COMPLETELY FREE LATTICES GENERATED BY PARTIALLY ORDERED SETS

BY
R. A. DEAN

1. **Introduction.** A description of the free lattice, FL(n), generated by n unordered elements has been given by P. M. Whitman [5; 6] for any cardinal, n. A description of a free lattice, FL(P), generated by any partially ordered set, P, has been given by R. P. Dilworth [2]. This lattice has three chief properties: (1) P is embedded(1) in FL(P), (2) least upper and greatest lower bounds existing for pairs of elements in P are preserved in FL(P), and (3) any lattice providing a minimal embedding for P with property (2) is a lattice homomorphic image of FL(P). In that same paper Dilworth called the completely free lattice, CF(P), generated by P, the lattice whose three chief properties are: (1′) P is embedded in CF(P), (2′) the ordering of P is preserved in CF(P), and (3′) any lattice providing a minimal embedding for P with property (2′) is a lattice homomorphic image of CF(P). Thus in CF(P), the only least upper and greatest lower bounds which are preserved are those between comparable elements. The present paper investigates these completely free lattices. In §§2 and 3 the techniques of Whitman [5; 6] are applied to these lattices and many of his results are easily extended to this case. In particular, the word problem is solved in these lattices, a canonical form is shown to exist for each word, and necessary and sufficient conditions are given for a finite subset of CF(P), considered as a partially ordered set, to generate a completely free sublattice.

When P has no bounds, other than between comparable elements, to be preserved, CF(P) and FL(P) are identical and in this event the solution to the word problem in CF(P) becomes very useful and provides a decision method easier to employ than that given by T. Evans [3]. Moreover the class of lattices for which the word problem is solved by the present method contains some lattices to which Evans' method is inapplicable. An example is the lattice FL(P) when P consists of any number of disjoint infinite chains.

In §4 two examples are given in which the decision method is particularly effective. The first partially ordered set considered is the set P consisting of two disjoint chains of two elements each, $t>u$ and $v>w$. The second set Q consists of two disjoint chains, one of four elements, $a>b>c>d$, and the other of a single element, e. CF(P) is identical with FL(P), and CF(Q) is identical with FL(Q). Sorkin [4] showed that these lattices contain chains

Presented to the Society, September 2, 1955; received by the editors November 14, 1955.

(1) For the definition of this term and others appearing in the introduction, see §2.

of infinite length. Here it is shown that FL(P) is a sublattice of FL(Q), and infinite chains are constructed in FL(P).

2. **CF**(P). Let P be a partially ordered set with elements[2] p_i and order relation $\geqq$. Words or lattice polynomials on the elements p_i and their lengths are defined inductively by Definition 1.

DEFINITION 1. (i) For all i, p_i is a word of length 1.

(ii) If A and B are words of length $\lambda(A)$ and $\lambda(B)$ respectively then $A\cup B$ and $A\cap B$ are words of length $\lambda(A)+\lambda(B)$.

Next the relation ($\geqq$) is extended to the set of words formed from the p_i.

DEFINITION 2. Let A and B be two words on the p_i. $A\geqq B$ if and only if one or more of the following hold[3]:

(i) $A\equiv p_i$ and $B\equiv p_j$ and $p_i\geqq p_j$ in P.

Proceeding recursively:

(ii) (2.1) $A\equiv A_1\cup A_2$ and $A_1\geqq B$ or $A_2\geqq B$,

(2.2) $A\equiv A_1\cap A_2$ and $A_1\geqq B$ and $A_2\geqq B$,

(2.3) $B\equiv B_1\cup B_2$ and $A\geqq B_1$ and $A\geqq B_2$,

(2.4) $B\equiv B_1\cap B_2$ and $A\geqq B_1$ or $A\geqq B_2$.

LEMMA 1. *$A\geqq B_1\cup B_2$ implies $A\geqq B_1$ and B_2. Dually $A_1\cap A_2\geqq B$ implies A_1 and $A_2\geqq B$.*

Proof. A proof is given for the first half of the lemma and the second half then follows by duality. The proof proceeds by induction on $\lambda(A)$. When $\lambda(A)=1$, $A\equiv p_i$, for some i. Thus $p_i\geqq B_1\cup B_2$ and from Definition 2, the only applicable rule is (2.3). Assuming the result when $\lambda(A)<k$, let $\lambda(A)=k$.

Case 1. $A\equiv A_1\cup A_2$. From Definition 2, $A\geqq B_1\cup B_2$ implies either (2.1) or (2.3) holds. If (2.3) holds the result is true, thus assume that (2.1) holds, and without loss of generality, that $A_1\geqq B_1\cup B_2$. Then by induction $A_1\geqq B_i$, $i=1$ and 2, and by (2.1) again $A_1\cup A_2\geqq B_i$, $i=1$ and 2.

Case 2. $A\equiv A_1\cap A_2$. From Definition 2, $A\geqq B_1\cup B_2$ implies either (2.2) or (2.3) holds. If (2.3) holds the result is true; thus assume that (2.2) holds. Hence for $i=1$ and 2, $A_i\geqq B_1\cup B_2$. Hence by induction, $A_i\geqq B_j$, for $j=1$ and 2. Hence by (2.2), $A_1\cap A_2\geqq B_j$, for $j=1$ and 2.

