of the $X^*(\tau)$ then $\{X(\tau+T); \tau \in I_\infty\}$ is a Brownian motion in which the initial position is independent of the later motion.*

We have simplified the statement by assuming that the stopping time T is always finite and does not involve the auxiliary variable U and by asserting only a part of the independence of the future and the past. The phrase " T is measurable on the sample space of the $X^*(\tau)$ " is equivalent to "either T can be defined as in §2.4 (with f satisfying the first part of the condition in §2.1

and not involving u) or T is equal almost everywhere to a random variable so defined."

3. Complements and examples. We first sketch the changes to be made in adapting the preceding sections to processes with stationary independent increments, and then discuss two examples of stopping times for Brownian motion. The first example presents an instance in which the auxiliary variable actually occurs. The second is the "first passage time" and is used throughout the rest of this paper. Its treatment requires the weakening of the conditions of §2.1 which is carried out in §3.3.

3.1. In [2] Doob proves that a process in R^n with stationary independent increments can be so normalized that the sample functions are continuous on the right. (His proof for R^1 extends without change.) For the normalized process Theorem 2.5 and its proof hold with a few modifications of the definitions. Take \mathcal{X}_α^n to be the set of functions from I_α to R^n which are continuous on the right and define $\mathcal{B}(\mathcal{X}_\alpha^n)$ as in §1.5 to be the Borel field generated by the sets $A(j, \tau, \gamma, \delta) = \{x \mid \gamma < x_j(\tau) < \delta\}$ with j, τ, γ, δ fixed but arbitrary.

The mapping $\phi: \mathcal{X}_\infty^n \times I_\infty \rightarrow \mathcal{X}_\infty^n$ introduced in §1.7 still has the property that $\phi^{-1}(B)$ belongs to $\mathcal{B}(\mathcal{X}_\infty^n) \times \mathcal{B}(I_\infty)$ whenever B belongs to $\mathcal{B}(\mathcal{X}_\infty^n)$. It suffices to verify this statement for $B = A(j, \tau, \gamma, \delta)$, in which case the proof goes this way: Each set

$$E(k, l, m) = \{(x, \beta) \mid (m-1)/l \leq \beta < m/l, \gamma + 1/k < x_j(m/l) < \delta - 1/k\}$$

belongs to $\mathcal{B}(\mathcal{X}_\infty^n) \times \mathcal{B}(I_\infty)$ and moreover

$$\begin{aligned} \phi^{-1}(A(j, \tau, \gamma, \delta)) &= \{(x, \beta) \mid \gamma < x_j(\beta + \tau) < \delta\} \\ &= \bigcup_k \bigcup_p \bigcap_{l > p} \bigcup_m E(k, l, m) \end{aligned}$$

because each element of \mathcal{X}_∞^n , considered as a function, is continuous on the right.

The notion of Wiener process has an obvious analogue. Now the proofs in §§2.6–2.8 are valid without change; the only change in §2.9 is that A must first be assumed to be an interval of continuity for the analogue of the Wiener process.

3.2. The equation $v_\tau = \Delta v/2 - k(\tau)v$ on $I_\infty \times R^n$, where Δ is the ordinary Laplacian and k a positive Borel measurable function on R^n can be studied in terms of Brownian motion [7]. The appropriate stopping time is defined in this manner: Let $f(x, u)$ be the least value of τ such that $\int_0^\tau k(x(\sigma))d\sigma = u$, or ∞ if there is no such τ . Then $T(\omega) = f(X(\omega), U(\omega))$, where U is a positive random variable with the distribution function $\mathcal{P}\{U(\omega) < u\} = 1 - e^{-u}$ for $u \geq 0$. The physical interpretation is this: A particle wanders according to Brownian motion and is subject to extinction, the probability of extinction in the time interval $(\tau, \tau + d\tau)$ being approximately $k(\tau)d\tau$ if the particle finds itself at the point τ at time τ ; then $T(\omega)$ is the lifetime of the particle ω .

3.3. We shall prepare for the next example by weakening the conditions of §2.1. Let \mathcal{U} be a space R^k and let (f_m, g_m) be a sequence of pairs of functions such that Theorem 2.5 holds for each pair (f_m, g_m) , the limits

$$f(x, u) = \lim f_m(x, u) \quad \text{and} \quad g(x, u) = \lim g_m(x, u)$$

exist for all x and u , and $f(x, u)$ is infinite if and only if each $f_m(x, u)$ is infinite.

The functions f and g need not satisfy §2.1; however, Theorem 2.5 remains true. To see this define T_m, V_m, W_m in terms of the pair (f_m, g_m) ; then $T_m(\omega), V_m(\omega), W_m(\omega)$ tend to $T(\omega), V(\omega), W(\omega)$ for every ω in Ω' . The argument of §2.9 now establishes equation (1) of §2 whenever A is chosen as in §2.9 and B is an interval of continuity for the distribution of V . It is then easy to prove that the equation holds generally.

We shall use this result in the next number; there all the g_m are the same and the argument sketched above becomes a mere repetition of §2.9.

The argument can be modified to allow exceptional sets of measure 0 or to permit the sets on which the f_m are infinite to vary with m .

3.4. Let E be a closed set in R^n and X a Brownian motion. Take $T(\omega)$ to be the infimum of those strictly positive τ for which $X(\tau) \in E$, or ∞ if there are no such τ . We assume that Ω' , the set on which T is finite, has probability greater than 0. (This condition is independent of the choice of X .) Let V be the random point $V(\omega) = (X(T(\omega), \omega), T(\omega))$ of $R^n \times R^1$ over Ω' .

We verify that Theorem 2.5 is true with these definitions. For x in \mathcal{X}_∞^n take $f(x)$ to be the infimum of the strictly positive τ for which $x(\tau) \in E$, or ∞ if there are no such τ , and let $f_m(x)$ be the maximum of $1/m$ and $f(x)$. The f_m decrease to f , and it is clear that $T(\omega) = f(X(\omega))$. Let E_k be the set of points in R^n at a distance not greater than $1/k$ from E and let α be greater than $1/m$. Then

$$\{x \mid f_m(x) \leq \alpha\} = \bigcup_l \bigcap_k \bigcup_\rho \{x \mid x(\rho) \in E_k\},$$

where ρ runs through the rationals in $(1/l, \alpha)$, and k and l run through the natural numbers. Since each set $\{x \mid x(\rho) \in E_k\}$ is a Borel set in \mathcal{X}_∞^n , this representation of the set on which f_m is not greater than α shows that f_m is Borel measurable. Consequently so is f .

If g is the function

$$g(x) = \begin{cases} (x(f(x)), f(x)) & \text{if } f(x) < \infty, \\ (x(0), 0) & \text{if } f(x) = \infty \end{cases}$$

from \mathcal{X}_∞^n to $R^n \times R^1$, then $g(X(\omega)) = V(\omega)$ for each ω in Ω' and each pair (f_m, g) satisfies §2.1. According to the preceding number Theorem 2.5 holds in the present situation.

It is the minimum of this stopping time and the one defined in §3.2 which

is used in studying the equation $v_r = \Delta v/2 - k(r)v$ on $R^n - E$ (see [6]).

3.5. We remark that the foregoing result still holds for an arbitrary Borel set E . The functions f_m need not be measurable then, but if \mathcal{X}_∞^n is given the compact-open topology they have the property that $f_m^{-1}(A)$ is an analytic set in \mathcal{X}_∞^n whenever A is an open set in I_∞ . This property serves as well as the measurability we have used, in proving that $f_m(X(\omega))$ and $g(X(\omega))$ are random points, provided the probability field is complete.

