ON THE DETERMINATION OF THE PHASE OF A FOURIER INTEGRAL, I[1]

BY

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1. **Introduction.** Suppose $\phi(t)$ is a complex valued function on $(-\infty, \infty)$ and let $\hat{\phi}(x)$ denote its Fourier transform. A question which arises in various physical applications, and which has an intrinsic interest in its own right is: To what extent does the modulus of $\hat{\phi}$ determine ϕ? It is inevitable that some a priori conditions be imposed in order to obtain reasonably determinate results, and we shall establish the following

THEOREM 1. *Let* $\mathcal{C}(\mathrm{a})$ *be the class of all functions* ϕ *fulfilling the following conditions*:

α. $\phi \in L^1 \cap L^2$, *where* L^1 *and* L^2 *are the usual Lebesgue function spaces on* $(-\infty, \infty)$,

β. $\phi(t)$ *vanishes almost everywhere for* $t<0$[2],

γ. $\hat{\phi}(x) \neq 0$, $-\infty < x < \infty$,

δ. $\mathrm{a}(x)$ *is a fixed function such that* $|\hat{\phi}(x)| = \mathrm{a}(x)$, $-\infty < x < \infty$. *Then if* ϕ_1 *and* ϕ_2 *belong to* $\mathcal{C}(\mathrm{a})$ *there subsists a relation between them of the form*

$$e^{ic_1+ib_1x}B_1(x)\hat{\phi}_1(x) = e^{ic_2+ib_2x}B_2(x)\hat{\phi}_2(x), \tag{1}$$

where c_1, c_2, b_1, b_2, *are real numbers,* $b_1 \geqq 0$, $b_2 \geqq 0$ *and* $B_1(x)$, $B_2(x)$ *are limits as* $y \to 0+$ *of certain Blaschke products in the upper half-plane.* $B_1(x)$ *and* $B_2(x)$ *are holomorphic functions of modulus identically* 1.

Thus ϕ and $\hat{\phi}$ are both partly specified, and the data fall short of determining ϕ (or $\hat{\phi}$) uniquely to the extent of the arbitrariness of the zeros occurring in the Blaschke products, a complex number of modulus 1, and a pure oscillation. The zeros of B_1 and B_2 are not entirely arbitrary (subject, of course, to the convergence of the products), for the regularity of B_1 and B_2 on the real axis excludes the existence of a cluster point of zeros at a real point.

The method of proof depends in an essential way upon a canonical representation of certain holomorphic functions of one variable in the upper half-plane, and the multi-dimensional case remains untouched. This is perhaps

Presented to the Society, October 27, 1956; received by the editors December 5, 1955.

(1) The research in this paper was supported jointly by the Army, Navy and Air Force under contract with the Lincoln Laboratory, Massachusetts Institute of Technology.

(2) $t<0$ can be replaced by $t<t_0=t_0(\phi)$, for a translation reduces the latter case to the former; the conclusion is altered only by the addition of $t_0(\phi_j)$ to b_j, $j=1, 2$.

unfortunate from the standpoint of the crystallographers. (See §6, Example II.)

It would be possibly interesting to determine what happens if $\hat{\phi}$ is allowed to vanish.

2. **Statement of several known facts.** We begin by recalling certain facts about Blaschke products in the upper half-plane. Suppose $a_1, a_2, \cdots$ is a sequence of complex numbers with $\operatorname{Im} a_k > 0$, and

$$\sum_{k=1}^{\infty} \frac{\operatorname{Im} a_k}{1 + |a_k|^2} < \infty. \tag{2}$$

Condition (2) is necessary and sufficient that the Blaschke product,

$$\left(\frac{z-i}{z+i}\right)^n \prod_{k=1}^{\infty} \frac{|a_k - i|}{a_k - i} \cdot \frac{|a_k + i|}{a_k + i} \cdot \frac{z - a_k}{z - \bar{a}_k},$$

(where $z = x + iy$ and n = non-negative integer) be convergent for $y > 0$ to a holomorphic function $B(z)$. Then $|B(z)| < 1$ and $\lim_{y \to 0} |B(x+iy)| = 1$ for almost all x. If there are no zeros we set $B(z) \equiv 1$. Such a function has a simple characterization, embodied in the following theorem.