LEMMA 2. (1) *$A\equiv A_1\cup A_2\geqq B_1\cap B_2\equiv B$ implies one of the following*:

(i) *$A\geqq B_i$, for $i=1$ or 2,*

(ii) *$A_j\geqq B$, for $j=1$ or 2.*

(2) *$p\geqq B_1\cap B_2$ implies $p\geqq B_i$, for $i=1$ or 2.*

(3) *$A_1\cup A_2\geqq p$ implies $A_j\geqq p$ for $j=1$ or 2.*

Proof. (1), (2) and (3) simply list the possibilities afforded by Definition 2.

---

(2) In this paper p, subscripted or not, will always denote an element of the partially ordered set P.

(3) ($\equiv$) denotes logical identity.

LEMMA 3. *The relation* ($\geqq$) *on the words on the* p_i *is reflexive and transitive.*

Proof. PART I. $A \geqq A$. The proof is by induction on $\lambda(A)$. For $\lambda(A)=1$, $A \equiv p_i$ and the result follows from that property for the set P. Assume therefore that the result is verified when $\lambda(A)<k$, let $\lambda(A)=k$.

Case 1. $A \equiv A_1 \cup A_2$. By induction hypothesis $A_1 \geqq A_1$. By (2.1), $A_1 \cup A_2 \geqq A_1$. Similarly, $A_1 \cup A_2 \geqq A_2$. Thus by (2.3), $A_1 \cup A_2 \geqq A_1 \cup A_2$.

Case 2. $A \equiv A_1 \cap A_2$. This case is the dual of Case 1.

PART II. $A \geqq B$ and $B \geqq C$ implies $A \geqq C$. The proof is by induction on $\lambda(A)+\lambda(B)+\lambda(C)=\lambda$. When $\lambda=3$, $A \equiv p_i$, $B \equiv p_j$, $C \equiv p_k$ and the result follows from that property of the set P. Assume therefore that the result is verified when $\lambda<k$ and that $\lambda=k$.

Case 1.1.1. $B \equiv p_i$, $A \equiv p_j$, $C \equiv C_1 \cup C_2$. From $p_i \geqq C_1 \cup C_2$ and Lemma 1, $p_i \geqq C_1$ and C_2. Then, by induction hypothesis, $p_j \geqq C_1$ and C_2, thus $p_j \geqq C_1 \cup C_2$ by (2.3).

Case 1.1.2. $B \equiv p_i$, $A \equiv p_j$, $C \equiv C_1 \cap C_2$. From $p_i \geqq C_1 \cap C_2$ and Lemma 2, $p_i \geqq C_1$ or C_2. Then, by induction hypothesis, $p_j \geqq C_1$ or C_2, thus $p_j \geqq C_1 \cap C_2$ by (2.4).

Case 1.2. $B \equiv p_i$, $A \equiv A_1 \cup A_2$. From $A_1 \cup A_2 \geqq p_i$ and Lemma 2, A_1 or $A_2 \geqq p_i$. Then, by induction hypothesis, A_1 or $A_2 \geqq C$, thus $A_1 \cup A_2 \geqq C$ by (2.1).

Case 1.3. $B \equiv p_i$, $A \equiv A_1 \cap A_2$. From $A_1 \cap A_2 \geqq p_i$ and Lemma 1, A_1 and $A_2 \geqq p_i$. Then, by induction hypothesis, A_1 and $A_2 \geqq C$, thus $A_1 \cap A_2 \geqq C$ by (2.2).

Cases 1.1.1, 1.1.2, 1.2, and 1.3 exhaust all possible forms with $B \equiv p_i$.

Case 2.1. $B \equiv B_1 \cup B_2$, $C \equiv p_i$. From $A \geqq B_1 \cup B_2$ and Lemma 1, $A \geqq B_1$ and B_2. From $B_1 \cup B_2 \geqq p_i$ and Lemma 2, B_1 or $B_2 \geqq p_i$. Then, by induction hypothesis, $A \geqq p_i$.

Case 2.2. $B \equiv B_1 \cup B_2$, $C \equiv C_1 \cup C_2$. From $B \geqq C_1 \cup C_2$ and Lemma 1, $B \geqq C_1$ and C_2. Then, by induction hypothesis, $A \geqq C_1$ and C_2, thus $A \geqq C_1 \cup C_2$ by (2.3).

Case 2.3. $B \equiv B_1 \cup B_2$, $C \equiv C_1 \cap C_2$. From $B_1 \cup B_2 \geqq C_1 \cap C_2$ and Lemma 2, $B \geqq C_1$ or $B \geqq C_2$ or $B_1 \geqq C$ or $B_2 \geqq C$. If either of the first two alternatives holds, by induction hypothesis, $A \geqq C_1$ or C_2, hence $A \geqq C_1 \cap C_2$ by (2.4). If either of the last two alternatives holds, consider $A \geqq B_1 \cup B_2$. From Lemma 1, $A \geqq B_1$ and B_2. Hence whichever of the last two alternatives holds, the induction hypothesis yields $A \geqq C$.

Cases 2.1, 2.2 and 2.3 exhaust all possible forms with $B \equiv B_1 \cup B_2$. The forms with $B \equiv B_1 \cap B_2$ are duals of these cases.

Now the relation ($\geqq$) on the words may be extended to a partial ordering in the usual(4) way.

---

(4) For example, see Birkhoff [**1**, p. 4].

DEFINITION 3. Two words A and B are equal if $A \geqq B$ and $B \geqq A$, written $A = B$.

It should now be verified that this is an equivalence relation and all further work should be carried out on the resulting equivalence classes. The elements of the lattice CF(P) will be these equivalence classes and the elements can only be represented by a word in that equivalence class. However no confusion will result if the terms "word" and "element" are used interchangeably. The formal details are left to the reader.