4. **The stopped Brownian motion.** We study the transition and absorption probabilities of a Brownian motion in R^n which is stopped at the moment it hits a given set. Let us fix the notation.

E is a closed set in R^n , W a Wiener process in R^n , and r a variable point of R^n . Denote by X_r the Brownian motion

$$(1) \quad X_r(\tau, \omega) = r + W(\tau, \omega), \quad \tau \geq 0, \omega \in \Omega,$$

and by $T_r(\omega)$ the infimum of those strictly positive τ for which $X_r(\tau, \omega) \in E$ (or ∞ if there are no such τ). We assume that Ω'_r , the set on which T is finite, has probability greater than 0; this condition does not depend on r (see [3]). On Ω'_r define

$$(2) \quad Y_r(\omega) = X_r(T_r(\omega), \omega),$$

$$(3) \quad W'_r(\tau, \omega) = X_r(\tau + T_r(\omega), \omega) - X_r(T_r(\omega), \omega), \quad \tau \geq 0,$$

so that Y_r is a random point of R^n and W'_r a random point of \mathcal{W}_∞^n over Ω'_r . According to §3.4, W'_r and the random point (Y_r, T_r) of $R^n \times I_\infty$ are independent over Ω'_r and W'_r is a Wiener process.

4.1. We shall use the following results, which are proved in [3]. For every r the probability that $T_r(\omega) = 0$ is either 0 or 1. Of course the probability is 0 if r lies outside E . A point r of E is *regular* if the probability is 1, *irregular* if it is 0. The irregular points of E form a set which is negligible in the sense that no Brownian motion has a positive probability of hitting it at some positive time; one easily verifies that a negligible set has Lebesgue measure 0.

The measure

$$(4) \quad \mu(r, A) = \mathcal{P}\{\omega \in \Omega'_r, Y_r(\omega) \in A\},$$

defined for the Borel subsets A of E , is concentrated on the boundary of E ; for each A it is harmonic in $R^n - E$. If f is a bounded Borel measurable function on E then the function

$$(5) \quad F(r) = \int_{\Omega'_r} f(Y_r(\omega)) d\omega = \int_E f(s) \mu(r, ds)$$

is bounded on R^n and harmonic on $R^n - E$. If E is compact then as $|r|$ becomes large $F(r)$ approaches a limit, which is 0 if $n > 2$. See [3] for the sense in which F is the unique solution of the Dirichlet problem on $R^n - E$ with

boundary values f on E . Note that (5) defines $F(r)$ for every r in R^n .

The reference to [3] is not quite exact, for there Doob discusses a Brownian motion on a connected open set D which is stopped at the moment it hits the boundary of D . However, the translation to the present situation is easy.

$R^n - E$ has a countable number of connected components. Let r be an irregular point of E . It follows at once from Doob's results that there is exactly one component D with the properties: (i) r is a boundary point of D and (ii) for almost all ω there is a positive $\alpha(\omega)$ such that $X_r(\tau, \omega)$ lies in D for $0 < \tau < \alpha(\omega)$. We shall say that r belongs to D ; it will turn out that r behaves very much like an interior point of D .

4.2. The transition probability density of Brownian motion in R^n is

$$(6) \quad p(\tau, r, s) = 1/(2\pi\tau)^{n/2} \exp(-|r-s|^2/2\tau), \quad \tau > 0.$$

It is symmetric in r, s and satisfies

$$(7) \quad \int_{R^n} p(\sigma, r, t) p(\tau, t, s) dt = p(\sigma + \tau, r, s), \quad \sigma, \tau > 0,$$

$$(8) \quad \frac{\partial}{\partial \tau} p(\tau, r, s) = \frac{1}{2} \Delta_s p(\tau, r, s), \quad \tau > 0,$$

where dt is the element of Lebesgue measure and Δ_s is the Laplacian operating on the variable s .

4.3. For r in R^n and $0 < \xi < \eta < \infty$ define

$$P(r, \xi, \eta) = \mathcal{P}\{X_r(\tau, \omega) \in E \text{ for some } \tau \in [\xi, \eta]\}.$$

It follows from the equation

$$P(r, \xi, \eta) = \int_{R^n} p(\epsilon, r, s) P(s, \xi - \epsilon, \eta - \epsilon) ds, \quad 0 < \epsilon < \xi,$$

that $P(r, \xi, \eta)$ is continuous in r . Since $P(r, \xi, \eta)$ is an increasing function of η and a decreasing function of ξ , the limits

$$P(r, \tau) = \lim_{\xi \rightarrow 0} P(r, \xi, \tau) = \mathcal{P}\{T_r(\omega) \leq \tau\},$$

$$P(r) = \lim_{\tau \rightarrow \infty} P(r, \tau) = \mathcal{P}\{T_r(\omega) < \infty\}$$

are lower semi-continuous in r . If r is a regular point of E then $P(r, \tau) = 1$ for every positive τ ; thus, given the positive numbers ϵ and τ there is a neighborhood U of r such that $P(s, \tau) > 1 - \epsilon$ for every s in U .

4.4. For r in R^n define the measure

$$Q(r, A, B) = \mathcal{P}\{T_r(\omega) \in A, Y_r(\omega) \in B\}, \quad A \subset I_\infty, B \subset E,$$

on the Borel field of $I_\infty \times E$. It is clear that $Q(r, I_r, E) = P(r, \tau)$.

Given a regular point r of E , a neighborhood V of r in E , and two positive numbers ϵ and σ , one can find a neighborhood U of r in R^n such that the function $Q(s, I_\sigma, V)$ exceeds $1 - \epsilon$ for all s in U . This follows from the last sentence of the preceding number: Take δ positive so that V includes all points of E distant less than 3δ from r , then choose τ less than σ so that

$$\mathcal{P} \left\{ \max_{0 \leq \eta \leq \tau} |W(\eta)| > \delta \right\} < \epsilon/2.$$

Let U be a neighborhood of r not including points at a distance greater than δ from r and so small that $P(s, \tau) > 1 - \epsilon/2$. It is easy to see that $Q(s, I_\tau, V)$ exceeds $1 - \epsilon$ for s in U , a result slightly stronger than the one to be proved.

4.5. For r in R^n and $\tau > 0$ define the measure

$$q(\tau, r, A) = \mathcal{P} \{ T_r(\omega) \geq \tau, X_r(\tau, \omega) \in A \}$$

on the Borel field of R^n . Then p, q, Q satisfy the relation

$$(9) \quad \int_A p(\tau, r, s) ds = q(\tau, r, A) + \int_A ds \int_{J_r \times E} p(\tau - \sigma, t, s) Q(r, d\sigma, dt)$$

where J_r is the interval $0 \neq \sigma < \tau$. To see this let ϕ be the function

$$\phi(x, \alpha) = \begin{cases} 1 & x(\alpha) \in A, \\ 0 & x(\alpha) \notin A \end{cases}$$

on $X_\infty^n \times I_\infty$ and let Λ be the set on which $T_r(\omega) < \tau$. For s in R^n and any Wiener process W

$$\mathcal{E} \{ \phi(s + W(\omega), \alpha) \} = \int_A p(\alpha, s, t) dt.$$

Note also that $\phi(X_r(\omega), \tau) = \phi(Y_r(\omega) + W'_r(\omega), \tau - T_r(\omega))$ for ω in Λ . Hence by Theorem 2.5

$$\begin{aligned} \int_A p(\tau, r, s) ds &= \int_\Omega \phi(X_r(\omega), \tau) d\omega \\ &= \int_{\Omega - \Lambda} \phi(X_r(\omega), \tau) d\omega + \int_\Lambda \phi(Y_r(\omega) + W'_r(\omega), \tau - T_r(\omega)) d\omega \\ &= q(\tau, r, A) + \int_\Lambda d\omega \int_A p(\tau - T_r(\omega), Y_r(\omega), s) ds \end{aligned}$$

and this is (9) except for the notation.

