MEANS ON SEMIGROUPS AND THE HAHN-BANACH EXTENSION PROPERTY

BY

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I. **Introduction.** In this paper generalizations of the Hahn-Banach theorem on the extension of linear functions are proved. The restriction that the range space be the real field is removed, and the condition that the extension be fixed with respect to certain semi-groups of operators is imposed. Another problem considered which is very closely related to Hahn-Banach extensions is the existence of monotone, distributive extensions of functions which preserve invariance with respect to certain semigroups of operators.

It was shown in [10](1) that when certain necessary restrictions are placed on the range space, the class of semigroups which permit the two types of invariant extensions is contained in the class of functions which have invariant means definable on the associated Banach space of bounded, real-valued functions defined on these semigroups. Further it was shown that every semigroup *known* to have an invariant mean also permitted the two types of extensions.

When structure in addition to the minimal required structure is placed on the range space, e.g., when the space is an ordered subspace of the conjugate space of an ordered linear space with reproducing cone, a semi-group which has an invariant mean also has the two extension properties relative to these particular range spaces. Many of the standard function spaces satisfy these conditions. In addition this paper will exhibit conditions which guarantee the continuity of these extensions in the case when linear topological spaces are considered and make some application to real-valued functions.

§II outlines definitions and results preceding this paper. §III contains theorems on the existence of invariant extensions for a given semigroup G with an invariant mean and a given range space V. Various extra conditions added to the known necessary condition that V be a boundedly complete vector lattice are sufficient for these extensions. Some converse results are also given under which the existence of some sort of invariant extension implies that G has an invariant mean. §IV contains examples of spaces satisfying the conditions imposed on the space V. §V discusses continuity of the extensions when order and topology are both present. §VI presents applications of the theorems of §§III and V to some of the examples of §IV.

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(1) The numbers in brackets refer to the bibliography at the end of the paper.

II. **Background material.** For reference to background material not specifically introduced in this section see [10]. All linear spaces are assumed to have the real field R as the scalar field.

A linear space V, and a binary relation (ordering) on V denoted by $\geqq$ (read greater than or equal to) is an *ordered linear space* (*ols*) if and only if (1) $\geqq$ is transitive and reflexive, (2) if $x \geqq y$, then $x+z \geqq y+z$ and $tx \geqq ty$, for every $z \in V$ and non-negative real number t. A set C in a linear space is a *cone* if and only if $x, y \in C$, $t \in R$, $t \geqq 0$, imply $x+y \in C$ and $tx \in C$.

A cone C determines an ordering which makes C an ols: $x \geqq y \cdot \equiv \cdot x-y \in C$. Conversely the set of elements $\geqq 0$ determines a cone in an ols and this cone in turn determines the original ordering.

A cone C is *sharp* if and only if x, $-x \in C$ imply $x=0$. This is equivalent to the statement: $x \geqq y \geqq x$ imply $x=y$. A sharp cone will be called an *s-cone*.

Given a cone $K \subseteq V$, an ols, consider any subspace V^0 of the space of distributive functions $V^\sharp$ of V. Define the *induced cone* of K^0 of V^0: $K^0 = \{f \in V^0: f(v) \geqq 0, v \in K\}$. It follows that K^0 is a cone.

Definitions and basic properties of upper bound, least upper bound (sup), etc. are assumed [10]. An ols V has the (*finite*) *least upper bound property* [(*finite*) *LUBP*], or equivalently, has a (*minihedral*) *fully minihedral* cone, if and only if every (finite) set of elements with an upper bound has a least upper bound.

A function F from an ols Y to an ols V is *monotone* if and only if $y \geqq y'$ implies $F(y) \geqq F(y')$. F is *non-negative* if and only if $y \geqq 0$ implies $F(y) \geqq 0$. If F is distributive, F is non-negative if and only if it is monotone. The following theorem is of basic importance. It is due to M. M. Day [2].

THEOREM A. *Consider an ols V, then statments* 1, 2, 3, 4 *are equivalent and they imply* 5.

1. *V has the LUBP.*

2. *Given sets A and B in V such that $A \geqq B$ (i.e. $a \geqq b$, $a \in A$, $b \in B$), there exists $v \in V$ such that $A \geqq v \geqq B$.*

3. *V has the monotone extension property (MEP). That is if X is an ordered linear subspace of an ols Y with positive cone C, such that* (a) *X has order induced by C (i.e. the positive cone of X is $X \cap C$),* (b) $(y+X) \cap C \neq \varnothing \cdot \equiv \cdot (-y+X) \cap C \neq \varnothing$, *then every monotone, distributive function f from X to V has a monotone, distributive extension F from Y to V.*

4. *V has the monotone projection property (MPP). That is, if* (a) *V is contained in an ols Y with positive cone C,* (b) *the order in V is induced by C,* (c) $(y+V) \cap C \neq \varnothing \cdot \equiv \cdot (-y+V) \cap C \neq \varnothing$, *then there exists a monotone, distributive projection P from Y onto V.*

5. *V has the Hahn-Banach extension property (HBEP). That is, if* (a) *X is a subspace of a linear space Y,* (b) *p is a positive-homogeneous, subadditive function from Y to V,* (c) *f is a distributive function from X to V such that*

$f(x) \leqq p(x)$, $x \in X$, then there exists a distributive extension F of f defined from Y to V such that $F(y) \leqq p(y)$, for all $y \in Y$.

A useful criterion for determining whether the hypothesis $y+X$ meets C in the above theorem is satisfied, is the existence of a vector interior point in C which meets X. In this situation every translate of X meets C. A vector interior point in C also guarantees that the interior (in the vector topology sense) is sharp. For convenience, however, when sharpness is required the cones will be assumed sharp. It should be noted that the vector topology about to be introduced does not determine a linear topological space unless some additional conditions, such as convexity, are placed on the neighborhoods.