(A) [1] *Suppose* 1. *$F(z)$ is holomorphic for $y > 0$, $|F(z)| \leqq 1$, and* 2.

$$\lim_{y \to 0} \int_{-\infty}^{\infty} \frac{\log |F(x + iy)|}{1 + x^2} dx = 0.$$

Then $F(z)$ is of the form

$$F(z) = e^{ic + i\beta z} B(z),$$

where c and β are real, $\beta \geqq 0$, and $B(z)$ is the Blaschke product formed with the zeros of F. Conversely, if $B(z)$ is any Blaschke product in the upper half-plane, 2. *holds with F replaced by B.*

Subsequent arguments rest upon a number of more or less well-known theorems and formulae, which we proceed to assemble. These need not be read before they are referred to later on.

(B) [2, pp. 18–20]. *If ϕ belongs to $L^2(-\infty, \infty)$ and vanishes on a half-line, then*

$$\int_{-\infty}^{\infty} \frac{|\log |\hat{\phi}(x)||}{1 + x^2} dx < \infty.$$

(C) [3, Theorem 1, p. 643]. *If $\phi(t)/(1+t^2)$ belongs to $L^1(-\infty, \infty)$ and is continuous at x_0, then*

$$u(x, y) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{y^2 + (x - t)^2} \phi(t) dt$$

is harmonic for $y>0$ *and* $\lim_{y\to 0} u(x_0, y)=\phi(x_0)$.

An examination of the proof of (C) will disclose that the limit is uniform with respect to x in closed intervals interior to intervals of continuity of ϕ.

(D) [4, p. 106, Theorem IX for $p=2$]. *Let* $\Phi(z)$ *be a holomorphic function for* $y>0$ *subject to*

$$\int_{-\infty}^{\infty} |\Phi(x+iy)|^2 dx \leqq M < \infty, \tag{3}$$

where M *is independent of* y. $\Phi(z)$ *can then be represented in the form*

$$\Phi(z) = e^{ic+i\beta z}B(z)D(z)G(z), \tag{4}$$

where

(i) c *is a real number,*

(ii) β *is a non-negative real number,*

(iii) $B(z)$ *is the (convergent) Blaschke product formed with the zeros of* Φ,

(iv) $$D(z) = \exp\left(\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1+tz}{t-z} \frac{\log |\Phi(t)|}{1+t^2} dt\right),$$

where $\Phi(x)=\lim_{y\to 0}\Phi(x+iy)$ *almost everywhere, and*

$$\int_{-\infty}^{\infty} \frac{|\log |\Phi(x)||}{1+x^2} dx < \infty, \qquad \int_{-\infty}^{\infty} |\Phi(x)|^2 dx < \infty,$$

(v) $$G(z) = \exp\left(\frac{i}{\pi}\int_{-\infty}^{\infty} \frac{1+tz}{t-z} dE(t)\right),$$

where $E(t)$ *is a real, bounded, increasing function with derivative* $E'(t)=0$ *almost everywhere. Conversely, every function of the form* (4) *is holomorphic for* $y>0$ *and satisfies* (3).

(E) [5, p. 44]. *If* $\Phi(z)$ *is holomorphic for* $y>0$ *and in the neighborhood of every point on the real axis satisfies* lim sup $|\Phi(z)|\leqq 1$, *then either*

(α) *the modulus* $|\Phi(z)|$ *tends to* $+\infty$ *so rapidly that*

$$\liminf_{r\to\infty} \frac{\log M(r)}{r} > 0,$$

where $M(r)=\mathrm{Max}_{|z|=r} |\Phi(z)|$, *or*

(β) $|\Phi(z)|\leqq 1$ *for* $y>0$.

(F) [6, p. 152]. *If both the upper and lower symmetrical derivates of a function of bounded variation are everywhere finite, then the function is absolutely continuous.*

3. **Proof of the Theorem 1.** Let ϕ denote any function of the class $\mathcal{C}(a)$. Define a holomorphic function of $z=x+iy$ for $y>0$ by

$$\Phi(z) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty e^{i(x+iy)t} \phi(t) dt.$$

We shall refer to Φ as the *holomorphic extension of* $\hat{\phi}$. Then by the Parseval identity and Schwarz inequality:

$$\int_{-\infty}^{\infty} | \Phi(x + iy) |^2 dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \int_0^\infty e^{ixt - y|t|} \phi(t) dt \right|^2 dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \left(\frac{2}{\pi}\right)^{1/2} \frac{y}{y^2 + (x + \lambda^2)} \overline{\hat{\phi}(\lambda)} d\lambda \right|^2 dx$$

$$\leqq \frac{1}{\pi^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{y}{y^2 + (x + \lambda)^2} d\lambda \int_{-\infty}^{\infty} \frac{y}{y^2 + (x + \lambda)^2} | a(\lambda) |^2 d\lambda$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{y}{y^2 + (x + \lambda)^2} | a(\lambda) |^2 d\lambda = \frac{1}{\pi} \int_{-\infty}^{\infty} | a(\lambda) |^2 d\lambda < \infty.$$

By (D),

$$\Phi(z) = e^{ic + i\beta z} B(z) D(z) G(z), \tag{4}$$

where c, β, B, D, G have the properties mentioned in §2. The next paragraph is devoted to showing that $G(z) \equiv 1$; that is, $E(t) \equiv \text{const}$.