The details of later applications of Theorem 2.5 are similar to those above and will be omitted.

4.6. Since the left member of (9) is absolutely continuous so also is the

set function $q(\tau, r, A)$. Thus $q(\tau, r, A) = \int_A q(\tau, r, s) ds$ for some point function, and

$$(10) \quad q(\tau, r, s) = p(\tau, r, s) - \int_{J_{\tau \times E}} p(\tau - \sigma, t, s) Q(r, d\sigma, dt)$$

for almost all s . We take this equation to define $q(\tau, r, s)$ for every s in R^n . The integral on the right is lower semi-continuous in s by Fatou's lemma; so $q(\tau, r, s)$ is upper semi-continuous in s and consequently never negative. Also $q(\tau, r, s) \leq p(\tau, r, s)$. If r is a regular point of E then $Q(r, A, B)$ attributes measure 1 to the point $(0, r)$, the integral in (10) reduces to $p(\tau, r, s)$, and $q(\tau, r, s) = 0$ for all s .

If r and s are in distinct components of $R^n - E$ then $q(\tau, r, s)$ vanishes. This follows for almost all s from the probability interpretation; since $q(\tau, r, s)$ is continuous in s on $R^n - E$ (see §4.9) the statement is true without exception.

4.7. Probability arguments usually establish a statement concerning $q(\tau, r, s)$ only for "almost all s ." In order to pass from "almost all" to "all" we show that

$$(11) \quad q(\tau, r, s) = \lim_{\epsilon \searrow 0} \int_{R^n} q(\tau - \epsilon, r, t) p(\epsilon, t, s) dt.$$

To prove this relation, first replace $q(\tau - \epsilon, r, t)$ by the right member of (10) with $\tau - \epsilon$ written for τ and t for s ; then (7) implies

$$(12) \quad \int_{R^n} q(\tau - \epsilon, r, t) p(\epsilon, t, s) dt = p(\tau, r, s) - \int_{J_{\tau - \epsilon \times E}} p(\tau - \sigma, t, s) Q(r, d\sigma, dt)$$

and the right member obviously tends to $q(\tau, r, s)$. One obtains another proof that $q(\tau, r, s)$ is upper semi-continuous by noting that the right member of (12) is continuous in s and decreases with ϵ .

We use (11) to prove

$$(13) \quad \int_{R^n} q(\sigma, r, t) q(\tau, t, s) dt = q(\sigma + \tau, r, s).$$

The probability interpretation shows that for a set A in R^n

$$\int_A ds \int_{R^n - E} q(\sigma, r, t) q(\tau, t, s) dt = \int_A q(\sigma + \tau, r, s) ds.$$

Here the integration on t may be extended over R^n , because $q(\tau, t, s) = 0$ if t is a regular point of E and the irregular points form a set of measure 0. It is now obvious that (13) is true for almost all s . This being so, we have

$$\int_{R^n} q(\sigma, r, t) dt \int_{R^n} q(\tau - \epsilon, t, u) p(\epsilon, u, s) du = \int_{R^n} q(\sigma + \tau - \epsilon, r, u) p(\epsilon, u, s) du$$

for every s . This relation becomes (13) as ϵ decreases to 0; the passage to the limit under the integral sign (in the first member) is justified by the majorization

$$\int_{R^n} q(\tau - \epsilon, t, u) p(\epsilon, u, s) du \leq \max_{u \in R^n} p(\tau, t, s).$$

In (13) the integration on t may be extended over $R^n - E$ rather than over R^n , indeed over the component of $R^n - E$ to which r belongs (the empty set if r is a regular point of E).

4.8. Differentiating $q(\tau, r, A)$ with respect to the measure $\int_A p(\tau, r, s) ds$ instead of Lebesgue measure gives a conditional probability relative to $X_r(\tau) = s$. Consequently

$$(14) \quad q(\tau, r, s) = p(\tau, r, s) P\{T_r(\omega) \geq \tau \mid X_r(\tau) = s\}$$

for almost all s . We use this interpretation to prove that $q(\tau, r, s)$ is symmetric in r and s .

For every positive α let Z_α be the random point

$$Z_\alpha(\sigma, \omega) = W(\sigma, \omega) - \frac{\sigma}{\alpha} W(\alpha, \omega), \quad 0 \leq \sigma \leq \alpha,$$

of \mathcal{X}_α^n . Then Z_α is a Markoff process with the properties:

- (i) Z_α and $W(\alpha)$ are independent;
- (ii) Z'_α , defined by $Z'_\alpha(\sigma) = Z_\alpha(\alpha - \sigma)$ for $0 \leq \sigma \leq \alpha$, is distributed like Z_α on \mathcal{X}_α^n ;
- (iii) the density function of $Z_\alpha(\alpha - \epsilon)$ on R^n is $p(\epsilon - \epsilon^2/\alpha, 0, t)$;
- (iv) $Z_{\alpha-\epsilon}(\sigma) = Z_\alpha(\sigma) - \sigma/(\alpha - \epsilon) Z_\alpha(\alpha - \epsilon)$ for $0 \leq \sigma \leq \alpha - \epsilon$;
- (v) $Z_{\alpha-\epsilon}$ and $Z_\alpha(\alpha - \epsilon)$ are independent. These facts, which are well known, follow at once from Paul Lévy's beautiful construction of the Wiener process in [5].

Property (i) implies

$$\begin{aligned} P\{T_r \geq \tau \mid X_r(\tau) = s\} \\ &= P\{X_r(\sigma) \notin E, 0 < \sigma < \tau \mid X_r(\tau) = s\} \\ &= P\{r + Z_r(\sigma) + (\sigma/\tau)(s - r) \notin E, 0 < \sigma < \tau \mid W(\tau) = s - r\} \\ &= P\{r + Z_r(\sigma) + (\sigma/\tau)(s - r) \notin E, 0 < \sigma < \tau\} \end{aligned}$$

for almost all s . Let us denote the last member, which is an ordinary probability and therefore defined unambiguously, by $R(\tau, r, s)$. Then (14) becomes

$$(15) \quad q(\tau, r, s) = p(\tau, r, s) R(\tau, r, s).$$

We are going to show that this equality holds for every s .