A set U in a linear space V is a *vector neighborhood* of $x \in V$ if and only if for every line $L(y) = \{x+ty : t \in R\}$ through x, there exists a segment $L'(y) = \{x+ty : |t| \leqq \epsilon, \epsilon > 0\}$ which is contained in U. The real number ϵ in general depends upon y. Hence a point x in a cone C is a *vector interior point* of C if there exists a vector neighborhood of x which is contained in C. The following theorems are stated without proof [2; 5].

THEOREM B. *If u is a vector interior point of an s-cone C, then the vector neighborhood of x which is contained in C can be represented as $\{u+ty : -su \leqq y \leqq su,\ s = 1/t,\ |t| \leqq \epsilon,\ \epsilon > 0\}$.*

THEOREM C. *If u is in a subspace X of an ordered linear space Y, and if u is also a vector interior point of the cone C of Y, then every translation of X by an element of Y meets C.*

THEOREM D. *If Y is any linear topological space and U is a neighborhood of a point in Y in the given topology, then there exists a vector neighborhood of the point which is contained in U.*

THEOREM E. *If Y is an ordered linear topological space such that the cone C has an interior point, then C^*, the induced cone in the conjugate space Y^* of Y, is nontrivial (i.e., there exists a nonzero element in the set).*

Another useful bit of information is the relationship between ordered linear spaces which possess minihedral cones with vector interior points and normed spaces.

THEOREM F. *If V is an ordered linear space with sharp minihedral cone K, such that K contains a vector interior point u, then V can be made into a normed space, where $\|v\| = \inf \{s : = su \leqq v \leqq su,\ s \in R\}$ for every v in V. The unit sphere is the interval $I = \{v \in V : -u \leqq v \leqq u\}$. The cone is not necessarily closed with respect to this norm.*

The following theorems are concerned with ordered linear spaces which are also normed spaces [7].

THEOREM G. *If V is an ordered normed space with s-cone K, such that K contains a normed interior point u, then for every $f \neq 0$ in K^*, the positive cone in V^*, $f(u) \geqq r\|f\|$, where r is the radius of the sphere about u which is contained in K.*

THEOREM H. *If K is a cone in an ordered normed space V, then x is in the closure of K if and only if $f(x) \geqq 0$ for every f in K^*.*

Given an ordered normed space with positive s-cone K, the concept of what might be termed uniform sharpness is defined. A closed s-cone K contained in an ordered normed space V is *normal* if and only if there exists a positive real number δ, such that for every v', $v \in K$ such that $\|v\| = \|v'\| = 1$, then $\|v+v'\| \geqq \delta$.

THEOREM I. *A norm-closed s-cone K with norm-interior point u in an ordered normed space V is normal, if and only if $I = \{v \in V: u \geqq v \geqq -u\}$ is bounded in norm.*

An ols V has *reproducing* cone K if and only if for every $v \in V$ there exists v_1, $v_2 \in K$ such that $v = v_1 - v_2$.

THEOREM J. *A norm-closed, s-cone K in an ordered normed space V is normal if and only if the induced cone K^* in the conjugate space V^* is reproducing.*

Consider a semigroup G, and the Banach space $M(G)$ of bounded real-valued functions on G, then G has a [*left*] (*right*) *invariant mean* if there exists an element μ of norm 1 in the conjugate space $M(G)^*$ of $M(G)$, such that $\mu(e) = 1$, where e is the constant 1 function in $M(G)$, and μ is invariant with respect to [left] (right) both left and right regular representations of G on $M(G)$. (I.e. $\mu(L_g f) = \mu(R_g f) = \mu(f)$, where L_g is the left and R_g is the right regular representation. That is $(L_g f)(g') = f(gg')$, $f \in M(G)$, g, $g' \in G$, etc.)

For properties of invariant means see [3]. In particular, a semigroup with a left invariant mean and a right invariant mean has an invariant mean. Also, considering the positive cone of $M(G)$ to be the collection of those functions which are pointwise non-negative, a mean is monotone. It is also mentioned the only class of groups known not to have an invariant mean are groups which contain free non-abelian subgroups. Further, a homomorphic (anti-homomorphic) image of a semigroup with an invariant mean has an invariant mean.

III. Extension theorems.

DEFINITION 1. The pair $[\mathfrak{G}, V]$, where

i. $\mathfrak{G}$ is an abstract semigroup,

ii. V is an ols with the LUBP and whose positive cone is sharp, has the *Hahn-Banach extension property* (*HBEP*) if and only if for every collection $[Y, X, G, p, f]$, where

(a) Y is a linear space,

(b) X is a subspace of Y,

(c) G is a representation of $\mathfrak{G}$ on Y (i.e., a homomorphic or anti-homomorphic image of $\mathfrak{G}$ in the distributive operators on Y) such that $gx \in X$ for all $x \in X$ and $g \in G$,

(d) p is a positive-homogeneous, subadditive function from Y to V such that $p(gy) \leqq p(y)$ for each $y \in Y$ and $g \in G$,

(e) f is a distributive function from X to V such that $f(x) \leqq p(x)$ and $f(gx) = f(x)$ for every $x \in X$ and $g \in G$,

there exists a distributive extension F of f with domain Y and range V such that $F(y) \leqq p(y)$ and $F(gy) = F(y)$ for every $y \in Y$ and $g \in G$.