In the first place,

$$| D(z) | = \exp\left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{y^2 + (t - x^2)} \log a(t) dt\right),$$

because $\Phi(x+iy) \to \hat{\phi}(x)$ as $y \to 0$, for every x. By (B) and (C) it follows that

$$| D(z) | \to a(x) \qquad \text{as } y \to 0, \ -\infty < x < \infty.$$

Since we have $|B(z)| < 1$, (4) implies:

$$a(x) = \lim | \Phi(x + iy) | \leqq \liminf \left| \frac{\Phi(x + iy)}{B(x + iy)} \right|$$

$$= a(x) \liminf \exp\left(-\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{y(1 + t^2)}{y^2 + (x - t)^2} dE(t)\right).$$

Therefore, as $a(x) \neq 0$,

$$\limsup \int_{-\infty}^{\infty} \frac{y(1 + t^2)}{y^2 + (x - t)^2} dE(t) < \infty, \ -\infty < x < \infty.$$

A fortiori,

$$\limsup \int_{-\infty}^{\infty} \frac{y}{y^2 + (x - t)^2} dE(t) < \infty, \qquad -\infty < x < \infty. \tag{5}$$

Since $E(x+t) - E(x-t)$ is an increasing function of t, we can write:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{y}{y^2 + (x - t^2)} dE(t) &= \int_0^{\infty} \frac{y}{y^2 + t^2} d[E(x + t) - E(x - t)] \\ &\geqq \int_0^{y} \frac{y}{y^2 + t^2} d[E(x + t) - E(x - t)] \\ &\geqq \frac{E(x + y) - E(x - y)}{2y}, \qquad -\infty < x < \infty. \end{aligned}$$

Therefore in view of (5), the upper (and lower) symmetrical derivates of E are everywhere finite. By (F), E is absolutely continuous. Since $E' = 0$ almost everywhere, $E \equiv \text{const.}$, $G \equiv 1$.

Hence, if ϕ belongs to the class $\mathcal{C}(a)$, we have

$$\Phi(z) = e^{ic + i\beta z} B(z) D(z). \tag{6}$$

From (6), together with the condition $\hat{\phi}(x) \neq 0$, $-\infty < x < \infty$, it follows that $B(z)$ cannot have zeros which cluster at a finite point of the real axis. But then $B(z)$ is uniformly convergent in a rectangle $|y| \leqq \delta$, $a \leqq x \leqq b$, provided δ is so small that this rectangle is free of zeros(³). Denoting the zeros of $B(z)$ by $a_k = x_k + iy_k$, this follows at once from the identity

$$\frac{|a_k + i|}{a_k + i} \cdot \frac{|a_k - i|}{a_k - i} \cdot \frac{z - a_k}{z - \bar{a}_k} = 1 - \frac{iy_k}{1 + x_k^2 + y_k^2} \cdot \frac{(\overline{i + a_k})(i + z)}{z - \bar{a}_k} \cdot c_k,$$

where

$$c_k = \frac{1 + |(i - a_k)/(i + a_k)|^2}{1 + |(i - a_k)/(i + a_k)|} \cdot \left\{1 + \frac{i - z}{i + z} \cdot \frac{i + a_k}{i - a_k} \cdot \left|\frac{i - a_k}{i + a_k}\right|\right\}.$$

Therefore $B(z)$ is holomorphic for $y \geqq 0$. By (6),

$$\lim_{y \to 0} D(z) \equiv D(x), \qquad -\infty < x < \infty,$$

exists as a continuous function, and the conclusion of Theorem 1 follows at once from (6), written for ϕ_1 and ϕ_2 with $y = 0$.

REMARK. It can be shown that

$$D(x) = a(x) \exp iH(x),$$

where

(³) I owe this remark to a referee.

$$H(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 + tx}{t - x} \cdot \frac{\log a(t)}{1 + t^2} dt,$$

the integral being taken as a principal value at $t = x$. We omit the demonstration of this fact.