THEOREM 1. *The partially ordered set obtained from the set of words on a partially ordered set P by Definitions 2 and 3 is a lattice in which $A \cup B$ and $A \cap B$ are the least upper and greatest lower bounds, respectively, of A and B. This lattice will henceforth be denoted $CF(P)$.*

Proof. $A \cup B \geqq A$ and B by Definition 2. Let $C \geqq A$ and B. Then by Definition 2, $C \geqq A \cup B$. Hence $A \cup B$ is the least upper bound of A and B. A dual proof shows that $A \cap B$ is their greatest lower bound.

Because the definition of equality depends only on ($\geqq$) and because the criteria for ($\geqq$) is recursive and all words have finite length it is easily seen that we have proved

THEOREM 2. *The word problem, i.e. the problem of deciding in a finite number of steps whether two given words are equal, is solved in $CF(P)$.*

DEFINITION 4. A partially ordered set P with elements p_i is said to be embedded in a lattice L, if L possesses a subset of elements s_i such that $s_i \leftrightarrow p_i$ is a 1-1 correspondence with the property that $p_i \leqq p_j$ implies $s_i \leqq s_j$. The embedding is called minimal if the sublattice generated by the s_i is L.

THEOREM 3. *P is minimally embedded in $CF(P)$ and moreover any lattice in which P can be minimally embedded is a homomorphic image of $CF(P)$.*

Proof. The first part of the theorem follows immediately from the construction of CF(P). To prove the second part, let L be any lattice in which P is embedded. Let P have elements p_i, the corresponding subset of L, elements s_i, and let $p_i \leftrightarrow s_i$ be the required correspondence. By assumption $S = \{s_i\}$ generates L. The correspondence $p_i \leftrightarrow s_i$ induces a natural correspondence between words on the p_i and words on the s_i, hence a correspondence between the elements of CF(P) and L. If $f(x_1, \cdots, x_n)$ denotes a lattice polynomial or word on indeterminates $x_1, \cdots, x_n$, the mapping described is

$$f(p_{i_1}, \cdots, p_{i_n}) \leftrightarrow f(s_{i_1}, \cdots, s_{i_n}) \tag{M}$$

or simply

$$f(P) \leftrightarrow f(S)$$

for any substitution $p_{i_1}, \cdots, p_{i_n}$ for $x_1, \cdots, x_n$. Now $f(s_{i_1}, \cdots, s_{i_n})$ is some element a of L. It will be shown that the mapping

$$\text{(N)} \qquad f(p_{i_1}, \cdots, p_{i_n}) \to a$$

obtained in this way is a homomorphism of CF(P) onto L. Clearly every word in CF(P) has an image in L and since S generates L, every element in L is expressible as a word on the s_i, hence has a mate in CF(P).

It will be shown first that the mapping (M) preserves the order relation $(\geqq)$ on the elements of CF(P). That is, if $f(P)$ and $g(P)$ are two words, and $f(P) \geqq g(P)$ in CF(P), then $f(S) \geqq g(S)$ in L. The proof is by induction on $\lambda[f(P)]+\lambda[g(P)]=\lambda$. When $\lambda=2$, $f(P)\equiv p_i$, $g(P)\equiv p_j$ and $p_i \geqq p_j$ implies $s_i \geqq s_j$ by the embedding property. Assuming the result for $\lambda<k$, let $\lambda=k$.

Case 1. $f(P)\equiv f_1(P)\cap f_2(P)$. By Lemma 1, $f_1(P)$ and $f_2(P)\geqq g(P)$. Hence by induction hypothesis $f_1(S)$ and $f_2(S)\geqq g(S)$. Since $f_1(S)$, $f_2(S)$ and $g(S)$ are elements of a lattice L, $f_1(S)\cap f_2(S)\geqq g(S)$. But $f(P)\to f(S)\equiv f_1(S)\cap f_2(S)$. The case $g(P)=g_1(P)\cup g_2(P)$ is the dual of Case 1.

Case 2. $f(P)\equiv p_i$ and $g(P)=g_1(P)\cap g_2(P)$. From Lemma 2, $p_i\geqq g_1(P)$ or $g_2(P)$. Hence, by induction hypothesis, $s_i\geqq g_1(S)$ or $g_2(S)$ and thus $s_i\geqq g_1(S)\cap g_2(S)$. But $g(P)\to g(S)\equiv g_1(S)\cap g_2(S)$. Thus $s_i\geqq g(S)$. The case $g(P)\equiv p_i$ and $f(P)=f_1(P)\cup f_2(P)$ is the dual of Case 2.

Case 3. $f(P)\equiv f_1(P)\cup f_2(P)$ and $g(P)\equiv g_1(P)\cap g_2(P)$. By Lemma 2 four possibilities must be examined. Suppose $f(P)\geqq g_1(P)$. By induction hypothesis $f(S)\geqq g_1(S)$ and thus $f(S)\geqq g_1(S)\cap g_2(S)$. The other possibilities are handled similarly.

Since (M) preserves the order in CF(P) it is clear that (N) does also. Thus, equality in CF(P) implies equality in L, conversely distinct elements in L cannot correspond to equal words in CF(P).

It will now be shown that the correspondences (M) and (N) preserve unions and intersections in CF(P). Let $f(P)\to f(S)=a$, $g(P)\to g(S)=b$ and $f(P)\cup g(P)=h(P)\to h(S)=c$. It is to be shown that $c=a\cup b$. In any event $f(P)\cup g(P)\to f(S)\cup g(S)=a\cup b$. Since $f(P)\cup g(P)=h(P)$, the preceding paragraph shows that $h(S)=a\cup b$. Similarly intersections are preserved.

