AN INVERSION OF THE LAPLACE AND STIELTJES TRANSFORMS UTILIZING DIFFERENCE OPERATORS

BY

R. S. PINKHAM

INTRODUCTION

According to a theorem of Hausdorff every completely monotonic sequence,

$$(1) \qquad (-1)^k \Delta^k \mu_s \geqq 0 \qquad (k = 0, 1, \cdots; s = 0, 1, \cdots),$$

has the form

$$(2) \qquad \mu_s = \int_0^1 t^s d\alpha(t)$$

with $\alpha(t)$ nondecreasing. Now consider the following "moment problem": to determine $\phi(t)$ from μ_s where

$$(3) \qquad \mu_s = \int_0^1 t^s \phi(t) dt \qquad (s = 0, 1, \cdots).$$

D. V. Widder [6] has shown that the inversion of (3) may be accomplished by the use of an operator whose general form is that of the left side of (1). Thus if one wished to solve the analogous "moment problem" for the Stieltjes transform one might begin by searching for a theorem analogous to that of Hausdorff in the hope that it would indicate the general form of the operator. Now mimicking the proof of D. V. Widder's Theorem 10.1 [7] one can prove the analog of Hausdorff's theorem, and the analog of (1) is

$$(4) \qquad (-1)^k \Delta^{n+k} s^n \mu_s \geqq 0 \qquad \begin{Bmatrix} k = 1, 2, \cdots \\ n = 1, 2, \cdots \\ s = 1, 2, \cdots \end{Bmatrix}.$$

Thus in trying to invert

$$(5) \qquad \mu_s = \int_0^\infty \frac{\phi(t)}{s + t} dt \qquad (s = 1, 2, \cdots)$$

one would try operators of the form (4). Define $P_{n,y}(\mu)$ by

$$P_{n,y}(\mu) = d_n (-1)^{n-p} \underset{s=0}{\Delta^n s^p \mu_s} \qquad (y > 0; s = 1, 2, \cdots)$$

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where $p=[y \log n]$ (the greatest integer contained in $y \log n$), $d_n = (\log n)^{p+1}/p!\Gamma(y)$, and $\Delta^n_{s=0}$ denotes the nth forward difference on s with mesh one, evaluated at $s=0$. Applying $P_{n,y}$ to (5) one finds

$$P_{n,y}(\mu) = d_n \int_0^\infty \frac{t^p n!}{t(t+1)\cdots(t+n)} \phi(t)dt.$$

But

$$\frac{n^t n!}{t(t+1)\cdots(t+n)} \to \Gamma(t) \qquad (n \to \infty);$$

hence for n large we have approximately

$$P_{n,y}(\mu) = \frac{r^{ry+1}}{\Gamma(ry+1)} \int_0^\infty e^{-rt} t^{ry} \left\{\frac{\Gamma(t)}{\Gamma(y)} \phi(t)\right\} dt$$

where $r=\log n$. The sequence of functions

$$\frac{r^{ry+1}}{\Gamma(ry+1)} e^{-rt} t^{ry} \qquad (n = 1, 2, \cdots)$$

is a Fejér sequence of kernels concentrated at $t=y$ and it is thus intuitively clear that

$$\lim_{n\to\infty} P_{n,y}(\mu) = \frac{\Gamma(t)}{\Gamma(y)} \phi(t)\Big|_{t=y} = \phi(y).$$

The "continuous" version of this operator reads as follows:

$$T_{n,y}(f) = c_n(-1)^{n-m} \underset{x=0}{\Delta^n} x^m f(xy) \qquad (n = 1, 2, \cdots),$$

where $m=[\log n]$ and $c_n=(\log n)^{m+1}/m!$. Applying $T_{n,y}$ to

$$f(x) = \int_0^\infty \frac{\phi(t)}{x+t} dt \tag{6}$$

one has, proceeding as before, the approximate equation

$$T_{n,y}(f) = \frac{(\log n)^{m+1}}{m!} \int_0^\infty e^{-mt} t^m \{\phi(yt)\Gamma(t)\} dt.$$