4. On the existence of "zero-free" solutions. Formula (1) of Theorem 1 exhibits a relation between any *pair* of solutions of our phase problem. In this section we seek a single particularly simple solution; to wit, one whose holomorphic extension to the upper half-plane is free of zeros, so that $B(z) \equiv 1$ in (6). Such a "zero-free" solution always exists in $\mathcal{C}(a)$ whenever a solution exists in $\mathcal{C}(a)$ whose holomorphic extension has only a finite number of zeros, and we shall show that this is still true whenever some solution has a holomorphic extension with sufficiently sparse zeros, though possibly infinitely many.

The precise condition of sparseness of zeros in the upper half-plane which we take as an hypothesis is that the product $\prod_\nu \exp (2i \arg a_\nu)$ be convergent, where $\{a_\nu\}$ is the set of all zeros of the holomorphic extension of some solution ϕ, i.e., $\phi \in \mathcal{C}(a)$, the limit being independent of the enumeration of the set $\{a_\nu\}$. Then the zeros must be on the whole rather close to the x-axis, and since we have proved that there cannot exist a finite cluster point of zeros we must have $|\operatorname{Re} a_\nu| \to +\infty$. Using this fact we conclude that

$$\prod_\nu \exp (i[\arg (a_\nu + i) + \arg (a_\nu - i)])$$

and

$$\prod_\nu \exp (2i \arg (x - a_\nu)) = \prod_\nu \frac{x - a_\nu}{x - \bar{a}_\nu}$$

are convergent products, the latter being so uniformly with respect to finite x-intervals. It follows that the Blaschke product $B(z)$ associated with ϕ can be written in the form

$$B(z) = e^{i\eta} \cdot \prod_n \frac{z - a_n}{z - \bar{a}_n}, \qquad y \geqq 0, \ \eta \text{ real}.$$

A "zero-free" solution, q, if it exists in $\mathcal{C}(a)$ must, by (6), be of the form

$$\hat{q}(x) = e^{i\alpha + i\beta x} D(x), \qquad D(x) = \lim_{y \to 0} D(x + iy).$$

We can afford to take $\alpha = \beta = 0$ as the factors $e^{i\alpha + i\beta x}$ are irrelevant for our present considerations, since they correspond to trivial transformations of q. Therefore, taking account of the above assumption on the distribution of the zeros $\{a_n\}$ of the holomorphic extension of $\hat{\phi}(x)$, we have, by Theorem 1, the necessary condition,

$$\hat{\phi}(x) = \prod_n \frac{x - a_n}{x - \bar{a}_n} \cdot \hat{q}(x), \qquad \hat{q}(x) = \prod_n \frac{x - \bar{a}_n}{x - a_n} \cdot \hat{\phi}(x).$$

We proceed to show that such a function q does indeed exist in $\mathcal{C}(a)$. Put

$$\sigma_n(t) = \begin{cases} -2y_n e^{-ia_n t} & \text{if } t \leqq 0, \\ 0 & \text{if } t > 0, \end{cases}$$

where $a_n = x_n + iy_n$. Then

$$\frac{x - \bar{a}_n}{x - a_n} \cdot \hat{\phi}(x) = \widehat{T_n\phi}(x),$$

where

$$T_n\phi = \sigma_n * \phi + \phi,$$

" $*$ " denoting convolution in L^1.

Regarding σ_n and ϕ as functions on $(-\infty, \infty)$, we have σ_n and ϕ belonging to L^1. Hence $\sigma_n * \phi \in L^1$, $T_n\phi \in L^1$. Since $|\hat{\phi}(x)| = |(T_n\phi)^\frown(x)|$ and $\phi \in L^2$, it follows that $T_n\phi \in L^2$. By definition,

$$\begin{aligned} T_n\phi(t) &= \int_{-\infty}^{\infty} \phi(u)\sigma_n(t - u)du + \phi(t) \\ &= \int_{\max(0,t)}^{\infty} \phi(u)\sigma_n(t - u)du + \phi(t); \end{aligned}$$

and if $t < 0$, this is

$$T_n\phi(t) = -2y_n e^{-ia_n t} \int_0^{\infty} e^{ia_n u}\phi(u)du = 0,$$

because a_n is a zero of the holomorphic extension of $\hat{\phi}$. Obviously

$$\widehat{T_n\phi}(x) \neq 0$$

for all real x. Hence $T_n\phi \in \mathcal{C}(a)$. It follows at once that

$$\prod_{n=1}^{N} \frac{x - \bar{a}_n}{x - a_n} \cdot \hat{\phi}(x)$$

is the Fourier transform $\hat{\psi}_N(x)$ of the function

$$\psi_N(t) = T_1 T_2 \cdots T_N \phi(t),$$

and $\psi_N \in \mathcal{C}(a)$. Clearly, $\hat{\psi}_N(x)$ converges pointwise everywhere as $N \to \infty$ to the function