Definition 2. The pair $[\mathfrak{G}, V]$, where

i. $\mathfrak{G}$ is an abstract semigroup,

ii. V is an ols with the LUBP and s-cone K, has the *monotone extension property* (*MEP*) if and only if every collection $[Y, C, X, G, f]$, where

(a) Y is an ols with cone C,

(b) X is an ordered subspace of Y with the induced ordering and such that $y + X$ meets C for every $y \in Y$,

(c) G is a representation of $\mathfrak{G}$ on Y such that $gz \in C$ and $gx \in X$ for every $g \in G$, $x \in X$ and $z \in C$,

(d) f is a monotone distributive function from X to V such that $f(gx) = f(x)$ for each $x \in X$ and $g \in G$,

there exists a monotone, distributive extension F of f with domain Y and range V such that $F(gy) = F(y)$ for all $y \in Y$ and $g \in G$.

The condition that K be an s-cone in the two definitions and that $y + X$ meets C for every y in Definition 2 are stronger than the corresponding conditions in Theorem A, but they appear necessary for some of the theorems which will be proved.

Theorem 1. *$[\mathfrak{G}, V]$ has the HBEP if and only if it has the MEP.*

The proof of this theorem is identical with the proof of Theorem 1 in [10] and hence will not be reproduced. By virtue of this theorem consideration can be restricted to considering pairs $[\mathfrak{G}, V]$ with the MEP. The proofs of the following theorems could also use the HBEP but there is neither advantage nor disadvantage in this procedure.

It is remarked that if the condition in Definition 1 that $p(gy) \leqq p(y)$ for each y in Y and all g in G is changed to $p(gy) \leqq Np(y)$ for each y and all g, where N is some fixed positive real number independent of y and g, and the condition that the extension of F of f is dominated by p is also changed so that $F(y) \leqq Np(y)$ for all y in Y, then all the theorems concerned with the HBEP under this change of definition are valid. For replace p by p', where

$p'(y) = \sup \{p(hy) : h \in G$ or h is the identity operator$\}$, then p' is positive-homogeneous and subadditive, $p(y) \leqq p'(y)$, and $p'(gy) \leqq p'(y)$ for all y in Y and g in G.

THEOREM 2. (1) *Let* W *be an ols with reproducing cone* K', (2) *let* V *be an ordered subspace of* $W^{\#}$, *the space of distributive functionals on* W, (3) *let* K *be the cone in* V *such that* K *is induced by* K', K *is sharp, and with respect to this induced ordering* V *has the LUBP*, (4) *let* $\mathfrak{G}$ *be an abstract semigroup with an invariant mean, then* $[\mathfrak{G}, V]$ *has the MEP.*

Proof. 1. Let $[Y, X, C, G, f]$ be as in Definition 2 where G is a representation of $\mathfrak{G}$. Since a homomorphism (anti-homomorphism) of a semigroup with invariant mean also has an invariant mean, G considered as an abstract semigroup has an invariant mean.

2. Since V has an LUBP, there exists a monotone, distributive extension F' of f to all Y with values in V.

3. $[F'(gy)](w)$ is a real number for fixed $y \in Y$, $w \in W$ and $g \in G$. Hence for fixed y and w, $[F'(gy)](w)$ can be considered a function from G to the real numbers. $[F'(gy)](w)$ is a bounded function on G. For given $y \in Y$, there exists $x, x' \in X$ such that $y+x$, $-y+x' \in C$. Therefore, $F'(y+x)$ and $F'(x'-y)$ are in K since F' is monotone. Hence, $-f(x) \leqq F'(y) \leqq f(x')$. Since g is monotone and $f(gu) = f(u)$ for each $g \in G$ and $u \in X$, $-f(x) \leqq F'(gy) \leqq f(x')$ for every $g \in G$. Consider $w \in W$, then $w = w' - w''$, where w', $w'' \in K'$. Then, since K is the cone induced by K', $-[f(x)](w''') \leqq [F'(gy)](w''') \leqq [f(x')](w''')$, where w''' is w' or w''. Thus, $-[f(x)](w') - [f(x')](w'') \leqq [F'(gy)](w) \leqq [f(x')](w') + [f(x)](w'')$ for all $g \in G$, and boundedness is proved.

4. Since G has an invariant mean, μ, define F'' as follows: $[F''(y)](w) = \mu\{[F'(gy)](w)\}$ for each $y \in Y$ and $w \in W$. It is easily verified that $F''(y)$ is in $W^{\#}$ for each $y \in Y$. It is also easily verified that F'' is a distributive function from Y to $W^{\#}$. Further, since μ is an invariant mean, it follows that $F''(gy) = F''(y)$ for all $g \in G$ and $y \in Y$.

5. Define V'' to be the subspace of $W^{\#}$ generated by V and the set $\{F''(y) : y \in Y\}$. Then by definition of V'', F'' is a distributive map from Y to V''. Let K'' be the cone induced by K' in V''. The function F'' is monotone with respect to the cone K''. For if $y \in C$, then $gy \in C$ for all $g \in G$. Thus $F'(gy)$ is in K for all g, since F' is monotone. Since K is induced by the cone K' of W, $[F'(gy)](w) \geqq 0$ for all $g \in G$, $y \in C$, and $w \in K'$. Therefore, since μ is monotone from $M(G)$ to the real numbers and $[F'(gy)](w)$ is a positive element in $M(G)$, it follows that $[F''(y)](w) \geqq 0$ for all $w \in K'$. Therefore $F''(y) \in K''$ and F'' is monotone.

6. It is immediate that F'' is an extension of f with values in V''.

7. Note that $K = V \cap K''$. Also $(v'' + V) \cap K'' \neq \phi$ for every $v'' \in V''$. This statement is trivial if $v'' \in V$. Therefore consider $v'' = F''(y)$. For each $y \in Y$, there exists an $x \in X$ such that $x + y \in C$. Therefore, using the monotonicity of

F'', $F''(y+x)=F''(y)+f(x)=F''(y)+v\in K''$. Thus V, K, V'' and K'' satisfy the hypotheses of Theorem A-4. Hence, there exists a monotone, distributive projection P fron V'' onto V.