THEOREM 4. *FL(P) and CF(P) are identical if and only if P has the following two properties*:

(i) $p_i=l.u.b.\ (p_j, p_k)$ *if and only if* $i=j$ *or* k.

(ii) $p_i=g.l.b.\ (p_j, p_k)$ *if and only if* $i=j$ *or* k.

Proof. *Necessity*. Suppose that FL(P) and CF(P) are identical and that $p_i=$l.u.b. (p_j, p_k). Then $p_i=p_j\cup p_k$ in FL(P), since it is to preserve all existing bounds of pairs of elements. But in CF(P), $p_i=p_j\cup p_k$ implies $p_j\cup p_k\geqq p_i$, hence, $p_j\geqq p_i$ or $p_k\geqq p_i$, while on the other hand, $p_i\geqq p_j$ and p_k. Thus $p_i=p_j$ or p_k, i.e. $i=j$ or k. (ii) is established in a dual way.

Sufficiency. Since the elements of FL(P) and CF(P) are represented by the same set of words it suffices to prove that equality of words in FL(P) is equivalent to equality in CF(P). Since FL(P) provides a minimal embedding for P, by Theorem 3, CF(P)$\rightarrow$FL(P), hence equality in CF(P) implies equality in FL(P).

The converse is established by showing that for words A, B, $A \geqq B$ in FL(P) implies $A \geqq B$ in CF(P). The notation used in this proof follows Dilworth's [2, p. 126].

LEMMA 4. *If $v(A)$ exists, i.e. $A = v(A) = p_i$, for some i, in $FL(P)$, $A = p_i$ in $CF(P)$.*

Proof. The proof is by induction on $\lambda(A)$. When $\lambda(A) = 1$, $A \equiv p_i$ and $v(A) = p_i$ in FL(P), hence $A = p_i$ in CF(P). If $A \equiv A_1 \cup A_2$ and $v(A)$ exists, then $v(A_1)$, $v(A_2)$ exist and l.u.b. $[v(A_1), v(A_2)]$ exists in P and equals $v(A)$. Let $v(A_1) = p_j$, $v(A_2) = p_k$. Then l.u.b. $(p_j, p_k) = p_j$ or p_k, from the assumption (i) on P. Without loss of generality, let $p_j =$l.u.b. (p_j, p_k) so that $p_j \geqq p_k$. Thus in CF(P), by induction, $A_1 = p_j$, $A_2 = p_k$ and thus in CF(P), $A_1 \cup A_2 = p_j \cup p_k = p_j$. Thus $A = p_j$ in CF(P) as was to be proved. A dual proof handles the other case when $A \equiv A_1 \cap A_2$.

Now suppose $A \geqq B$ in FL(P). By virtue of (iii) in Dilworth's definition (1.5) $A \geqq B(n)$. Proceeding now by induction on n, let $A \geqq B(1)$. If $A \equiv B$ there is nothing to prove. If $v(A)$, $v(B)$ exist and $v(A) \geqq v(B)$ in P, then in CF(P), by Lemma 4, $A = v(A)$, $B = v(B)$, then $A \geqq v(A) \geqq v(B) \geqq B$ in CF(P). If $A \geqq B(n)$ one of (1)–(5) must hold. Induction hypothesis gives the corresponding result in CF(P), and either the transitivity of $\geqq$ or one of (2.1)–(2.4) in our Definition 2 implies $A \geqq B$ in CF(P).

In what follows it is convenient to consider expressions of the form $A_1 \cup \cdots \cup A_m$, m finite. While such expressions are not words as defined they shall be used here to represent a word of the form

$$(\cdots (A_1 \cup A_2) \cup \cdots \cup A_{m-1}) \cup A_m$$

or any word obtained from this form by use of the associative and commutative laws. In the sequel ($\equiv$) shall be used to denote not only logically equivalent words, but also words which become logically equivalent when appropriately operated upon by the associative and commutative laws alone, writing $A_1 \cup \cdots \cup A_n$ for any one such word, more frequently $\bigcup_i A_i$, tacitly assuming that only a finite number of A_i are involved.

Lemma 1 and (1) of Lemma 2 then may be extended by an easy induction to any finite number of terms, rather than just 2. Much use will be made of these two lemmas and special reference to them will be omitted.

3. Canonical forms and sublattices in CF(P).

THEOREM 5. *Let P be any partially ordered set. In $CF(P)$, if $A \equiv \bigcup_i A_i$, $A_i \equiv \bigcap_j a_{ij}$ or $A_i \equiv p$, an element of P, then there is a shorter word B equal to A if and only if one or more of the following hold:*

(1) *For some i, A_i is equal to a shorter word, A_i'.*
(2) $A_h \geqq A_k$ *for some h and k, $h \neq k$.*
(3) $A \equiv \bigcup_i A_i \geqq a_{hk}$ *for some h and k where $A_h \not\equiv p$.*

Proof. (1) and (2) are clearly sufficient. To see that (3) is sufficient consider $B \equiv \bigcup_{i \neq h} A_i \cup a_{hk}$. B has shorter length than A since $\lambda(a_{hk}) < \lambda(A_h)$ as $A_h \not\equiv p$. $A \geqq B$ since $A \geqq a_{hk}$ by hypothesis, and $A \geqq \bigcup_{i \neq h} A_i$. Conversely $B \geqq A$ since $a_{hk} \geqq \bigcap_j a_{hj} \equiv A_h$ and therefore $\bigcup_{i \neq h} A_i \cup a_{hk} \geqq \bigcup_{i \neq h} A_i \cup A_h \equiv A$. Thus $A = B$.