$$\tilde{\psi}(x) = \prod_n \frac{x - \bar{a}_n}{x - a_n} \cdot \hat{\phi}(x),$$

and since $|\tilde{\psi}(x)| = |\hat{\phi}(x)|$, $\tilde{\psi}$ must belong to L^2, and hence

$$\tilde{\psi} = \hat{q}, \qquad q \in L^2.$$

Such q is in the first place defined only to within the class of functions differing from it on null sets. We shall show that under the present circumstances q can be chosen in $\mathfrak{C}(a)$.

First, recall that $\hat{\psi}_N(x) \to \hat{q}(x)$ uniformly on finite x-intervals and that $|\hat{\psi}_N(x)| = |\hat{q}(x)|$ almost everywhere. Then, given $\epsilon > 0$, split as follows:

$$\int_{-\infty}^{\infty} |\psi_N(x) - q(x)|^2 dx = \int_{-\infty}^{-A} + \int_{-A}^{A} + \int_{A}^{\infty},$$

choosing A independent of N so that the first and third integrals on the right add up to less than ϵ. With such A fixed, for all N sufficiently large the second integral is less than ϵ. Hence $\|\hat{\psi}_N - \hat{q}\|_2 = \|\psi_N - q\|_2 \to 0$. Hence $q(t) = 0$ for almost all $t < 0$, and we can redefine q so that it is identically 0 for $t < 0$.

Clearly, $\hat{q}(x) \neq 0$, $-\infty < x < \infty$.

Since $\{\psi_N\}$ is a Cauchy sequence in L^2, there exists a subsequence N_ν such that $\{\psi_{N_\nu}(t)\}$ is a Cauchy sequence of complex numbers for almost all t: $|\psi_{N_\nu}(t) - \psi_{N_\mu}(t)| \to 0$ as $\nu, \mu \to \infty$. Application of the diagonal process yields a further subsequence of $\{N_\nu\}$ (which we denote by the same notation) such that for a countable dense set of t's

$$|\psi_{N_\nu}(t) - \psi_{N_\mu}(t)| \downarrow 0 \text{ as } \nu, \mu \to \infty. \tag{7}$$

Since the term $\phi(t)$ cancels out in $\psi_{N_\nu}(t) - \psi_{N_\mu}(t)$, this difference is a continuous function of t, and therefore (8) holds for all t. By the theorem of Lebesgue on integrating monotonic sequences, it follows that

$$\int_0^{\infty} |\psi_{N_\nu}(t) - \psi_{N_\mu}(t)| \, dt \to 0, \qquad (\nu, \mu \to \infty).$$

Therefore ψ_{N_ν} tends to some function ψ_0 in L^1. But ψ_{N_ν} tends to q in L^2. Therefore $\psi_0(t) = q(t)$ almost everywhere, and hence $q \in L^1$. This completes the proof that $q \in \mathfrak{C}(a)$. We can thus assert

THEOREM 2. *If some function $\phi \in \mathfrak{C}(a)$ is such that the zeros $\{a_\nu\}$ of the holomorphic extension of $\hat{\phi}$ satisfy the condition:*

$\prod_\nu e^{2i \arg a_\nu}$ *is convergent to a limit which is independent of the enumeration of $\{a_\nu\}$,*

then there exists a solution $q \in \mathfrak{C}(a)$ whose Fourier transform is given by

$$\hat{q}(x) = e^{i\alpha + i\beta x} D(x), \qquad -\infty < x < \infty,$$

where α, β are real and $D(x)$ is a continuous function given by

$$D(x) = \lim_{y\to 0} \exp \left\{\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1+tz}{t-z}\,\frac{\log \mathrm{a}(t)}{1+t^2}\,dt\right\}.$$

5. **On the arbitrariness of $B(z)$.** We have seen that for functions ϕ in the class $\mathcal{C}(a)$ the canonical representation of the holomorphic extension of $\widehat{\phi}$ contains a certain Blaschke product, the limit function of which contributes to the phase of $\widehat{\phi}(x)$. It is natural to inquire whether the special properties of ϕ restrict the Blaschke products appearing, beyond the necessity of continuous boundary values. Explicitly, suppose $B(z)$ is any Blaschke product with continuous boundary values $B(x) = \lim B(x+iy)$.