8. Define F: $F(y)=P(F''(y))$ for all $y\in Y$. This is the desired extension and the theorem is proved.

THEOREM 3. *If* (1) *V is an ordered normed space with the LUBP and with norm-closed, normal s-cone K,* (2) *$\mathfrak{G}$ is an abstract semigroup with invariant mean, then the pair $[\mathfrak{G}, V]$ has the MEP.*

Proof. (1) The natural linear map Q from V to V^{**} (the second conjugate space of V): $(Qv)(f)=f(v)$ for every f in V^*, is an isometry into and in this case, the closure of K implies that Q and Q^{-1} (on QV) are order preserving mappings between V and QV, where the cone in QV is the natural cone induced by K^* in V^* which is in turn induced by K in V. This will be proved in the following lemma.

LEMMA 1. *Let V be an ordered normed space with norm closed cone K, then*

(1) *The natural map Q from V to V^{**} is monotone* (*with respect to the induced ordering in V^{**}*).

(2) *QK is the induced cone in QV.*

(3) *Q^{-1} is monotone from QV to V.*

(4) *If V has the LUBP then QV has the LUBP.*

Proof of lemma. From Theorem H, v is in cone K if and only if $f(v)\geqq 0$ for every $f\in K^*$. Therefore $Qv(f)\geqq 0$ for every $f\in K^*$ if and only if v is in K. That is Qv is in the induced cone in QV if and only if $v\in K$. Thus QK is the induced cone, and Q and Q^{-1} are monotone.

Now $v_1\leqq v_2\leqq v_3$ if and only if $Qv_1\leqq Qv_2\leqq Qv_3$. Thus from Theorem A-2, QV has the LUBP if and only if V has.

(2) Since K is normal, by Theorem J, K^* is reproducing. Hence QV satisfies the hypothesis of Theorem 2 and $[\mathfrak{G}, QV]$ has the MEP.

(3) Consider Y, X, C, G, and f as in Definition 2. Define f' from X to QV: $f'(x)=Q(f(x))$ for all $x\in X$. Hence by (2) there exists a monotone, distributive extension of F' of f' to all of Y such that $F'(gy)=F'(y)$ for all $g\in G$ and $y\in Y$. Define F from Y to V: $F(y)=Q^{-1}(F'(y))$. This is clearly the desired extension and $[\mathfrak{G}, V]$ has the MEP.

The continuity of the extensions will be considered in § IV.

THEOREM 4. *Consider V an ordered normed space with the LUBP with s-cone K such that K is closed and has a norm-interior point u, and such that the interval $I=\{v: -u\leqq v\leqq u\}$, is bounded in norm. Then if $\mathfrak{G}$ is a semigroup with an invariant mean, $[\mathfrak{G}, V]$ has the MEP.*

Proof. By Theorem I, the above hypotheses imply K is normal and by Theorem 3 $[\mathfrak{G}, V]$ has the MEP.

Given an ols V with the finite LUBP and a sharp cone, then if the cone of V contains a vector interior point u, V can be made into a normed space, where

$$\|v\| = \inf_t \{ |t| : -tu \leqq v \leqq tu\} \text{ and the unit sphere is}$$
$$I = \{v: -u \leqq v \leqq u\} \text{ (Theorem F).}$$

LEMMA 2. *Let V be an ols with the finite LUBP such that u is a vector interior point of the s-cone K. Further suppose that K is closed with respect to the norm determined by this interior point. Then with respect to this norm, K is normal.*

Proof. By Theorem I, a closed s-cone with normed interior point u' is normal if and only if the interval $I' = \{v: -u' \leqq v \leqq u'\}$ is bounded in norm. Note that u' and $-u'$ are contained in the sphere tI, for some positive real number t. Hence, $-tu \leqq -u' \leqq v \leqq u' \leqq tu$. Therefore $I' \subseteq tI$, and K is normal.

The property of a vector interior point in a cone, and the above lemma imply the following theorem.

THEOREM 5. *If V is an ols with the LUBP and has s-cone K with vector interior point u, and K is closed with respect to the norm determined by this interior point, then if $\mathfrak{G}$ is any semi-group with invariant mean, the pair $[\mathfrak{G}, V]$ has the MEP.*

Proof. V can be considered an ordered normed space with the LUBP with closed, normal s-cone K, and Theorem 4 applies.

The next theorem is concerned with extensions where V is not the range space but part of the domain space. It serves as an example of other such theorems which could be stated.

THEOREM 6. (1) *Let Y be an ols with cone C such that V is an ordered subspace of Y, V has the LUBP, and every translate of V meets C.* (2) *Let L be an ols with reproducing cone.* (3) *Let G be a semigroup of operators on Y such that* (a) *each g in G maps V into itself and C into itself, and* (b) *G, considered as an abstract semigroup has an invariant mean. Then if f is a monotone, distributive function from V to $L^\#$ such that $f(gv) = f(v)$ for all g in G and v in V, there exists a monotone, distributive extension F of f from Y to L, such that $F(gy) = F(y)$ for all y in Y and g in G.*

Proof. By the Theorem A-4 there exists a monotone, distributive projection P from Y onto V. Define F' so that $F'(y) = f(Py)$. This is a monotone, distributive extension of f. Consider $[F'(gy)](z)$, where z is in L. This is a real-valued, bounded function on G for fixed z and y. Since there exists an invariant mean μ on G, define F, the desired extension: $[F(y)](z) = \mu([F'(gy)](z))$ for each z in L and y in Y.