And it is again intuitively clear that

$$\lim_{n\to\infty} T_{n,y}(f) = \phi(yt)\Gamma(t)\,|_{t=1} = \phi(y). \tag{7}$$

Having an inversion of the Stieltjes transform leads one to attempt an inversion of the Laplace transform. Write (6) in the form

$$f(x) = \int_0^\infty e^{-xu}F(u)du$$

where

$$F(u) = \int_0^\infty e^{-ut}\phi(t)dt. \tag{8}$$

Proceeding formally

$$\begin{aligned} T_{n,y}(f) &= \int_0^\infty T_{n,y}(e^{-xu})F(u)du \\ &= \frac{(\log n)^{m+1}}{m!}\frac{1}{y}\int_0^\infty F\left(\frac{u}{y}\right)\frac{d^m}{du^m}(1-e^{-u})^n du. \end{aligned}$$

One thus has from (7)

$$\lim_{n\to\infty}\frac{(\log n)^{m+1}}{m!}\frac{1}{y}\int_0^\infty F\left(\frac{u}{y}\right)\frac{d^m}{du^m}(1-e^{-u})^n du = \phi(y); \tag{9}$$

i.e., a new inversion of (8).

These inversion formulas may be used in the standard way to obtain representation theories, and in the case of (9) we state a pair of sample theorems.

I should like to express my thanks to D. V. Widder for his careful guidance of this work in thesis form, and to the referee of the present paper, whose suggestions have resulted in a measurable increase in clarity and elegance.

THEOREM 1.

1. $y > 0$; $r \geqq 0$; $R > y$;

2. $\int_R^\infty t^{-r}\phi(t)dt$ *exists*;

3. $p = [y \log n]$ $\qquad (n = 1, 2, \cdots)$;

$\Rightarrow$

$$I_n = \frac{(\log n)^{p+1}}{p!}\int_R^\infty \frac{t^p n!}{t(t+1)\cdots(t+n)}\phi(t)dt = o(1) \qquad (n \to \infty).$$

Let $c_n = (\log n)^{p+1} n!/p!$ and $k_n(t) = t^{r+p}[t(t+1)\cdots(t+n)]^{-1}$. Then

$$I_n = c_n\int_R^\infty k_n(t)t^{-r}\phi(t)dt.$$

Now

$$\frac{k_n'(t)}{k_n(t)} = \frac{r+p}{t} - \frac{1}{t} - \frac{1}{t+1} - \cdots - \frac{1}{t+n}. \tag{1}$$

Hence if we define $g_n(\lambda)$ by

$$g_n(\lambda) = r + p - 1 - e^{-\lambda} - e^{-2\lambda} - \cdots - e^{-n\lambda},$$

we shall have

$$\frac{k_n'(t)}{k_n(t)} = \int_0^\infty e^{-t\lambda} g_n(\lambda) d\lambda.$$

But for n large $g_n(\lambda)$ has exactly one change of sign. Thus by Pólya and Szegö [4, book two, problem 80, p. 50] we are assured that $k_n'(t)/k_n(t)$ has at most one zero in $0<t<\infty$. But for n large (1) is positive for t small and positive and is negative for t large and positive. Thus (1) has exactly one zero in $0<t<\infty$. Hence if (1) is negative for $t=t_0$, then it will remain so for all $t \geqq t_0$.

Now

$$\frac{r+p}{t} \sim \frac{y \log n}{t} \qquad (n \to \infty),$$

and

$$\frac{1}{t} + \frac{1}{t+1} + \cdots + \frac{1}{t+n} \sim \log n \qquad (n \to \infty).$$

Whence at $t=R$, (1) is negative for all sufficiently large n. Taking $t_0=R$ we have that $k_n(t) \in \downarrow [R, \infty)$ if n be large.

Clearly then I_n exists; in fact by the mean-value theorem

$$I_n = c_n k_n(R) \int_R^\xi t^{-r} \phi(t) dt.$$

Whence

$$I_n = O[c_n k_n(R)] \qquad (n \to \infty).$$

Now by Stirling's formula

$$c_n k_n(R) = \exp\{[(y-R) + y(\log R - \log y)] \log n + o(\log n)\} \qquad (n \to \infty).$$

But

$$(y-R) + y(\log R - \log y) = y\left[\left(1 - \frac{R}{y}\right) + \log \frac{R}{y}\right] < 0$$

since

$$1 - z + \log z < 0 \qquad (z \neq 1).$$

Thus

$$c_n k_n(R) \to 0 \qquad (n \to \infty),$$

and the proof is complete.

THEOREM 2.