Q_1: *Does there exist $\psi \in \mathcal{C}(a)$ such that*

$$\widehat{\psi}(x) = B(x)a(x) \exp i\tilde{H}(x), \qquad -\infty < x < \infty?$$

The function $H(x)$ is defined in the remark at the end of §3. A more specific question is

Q_2. *Given $\phi \in \mathcal{C}(a)$ and $B(z)$ as above, does there exist $\psi \in \mathcal{C}(a)$ such that $\psi(x) = B(x)\phi(x)$, $-\infty < x < \infty$?*

We are unable to answer these questions. Their difficulty stems from the requirement that ψ belong to L^1. That such ψ exists in L^2 and vanishes for negative arguments is almost trivial. We consider Q_1. The function $B(x)a(x) \cdot \exp iH(x)$ is the boundary function of a function $\Phi(z)$ in the Hardy class H_2; that is Φ is holomorphic for $y>0$,

$$\int_{-\infty}^{\infty} |\Phi(x+iy)|^2 dx \leq M < \infty, \ M \text{ independent of } y,$$

$$\Phi(x) = \lim_{y\to 0+} \Phi(x+iy) = B(x)a(x) \exp iH(x),$$

almost everywhere. Therefore ([7, Theorem 93, p. 125]),

$$\begin{aligned} \Phi(z) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\Phi(u)}{u-z}\,du, \qquad (y>0) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(u)\overline{\left[\frac{1}{-i(u-\bar{z})}\right]}du \\ &= \int_0^{\infty} e^{izt}\psi(t)dt, \end{aligned}$$

where

$$\Phi(x) = \underset{A\to\infty}{\mathrm{l.i.m.}} \int_{-A}^{A} e^{ixt}\psi(t)dt, \qquad \psi \in L^2(-\infty, \infty).$$

On the other hand

$$\Phi(x) = \underset{y\to 0}{\text{l.i.m.}}\ \Phi(x+iy), \tag{8}$$

and

$$\left\{\int_{-\infty}^{\infty}\left|\Phi(x) - \int_0^A e^{ixt}\psi(t)dt\right|^2 dx\right\}^{1/2}$$

$$\leqq \left\{\int_{-\infty}^{\infty}\left|\Phi(x) - \int_0^{\infty} e^{ixt-yt}\psi(t)dt\right|^2 dx\right\}^{1/2}$$

$$+ \left\{\int_{-\infty}^{\infty}\left|\int_0^{\infty} e^{ixt-yt}\psi(t)dt - \int_0^A e^{ixt}\psi(t)dt\right|^2 dx\right\}^{1/2} = J_1 + J_2.$$

By (8), $J_1 = o(1)$, $y \to 0$. By Plancherel, for $A > 0$,

$$\int_{-\infty}^{\infty}\left|\int_A^{\infty} e^{ixt-yt}\psi(t)dt\right|^2 dx = 2\pi\int_A^{\infty}|\psi(t)|^2 e^{-2yt}dt.$$

Hence, if $\epsilon > 0$, we can find A_0 such that $A > A_0$ implies that the last integral is $< \epsilon$. Fixing such A,

$$J_2 \leqq \left\{\int_{-\infty}^{\infty}\left|\int_0^A e^{ixt}(1 - e^{-yt})\psi(t)dt\right|^2 dx\right\}^{1/2} + \epsilon^{1/2}$$

$$= \left\{\int_0^A (1 - e^{-yt})^2 |\psi(t)|^2 dt\right\}^{1/2} + \epsilon^{1/2} = o(1) + \epsilon^{1/2}, \qquad y \to 0.$$

Therefore

$$\Phi(x) = \underset{A\to\infty}{\text{l.i.m.}} \int_0^A e^{ixt}\psi(t)dt,$$

and by the uniqueness of Fourier transforms we conclude that $\psi(t) = 0$ for $t < 0$ (after possibly altering ψ on a null set)(4). The same procedure is valid for Q_2.

6. **Examples.** Finally we shall exhibit a couple of examples which serve to illustrate Theorem 1.