A repetition of the argument leading to (15) yields

$$\begin{aligned}
 (16) \quad & \int_{R^n} q(\tau - \epsilon, r, t) p(\epsilon, t, s) dt \\
 &= p(\tau, r, s) \mathcal{P}\left\{r + Z_r(\sigma) + \frac{\sigma}{\tau}(s - r) \notin E, 0 < \sigma < \tau - \epsilon\right\}
 \end{aligned}$$

for almost all s , where ϵ is any number in the interval $(0, \tau)$. Here the left member is continuous in s . In view of (iv) and (iii)

$$\begin{aligned}
 & \mathcal{P}\left\{r + Z_r(\sigma) + \frac{\sigma}{\tau}(s - r) \in E, 0 < \sigma < \tau - \epsilon\right\} \\
 &= \mathcal{P}\left\{r + Z_{\tau-\epsilon}(\sigma) + \frac{\sigma}{\tau-\epsilon}Z_r(\tau - \epsilon) + \frac{\sigma}{\tau}(s - r) \notin E, 0 < \sigma < \tau - \epsilon\right\} \\
 &= \int_{R^n} A(t) p\left(\epsilon - \epsilon^2/\tau, t, \frac{\tau - \epsilon}{\tau}s + \frac{\epsilon}{\tau}r\right) dt
 \end{aligned}$$

where $A(t)$ is the conditional probability of the event

$$r + Z_{\tau-\epsilon}(\sigma) + \frac{\sigma}{\tau - \epsilon}(t - r) \notin E, \quad 0 < \sigma < \tau - \epsilon,$$

under the condition that

$$Z_r(\tau - \epsilon) + \frac{\tau - \epsilon}{\tau}s + \frac{\epsilon}{\tau}r = t.$$

By (v) this probability is just $R(\tau - \epsilon, r, t)$ for almost all t , so that $A(t)$ does not involve s . Hence the last integral is continuous in s and (16) holds for all s . Now (15) is proved for all s by letting ϵ approach 0 in (16) and taking (11) into account.

Finally, (ii) implies $R(\tau, r, s) = R(\tau, s, r)$. Consequently $q(\tau, r, s)$ is symmetric in r and s .

4.9. We prove

$$(17) \quad \frac{\partial q}{\partial \tau}(\tau, r, s) = \frac{1}{2} \Delta_s q(\tau, r, s), \quad \tau > 0, r \in R^n, s \in R^n - E.$$

For $|t - s|$ bounded below by a positive constant, $p(\tau - \sigma, t, s)$ is a well behaved function of t, s and σ (with $0 \leq \sigma < \tau$). Thus the integral in (10) is a smooth function of s on $R^n - E$ and differentiations with respect to s may be interchanged with the integration over $J_\tau \times E$. This gives

$$(18) \quad \frac{1}{2} \Delta_s q(\tau, r, s) = \frac{1}{2} \Delta_s p(\tau, r, s) - \int_{J_\tau \times E} \frac{1}{2} \Delta_s p(\tau - \sigma, t, s) Q(r, d\sigma, dt)$$

for s in $R^n - E$. Let us also compute $\partial q / \partial \tau$. If $\epsilon > 0$ then

$$\begin{aligned}
 (19) \quad & \frac{1}{\epsilon} \left[\int_{J_{\tau+\epsilon} \times E} p(\tau + \epsilon - \sigma, t, s) Q(r, d\sigma, dt) - \int_{J_{\tau} \times E} p(\tau - \sigma, t, s) Q(r, d\sigma, dt) \right] \\
 &= \int_{J_{\tau} \times E} \frac{1}{\epsilon} [p(\tau + \epsilon - \sigma, t, s) - p(\tau - \sigma, t, s)] Q(r, d\sigma, dt) \\
 &\quad + \frac{1}{\epsilon} \int_{J_{\epsilon} \times E} p(\sigma, t, s) Q(r, \tau - d\sigma, dt).
 \end{aligned}$$

If δ is the distance from s to E then

$$p(\sigma, t, s) \leq \frac{1}{(2\pi\epsilon)^{n/2}} e^{-\delta^2/2\epsilon}, \quad 0 \leq \sigma \leq \epsilon, t \in E,$$

for sufficiently small ϵ , so that in (19) the last term on the right vanishes with ϵ . In the first term on the right the limit on ϵ may be carried under the integral sign. Thus differentiation of (10) with respect to τ gives

$$(20) \quad \frac{\partial q}{\partial \tau}(\tau, r, s) = \frac{\partial p}{\partial \tau}(\tau, r, s) - \int_{J_{\tau} \times E} \frac{\partial p}{\partial \tau}(\tau - \sigma, t, s) Q(r, d\sigma, dt)$$

for s in $R^n - E$. Now (17) follows from (18), (20), and (8).

4.10. The following facts will be useful in the next section.

$$\begin{aligned}
 (21) \quad & 0 \leq p(\tau, r, s) - q(\tau, r, s) \\
 &= \int_{J_{\tau} \times E} p(\tau - \sigma, t, s) Q(r, d\sigma, dt) \\
 &\leq \frac{1}{(2\pi\tau)^{n/2}} e^{-\delta^2/2\tau}
 \end{aligned}$$

for small values of τ , where δ is the distance from s to E .

In particular, if $r \in R^n - E$ then

$$(22) \quad q(\tau, r, r) \sim p(\tau, r, r) = 1/(2\pi\tau)^{n/2}, \quad \tau \rightarrow 0.$$

This relation is also true if r is an irregular point of E . The simplest proof is probably conducted in terms of the process introduced in §4.8. According to (15) we must prove that $R(\tau, r, r) \rightarrow 1$. From the definition of Z_r and property (i)

$$\int_{R^n} R(\tau, r, s) p(\tau, r, s) ds = \mathbb{P}\{X_r(\sigma) \notin E, 0 < \sigma < \tau\},$$

where the right side tends to 1 because r is irregular. Thus

$$\int_{R^n} [1 - R(\tau, r, s)] p(\tau, r, s) ds \rightarrow 0, \quad \tau \rightarrow 0.$$

On the other hand

$$(23) \quad \mathcal{P}\{r + Z_{2\tau}(\sigma) \notin E, 0 < \sigma < \tau\} = \int_{R^n} R(\tau, r, s) p(\tau/2, r, s) ds$$

by (iii), (iv), and (v). Since $p(\tau/2, r, s) \leq 2^{n/2} p(\tau, r, s)$, also

$$\int_{R^n} [1 - R(\tau, r, s)] p(\tau/2, r, s) ds \rightarrow 0, \quad \tau \rightarrow 0,$$

or, what is the same, the right member of (23) tends to 1. From (ii) of §4.8 it follows that $\mathcal{P}\{r + Z_{2\tau}(\sigma) \notin E, \tau < \sigma < 2\tau\}$ also tends to 1. Finally the probability that $r + Z_{2\tau}(\tau)$ belongs to E^2 tends to 1, because r is an irregular point of E . These three facts imply that $R(2\tau, r, r)$ approaches 1, which is what had to be proved.

5. The Green's function of $R^n - E$. We shall prove that

$$(1) \quad G(r, s) \equiv \int_0^\infty q(\tau, r, s) d\tau, \quad r, s \in R^n$$

is the Green's function of $R^n - E$. Note that (1) defines $G(r, s)$ for all r and s in R^n and that $G(r, s)$ is symmetric. If f is a non-negative Borel measurable function on R^n then

$$(2) \quad \begin{aligned} \int_\Omega d\omega \int_0^{T_r(\omega)} f(X_r(\tau, \omega)) d\tau &= \int_0^\infty d\tau \int_{R^n} q(\tau, r, s) f(s) ds \\ &= \int_{R^n} G(r, s) f(s) ds. \end{aligned}$$

It is this interpretation which enables one to use probability arguments to establish a property of G .

We shall deal only with the case $n=2$. For $n>2$ the proofs can be simplified because then $\int_0^\infty p(\tau, r, s) d\tau$ is finite if $r \neq s$, and for $n=1$ straightforward calculation gives the facts at once.

From now on we assume E to be compact. Once this case is settled the results for unbounded E are obtained immediately by considering larger and larger bounded portions of E .