1. $y > 0$; $s \geqq 0$; $0 < \epsilon < y$;

2. $\int_0^\epsilon t^s \phi(t)dt$ *exists*;

3. $p = [y \log n] \qquad (n = 1, 2, \cdots);$

$\Rightarrow$

$$I_n = \frac{(\log n)^{p+1}}{p!} \int_0^\epsilon \frac{t^p n!}{t(t+1) \cdots (t+n)} \phi(t)dt = o(1) \qquad (n \to \infty).$$

With the notation of the previous theorem

$$I_n = c_n \int_0^\epsilon h_n(t) t^s \phi(t)dt,$$

where $h_n(t) = t^{p-s}[t(t+1) \cdots (t+n)]^{-1}$. Proceeding in exactly the same manner as before, one finds that $h_n(t)$ is increasing in $[0, \epsilon]$ for n sufficiently large. Thus by the mean-value theorem

$$I_n = c_n h_n(\epsilon) \int_\xi^\epsilon t^s \phi(t)dt;$$

i.e.,

$$I_n = O[c_n h_n(\epsilon)] \qquad (n \to \infty).$$

But

$$c_n h_n(\epsilon) = \exp \{ [(y - \epsilon) + y(\log \epsilon - \log y)] \log n + o(\log n) \} \quad (n \to \infty),$$

and the proof is completed as in Theorem 1.

DEFINITION. *The Lebesgue set of a function $\phi(t)$ will be denoted $\lambda(\phi)$.*

THEOREM 3.

1. $\phi(t) \in L(1/T, T)$ *for all* $T > 0$;

2. $y > 0$; $r \geqq 0$; $s \geqq 0$;

3. $\int_1^\infty t^{-r} \phi(t)dt$ *exists*;

4. $\int_0^1 t^\epsilon \phi(t)dt$ *exists;*

5. $p = [y \log n]$ $\qquad (n = 1, 2, \cdots);$

6. $y \in \lambda(\phi)$

$$I_n = \frac{(\log n)^{p+1}}{p!} \frac{1}{\Gamma(y)} \int_0^\infty \frac{t^p n!}{t(t+1) \cdots (t+n)} \phi(t)dt \to \phi(y) \qquad (n \to \infty).$$

By Theorems 1 and 2 it will be sufficient to show that

$$d_n \int_\epsilon^R G_n(t) t^p n^{-t} \phi(t)dt \to \phi(y) \qquad (n \to \infty),$$

where $0 < \epsilon < y < R$,

$$d_n = \frac{(\log n)^{p+1}}{p!} \frac{1}{\Gamma(y)}$$

and

$$G_n(t) = \frac{n^t n!}{t(t+1) \cdots (t+n)} \cdot$$

Now by K. Knopp [3, pp. 440–441] we have that if $\epsilon \leqq t \leqq R$, then

$$|G_n(t) - \Gamma(t)| = O\left(\frac{1}{n}\right) \qquad (n \to \infty).$$

Thus

$$(2) \quad \begin{aligned} & d_n \int_\epsilon^R t^p n^{-t} |G_n(t)\phi(t) - \Gamma(t)\phi(t)| dt \\ & \leqq O\left(\frac{1}{n}\right) d_n \int_\epsilon^R t^p n^{-t} |\phi(t)| dt \qquad (n \to \infty). \end{aligned}$$

But by the mean-value theorem and Stirling's formula

$$d_n \int_\epsilon^R t^p n^{-t} |\phi(t)| dt = d_n \xi^p n^{-\xi} \int_\epsilon^R |\phi(t)| dt = O((\log n)^{1/2}) \qquad (n \to \infty).$$

Therefore the right-hand side of (2) tends to zero as $n \to \infty$. Hence we must show

$$d_n \int_\epsilon^R t^p n^{-t} \Gamma(t) \phi(t)dt \to \phi(y) \qquad (n \to \infty).$$

Since

$$d_n \int_0^\infty t^p n^{-t} \Gamma(y) dt = 1$$

it will be sufficient to show that

$$J_n = d_n \int_\epsilon^R t^p n^{-t} [\Gamma(t)\phi(t) - \Gamma(y)\phi(y)] dt = o(1) \qquad (n \to \infty).$$

If

$$\alpha(t) = \int_y^t | \Gamma(u)\phi(u) - \Gamma(y)\phi(y) | du,$$

then

$$| J_n | \leqq d_n \int_\epsilon^R t^p n^{-t} d\alpha(t) \leqq M d_n \int_\epsilon^R t^{y \log n} n^{-t} d\alpha(t).$$

But by Hypothesis 6,

$$\left| \int_\epsilon^R \alpha(t) [n^{-t} t^{y \log n}]' dt \right| \leqq o(1) \int_\epsilon^R (y - t) [n^{-t} t^{y \log n}]' dt = o(1) \qquad (n \to \infty);$$

hence

$$| J_n | \leqq o(1) \qquad (n \to \infty),$$

and the theorem is proved.