I. The first example is a generalization of one which was pointed out to me by Professor W. Rudin before Theorem 1 was found. Suppose that $\phi \in \mathcal{C}(a)$, and furthermore $\phi(t) = 0$ for $t < n$ where n is a positive integer. Let $a_1, a_2, \cdots, a_n$ be arbitrary real numbers, and let λ be any nonreal complex number. Then the function

$$\phi_\lambda(t) = \lambda\phi(t) + \sum_{k=1}^{n} a_k(\phi(t+k) + \phi(t-k))$$

(4) This result can also be derived by use of (D), (iv), in conjunction with a theorem of Paley and Wiener [2, Theorem XII].

belongs to $\mathfrak{C}(a)$, and its Fourier transform is

$$F_\lambda(x) = \left(\lambda + 2\sum_{k=1}^{n} a_k \cos kx\right)\hat{\phi}(x).$$

Hence

$$F_\lambda(x) = \frac{P(e^{ix}) - \lambda}{P(e^{ix}) - \bar{\lambda}} F_{\bar{\lambda}}(x), \tag{9}$$

where we have put

$$P(e^{ix}) = -2\sum_{k=1}^{n} a_k \cos kx.$$

Replacing x by $z=x+iy$, $y>0$, in (9) we obtain the corresponding relation for the holomorphic extensions of F_λ and $F_{\bar{\lambda}}$. Since $\operatorname{Im}\lambda\neq 0$, $P(w)-\lambda$ vanishes for $2n$ nonreal values of w, none of which can be of modulus 1, and which must therefore be of the form $w_1, w_2, \cdots, w_n, 1/w_1, 1/w_2, \cdots, 1/w_n$, where $|w_1|<1, \cdots, |w_n|<1$. Therefore the only values of z in the upper half-plane for which $P(e^{iz})-\lambda$ vanishes are

$$z = a_n^{(k)} \equiv -i\log|w_k| + \arg w_k + 2\pi n, \quad n = 0, \pm 1, \cdots, k = 1, 2, \cdots, n.$$

Hence the Blaschke product $B_1(z)$ formed with these zeros is convergent. Likewise $P(e^{iz})-\bar{\lambda}=0$, $y>0$, if and only if

$$z = b_n^{(k)} \equiv -i\log|w_k| - \arg w_k + 2\pi n, \quad n = 0, \pm 1, \cdots, k = 1, 2, \cdots, n,$$

so that the corresponding Blaschke product $B_2(z)$ is defined. Put, for $y>0$,

$$Q(z) = \frac{B_2(z)}{B_1(z)} \cdot \frac{P(e^{iz}) - \lambda}{P(e^{iz}) - \bar{\lambda}}.$$

Then $Q(z)$ is holomorphic and fails to vanish for $y>0$. We shall show that $Q(z)$ is bounded in the strip $|x|\leqq 2\pi$, $y>0$. By periodicity it will then follow that $Q(z)$ is bounded in the entire upper half-plane. (Note that $B_1(z)$, $B_2(z)$ as well as $P(e^{iz})$ are periodic with period 2π.) Put

$$K = 1 + \max_{k=1,\dots,n}\{-\log|w_k|\}.$$

Then for $y\geqq K$, $|x|\leqq 2\pi$, $(P(e^{iz})-\lambda)/P(e^{iz})-\bar{\lambda})$ is bounded, and in the rectangle

$$|x| \leqq 2\pi, \qquad 0 < y \leqq K$$

$Q(z)$ is bounded. It remains to prove that $B_2(z)/B_1(z)$ is bounded for $z\in S_K$: $y\geqq K$, $|x|\leqq 2\pi$, and it is no real loss in generality to take $k=1$, so that the zeros $a_n^{(k)} \equiv a_n$, $b_n^{(k)} \equiv b_n$ lie on a single horizontal line in the upper half-plane. Thus

$$\left|\frac{B_2(z)}{B_1(z)}\right| = \prod_n \left|\frac{z - b_n}{z - \bar{b}_n} \cdot \frac{z - \bar{a}_n}{z - a_n}\right|.$$

Let us pair the adjacent zeros lying to the left of S_K into pairs (b_n, a_n) where b_n lies nearer to S_K than the adjacent a_n, and let $\prod_1$ denote the partial product containing the zeros so paired. For $z \in S_K$,

$$\left|\frac{z - b_n}{z - a_n}\right| \leqq \left|\frac{z - \bar{b}_n}{z - \bar{a}_n}\right|,$$

or

$$\left|\frac{z - b_n}{z - \bar{b}_n} \cdot \frac{z - \bar{a}_n}{z - a_n}\right| \leqq 1.$$

Hence $\prod_1 \leqq 1$. We pair the zeros lying to the right of S_K in a similar way and form $\prod_2$, $\prod_2 \leqq 1$. This leaves out of account approximately seven zeros which lie nearest the y-axis, but the partial product $\prod_3$ involving these is clearly bounded for $z \in S_K$. Hence

$$\left|\frac{B_2(z)}{B_1(z)}\right| = \prod_1 \prod_2 \prod_3$$

is bounded for $z \in S_K$.