What makes $n=2$ especially difficult is the proof that G is finite for $r \neq s$. We shall first augment E by a large circumference; inside the circumference the integral defining G converges rapidly. We shall use without mention the fact that T_r is finite with probability 1.

5.1. Let C be a circumference of radius ρ (greater than 1) enclosing E . Let $E' = E \cup C$ and define T'_r , $q'(\tau, r, s)$, $G'(\tau, s)$ and so forth in terms of E' . Every point of C is a regular point of E' .

Until §5.3 we consider only points of D , the interior of C .

We first show that $q'(\tau, r, s)$ decreases exponentially in τ . It is clear that

$$\max_{\tau \geq \rho^2} q'(\tau, r, s) \leq \max_{\tau \geq \rho^2} p(\tau, r, s) = \frac{1}{2\pi\rho^2}.$$

If $\tau > 2\rho^2$ we write $\tau = k\rho^2 + \sigma$ with $\rho^2 \leq \sigma < 2\rho^2$ and have

$$\begin{aligned} q'(\tau, r, s) &= \int_D q'(\sigma, r, t) q'(k\rho^2, t, s) dt \\ &\leq \int_D \frac{1}{2\pi\rho^2} \max_t q'(k\rho^2, t, s) dt \\ &= \frac{1}{2} \max_t q'(k\rho^2, t, s). \end{aligned}$$

Applied to $\tau = (k+1)\rho^2$ this inequality shows that

$$\max_{r, s \in D} q'((k+1)\rho^2, r, s) \leq \frac{1}{2} \max_{r, s \in D} q'(k\rho^2, r, s).$$

Consequently $q'(k\rho^2, r, s) \leq 2^{1-k}/(2\pi\rho^2)$. The inequality then implies the existence of a positive γ such that

$$(3) \quad q'(\tau, r, s) < e^{-\gamma\tau}, \quad \tau \geq \rho^2; r, s \in D.$$

For $\tau \leq \rho^2$ there is the obvious majorization

$$(4) \quad q'(\tau, r, s) \leq p(\tau, r, s).$$

Now set $G'(r, s) = \int_0^\infty q'(\tau, r, s) d\tau$. The majorizations of $q'(\tau, r, s)$ ensure that the integral is finite for s different from r . Also $G'(r, r) = 0$ if r is a regular point of E , for then $q'(\tau, r, s)$ vanishes identically in τ and s ; and $G'(r, r) = \infty$ if r is an irregular point of E or a point of $D - E$ by the asymptotic relation (22) of §4.10.

5.2. It is well known that if α is finite and positive

$$(5) \quad \int_0^\alpha p(\tau, r, s) d\tau = \frac{1}{\pi} \ln \frac{1}{|r - s|} + f_\alpha(r - s), \quad r, s \in R^2,$$

where $f_\alpha(t)$ is a continuous function of the point t . Thus (2) and (3) imply

$$(6) \quad G'(r, s) \leq \frac{1}{\pi} \ln \frac{1}{|r - s|} + K, \quad r, s \in D,$$

for some constant K .

$G'(r, s)$ is upper semi-continuous in s (for r, s in D). If r is a regular point of E then $G'(r, s) = 0$ for all s . If r is an irregular point of E or a point of $D - E$ then $G'(r, r) = \infty$, so that $G'(r, s)$ is upper semi-continuous at $s = r$. We

are left to consider continuity at a point s distinct from r . Then in a neighborhood U of s which is at a positive distance from r the majorizations of q' above provide a majorization $q'(\tau, r, s') < f(\tau)$ for all τ and all s' in U , with f an integrable function. Now Fatou's lemma and the upper semi-continuity of q' yield

$$\begin{aligned} \limsup_{s' \rightarrow s} \int_0^\infty q'(\tau, r, s') d\tau &\leq \int_0^\infty \limsup_{s' \rightarrow s} q'(\tau, r, s') d\tau \\ &\leq \int_0^\infty \dot{q}'(\tau, r, s) d\tau. \end{aligned}$$

Since the upper limit may be replaced by limit and the inequalities by equalities when s belongs to $D-E$, the argument proves also that $G'(r, s)$ is continuous in s on $D-E-\{r\}$.

If $r \in D-E$ then $\pi G'(r, s) + \ln |r-s|$ is continuous at $s=r$. Equation (5) and the inequality at the beginning of §4.10 together imply that the function $\pi \int_0^1 q'(\tau, r, s) d\tau + \ln |r-s|$ is continuous at $s=r$. Also $\int_1^\infty q'(\tau, r, s) d\tau$ is continuous at $s=r$ by the same argument as in the preceding paragraph.

Before establishing other properties of G' we make a remark. Even in the general case—that is, without adjoining C to E —for every r and s in R^2

$$\begin{aligned} \int_{R^2} G(r, t) q(\epsilon, t, s) dt &\equiv \int_0^\infty d\tau \int_{R^2} q(\tau, r, t) q(\epsilon, t, s) dt \\ &\equiv \int_\epsilon^\infty q(\tau, r, s) d\tau \end{aligned}$$

increases to $G(r, s)$ as ϵ decreases to 0. If we are dealing with G' and if either r or s lies in D the integration may be extended over D instead of R^2 .

If $s \in D$ then $G'(r, s)$ is subharmonic in r on $D-\{s\}$ and harmonic in r on $D-E-\{s\}$. Let r be a point of $D-\{s\}$ and C' a circumference lying in D and centered at r which separates r from s . Denote by $S(\omega)$ the least τ such that $X_r(\tau) \in C'$ (or ∞ if there is no such τ), by Λ the set on which S is finite. Then $P\{\Lambda\} = 1$ and the random point $Z(\omega) = X_r(S(\omega), \omega)$, defined on Λ , is uniformly distributed on C' . We shall use Theorem 2.5 with S as the stopping time. Let us first assume that no point of E lies on or inside C' . Then $S(\omega)$ does not exceed $T'_r(\omega)$ and

$$\begin{aligned} \int_D G'(r, t) q'(\epsilon, t, s) dt &= \int_\Omega d\omega \int_0^{T'_r(\omega)} q'(\epsilon, X_r(\tau, \omega), s) d\tau \\ (7) \qquad &= \int_\Omega d\omega \int_0^{S(\omega)} q'(\epsilon, X_r(\tau, \omega), s) d\tau \\ &\quad + \int_\Lambda d\omega \int_{S(\omega)}^{T'_r(\omega)} q'(\epsilon, X_r(\tau, \omega), s) d\tau. \end{aligned}$$

The first term of the last member may be written $\int_{D'} G''(r, t) q'(\epsilon, t, s) dt$, according to (2), where D' is the interior of C' and G'' is defined relative to C' (with E disregarded). Here $G''(r, t)$ is integrable over D' since there is a majorization like (6) for G'' , and $q'(\epsilon, t, s)$ tends to 0 with ϵ uniformly on D' because of the inequality in §4.10. So the first term approaches 0 with ϵ . As for the second term, Theorem 2.5 and equation (2) show that it is

$$\int_{\Lambda} d\omega \int_D G'(Z(\omega), t) q'(\epsilon, t, s) dt.$$

This tends to $\int_{\Lambda} G'(Z(\omega), s) d\omega$ as $\epsilon \rightarrow 0$ because the inner integral increases to $G'(Z(\omega), s)$. Finally

$$\begin{aligned} G'(r, s) &= \lim_{\epsilon \rightarrow 0} \int_D G'(r, t) q'(\epsilon, t, s) dt \\ &= \int_{\Lambda} G'(Z(\omega), s) d\omega, \end{aligned}$$

so that $G'(r, s)$ is equal to its average over C' , for Z is distributed uniformly on C' . Thus $G'(r, s)$ is harmonic in r on $D - E - \{s\}$.