Thus far it has been proved that if $\mathfrak{G}$ is a semigroup with invariant mean and V satisfies certain properties in addition to the LUBP, then $[\mathfrak{G}, V]$ has the two extension properties. The next theorems state that if $[\mathfrak{G}, V]$ has the extension property with respect to a suitable V then $\mathfrak{G}$ necessarily has an invariant mean. Each of the following theorems is stronger than the corresponding theorem in [10] which states that the class of semigroups with an invariant mean contains the class of semigroups with the MEP (HBEP).

THEOREM 7. *If there exists an ols V with the LUBP, with s-cone $K \neq 0$, such that V is a subspace of $W^\#$, the space of distributive functionals on an ols W with reproducing cone K', such that K is induced by K', then, if $[\mathfrak{G}, V]$ has the MEP, $\mathfrak{G}$ has an invariant mean.*

Proof. 1. Consider $v_0 \neq 0 \in K$. Consider $M(\mathfrak{G})$ the space of real-valued, bounded functions on $\mathfrak{G}$. Consider X a subspace of $M(\mathfrak{G})$: $\{te\colon t \in R,\ e$ the constant 1 function in $M(\mathfrak{G})\}$. Define the function f from X to $V\colon f(te) = tv_0$. The function f is clearly distributive. The function f is also monotone and invariant with respect to the left and right regular representations G on $M(\mathfrak{G})$. The cone C in $M(\mathfrak{G})$ is the set of all pointwise non-negative functions. The constant 1 function e is a norm-interior point of C. Hence every translate of X meets C. Further, the left and right representations map C into itself. Thus, since $[\mathfrak{G}, V]$ has the MEP, there exist monotone, distributive extensions F'_L and F'_R of f to all of $M(\mathfrak{G})$ with values in V which are respectively invariant with respect to the left and right representations on $M(\mathfrak{G})$.

2. There exists a $w_0 \in K'$ such that $v_0(w_0) > 0$, since K' is reproducing and $v_0 \neq 0$. Define F_L and F_R from $M(\mathfrak{G})$ to the real numbers:

$$F_L(m) = [F'_L(m)](w_0),$$
$$F_R(m) = [F'_R(m)](w_0), \text{ for every } m \text{ in } M(\mathfrak{G}).$$

F_L and F_R are clearly distributive, monotone and invariant respectively with respect to the left and right representations.

3. Define: $M_L = [F_L(e)]^{-1}F_L$, and $M_R = [F_R(e)]^{-1}F_R$. Then M_L and M_R are respectively right and left invariant means. By virtue of the theorem that a semi-group with left and right invariant means has an invariant mean, the proof is complete.

Theorems similar to Theorem 7 are stated, using the appropriate hypotheses of Theorems 3, 4, and 5.

THEOREM 8. *Let V satisfy the hypotheses of Theorem 3, and $K \neq 0$, then if $[\mathfrak{G}, V]$ has the MEP, $\mathfrak{G}$ has an invariant mean.*

Proof. Consider v_0, $M(G)$, X, and f as in Theorem 7. Then, F'_L and F'_R as in Theorem 7, exist. Define $F''_L = QF'_L$ and $F''_R = QF'_R$. The remainder of the proof proceeds as in Theorem 7, with F''_L and F''_R substituted for F'_L and F'_R in that theorem.

THEOREM 9. *Given an ols V satisfying the hypotheses of either Theorems 4 or*

5, *then, if* $[\mathfrak{G}, V]$ *has the MEP,* $\mathfrak{G}$ *has an invariant mean.*

The proof of this theorem is direct and does not reverse the steps in the proof of Theorems 4 and 5.

Proof. (1) Consider $M(\mathfrak{G})$ and X as in Theorem 7. Let u be an interior point of K. Define the distributive function f from X to V: $f(te)=tu$. Thus since $[\mathfrak{G}, V]$ has the MEP, there exist functions F'_R and F'_L from $M(\mathfrak{G})$ to V which are monotone, distributive, and invariant respectively with respect to the right and left regular representations.

(2) Since u is an interior of K, by Theorem A-4 there exists a monotone, distributive projection P from V onto $V'=\{tu: t\in R\}$.

(3) Define, $F_L=PF'_L$ and $F_R=PF'_R$. Note that $F_L(e)=F_R(e)=u$. Define T from V' to R: $T(tu)=t$. T is clearly monotone, distributive and $T(u)=1$.

(4) Define, $M_L=TF_L$ and $M_R=TF_R$. The functions M_L and M_R are respectively right and left invariant means of $\mathfrak{G}$, and thus $\mathfrak{G}$ has an invariant mean.

IV. **Examples.** In this section a listing of some of the standard function spaces will be presented along with some of their properties. These will furnish examples of spaces which permit invariant extensions relative to semigroups with invariant means.

Given a space of bounded continuous functions $C(S)$ from a topological space S to the real numbers, it is possible to define a cone $K=\{f\in C(S): f(s)\geqq 0,\ s\in S\}$. The space $C(S)$ is a Banach space with respect to the sup norm, and K contains a norm-interior point e, the constant 1 function. The sphere of radius 1 about e is contained in K. Kakutani [5] has proved the following theorem.

THEOREM K. *An ordered Banach space V whose cone K is sharp and contains a norm-interior point is isomorphic to a space of real-valued continuous functions V' on a compact Hausdorff space S, (i.e., there exists a* 1-1, *order preserving, homomorphic, homeomorphic map of V onto V'), if and only if V has the finite LUBP.*

That there exist spaces of continuous functions which do not have the LUBP is given by the example of the space of continuous function on the unit interval of the real line.

Thus, stronger conditions are required in order to guarantee the LUBP. Kelley [6], Nachbin [9], and Goodner [4] have proved the following theorems.