To prove the necessity, take B to be the shortest (and shorter than A) word equal to A.

Case 1. $B = p$. Thus $p = \bigcup_i A_i$, hence $p \geqq \bigcup_i A_i$ and $p \geqq A_i$ for all i. Conversely $\bigcup_i A_i \geqq p$ and hence $A_h \geqq p$ for some h. Thus $A_h \geqq p \geqq A_i$ for some h and all i, in particular condition (2) is satisfied.

Case 2. $B \equiv \bigcap_i B_i$. From $B \geqq A$ it follows that $B_i \geqq A_j$ for all i and j. From $A \geqq B$ it follows that either $A \geqq B_h$ for some h or $A_k > B$ for some k. When the first alternative holds, $\bigcap_i B_i \equiv B \geqq A \geqq B_h \geqq \bigcap_i B_i$, thus $A = B = B_h$, contrary to the choice of B as the shortest word equal to A. When the second alternative holds, $A \geqq A_k \geqq B \geqq A$. Hence $A = A_k$ and $A_k \geqq A_i$, for all i, in particular condition (2) is satisfied.

Case 3. $B \equiv \bigcup_r B_r$. It may be assumed $B_r \equiv \bigcap_s b_{rs}$ or $B_r \equiv p$. From $A \geqq B$ it follows that $\bigcup_i A_i \geqq B_r = \bigcap_s b_{rs}$ for all r. Thus either: (i) for all r, there exists an index $j(r)$ such that $A_{j(r)} \geqq B_r$, or (ii) the first alternative does not hold and for some r and t, $\bigcup_i A_i \geqq b_{rt}$ and $\lambda(b_{rt}) < \lambda(B_r)$. (For if the lengths were equal then $B_r \equiv b_{rt} \equiv p$ and for this r, there would have to be a $j(r)$ such that $A_{j(r)} \geqq B_r \equiv p$.) Similarly from $B \geqq A$, $\bigcup_r B_r \geqq A_i \equiv \bigcap_j a_{ij}$ for all i, two alternatives are possible: (iii) for all i, there exists an index $k(i)$ such that $B_{k(i)} \geqq A_i$ or (iv) the third alternative does not hold, and for some i and h, $\bigcup_r B_r \geqq a_{ih}$ and $\lambda(a_{ih}) < \lambda(A_i)$.

Suppose that alternative (ii) holds. Thus $\bigcup_i B_i \geqq b_{rt}$ and, as in the proof of the sufficiency of (3), $\bigcup_i B_i = \bigcup_{i \neq r} B_i \cup b_{rt}$, contrary to the choice of B as the shortest word equal to A. Suppose that alternative (iv) holds. Thus $\bigcup_j A_j \geqq a_{ih}$, which is condition (3).

Thus it may be assumed that (i) and (iii) hold simultaneously. Hence, combining, $B_r \leqq A_{j(r)} \leqq B_{k[j(r)]}$ for all r. If, for some r, $r \neq k[j(r)] = t$, then $B_r \leqq B_t$, $r \neq t$ and $B \equiv \bigcup_s B_s = \bigcup_{s \neq r} B_s$, contrary to the choice of B as the shortest word equal to A. Hence assume that $r = k[j(r)]$ for all r.

Now suppose that, for some r, there exist distinct indices $j_1(r)$ and $j_2(r)$ such that $A_{j_1(r)} \geqq B_r$ and $A_{j_2(r)} \geqq B_r$. Then from (3)

$$B_{k[j_1(r)]} \geqq A_{j_1(r)} \geqq B_r,$$

$$B_{k[j_2(r)]} \geqq A_{j_2(r)} \geqq B_r.$$

But $k[j_1(r)] = r = k[j_2(r)]$ as was just shown. Hence $A_{j_2(r)} = A_{j_1(r)}$ and in particular, condition (2) holds. Thus without loss of generality suppose in (i) that for every r there is a unique index $j(r)$.

Similarly, $A_i \leqq B_{k(i)} \leqq A_{j[k(i)]}$. If, for some i, $i \neq j[k(i)] = h$, then $A_i \leqq A_h$ and condition (2) holds. Hence assume $i = j[k(i)]$ for all i. If there are distinct indices $k_1(i)$ and $k_2(i)$ such that $B_{k_1(i)} \geqq A_i$ and $B_{k_2(i)} \geqq A_i$, then from (1)

$$A_{j[k_1(i)]} \geqq B_{k_1(i)} \geqq A_i,$$
$$A_{j[k_2(i)]} \geqq B_{k_2(i)} \geqq A_i.$$

But $i = j[k_1(i)] = j[k_2(i)]$ as was just shown, hence $B_{k_1(i)} = B_{k_2(i)}$ and therefore $B \equiv \bigcup_r B_r = \bigcup_{r \neq k_1(i)} B_r$, contradicting the choice of B as the shortest word equal to A. Hence, in (iii), for every i, there exists a unique index $k(i)$. Moreover from

$$A_i = A_{j[k(i)]} \geqq B_{k(i)} \geqq A_i;\ B_r = B_{k[j(r)]} \geqq A_{j(r)} \geqq B_r$$

it follows that $A_i = B_{k(i)}$ and $B_r = A_{j(r)}$, and $i \leftrightarrow k(i)$ is a 1-1 correspondence between the A_i's and B_r's. But since $\lambda(A) > \lambda(B)$ it follows that for some i, $\lambda(A_i) > \lambda(B_{k(i)})$, which is condition (1).