If points of E lie inside C' , in particular if $r \in E$, it may happen that $S(\omega)$ is greater than $T'_r(\omega)$ for certain ω . It is easy to see that (7) still holds with the second equality replaced by \leq . The argument then proves that $G'(r, s)$ is subharmonic on $D - \{s\}$.

5.3. Let C_1 be a circumference, with interior D_1 , which encloses E and is included in D . We shall express $G(r, s)$ in the form

$$(8) \quad G(r, s) = G'(r, s) + \int_{C_1} G'(t, s) \nu_r(dt), \quad r \in R^2, s \in D_1,$$

with ν_r a certain positive finite measure on C_1 . The properties of G can then be inferred from those of G' .

In order to justify the passage from "almost all" to "all" easily, we argue at first with the functions

$$(9) \quad G_{\rho}(r, s) = \int_0^{\infty} e^{-\rho\tau} q(\tau, r, s) d\tau$$

and G_{ρ}' , with ρ positive. It is clear that G_{ρ} increases to G as ρ tends to 0; also, G_{ρ} has a probability interpretation similar to that of G . Now, the integrand in (9) is less than $e^{-\rho\tau}$ for large values of τ , uniformly in r and s . Because of this the arguments of the preceding section apply to G_{ρ} ; they show that $G_{\rho}(r, s)$ is upper semi-continuous in s on R^2 and subharmonic in r on $R^2 - \{s\}$. Clearly $G_{\rho}(r, s)$ is symmetric.

For a given r in R^2 let

$S_1(\omega) = T'_1(\omega)$ be the "least" τ such that $X_\tau(\tau, \omega) \in E' = E \cup C$,

$S'_1(\omega)$ be the least $\tau > S_1(\omega)$ such that $X_\tau(\tau, \omega) \in C_1$,

. ,

$S_k(\omega)$ be the least $\tau > S'_{k-1}(\omega)$ such that $X_\tau(\tau, \omega) \in E'$,

$S'_k(\omega)$ be the least $\tau > S_k(\omega)$ such that $X_\tau(\tau, \omega) \in C_1$.

Since E is not negligible for the S_k are finite with probability 1; moreover $S_k(\omega)$ tends to ∞ if all $S_k(\omega)$ are finite, or else $X(\tau, \omega)$ would not be continuous in τ .

Let ν_k be the measure

$\nu_k(A, B) = \mathcal{P}\{S'_k(\omega) \in A, X_\tau(S'_k(\omega), \omega) \in B, X_\tau(\tau, \omega) \notin E \text{ for } 0 < \tau < S'_k(\omega)\}$

on the Borel sets of $I_\infty \times C_1$. If $\mu'(t, A)$ is harmonic measure relative to E' then $\nu_1(I_\infty, C_1) = \mu'(r, C)$ and

$$(10) \quad \nu_{k+1}(I_\infty, C_1) = \int_{C_1} \mu'(t, C) \nu_k(I_\infty, dt)$$

by Theorem 2.5 with stopping time S'_k . Observe that $\mu'(t, C)$ is a harmonic function on $D - E$ which is bounded by 1 and tends to 0 as t approaches a regular point of E . Therefore α , the maximum of $\mu'(t, C)$ for t on C_1 , is strictly less than 1, and (10) implies $\nu_{k+1}(I_\infty, C_1) \leq \alpha \nu_k(I_\infty, C_1)$. Accordingly the sum $\nu \equiv \sum \nu_k$ is a positive measure on $I_\infty \times C_1$ which has a total mass not greater than $\mu'(r, C)/(1 - \alpha)$. Note that ν vanishes if r is a regular point of E .

Consider a set A in D_1 with characteristic function ϕ . If $S_k(\omega) < \tau < S'_k(\omega)$ then either $T_r(\omega) < \tau$ or $X_\tau(\tau, \omega)$ lies outside A . Thus, using Theorem 2.5 with stopping time S'_k and setting $\epsilon_k(\omega) = 1$ if $S'_k(\omega) < T_r(\omega)$ and 0 otherwise, we have

$$\begin{aligned} \int_A G_\rho(r, s) ds &= \int_\Omega d\omega \int_0^{T_r(\omega)} e^{-\rho\tau} \phi(X_\tau(\tau, \omega)) d\tau \\ &= \int_\Omega d\omega \left[\int_0^{S_1} e^{-\rho\tau} \phi(X_\tau(\tau)) d\tau + \sum_k \epsilon_k(\omega) \int_{S'_k}^{S_{k+1}} e^{-\rho\tau} \phi(X_\tau(\tau)) d\tau \right] \\ &= \int_A G'_\rho(r, s) ds + \sum_k \int_\Omega \epsilon_k(\omega) e^{-\rho S'_k} d\omega \int_0^{S_{k+1}-S'_k} e^{-\rho\tau} \phi(X_\tau(\tau + S'_k)) d\tau \\ &= \int_A G'_\rho(r, s) ds + \sum_k \int_{I_\infty \times C_1} e^{-\rho\sigma} \nu_k(d\sigma, dt) \int_A G'_\rho(t, s) ds \\ &= \int_A G'_\rho(r, s) ds + \int_A ds \int_{I_\infty \times C_1} e^{-\rho\sigma} G'_\rho(t, s) \nu(d\sigma, dt). \end{aligned}$$

It follows that

$$(11) \quad G_\rho(r, s) = G'_\rho(r, s) + \int_{I_\infty \times C_1} e^{-\rho\sigma} G'_\rho(t, s) \nu(d\sigma, dt)$$

for almost all s in D_1 . The integral is upper semi-continuous in s on D_1 ; this is proved by using the semi-continuity of $G'_\rho(t, s)$ and Fatou's lemma, the majorization (6) providing the justification. The integral is subharmonic in s on D_1 because each $G'_\rho(t, s)$ is so. We have noted before that G_ρ and G'_ρ are upper semi-continuous in s on D and subharmonic on $D - \{r\}$. Consequently, the fact that (11) holds almost everywhere on D_1 implies that it holds for every s in D_1 except possibly for $s=r$. But if r is an irregular point of E or a point of $D-E$ both $G_\rho(r, r)$ and $G'_\rho(r, r)$ are infinite by (22) of §4.10, and if r is a regular point of E both members of (11) vanish.

Letting ρ approach 0 we obtain (8) by monotone convergence. The measure ν_r is $\nu_r(A) = \nu(I_\infty, A)$.

5.4. We list the properties of G .

(i) $G(r, s) = G(s, r)$ for r, s in R^2 .

(ii) $G(r, s)$ is upper semi-continuous in s .

(iii) $G(r, s)$, as a function of s , is subharmonic on $R^2 - \{r\}$ and harmonic on $R^2 - E - \{r\}$.

(iv) For r in $R^2 - E$ the function $\pi G(r, s) + \ln |r - s|$ is continuous in s at $s=r$.

(v) For every compact set A in R^2

$$(12) \quad G(r, s) \leq - (1/\pi) \ln |r - s| + K, \quad r \in R^2, s \in A,$$

with K depending only on A .

(vi) $G(r, r) = \infty$ if r is an irregular point of E or a point of $R^2 - E$, and $G(r, s) = 0$ for all s if r is a regular point of E .