LEMMA 4. *If $p > k$ and Δ^p is the pth forward difference on x with mesh* 1, *then*

$$\Delta^p \frac{x^k}{x + t} = (-1)^{p+k} \frac{t^k p!}{(x + t)(x + t + 1) \cdots (x + t + p)} .$$

Note that

$$\frac{x^k}{x + t} = (x + t)^{k-1} - \binom{k}{1} (x + t)^{k-2} + \cdots + \frac{(-1)^k t^k}{(x + t)},$$

and since $p > k$

$$\Delta^p \frac{x^k}{x + t} = (-1)^k t^k \Delta^p \frac{1}{x + t} = (-1)^{k+p} t^k \frac{p!}{(x + t)(x + t + 1) \cdots (x + t + p)} .$$

DEFINITIONS.

$$\Delta^p_{x=0} F(x) = \Delta^p F(x) \Big|_{x=0} = \sum_{j=0}^p \binom{p}{j} (-1)^{p+j} F(j); \tag{1}$$

$$T_n[f(x)] = T_n(f) = \frac{(\log n)^{m+1}}{m!} (-1)^{n-m} \underset{x=0}{\Delta^n} x^m f(x), \quad m = [\log n]; \tag{2}$$

$$T_{n,y}(f) = T_n[f(xy)] \qquad (n = 1, 2, \cdots). \tag{3}$$

THEOREM 4.

1. $\phi(t) \in L[0, R]$ *for all* $R>0$;
2. $f(x) = \int_0^\infty \frac{\phi(t)}{x+t} dt$ *exists for some* x;
3. $1 \in \lambda(\phi)$;

$\Rightarrow$

$$\lim_{n\to\infty} T_n(f) = \phi(1).$$

By Lemma 4

$$T_n(f) = \frac{(\log n)^{m+1}}{m!} \int_0^\infty \frac{t^m n!}{t(t+1)\cdots(t+n)} \phi(t) dt$$

provided the integral converges. It certainly does, for Hypothesis 2 implies that

$$\int_1^\infty \frac{\phi(t)}{t} dt$$

exists, whereas by 1

$$\int_0^1 \phi(t) dt$$

exists. Thus applying Theorem 3 with $y=1$

$$\lim_{n\to\infty} T_n(f) = \phi(1).$$

THEOREM 5.

1. $\phi(t) \in L[0, R]$ *for all* $R > 0$;
2. $f(x) = \int_0^\infty \frac{\phi(t)}{x+t} dt$ *exists for some* x;
3. $u \in \lambda(\phi), \qquad u > 0$;

$\Rightarrow$

$$\lim_{n\to\infty} T_{n,u}(f) = \phi(u).$$

Since

$$\int_0^\infty \frac{\phi(t)}{xu+t}\,dt = \int_0^\infty \frac{\phi(ut)}{x+t}\,dt$$

we have the desired result by Theorem 4.

One wonders whether this inversion formula for the Stieltjes transform may be interpreted operationally. Since the usual operators involve derivatives rather than differences, we might expect to obtain different results. The following argument shows that $T_{n,t}(f)$ may be interpreted as the familiar $-\sin \pi D/\pi$.

It is well known that

$$\int_0^\infty \frac{t^s}{1+t}\,dt = -\frac{\pi}{\sin \pi s} \qquad (-1 < s < 0);$$

or by changing variables of integration

$$-\frac{x^s\pi}{\sin \pi s} = \int_0^\infty \frac{t^s}{x+t}\,dt \qquad (-1 < s < 0).$$

Thus

$$\frac{(\log n)^{m+1}}{m!}(-1)^{n-m+1}\mathop{\Delta^n}_{x=0} x^{m+s}\frac{\pi}{\sin \pi s} = \frac{(\log n)^{m+1}}{m!}(-1)^{n-m+1}\int_0^\infty t^s \mathop{\Delta^n}_{x=0}\frac{x^m}{x+t}\,dt$$

$$= -\frac{(\log n)^{m+1}}{m!}\int_0^\infty \frac{t^{m+s}n!}{t(t+1)\cdots(t+n)}\,dt.$$