THEOREM L. *An ordered Banach space in whose cone K is sharp and has a norm-interior point has the LUBP if and only if it is isomorphic to a space of real-valued continuous functions on an extremally disconnected compact Hausdorff space. (A topological space is extremally disconnected if and only if the closure of every open set is open.)*

THEOREM M. *Given a Banach space V, then every linear function f defined on X, a subspace of a Banach space Y, to V has a linear extension F to all of Y such that $\|F\| = \|f\|$ if and only if V is isomorphic to a space of real-valued continuous functions on an extremally disconnected compact Hausdorff space.*

This implies that in an ordered Banach space the HBEP is not sufficient to guarantee norm-preserving extensions of linear functions. The difficulty here is that the suitable positive-homogeneous, subadditive function may not be definable which corresponds to $\|f\|\|y\|$ in the case where the range space is the real numbers.

A specific example of an ordered Banach space which satisfies Theorems L and M is $m(S)$, the space of bounded real-valued functions on a discrete index set S of any cardinality, where $\|f\| = \inf_{s\in S} |f(s)|$, $K = \{f \in m(S): f(s) \geqq 0, s \in S\}$. An interior point of K is the constant 1 function. In particular, R, the real line, which can be considered the space of functions on a single point, R^n, the real n dimensional vector space, considered as the space of functions on a discrete set of n points, have fully minihedral, closed, s-cones with interior. These spaces in addition have the property that they are conjugate spaces whose order is induced from below and such that the underlying spaces are reproducing. These spaces also have normal s-cones.

There exist examples of ordered linear spaces with fully minihedral cones which do not contain any interior points, even vector interior points, e.g., $c_0(S)$, the space of real-valued functions on a discrete directed system S, with the property that each function converges to 0 with respect to this directed system. The cone K in the space is defined as in $m(S)$, the norm is also defined as in $m(S)$. It is noted that c_0 is not a conjugate space, but c_0 has a normal, closed s-cone.

The space $l_p(S)$, $p>0$ of generalized sequences $\{x\}$ defined on S, an arbitrary index set, such that $\sum_{s\in S} |x(s)|^p < \infty$, has the LUBP with respect to the cone $K = \{x \in l_p: x(s) \geqq 0, s \in S\}$. K has no vector interior point. If $p \geqq 1$, then l_p is a Banach space and also the conjugate space of a Banach space.

Consider $L^p(a, b)$, $p>0$, the space of real-valued measurable functions defined on the bounded or unbounded interval $[a, b]$ of the real line, such that the pth powers of the absolute values of the functions are Lebesgue integrable. If $p \geqq 1$, L^p is a Banach space. If $p>1$, L^p is a conjugate space. L^1 is not a conjugate space. The cone in these spaces, $K = \{f \in L^p: f(s) \geqq 0$ for a.e. $s \in S\}$. As above K is fully minihedral and, if $[a, b]$ is unbounded, has no vector interior point. However, the cones in all these cases are normal.

L^∞, the space of essentially bounded Lebesgue measurable functions over the real numbers, is another example of a function space whose cone is fully minihedral and does contain a vector interior point. L^∞ is a Banach space. Further it is the conjugate space of L^1. The cone is defined as in L^P.

All finite dimensional linear topological spaces with respect to the usual co-ordinatewise ordering satisfy the conditions of Theorems 2, 3, 4 and 5.

Each Banach space with the LUBP mentioned in this section has a normal, closed s-cone. Hence Theorem 3 applies. Similarly, those Banach spaces mentioned which are conjugate spaces satisfy Theorem 2. However, no example has been presented of an ordered Banach space which is a conjugate space whose underlying space has reproducing cone, but which does not have a normal, closed s-cone.

V. **Continuity conditions.** If the ordered linear spaces in §III are assumed to be linear topological spaces, the question arises under what conditions are the extensions mentioned in that chapter continuous. Some conditions which guarantee continuity will be presented.

DEFINITION 3. Given an ordered linear topological space V, then V is *locally restricted* if and only if for every neighborhood U of 0 in V, there exists a neighborhood W of 0 contained in U such that for every a and b in W, the set $\{x: x\in V, a\leqq x\leqq b\}$ is contained in U.

All of the ordered normed spaces mentioned in § IV are locally restricted.

THEOREM 10. *If* (1) *Y is a linear topological space,* (2) *V is a locally restricted ordered linear topological space,* (3) *p is a positive-homogeneous, subadditive, continuous function from Y to V, and* (4) *F is a distributive function from Y to V such that $F(y)\leqq p(y)$ for every y in Y, then F is a continous function.*

Proof. Take all neighborhoods of 0 to be symmetric. Given a neighborhood $U(0)$ in V, consider $W(0)$ a neighborhood of 0 contained in V such that if a and b are in W, the interval $a\leqq x\leqq b$ is contained in U. Since p is continuous, there exists a neighborhood N of 0 in U such that $p(y)$ is contained in W for every y in N. Define $p'(y)=-p\,(Y-y)$ for all y in Y. This is a continuous function clearly. Hence, there exists a neighborhood $M(0)$ in Y such that $p'(y)$ is contained in W for every y in Y. Consider a neighborhood Q contained in $M\cap N$. Then if y is in Q we have $p'(y)\leqq F(y)\leqq p(y)$. Therefore $F(y)$ is in U for every y in Q. Thus F is a continuous function.

THEOREM 11. *If* (1) *Y is an ordered linear topological space with cone C,* (2) *V is a locally restricted ordered linear topological space with cone K,* (3) *F is a monotone, distributive function from Y to V such that there exists a neighborhood of an element z in Y, $M(z)$, with the property that $F(y')$ is in K for all y' in $M(z)$, then F is a continuous function.*

Proof. 1. Assume all neighborhoods in Y are symmetric. $M(z)=z+N(0)$, where N is a symmetric neighborhood of 0. Thus for every y in $N(0)$, $F(y+z)$ is in K and $F(-y+z)$ is in K. Hence $-F(z)\leqq F(y)\leqq F(z)$ for all y in $N(0)$.