The dual theorem is of course valid.

COROLLARY 1. *If $A = B$ then either $A \equiv B$ or there exists a C such that $A = B = C$ and $\lambda(C) < \lambda(A)$ or $\lambda(B)$.*

Proof. It may be supposed that $\lambda(A) = \lambda(B)$, for if not, take C to be the shorter word. The proof is by induction on $\lambda(A) = \lambda(B) = \lambda$. When $\lambda = 1$, $A = B$ implies $A \equiv B \equiv p$. Assuming the corollary when $\lambda < k$, let $\lambda = k$. Taking B as in the theorem, it is easily seen that a word shorter than A or shorter than B is produced, except perhaps in the last case where it might occur that $\lambda(A) = \lambda(B)$ and $A_i = B_{k(i)}$ all (i). But then, by induction either a shorter word C_i exists or by application of the commutative and associative laws A_i may be derived from $B_{k(i)}$. If the latter alternative holds for all i, A can be derived from B by application of the associative and commutative laws alone. If the former alternative holds for one index i, replacing A_i or $B_{k(i)}$ by C_i gives a word shorter, but equal to A and B.

DEFINITION 5. A word W is canonical if $A = W$ implies $\lambda(A) \geqq \lambda(W)$.

COROLLARY 2. *If $\bigcup_j A_j = \bigcup_i B_i$ and $\bigcup_i B_i$ is canonical, then for every i there exists a j such that $B_i \leqq A_j$.*

Proof. If $B_i \equiv p$, then the result follows from the extended form of Lemma 2. If $B_i \equiv \bigcap_r b_{ir}$, consider $\bigcup_j A_j \geqq B_i \equiv \bigcap_r b_{ir}$. Either $A_j \geqq B_i$ for some j or $\bigcup_j A_j \geqq b_{ir}$ for some i. The first case is the desired conclusion. The second yields the condition $\bigcup_i B_i \geqq b_{ir}$, which, by Theorem 5, is sufficient to construct a word shorter than $\bigcup_i B_i$ and equal to it.

COROLLARY 3. *If $A = \bigcup_i A_i = \bigcap_j B_j = B$ and $\bigcap_j B_j$ is canonical, then $\bigcup_i A_i = A_h$ for some h.*

Proof. For all i and j, $B_j \geqq A_i$ as $B \geqq A$. From $A \geqq B$, $\bigcup_i A_i \geqq \bigcap_j B_j$. From

Lemma 2, either $A \geqq B_j$, for some j, or $A_h \geqq B$. If the first case holds, $B \geqq A \geqq B_j \geqq B$ and $B = B_j$, but B was assumed canonical. Therefore $A \geqq A_h \geqq B \geqq A$ and $A \equiv \bigcup_i A_i = A_h$.

Dual corollaries hold of course.

LEMMA 5. *Let P be a partially ordered set with elements p_i. In $CF(P)$, for any index j and for any word A,*

(1) $p_j \leqq A$ *or*

(2) *there exists a finite subset of indices R such that R has no indices in common with $S_j = [k \mid p_j \leqq p_k$ in $P]$ and $A \leqq \bigcup_{i \in R} p_i$.*

(1) *and* (2) *are mutually exclusive.*

Proof. To show that (1) and (2) must be mutually exclusive let R be any subset of indices disjoint from S_j and suppose that (1) and (2) both hold. Thus $p_j \leqq \bigcup_{i \in R} p_i$, which implies that there exists an index $i \in R$ such that $p_j \leqq p_i$. But then $i \in S_j$, contrary to the supposed disjointness of R and S_j.

That (1) or (2) must hold is proved by induction on $\lambda(A)$. When $\lambda(A) = 1$, $A \equiv p_k$. Now either $p_j \leqq p_k$ or $p_j \nleqq p_k$. In the second case let R consist of k alone and (2) holds trivially. When $A \equiv A_1 \cup A_2$ and $p_j \nleqq A_1 \cup A_2$ then $p_j \nleqq A_1$ and $p_j \nleqq A_2$. Hence by induction there exist finite index sets R_1 and R_2 each disjoint from S_j such that $A_1 \leqq \bigcup_{i \in R_1} p_i$ and $A_2 \leqq \bigcup_{i \in R_2} p_i$. Let R be the (finite) set composed of the indices in R_1 and R_2. Clearly R is disjoint from S_j and $A_1 \leqq \bigcup_{i \in R_1} p_i \leqq \bigcup_{i \in R} p_i$ while $A_2 \leqq \bigcup_{i \in R_2} p_i \leqq \bigcup_{i \in R} p_i$. Thus $A_1 \cup A_2 \leqq \bigcup_{i \in R} p_i$. When $A \equiv A_1 \cap A_2$ and $p_j \nleqq A_1 \cap A_2$ then $p_j \nleqq A_h$, $h = 1$ or 2. Then by induction there exists a finite index set R disjoint from S_j such that $A_h \leqq \bigcup_{i \in R} p_i$ and so $A_1 \cap A_2 \leqq \bigcup_{i \in R} p_i$. A dual lemma holds of course. When P is a finite set, the subset R may always be chosen to be all subscripts not in S_j.