Now by employing Theorem 3 we have

$$\frac{(\log n)^{m+1}}{m!}(-1)^{n-m}\mathop{\Delta^{n-1}}_{x=0} x^{m+s} \to -\frac{\sin \pi s}{\pi} \qquad (n \to \infty). \tag{3}$$

Let

$$F(w) = f(e^w), \qquad \Phi(w) = \phi(e^w);$$

then the statement

$$\lim_{n\to\infty} T_{n,y}(f) = \phi(y)$$

is operationally the assertion

$$\lim_{n\to\infty}\frac{(\log n)^{m+1}}{m!}(-1)^{n-m}\mathop{\Delta^n}_{x=0} x^{m+D}F(w) = \Phi(w);$$

or by (3)

$$-\frac{\sin \pi D}{\pi} F(w) = \Phi(w)$$

in accord with the Hirschman-Widder convolution theory.

THEOREM 6.

1. *$\alpha(t)$ is a normalized function of bounded variation in $(0 \leqq t \leqq R)$ for every positive R;*

2. $f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t}$ *converges;*

$\Rightarrow$

$$\lim_{n\to\infty} \int_0^t T_{n,u}(f)du = \alpha(t) - \alpha(0+).$$

If

$$K_n(t) = \frac{(\log n)^{m+1}}{m!} n!t^m\left[t(t+1)\cdots(t+n)\right]^{-1},$$

then

$$T_{n,u}(f) = \int_0^\infty \frac{1}{u} K_n\left(\frac{y}{u}\right) d\alpha(y).$$

Since we may assume $\alpha(0)=0$,

$$T_{n,u}(f) = -\int_0^\infty \frac{\alpha(y)}{u} \frac{\partial}{\partial y} K_n\left(\frac{y}{u}\right) dy.$$

But by Euler's theorem on homogeneous functions

$$-\frac{1}{u}\frac{\partial}{\partial y} K_n\left(\frac{y}{u}\right) = \frac{1}{y}\frac{\partial}{\partial u} K_n\left(\frac{y}{u}\right).$$

Thus

$$T_{n,u}(f) = \int_0^\infty \frac{\alpha(y)}{y} \frac{\partial}{\partial u} K_n\left(\frac{y}{u}\right) dy.$$

This last integral clearly converges uniformly for $\epsilon \leqq u \leqq t$. Whence we may integrate under the integral sign to obtain

$$\int_\epsilon^t T_{n,u}(f)du = \int_0^\infty \frac{\alpha(y)}{y} K_n\left(\frac{y}{t}\right) dy - \int_0^\infty \frac{\alpha(y)}{y} K_n\left(\frac{y}{\epsilon}\right) dy$$
$$= I_n' + I_n''.$$

$$I_n' = \int_0^\infty \frac{\alpha(tx)}{x} K_n(x)dx.$$

$$I_n'' = -\int_0^\infty \frac{\alpha(\epsilon x)}{x} K_n(x)dx.$$

$$I_n'' = -\int_0^1 \frac{\alpha(\epsilon x)}{x} K_n(x)dx - \int_1^\infty \frac{\alpha(\epsilon x)}{x} K_n(x)dx$$
$$= J_n' + J_n''.$$

Since $\alpha(\epsilon x)/x$ is bounded for $0 \leqq \epsilon \leqq 1$ and $1 \leqq x < \infty$, we may employ Lebesgue's limit theorem to obtain

$$\lim_{\epsilon \to 0+} J_n'' = -\alpha(0+) \int_1^\infty \frac{1}{x} K_n(x)dx.$$

Noting that $\alpha(\epsilon x)$ is bounded for $0 \leqq \epsilon \leqq 1$ and $0 \leqq x \leqq 1$, we find in a similar fashion that

$$\lim_{\epsilon \to 0+} J_n' = -\alpha(0+) \int_0^1 \frac{1}{x} K_n(x)dx.$$

And by Theorem 3

$$\lim_{\epsilon \to 0+} I_n'' = -\alpha(0+) + o(1) \qquad (n \to \infty),$$

$$I_n' = \alpha(t) + o(1) \qquad (n \to \infty).$$

Hence

$$\lim_{n \to \infty} \int_{0+}^t T_{n,u}(f)du = \alpha(t) - \alpha(0+).$$

To complete the proof it will be sufficient to show that

$$\int_{0+}^t T_{n,u}(f)du = \int_0^t T_{n,u}(f)du.$$

Let $f(x) = f_1(x) + f_2(x)$ where

$$f_1(x) = \int_0^1 \frac{d\alpha(t)}{x+t}.$$

Then by interchanging the order of integration

$$\int_0^1 |T_{n,u}(f_1)| du \leqq \int_0^1 |d\alpha(t)| \int_0^1 \frac{1}{u} K_n\left(\frac{t}{u}\right) du.$$