Therefore $|Q(z)| < M_0$ for $y > 0$. Putting $M(r) = \mathrm{Max}_{|z|=r} |Q(z)|$,

$$\liminf_{r \to \infty} \frac{\log M(r)}{r} \leqq 0.$$

But

$$\limsup_{y \to 0} |Q(z)| \leqq 1, \qquad -\infty < x < \infty.$$

Hence it follows by the Phragmèn-Lindelöf theorem (E) that

$$|Q(z)| \leqq 1 \text{ for } y > 0. \tag{10}$$

Next we show that

$$\int_{-\infty}^{\infty} \frac{\log |Q(z)|}{1 + x^2} dx \to 0, \ y \to 0. \tag{11}$$

According to (A),

$$\int_{-\infty}^{\infty} \frac{\log |B_j(x)|}{1 + x^2} dx \to 0, \ y \to 0, \qquad j = 1, 2.$$

By the periodicity of $(P(e^{ix})-\lambda)/(P(e^{ix})-\bar{\lambda})$ we have

$$\int_{-\infty}^{\infty}\left|\log\left|\frac{P(e^{iz})-\lambda}{P(e^{iz})-\bar{\lambda}}\right|\right|\frac{dx}{1+x^2}=\sum_{k=-\infty}^{\infty}\int_{2\pi k}^{2\pi(k+1)}\left|\log\left|\frac{P(e^{iz})-\lambda}{P(e^{iz})-\bar{\lambda}}\right|\right|\frac{dx}{1+x^2}$$

$$=\int_0^{2\pi}\sum_{k=-\infty}^{\infty}\frac{1}{1+(x+2k\pi)^2}\cdot\left|\log\left|\frac{P(e^{iz})-\lambda}{P(e^{iz})-\bar{\lambda}}\right|\right|dx,$$

which clearly tends to 0 as $y\to 0$. By (10) and (11), since Q has no zeros, (A) implies that $Q(z)$ must be of the form

$$Q(z) = e^{ic+i\beta z}, \qquad c \text{ real}, \beta \geqq 0,$$

so that

$$\frac{P(e^{iz})-\lambda}{P(e^{iz})-\bar{\lambda}} = e^{ic+i\beta z}\cdot\frac{B_1(z)}{B_2(z)}.$$

Passing to the limit $y\to 0$ this shows that (9) can be written

$$B_2(x)F_\lambda(x) = e^{ic+i\beta x}B_1(x)F_{\bar{\lambda}}(x),$$

as required by Theorem 1.

II. In a rather well-known problem arising in crystallography the function ϕ would stand for electron density, and would be therefore non-negative. We shall show by a simple example that even if we assume $\phi\geqq 0$, in addition to $\phi\in\mathcal{C}(a)$, ϕ is *not* uniquely determined.

Let $b=\alpha+i\beta$ be a complex number such that $\beta>0$, $\alpha\neq 0$, $|4\beta/\alpha|<1$. Put $\phi_1(t)=e^{-\beta t}$ for $t\geqq 0$, $\phi_1(t)=0$ for $t<0$, and put $a(x)=|1/(ix-\beta)|$. Define $\phi_2(t)$ by the condition

$$\hat{\phi}_2(x) = \frac{x+\bar{b}}{x+b}\cdot\frac{x-b}{x-\bar{b}}\cdot\hat{\phi}_1(x), \qquad -\infty<x<\infty.$$

Then ϕ_1, $\phi_2\in\mathcal{C}(a)$, and a calculation shows that

$$\phi_2(t) = \phi_1(t) - 4\beta\int_0^t \cos\alpha(t-u)e^{-\beta(t-u)}\phi_1(u)du$$

$$+\frac{4\beta^2}{\alpha}\int_0^t \sin\alpha(t-u)e^{-\beta(t-u)}\phi_1(u)du$$

if $t\geqq 0$. If we substitute the particular ϕ_1 defined above, it turns out that

$$\phi_2(t) = e^{-\beta t}\left(\left(1-\frac{4\beta}{\alpha}\sin\alpha t\right)+\frac{4\beta^2}{\alpha^2}(1-\cos\alpha t)\right) \text{ if } t\geqq 0.$$

Thus $\phi_2(t)\geqq 0$.

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