Statement (i) is implied by the symmetry of $q(r, r, s)$, and (vi) by (22) of §4.10. The upper semi-continuity and the majorization (6) of G' allow one to apply Fatou's lemma (with upper limits) to the right member of (8) when s is in D_1 , and thus prove (ii) with s restricted to D_1 . (Note that $G'(r, s) = 0$ if $s \in D_1$ and $r \notin D$, so that no restrictions need be placed on r .) Likewise (8) and the properties of G' show that $G(r, s)$ is subharmonic in s on $D_1 - \{r\}$ and harmonic on $D_1 - E - \{r\}$. The integral in (8) is continuous in s on $D_1 - E$, so (iv) is true if $r \in D_1 - E$ by what we know of G' . If A is a compact set included in D_1 the majorization (6) of G' shows that $G'(t, s)$ is bounded for t on C_1 and s in A ; also $\nu_r(C_1) \leq 1/(1-\alpha)$ for every r in R^n , and $G'(r, s) = 0$ if $s \in A$ and $r \notin D$. These facts and (6), applied to the representation (8), prove (12) if $A \subset D_1$. Finally, statements (ii) through (v) are proved by letting D and D_1 swell to all of R^2 .

5.5. In the two-dimensional case we are treating there is also a Green's function with singularity at infinity. Take any point s in the unbounded component of $R^2 - E$ and define

$$H(r) = G(r, s) + \frac{1}{\pi} \ln |r - s| - \frac{1}{\pi} \int_{\Omega} \ln |Y_r(\omega) - s| d\omega.$$

The integral, considered as a function of r , is the solution of the Dirichlet problem on $R^2 - E$ with boundary values $\ln |r - s|$ on E . Hence the function $\pi H(r) - \ln |r - s|$ is bounded for large $|r|$; it follows that the difference, being harmonic, must approach a limit as $|r| \rightarrow \infty$. Clearly $H(r)$ vanishes if r is a regular point of E or a point of a bounded component of $R^2 - E$. It is also easy to see that $H(r)$ does not depend on the choice of s . (The calculation in the next section incidentally proves this fact.)

This is the definition of H which we shall use in §6. One arrives at a more natural definition on noting that $\int_C \ln |t - s| \zeta(ds)$, where ζ is the uniform distribution of mass 1 on a circumference C , does not depend upon the choice of t inside C . This being so, choose C to be a large circle enclosing E ; averaging $H(r)$ with respect to s over C gives

$$(13) \quad H(r) = \int_C G(r, t) \zeta(dt)$$

for every r inside C . Also

$$(14) \quad H(r) = \lim_{|s| \rightarrow \infty} G(r, s)$$

because of (13) and the fact that the limit exists.

The last equation shows that $H(r)$ formally has the same probability interpretation (2) as $G(s, r)$, but with the Brownian motion starting from the point at infinity. If one considers Brownian motion on the Riemann sphere this interpretation becomes rigorous.

6. Solutions of the heat equation. Let f be a bounded Borel measurable function on E (which we still assume compact) and define

$$(1) \quad F(\tau, r) = \int_{J_\tau \times E} f(s) Q(r, d\sigma, ds), \quad \tau > 0, r \in R^n.$$

We show first that F should be considered the bounded solution of the heat equation on $R^n - E$ with initial values 0 on $R^n - E$ and boundary values f on E . If $\mu(r, A) = Q(r, I_\infty, A)$ is harmonic measure relative to E then

$$(2) \quad F(\tau, r) \rightarrow F(r) = \int_E f(s) \mu(r, ds), \quad \tau \rightarrow \infty.$$

Our main result is that in two dimensions the difference $F(r) - F(\tau, r)$ is asymptotic to $2\pi F(\infty)H(r)/\ln \tau$, where $F(\infty)$ is the limit of $F(r)$ as $|r| \rightarrow \infty$.

6.1. Theorem 2.5 with stopping time τ proves that

$$(3) \quad \mu(r, A) = Q(r, I_\tau, A) + \int_{R^n} \mu(s, A) q(\tau, r, s) ds.$$

It follows that

$$(4) \quad F(r) = F(\tau, r) + \int_{R^n} F(s) q(\tau, r, s) ds.$$

Here we may differentiate under the integral sign with respect to τ and r , provided r lies outside E . Since $F(r)$ is harmonic on $R^n - E$ and $q(\tau, r, s)$ satisfies the heat equation on $R^n - E$, we obtain

$$(5) \quad \frac{\partial F}{\partial \tau}(\tau, r) = \frac{1}{2} \Delta F(\tau, r), \quad \tau > 0, r \in R^n - E.$$

It is clear from (1) that $F(\tau, r) = f(r)$ for all τ if r is a regular point of E . Also $F(\tau, r)$ tends to 0 with τ if r is an irregular point of E or a point of $R^n - E$. The inequality at the end of §4.4 shows that $F(\tau, r') \rightarrow f(r)$ if r' approaches a regular point r of E at which f is continuous; it is proved in [4] that $F(\tau, r)$ is the unique solution of a boundary value problem for the heat equation.

6.2. THEOREM. *If $n = 2$ then*

$$(6) \quad \lim_{\tau \rightarrow \infty} [F(r) - F(\tau, r)] \ln \tau = 2\pi F(\infty) H(r), \quad r \in R^2.$$

The theorem is trivial for a point r at which $H(r)$ vanishes. Such a point is either a regular point of E , and then $F(r) - F(\tau, r) = 0$; or it is a point (possibly an irregular point) belonging to a bounded component of $R^2 - E$, and then $F(\tau, r)$ approaches $F(r)$ exponentially. (To see this, first show by the argument of §5.1 that $\int q(\tau, r, s) ds$ decreases exponentially.)

The proof of the theorem occupies the next three numbers. First we establish (6) by a Tauberian argument when f , and hence F , is identically 1; the theorem is then a statement about $Q(r, J_r, E)$. The general result follows easily. The proof for the simple case can be extended to cover the general one, but at the cost of obscuring the core of the argument.

6.3. From now on we assume that r is a point of $R^2 - E$ or an irregular point of E . Fix a disc A in the component of $R^2 - E$ to which r belongs and at a positive distance from r ; let $|A|$ be the area and ϕ the characteristic function of A . For $\rho > 0$

$$\begin{aligned} & \int_{\Omega} d\omega \int_0^{T_r(\omega)} e^{-\rho\tau} \phi(X_r(\tau, \omega)) d\tau \\ &= \int_{\Omega} d\omega \int_0^{\infty} e^{-\rho\tau} \phi(X_r(\tau, \omega)) d\tau - \int_{\Omega} d\omega \int_{T_r(\omega)}^{\infty} e^{-\rho\tau} \phi(X_r(\tau, \omega)) d\tau \\ &= \int_A ds \int_0^{\infty} e^{-\rho\tau} p(\tau, r, s) d\tau - \int_{\Omega} e^{-\rho T_r(\omega)} d\omega \int_0^{\infty} e^{-\rho\tau} \phi(X_r(\tau + T_r(\omega), \omega)) d\tau, \end{aligned}$$

an equation we abbreviate to $B = B_1 - B_2$.