But

$$\int_0^1 \frac{1}{u} K_n\left(\frac{t}{u}\right) du = \int_t^\infty \frac{1}{y} K_n(y) dy.$$

Also note

$$\begin{aligned} T_{n,u}(f_2) &= \int_1^\infty \frac{1}{u} K_n\left(\frac{y}{u}\right) d\alpha(y) \\ &= -\frac{\alpha(1)}{u} K_n\left(\frac{1}{u}\right) - \int_1^\infty \alpha(y) \frac{\partial}{\partial y}\left\{\frac{1}{y} \frac{(y/u)^{m+1}}{(y/u)\cdots(y/u+n)}\right\} dy, \end{aligned}$$

since $\alpha(y)/y$ is bounded $(1 \leqq y < \infty)$. But this last integral is equal to

$$\int_1^\infty \frac{\alpha(y)}{y} \frac{\partial}{\partial y}\left\{\frac{(y/u)^{m+1}}{(y/u)\cdots(y/u+n)}\right\} dy - \int_1^\infty \frac{\alpha(y)}{y^2} \frac{(y/u)^{m+1}}{(y/u)\cdots(y/u+n)} dy.$$

Suppose $0 \leqq u < 1$. Then

$$\int_1^\infty \left|\frac{\alpha(y)}{y^2}\right| \frac{(y/u)^{m+1}}{(y/u)\cdots(y/u+n)} dy \leqq M \int_1^\infty \frac{y^{m+1}}{y\cdots(y+n)} dy,$$

and since $y^{m+1}[y(y+1)\cdots(y+n)]^{-1}$ is decreasing for $y > 1$ and n large

$$\int_1^\infty \left|\frac{\alpha(y)}{y}\right| \left|\frac{\partial}{\partial y}\left\{\frac{(y/u)^{m+1}}{(y/u)\cdots(y/u+n)}\right\}\right| dy \leqq M.$$

Thus $T_{n,u}(f)$ is integrable in a neighborhood of the origin, and therefore

$$\int_{0+}^t T_{n,u}(f) du = \int_0^t T_{n,u}(f) du,$$

which completes the proof.

We now proceed to invert the Laplace transform as indicated in the introduction.

LEMMA 7.

1. $g_m(x) = \dfrac{d^m}{dx^m}(1 - e^{-x})^n$;

2. $n > m$;

$\Rightarrow$

$$g_m(\infty) = g_m(0) = 0 \qquad (m \geqq 1).$$

$$g_m(x) = \sum_{j=0}^n (-1)^j \binom{n}{j} (-j)^m e^{-jx}.$$

Clearly

$$g_m(\infty) = 0 \qquad (m \geqq 1).$$

But

$$g_m(0) = (-1)^{n+m} \sum_{j=0}^{n} (-1)^{n+j} \binom{n}{j} j^m.$$

Or

$$\begin{aligned} g_m(0) &= (-1)^{n+m} \sum_{j=0}^{n} (-1)^{n+j} \binom{n}{j} (x + j)^m \Big|_{x=0} \\ &= (-1)^{n+m} \Delta^n x^m \Big|_{x=0} = 0. \end{aligned}$$

DEFINITION.

$$G_n(c, y) = \frac{(\log n)^{m+1}}{m!} \frac{e^{cy}}{y} \int_0^\infty F\left(\frac{x}{y} + c\right) \frac{d^m}{dx^m} (1 - e^{-x})^n dx$$

where

$$m = [\log n] \qquad (n = 1, 2, \cdots).$$

THEOREM 7.