THEOREM 6. *Let $T = \{t_i\}$ be any subset of $CF(P)$ with the property that, for any finite subset of indices R, $t_j \leqq \bigcup_{i \in R} t_i$ implies that there exists $k \in R$ such that $t_j \leqq t_k$ in T, and the dual property. Then the sublattice $L(T)$ of $CF(P)$ generated by T is lattice isomorphic to $CF(Q)$ where Q is a partially ordered set order isomorphic*(5) *to T.*

Proof. Let the elements of Q be denoted by q_i and the correspondence $q_i \leftrightarrow t_i$. It is clear that $L(T)$ provdes a minimal embedding for Q; hence $L(T)$ is a homomorphic image of $CF(Q)$ through the mapping generated by $q_i \rightarrow t_i$. To show that this is indeed one to one it only remains to be shown that equality of words in $L(T)$ implies the equality of the corresponding words in $CF(Q)$. Using the notation for words employed in Theorem 3, it suffices to show that for two words $f(T)$ and $g(T)$, $f(T) \leqq g(T)$ in $L(T)$ implies $f(Q) \leqq g(Q)$ in $CF(Q)$. The proof is by induction on $\lambda[f(Q)] + \lambda[g(Q)] = \lambda$.

(5) The elements of Q and T can be placed in a 1-1 correspondence so that order is preserved in both directions.

When $\lambda=2$, $f(T)\equiv t_i$, $g(T)\equiv t_j$ and $t_i\leqq t_j$ implies $q_i\leqq q_j$ because T and Q are assumed isomorphic. Assuming the result when $\lambda<k$, let $\lambda=k$.

Case 1. $f(T)\equiv t_j$. From Lemma 5, in CF(Q), either $q_j\leqq g(Q)$ or $g(Q) \leqq \bigcup_{i\in R} q_i$, for some finite set of indices R, disjoint from S_j. If the latter alternative holds then $g(T)\leqq \bigcup_{i\in R} t_i$ in $L(T)$, since $L(T)$ is a lattice homomorphic image of CF(Q). This, together with the assumption $t_j\leqq g(T)$ and the hypothesis of the theorem implies that $t_j\leqq t_i$ for some $i\in R$ contrary to the condition that R be disjoint from S_j. Hence the first alternative must hold, as was to be shown. The case $g(T)\equiv t_j$ is the dual of Case 1.

Case 2. $f(T)\equiv f_1(T)\cup f_2(T)\leqq g(T)$. Since $L(T)$ is a lattice $f_1(T)$ and $f_2(T)\leqq g(T)$. Then by induction, $f_1(Q)$ and $f_2(Q)\leqq g(Q)$, hence $f(Q)\equiv f_1(Q) \cup f_2(Q)\leqq g(Q)$. The case $g(T)\equiv g_1(T)\cap g_2(T)$ is the dual of Case 2.

Case 3. $f(T)\equiv f_1(T)\cap f_2(T)\leqq g_1(T)\cup g_2(T)\equiv g(T)$. Since $L(T)$ is a sublattice of CF(P) the words f, f_1, f_2, g_1, g_2 and g, up to now regarded as words on the t_i, may also be regarded as words on the p_i. Rewriting the condition of Case 3, $f(P)\equiv f_1(P)\cap f_2(P)\leqq g_1(P)\cup g_2(P)\equiv g(P)$. By Lemma 2, either $f(P) \leqq g_i(P)$, $i=1$ or 2, or $f_j(P)\leqq g(P)$, $j=1$ or 2. These words may again be considered as words on the t_i, $f(T)\leqq g_1(T)$, $i=1$ or 2 and $f_j(T)\leqq g(T)$, $j=1$ or 2, and applying the induction hypotheses to whichever condition holds, obtain $f(P)\leqq g(P)$.

The conditions of Theorem 6 are clearly necessary for $L(T)$ to be isomorphic to CF(Q), for in CF(Q), $q_j\leqq \bigcup_{i\in R} q_i$ implies $q_j\leqq q_k$, for some $k\in R$, and dually.

This section is concluded with a theorem giving sufficient conditions for a sublattice of CF(P) to be isomorphic to FL(P). The proof will require Lemma 5 stated for FL(P). This follows by the homomorphism CF(P) $\to$FL(P); however it is no longer true that (1) and (2) are mutually exclusive. The following lemma is also needed.

LEMMA 6. *In FL(P), for any subscript i and any word A either*

(1) $p_i\leqq A$ *or*

(2) *there exists a finite set V of pairs of indices (m, n) such that V has no pair in common with*

$$T_i = [(j, k)\,|\, l.u.b.(p_j, p_k) = p_h \text{ in } P,\ p_i \leqq p_h \text{ in } P]$$

and $A\leqq \bigcup_{(m,n)\in V} (p_m\cup p_n)$.

Proof. If $p_i\nleqq p_h$, then $(h, h)\notin T_i$. Thus if $p_i\nleqq A$, for the subset V use the indices of the finite set R given by Lemma 5 and take $V=[(h, h)\,|\, h\in R]$. Thus V and T_i are disjoint and $\bigcup_{h\in R} p_h=\bigcup_{(h,h)\in V} (p_h\cup p_h)=\bigcup_{(j,k)\in V} (p_j\cup p_k)$, and the result follows from Lemma 5, restated for FL(P).