2. Consider $U(0)$ in V, then there exists a neighborhood $W(0)$ contained in $U(0)$ such that if a and b are in W, the interval $a\leqq v\leqq b$ is contained in U. There exists a real number $t>0$ such that $F(tz)$ and $F(-tz)$ are contained in

$W(0)$ since V is a linear topological space. Therefore, since $-F(tz) \leqq F(ty) \leqq F(tz)$ for all y in $N(0)$, $F(ty)$ is in $U(0)$ for all y in $N(0)$. Consider a neighborhood $Q(0)$ in Y such that $Q(0)$ is contained in $tN(0)$. Such a neighborhood exists since multiplication by a nonzero scalar is a homeomorphism of a linear topological space onto itself. Then if x is in $Q(0)$, $(1/t)x$ is in $N(0)$ and hence $F(t(1/t)x) = F(x)$ is in $U(0)$ for all x in $Q(0)$, and F is continuous.

THEOREM 12. *In the last theorem, if the condition that F maps a neighborhood of Y into K is replaced by the condition that C contains an interior point, then every monotone, distributive function from Y to V is continuous.*

Proof. This theorem follows from Theorem 12 since every monotone distributive function maps the neighborhood of the interior point which is contained in C into K.

THEOREM 13. *Let Y be an ordered, normed space with positive cone C such that C contains an interior point u. Let $S(r, u)$ be a sphere of radius r about u contained in C. Let V be an ordered normed space such that if $v \geqq v' \geqq 0$, then $\|v\| \geqq \|v'\| \geqq 0$. Then, if F is a monotone distributive function from Y to V, F is continuous and $\|F\| \leqq 1/r\|F(u)\|$.*

Proof. If y is in Y and $\|y\| \leqq 1$, then $u \pm ry$ is in C. Hence $F(u-ry) \geqq 0$. Therefore $\|F(u)\| \geqq r\|F(y)\|$. Thus $\|F(u)\| \geqq r(\sup \{\|F(y)\| : \|y\| \leqq 1\}) \geqq r\|F\|$. This theorem is a generalization of a theorem of Kreĭn and Rutman.

Another condition for the continuity of a monotone, distributive function from one ordered Banach space to another is based on a communication from I. Kaplansky to M. M. Day. The ordered Banach spaces L^p and c_0 satisfy the properties of the spaces in the next theorem. The condition is a substitute for the existence interior point in the cone.

THEOREM 14. (1) *Let Y be an ordered Banach space with sharp cone C, such that* (a) *there exists a positive real number M such that for every x in Y with $\|x\| \leqq 1$, then $x = y - z$, where y and z are in C, and $\|y\| \leqq M$,* (b) *C is closed under norm-convergence of monotone increasing sequences.* (2) *Let V be an ordered normed space with positive cone K such that if $v \geqq v' \geqq 0$, then $\|v\| \geqq \|v'\|$.* (3) *Let F be a monotone, distributive function from Y to V. Then F is continuous.*

Proof. 1. Assume F is not continuous, then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in Y such that $\|x_n\| \leqq 1$ for all n and $\|F(x_n)\|$ approaches infinity as n increases. Thus $\|F(y_n) - F(z_n)\|$ approaches infinity as n increases and y_n and z_n are in C for each n. Further, each y_n can be chosen so that $\|y_n\| \leqq M$. Since $\|z_n\| - \|y_n\| \leqq \|y_n - z_n\| \leqq 1$, $\|z_n\| \leqq M+1$. Therefore either $\|F(y_n/M)\|$ approaches infinity with n or $\|F(z_n/(M+1))\|$ does, for if not $\|F(x_n)\|$ would be bounded.

2. Thus, there exists a sequence $\{w_n\}$ in C such that $\|w_n\| \leqq 1$ and $\|F(w_n)\| \geqq n2^n$ for all n.

3. The series $\sum(w_n/2^n)$ converges in norm to an element w in Y, for $\|\sum_{n=p}^{q}(w_n/2^n)\| \leqq \sum_{n=p}^{q}(\|w_n\|/2^n) \leqq \sum_{n=p}^{q} 1/2^n$ and thus approaches zero as p and q approach infinity. Hence, since Y is complete, w exists.

4. Define $u_n = \sum_{i=1}^{n}(w_i/2^i)$. Then, $u_p - u_q \geqq 0$, if $p \geqq q$. Hence, since C is closed under norm convergence of monotone increasing sequences, w is in C. Also, $w - w_k/2^k = \sum_{n \neq k}(w_n/2^n)$ is the C for each k. Therefore $\|F(w)\| \geqq \|F(w_k/2^k)\|$ for each k. Hence $\|F(w)\| \geqq \|F(w_k/2^k)\| \geqq k2^k/2^k = k$ for every integer k. This is impossible since $F(w)$ is in V and thus has finite norm. Therefore F is continuous.

VI. **Applications.** The next theorems are restatements of previous theorems where the space V is taken to be the real numbers, or a space of bounded functions. Theorem 15 is a generalization of a theorem of Banach [1] and of some theorems by A. P. Morse and R. P. Agnew [8]. It is concerned with Hahn-Banach extensions. Theorem 16 is a generalization of a theorem on monotone functionals of Kreĭn and Rutman [7]. Theorem 17 is concerned with a generalization of the concept of an invariant mean.