1. $F(x) = \int_0^\infty e^{-xt}\phi(t)dt$ *which converges for* $x > \sigma_c$;
2. $\phi(t) \in L[0, R]$ *for all positive* R;
3. $c > \sigma_c$; $y \in \lambda(\phi)$;

$\Rightarrow$

$$\lim_{n \to \infty} G_n(c, y) = \phi(y).$$

We first show that $G_n(c, y)$ exists. Utilizing the notation and result of Lemma 7, we have after repeated integration by parts

$$\int_0^\infty g_m(x) F\left(\frac{x}{y} + c\right) dx = (-1)^{m+1} \int_0^\infty F^{(m-1)}\left(\frac{x}{y} + c\right) \frac{d}{dx} (1 - e^{-x})^n dx.$$

But

$$\frac{d}{dx} (1 - e^{-x})^n \geqq 0 \qquad (x \geqq 0);$$

whence

$$\int_0^\infty \frac{d}{dx} (1 - e^{-x})^n F^{(m-1)}\left(\frac{x}{y} + c\right) dx \ll M \int_0^\infty \frac{d}{dx} (1 - e^{-x})^n dx = M,$$

since $F^{(m-1)}((x/y)+c)$ is bounded $(0 \leqq x < \infty)$.

Thus the operator does exist. Let

$$a_n = \frac{(\log n)^{m+1}}{m!} \frac{e^{cy}}{y};$$

then

$$\begin{aligned} G_n(c, y) &= a_n \int_0^\infty \frac{d}{dx}(1 - e^{-x})^n dx \int_0^\infty e^{-((x/y)+c)t}\left(\frac{t}{y}\right)^{m-1} \phi(t)dt \\ &= a_n \int_0^\infty \frac{d}{dx}(1 - e^{-x})^n dx \int_0^\infty e^{-xu}u^{m-1}ye^{-cuy}\phi(uy)du \\ &= a_n y \int_0^\infty \frac{d}{dx}(1 - e^{-x})^n dx \int_0^\infty xe^{-xu}\gamma(u)du \end{aligned}$$

where

$$\gamma(u) = \int_0^u e^{-cyv}v^{m-1}\phi(vy)dv.$$

Since there exists M such that

$$|\gamma(u)| \leqq M \qquad (0 \leqq u < \infty),$$

we may change the order of integration to obtain

$$\begin{aligned} G_n(c, y) &= a_n y \int_0^\infty \gamma(u)du \int_0^\infty xe^{-xu}\frac{d}{dx}(1 - e^{-x})^n dx \\ &= a_n y \int_0^\infty d\gamma(u) \int_0^\infty e^{-xu}\frac{d}{dx}(1 - e^{-x})^n dx \\ &= a_n y \int_0^\infty u d\gamma(u) \int_0^\infty e^{-xu}(1 - e^{-x})^n dx \\ &= a_n y \int_0^\infty u \frac{n!}{u(u+1)\cdots(u+n)} d\gamma(u), \end{aligned}$$

the last equation following from

$$\begin{aligned} \int_0^\infty e^{-xu}(1 - e^{-x})^n dx &= \int_0^\infty (-1)^n \Delta_u^n e^{-xu}dx \\ &= (-1)^n \Delta^n \int_0^\infty e^{-xu}dx \\ &= \frac{n!}{u(u+1)\cdots(u+n)}. \end{aligned}$$

Therefore

$$G_n(c, y) = a_n y \int_0^\infty e^{-cyu} \frac{u^m n!}{u(u+1)\cdots(u+n)} \phi(uy)du.$$

Or

$$G_n(c, y) = e^{cy} T_{n,y}(f)$$

where

$$f(x) = \int_0^\infty \frac{e^{-cu}\phi(u)}{x+u} du$$

provided $f(x)$ exists. But if

$$\eta(u) = \int_0^u e^{-cv}\phi(v)dv,$$

then

$$\int_0^R \frac{d\eta(u)}{x+u} = \frac{\eta(R)}{x+R} + \int_0^R \frac{\eta(u)}{(x+u)^2} du.$$

Since $\eta(u)$ is bounded, $f(x)$ does exist. Applying Theorem 5 we see that the proof is complete.

THEOREM 8.