The asymptotic expansion

$$\begin{aligned} K(\alpha) &= \frac{1}{2\pi} \int_0^\infty e^{-(\alpha\tau+1/\tau)} \frac{d\tau}{\tau} \\ &= \frac{1}{2\pi} \ln \frac{1}{\alpha} + \gamma + o(1) \end{aligned}$$

(as α decreases to 0) is well known. So

$$\begin{aligned} B_1 &= \int_A K\left(\frac{\rho}{2} \mid r - s \mid^2\right) ds \\ &= \frac{1}{\pi} \int_A \ln \frac{1}{\mid r - s \mid} ds + \mid A \mid \left\{ \frac{1}{2\pi} \ln \frac{1}{\rho} + \beta + o(1) \right\} \end{aligned}$$

as $\rho \rightarrow 0$. Here β is $\gamma + (2\pi)^{-1} \ln 2$.

By Theorem 2.5 with stopping time T_r

$$\begin{aligned} B_2 &= \int_\Omega e^{-\rho T_r(\omega)} d\omega \int_A ds \int_0^\infty e^{-\rho\tau} p(\tau, Y_r(\omega), s) ds \\ &= \int_\Omega e^{-\rho T_r(\omega)} d\omega \int_A K\left(\frac{\rho}{2} \mid Y_r(\omega) - s \mid^2\right) ds \\ &= \frac{1}{\pi} \int_A ds \int_\Omega e^{-\rho T_r(\omega)} \ln \frac{1}{\mid Y_r(\omega) - s \mid} d\omega \\ &\quad + \mid A \mid \left\{ \frac{1}{2\pi} \ln \frac{1}{\rho} + \beta + o(1) \right\} \int_\Omega e^{-\rho T_r(\omega)} d\omega. \end{aligned}$$

Thus $B_1 - B_2$ may be written

$$\begin{aligned} &\frac{1}{\pi} \int_A \ln \frac{1}{\mid r - s \mid} ds - \frac{1}{\pi} \int_A ds \int_\Omega e^{-\rho T_r(\omega)} \ln \frac{1}{\mid Y_r(\omega) - s \mid} d\omega \\ &\quad + \mid A \mid \left\{ \frac{1}{2\pi} \ln \frac{1}{\rho} + \beta \right\} \left\{ 1 - \int_\Omega e^{-\rho T_r(\omega)} d\omega \right\} + o(\mid A \mid). \end{aligned}$$

As $\rho \rightarrow 0$ the second term of this expression approaches

$$\pi^{-1} \int_A ds \int_\Omega \ln \mid Y_r(\omega) - s \mid d\omega$$

by dominated convergence and B approaches $\int_A G(r, s) ds$. Consequently, by the definition of $H(r)$ in §5.5,

$$(7) \quad \left\{ \frac{1}{2\pi} \ln \frac{1}{\rho} + \beta \right\} \left\{ 1 - \int_{\Omega} e^{-\rho T_r(\omega)} d\omega \right\} \rightarrow \frac{1}{|A|} \int_A H(r) ds = H(r)$$

as $\rho \rightarrow 0$.

6.4. Let us rewrite (7). First, β may be deleted because $\ln(1/\rho)$ becomes infinite. Next

$$\begin{aligned} 1 - \int_{\Omega} e^{-\rho T_r(\omega)} d\omega &= 1 - \int_0^{\infty} e^{-\rho\tau} d\tau P(\tau, r) \\ &= \rho \int_0^{\infty} [1 - P(\tau, r)] e^{-\rho\tau} d\tau \end{aligned}$$

where $P(\tau, r) = P\{T_r(\omega) < \tau\}$. Thus

$$(8) \quad \int_0^{\infty} [1 - P(\tau, r)] e^{-\rho\tau} d\tau \sim \frac{2\pi H(r)}{\rho \ln 1/\rho}, \quad \rho \rightarrow 0.$$

Since $\ln 1/\rho$ varies slowly, Karamata's Tauberian theorem [1, Satz 2, p. 208] implies

$$\int_0^{\tau} [1 - P(\sigma, r)] d\sigma \sim \frac{2\pi H(r)\tau}{\ln \tau}, \quad \tau \rightarrow \infty,$$

and since $1 - P(\tau, r)$ decreases in τ it follows that

$$(9) \quad \begin{aligned} 1 - P(\tau, r) &= P\{T_r \geq \tau\} \\ &\sim 2\pi H(r)/\ln \tau, \end{aligned} \quad \tau \rightarrow \infty.$$

This is the special case of the theorem.

6.5. Relation (9) is equivalent to

$$(10) \quad \int_{R^2} q(\tau, r, s) ds \sim 2\pi H(r)/\ln \tau, \quad \tau \rightarrow \infty.$$

We now write

$$\begin{aligned} F(r) - F(\tau, r) &= \int_{\tau}^{\infty} \int_E f(s) Q(r, d\sigma, ds) \\ &= \int_{R^2} q(\tau, r, t) dt \int_E f(s) \mu(t, ds) \\ &= \int_{R^2} F(t) q(\tau, r, t) dt \\ &= \int_{R^2-D} F(t) q(\tau, r, t) dt + \int_D F(t) q(\tau, r, t) dt, \end{aligned}$$

where D is a large disc. (In the first transformations we have used Theorem 2.5 with stopping time τ and obvious relations between Q and the harmonic measure μ relative to E .) Since $q(\tau, r, s)$ is not greater than $1/(2\pi\tau)$ the integral over D is majorized by $|D|M/\tau$, where M is a bound for F . (Note that this, for F identically 1, implies (10) with R^2 replaced by $R^2 - D$.) Take D so large that $|F(t) - F(\infty)| < \epsilon$ for t outside D . Then the integral over $R^2 + D$ lies between $(F(\infty) - \epsilon)$ and $(F(\infty) + \epsilon)$ times $\int_{R^2 - D} q(\tau, r, t) dt$; since ϵ is arbitrary Theorem 6.2 is proved.

6.6. In the preceding number we have used of F only the facts that it is bounded and that $F(r)$ has a limit as $|r| \rightarrow \infty$. Thus, if $g(r)$ is a bounded Borel measurable function on $R^2 - E$ which has a limit $g(\infty)$ as $|r| \rightarrow \infty$, the function

$$G(\tau, r) = \int_{R^2 - E} g(t) q(\tau, r, t) dt$$

has the asymptotic expression $2\pi H(r)g(\infty)/\ln \tau$. Now, it is easily seen that $F(\tau, r) + G(\tau, r)$ is the solution of the heat equation on $R^2 - E$ with boundary values $f(r)$ on E and initial values $g(r)$ on $R^2 - E$ (in the sense that it approaches $g(r)$ for almost all r in $R^2 - E$). So Theorem 6.2 could have been stated a little more generally.

BIBLIOGRAPHY

1. G. Doetsch, *Theorie und Anwendung der Laplace-Transformation*, Berlin, 1937.
2. J. L. Doob, *Stochastic processes*, New York, 1953.
3. ———, *Semimartingales and subharmonic functions*, Trans. Amer. Math. Soc. vol. 77 (1954) pp. 86–121.
4. ———, *A probability approach to the heat equation*, Trans. Amer. Math. Soc. vol. 80 (1955) pp. 216–280.
5. P. Lévy, *Le mouvement Brownien plan*, Amer. J. Math. vol. 62 (1940) pp. 487–550.
6. D. Ray, *On spectra of second order differential operators*, Trans. Amer. Math. Soc. vol. 77 (1954) pp. 299–321.
7. M. Rosenblatt, *On a class of Markoff processes*, Trans. Amer. Math. Soc. vol. 71 (1951) pp. 120–135.

CORNELL UNIVERSITY
ITHACA, N. Y.