THEOREM 15. A. *If (1) Y is a vector space with subspace X, (2) p is a positive-homogeneous, subadditive functional defined on Y, (3) f is a distributive functional defined on X such that $f(x) \leqq p(x)$ for all x in X, (4) G is a semi-group of operators from Y to Y such that gx is in X for every x in X and g in G, (5) $p(gy) \leqq p(y)$ for all y in Y and g in G, (6) $f(gx) = f(x)$ for all x in X and g in G, (7) G, considered as an abstract semigroup, has an invariant mean, then there exists a distributive extension F of f to all of Y such that $F(y) \leqq p(y)$ and $F(gy) = F(y)$ for all y in Y and g in G.*

B. *If, in addition, Y is a linear topological space, and p is continuous, then F is continuous.*

C. *If, in addition, Y is a normed space and $p(y) = \|f\|\|y\|$ for all y in Y, then there exists a linear extension F of f such that $\|F\| = \|f\|$ and $F(gy) = F(y)$ for all y in Y and g in G.*

Proof. A. The real numbers satisfy the conditions on the space V in Theorems 2 and 3. Hence either of these theorems apply.

B. Theorem 12 applies.

C. B applies directly.

THEOREM 16. A. *If (1) Y is an ordered linear space with cone C and subspace X such that $y + X$ meets C for every y in Y, (2) f is a monotone distributive functional defined on X, (3) G is a semigroup of operators on Y such that g maps X into X, C into C and $f(gx) = f(x)$ for all x in X and g in G, (4) G considered as an abstract semigroup has an invariant mean, then there exists a monotone, distributive functional F, an extension of f to all of Y, such that $F(gy) = F(y)$ for all g in G and y in Y.*

B. *If, in addition, Y is a linear topological space and C contains an interior* *point, or F has constant sign in some neighborhood of a point in Y, then F is continuous.*

C. *If, in addition, Y is an ordered, normed space with sharp cone C such that C has a norm-interior point u, such that u is in X and $\|u\|=1$, then $\|F\| \leqq 1/r\|f\|$, where r is the radius of the largest sphere about u contained in C. Thus, if $r=1$, then $\|F\|=\|f\|$.*

Proof. A. Since the real numbers satisfy the hypotheses for V in Theorem 2, Theorem 2 applies.

B. Theorems 10 or 11 apply.

C. By Theorem G, §II, the result follows.

THEOREM 17. *Consider $Y=Y(\mathfrak{G})$, a normed space of real-valued functions (sup norm) containing the constant 1 function e defined on a semigroup $\mathfrak{G}$ which possesses an invariant mean. Assume Y is ordered by cone $C=\{y: y(g)\geqq 0, g\in\mathfrak{G}\}$. Consider $V=V(S)$, a space of continuous functions on an extremally disconnected compact Hausdorff space S. Then there exists a monotone, distributive linear function M from Y to V such that $\|M\|=1$, $M(e)=u$, where u is the constant 1 function in V, and $M(R_g y)=M(L_g y)=M(y)$, where R_g and L_g are respectively elements of the right and left regular representations of G on Y.*

Proof. 1. Define the distributive, monotone function f from $X=\{te: t\in R$ (real numbers)$\}$ to $V: f(te)=tu$. Note that $y+X$ meets C since e is an interior point of C. Representations R_g and L_g leave X fixed and map C into itself. Hence, since $\mathfrak{G}$ has an invariant mean and V satisfies the conditions of Theorem 3, there exist monotone, distributive extensions M_R and M_L, of f which are invariant with respect to the right and left representations respectively.

2. Define M, a function from Y to V: $M(y)=M_L(y')$, where $y'(g)=M_R(L_g y)$ for each g in G. The function M is clearly monotone, distributive and an extension of f. Since $(L_g y)'(g')=M_R(L_{g'}L_g y)=M_R(L_{gg'}y)=y'(gg')=(L_g y')(g')$, it follows that $L_g y'=(L_g y)'$. Hence, $M(L_g y)=M(y)$ for every g and y. In addition, $M(R_g y)=M(y)$ for every g and y. This follows because $(R_g y)'(g')=M_R(L_{g'}R_g y)=M_R(R_g L_{g'}y)=y'(g')$ for each g'.

3. That $\|M\|=1$ follows because V is locally restricted, $\|e\|=1$, and the sphere of radius 1 about e is contained in C.

BIBLIOGRAPHY

1. S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932.

2. M. M. Day, *Notes on ordered linear spaces*, University of Illinois, 1950, unpublished.

3. ———, *Means for the bounded functions and ergodicity of the bounded representations of semigroups*, Trans. Amer. Math. Soc. vol. 69 (1950) pp. 276–291.

4. D. B. Goodner, *Projections in normed linear spaces*, Trans. Amer. Math. Soc. vol. 69 (1950) pp. 89–108.

5. S. Kakutani, *Concrete representations of abstract (M)-spaces*, Ann. of Math. vol. 42 (1941) pp. 523–537.

6. J. L. Kelley, *Banach spaces with the extension property*, Trans. Amer. Math. Soc. vol. 72 (1952) pp. 323–326.

7. M. G. Kreĭn and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, [Uspehi Matematicheskii Nauk (N.S.) vol. 3 (23) (1948) pp. 3–95] American Mathematical Society Translations, no. 26, 1950.

8. A. Morse and R. Agnew, *Extensions of linear functionals, with applications to limits, integrals, measures and densities*, Ann. of Math. vol. 39 (1938) pp. 20–30.

9. L. Nachbin, *A theorem of the Hahn-Banach type for linear transformations*, Trans. Amer. Math. Soc. vol. 68 (1950) pp. 28–46.

10. R. J. Silverman, *Invariant linear functions*, Trans. Amer. Math. Soc. vol. 81 (1956) pp. 411–424.

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