1. *$\alpha(t)$ is a normalized function of bounded variation in every* $(0, R)$, $R>0$;

2. $F(x) = \int_0^\infty e^{-xt}d\alpha(t)$ *exists* $x > \sigma_c$;

3. $c > \sigma_c$;

$\Rightarrow$

$$\lim_{n\to\infty} \int_0^t G_n(c, y)dy = \alpha(t) - \alpha(0+).$$

As before

$$G_n(c, y) = a_n \int_0^\infty \frac{d}{dx}(1-e^{-x})^n dx \int_0^\infty e^{-((x/y)+c)t} \left(\frac{t}{y}\right)^{m-1} d\alpha(t).$$

If

$$\beta(Z) = \int_0^Z e^{-ct}\left(\frac{t}{y}\right)^{m-1} d\alpha(t),$$

then

$$G_n(c, y) = a_n \int_0^\infty \frac{d}{dx}(1 - e^{-x})^n dx \int_0^\infty e^{-(x/y)t} d\beta(t)$$
$$= a_n \int_0^\infty \frac{d}{dx}(1 - e^{-x})^n dx \int_0^\infty \frac{x}{y} e^{-(x/y)t} \beta(t) dt.$$

We may interchange the order of integration as in Theorem 7. Hence

$$G_n(c, y) = a_n \int_0^\infty d\beta(t) \int_0^\infty e^{-(x/y)t} \frac{d}{dx}(1 - e^{-x})^n dx$$
$$= a_n \int_0^\infty \left(\frac{t}{y}\right) \frac{n!}{(t/y) \cdots ((t/y) + n)} d\beta(t)$$
$$= \frac{(\log n)^{m+1}}{m!} \frac{e^{cy}}{y} \int_0^\infty \frac{e^{ct}(t/y)^m n!}{(t/y) \cdots ((t/y) + n)} d\alpha(t)$$
$$= e^{cy} T_{n,y}(f)$$

where

$$f(x) = \int_0^\infty \frac{e^{-ct} d\alpha(t)}{x + t},$$

which certainly exists. The desired result now follows from Theorem 6.

In 1940 R. P. Boas and D. V. Widder [1] obtained an inversion of the Laplace transform which required no knowledge of the derivatives of the generating function. There the basic kernel was

$$\frac{\partial^k}{\partial x^k}(x^{2k-1} e^{-x}),$$

as opposed to our

$$\frac{\partial^m}{\partial x^m}(1 - e^{-x})^n.$$

There seems to exist no apparent relation between the two kernels except that they both stem from an inversion of the Stieltjes transform.

In [1] the authors develop general representation formulas for the Laplace transform. Their methods are sufficiently general to be readily applicable to the operator $G_n(c, y)$. We state two sample theorems.

THEOREM 9. *N.a.s.c. for $F(x)$ to have the representation*

$$F(x) = \int_0^\infty e^{-xt} \phi(t) dt$$

with

$$\int_0^\infty |\phi(t)|^p dt < \infty \qquad (p > 1)$$

are

1. $F(x) \in C(0, \infty)$;
2. $F(x) = O(x^{1-p/p})$ $\qquad (x \to \infty; x \to 0+)$;
3. $\int_0^\infty |G_n(0, y)|^p dy = O(1)$ $\qquad (n \to \infty)$.

THEOREM 10. *N.a.s.c. that*

$$F(x) = \int_0^\infty e^{-xt} d\alpha(t) \qquad (x > a)$$

with $\alpha(t) \in \uparrow [0, \infty)$ *are*

1. $F(x) \geqq 0$ *and* $F(x) \in C$ $\qquad (a < x < \infty)$;
2. $F(x) = O(1)$ $\qquad (x \to \infty)$;
3. $G_n(c, y) \geqq 0$ $\qquad (c > a; 0 < y < \infty)$;

for a sequence of n's tending to infinity.

In conclusion we give a solution of the "moment problem" for the Stieltjes transform.

DEFINITION. *Let* $\{\mu_s\}_{s=1}^\infty$ *be an arbitrary sequence of numbers, and let*

$$p = [y \log n], \qquad y > 0, n = 1, 2, \cdots.$$

Define the following operator

$$P_{n,y}(\mu) = \frac{(\log n)^{p+1}}{p!\Gamma(y)} (-1)^{n-p} \underset{s=0}{\Delta^n} s^p \mu_s.$$

THEOREM 11.

1. $\phi(t) \in L[0, R]$ *for all* $R > 0$;
2. $\mu_s = \int_0^\infty \frac{\phi(t)}{s+t} dt$ *exists* $(s = 1, 2, \cdots)$;
3. $y \in \lambda(\phi)$; $y > 0$;

$\Rightarrow$

$$\lim_{n \to \infty} P_{n,y}(\mu) = \phi(y).$$

The proof is immediate from Lemma 4 and Theorem 3.

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HARVARD UNIVERSITY,
CAMBRIDGE, MASS.
DUKE UNIVERSITY,
DURHAM, N. C.