THEOREM 7. *Let P be a partially ordered set of elements p_i. In CF(P) let there exist a subset S of elements s_j such that $p_i\leftrightarrow s_i$ is a* 1-1 *correspondence and*

(1) $s_k=s_i\cup s_j$ *if and only if* $p_k=l.u.b.\ (p_i, p_j)$, *and dually, and*

(2) $s_i \leqq \bigcup_{(j,k)\in T}(s_j \cup s_k)$, *where* T *is any finite collection of pairs of indices implies for some pair* (j, k) *in* T *and some* h, $s_h = s_j \cup s_k$ *and* $s_i \leqq s_h$ *in* $CF(P)$, *and dually.*

Then the sublattice $L(S)$ *of* $CF(P)$ *generated by* S *is lattice isomorphic to* $FL(P)$.

Proof. Clearly P is embedded in $L(S)$ and by condition (1), $L(S)$ preserves least upper and greatest lower bounds existing in P; hence $p_i \rightarrow s_i$ generates a homomorphism of FL(P) into $L(S)$, and equality of words on the p_i in FL(P) implies equality of the corresponding words on the s_i. To prove the converse it suffices to show, using the notation of Theorem 3, that $f(S) \leqq g(S)$ in $L(S)$ implies $f(P) \leqq g(P)$ in FL(P). The proof is by induction on $\lambda[f(P)] + \lambda[g(P)] = \lambda$. For $\lambda = 2$, $f(P) \equiv s_i$, $g(P) \equiv s_j$ and the result follows from condition (1). Assume the result for $\lambda < k$ and let $\lambda = k$.

Case 1. $f(S) \equiv s_i \leqq g(S)$. Consider p_i and $g(P)$. By Lemma 6, $p_i \leqq g(P)$ or $g(P) \leqq \bigcup_{(j,k)\in V}(p_j \cup p_k)$ for some finite subset V disjoint from T_i. If the latter condition holds then $g(S) \leqq \bigcup_{(j,k)\in V}(s_j \cup s_k)$ where V is disjoint from T_i', where now

$$T_i' = [(m, n)/ \exists\, t \ni : s_t = s_m \cup s_n \quad \text{and} \quad s_i \leqq s_t \text{ in } L(S)],$$

by condition (1) and the definition of T_i. This, together with the assumption $s_i \leqq g(S)$ and the hypothesis of the theorem, implies that there is a pair $(j, k) \in V$ such that $s_h = s_j \cup s_k$ and $s_i \leqq s_h$, or that $(j, k) \in T_i'$, contrary to the condition that V and T_i' are disjoint. Hence the first condition must hold, as was to be shown. The case $f(S) \leqq g(S) \equiv s_i$ is the dual.

The remainder of the proof is analogous to that given for Cases 2 and 3 in the proof of Theorem 6.

4. **Examples.** Consider the partially ordered set, P, consisting of four elements, t, u, v, and w, with $t > u$ and $v > w$ as its only order relations. By Theorem 4, CF(P) is identical with FL(P). Four infinite descending chains in FL(P) will now be constructed. Let $A_1 \equiv t$, $B_1 \equiv t \cup w$, $C_1 \equiv (t \cup w) \cap v$, $D_1 \equiv [(t \cup w) \cap v] \cup u$ and for $n \geqq 2$, $A_n \equiv D_{n-1} \cap t$, $B_n \equiv A_n \cup w$, $C_n \equiv B_n \cap v$, $D_n \equiv C_n \cup u$. Clearly $A_1 \geqq A_2 \geqq \cdots \geqq A_n \geqq \cdots$; $B_1 \geqq B_2 \geqq \cdots \geqq B_n \geqq \cdots$; $C_1 \geqq C_2 \geqq \cdots \geqq C_n \geqq \cdots$; $D_1 \geqq D_2 \geqq \cdots \geqq D_n \geqq \cdots$. The following lemma shows that all these containing relations are proper.

LEMMA 7. A_n, B_n, C_n, D_n *are in canonical form, for all* n.

Thus, since the lengths of the words are strictly increasing no equality could hold.

The proof of Lemma 7 is made by a straightforward induction on n, using Theorem 5 and its dual to show, in turn, that A_n, B_n, C_n, and D_n are in canonical form.

Consider the partially ordered set, Q, consisting of five elements, a, b, c, d, and e with $a > b > c > d$ as its only order relations. By Theorem 3, CF(Q) is identical with FL(Q). Consider the elements $T \equiv b$, $U \equiv c$, $V \equiv e \cup d$ and $W \equiv a \cap e$. The sublattice they generate is isomorphic to CF(P), where P is the set described above. This is proved by verifying the criteria of Theorem 6. It is easy to see that the set T, U, V, W is order isomorphic with P. The other criteria then reduce to verifying $b \nleq c \cup (e \cup d)$ and its dual which can be done by inspection.

Bibliography

1. G. Birkhoff, *Lattice theory*, rev. ed., Amer. Math. Soc. Colloquium Publications, vol. 25, New York, 1948.
2. R. P. Dilworth, *Lattices with unique complements*, Trans. Amer. Math. Soc. vol. 57 (1945) pp. 123–154.
3. T. Evans, *The word problem for abstract algebras*, J. London Math. Soc. vol. 26 (1951) pp. 64–71.
4. Yu. I. Sorkin, *Free unions of lattices*, Rec. Math. (Mat. Sbornik) N.S. (72) vol. 30 (1952) pp. 677–694 (Russian).
5. P. M. Whitman, *Free lattices*, I. Ann. of Math. (2) vol. 42 (1941) pp. 325–330.
6. ———, *Free lattices*, II. Ann. of Math. (2) vol. 43 (1942) pp. 104–105.

California Institute of Technology,
Pasadena, Calif.