

BOUNDARIES INDUCED BY NON-NEGATIVE MATRICES

BY

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1. Introduction. The object of this paper is to explore certain properties of positivity preserving operators on either functions or measures in a space E . Although our method applies to more general cases (cf. the end of this introduction) we take up explicitly only the case of a *denumerable* E so that the operators reduce to matrices. In this way the basic features of the theory will not be obscured by an unfamiliar formalism or by an a priori imposed topology.

Let E stand for the set of positive integers, and Π for a matrix with elements $\Pi(i, j) \geq 0$, where $i, j \in E$. Then Π acts as an operator by premultiplication on column-vectors, and by postmultiplication on row-vectors. It will be shown that the relevant properties of Π are intimately connected with the solutions of the infinite system of equations

$$(1.1) \quad x(i) = \sum_{j \in E} \Pi(i, j)x(j) \quad \text{or} \quad \Pi \mathbf{x} = \mathbf{x}$$

and the dual (or adjoint) system

$$(1.2) \quad \xi(j) = \sum_{i \in E} \xi(i)\Pi(i, j) \quad \text{or} \quad \xi = \xi\Pi.$$

Here \mathbf{x} and ξ stand for a column- or row-vector, respectively.

The matrix Π is called *stochastic* if all row sums are unity. Now if \mathbf{x} is a strictly positive⁽²⁾ solution of (1.1), the transformation

$$\Pi'(i, j) = \Pi(i, j)x(j)/x(i)$$

defines a stochastic matrix Π' and it will be shown (§14) that there exists an isomorphism between the solutions of (1.1) and the solutions of the transformed equation $\Pi' \mathbf{x}' = \mathbf{x}'$. For our purposes the matrices Π and Π' are in every respect equivalent, and there is no loss of generality in supposing that Π is stochastic. However, since we shall be dealing also with submatrices of Π , it is most convenient to suppose only that

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⁽²⁾ Strict positivity is assumed only to simplify formulations.

$$(1.3) \quad \Pi(i, j) \geq 0 \quad \sum_{j \in E} \Pi(i, j) \leq 1 \quad i, j \in E.$$

Such matrices will be called *sub-stochastic*. From now on we assume (1.3).

Probabilistically, a sub-stochastic matrix defines a *random walk*, and we shall explain the random walk interpretations of our boundaries, solutions, etc. However, our development is purely analytic and no probabilistic arguments or results are used.

It will be shown that to the typical solution x or ξ there corresponds a family of subsets of E contracting to the empty set. It is natural and useful to introduce these sets as neighborhoods of new points or sets. In this way E will be enlarged by a boundary corresponding to (1.1), and by an adjoint boundary corresponding to (1.2). Frequently these two topologies will be equivalent, but *there exist matrices Π such that the two topologies have no connection whatever* (cf. Example III, §17). The two boundaries can have arbitrary topological structures^(*). The boundaries will be compact Hausdorff spaces, but the enlarged space need *not* be compact.

In a subsequent paper these boundaries will be applied to the theory of the Kolmogorov differential equations in E . It will be shown that our boundaries play exactly the rôle of ordinary boundaries in potential theory (except that in the latter the two boundaries coincide because of the symmetry of the underlying operators). We shall also find analogues to the classical Green functions and boundary conditions⁽⁴⁾ of the theory of harmonic functions and diffusion.

Our development proceeds in *three* stages. We begin by a study of *bounded solutions* of (1.1). Each such solution is the difference of two positive solutions, and we are concerned with the solutions x such that $0 \leq x(i) \leq 1$. They form a convex set \mathfrak{P} and, at the same time, a linear lattice (§4). In §5 we introduce the basic notion of a *sojourn set*. To each such set there corresponds an element s_A of \mathfrak{P} , and these form the lattice \mathfrak{S} of *sojourn solutions*. Analytically they are the *extremals* of \mathfrak{P} in the sense of Krein and Milman (cf., for example, [2, Livre 5, Ch. 2]). This fact is proved in §10, but not used explicitly.

To each sojourn set A there corresponds a sojourn solution s_A , but to different sets there may correspond the same sojourn solution. Two sets are called *equivalent* if $s_A = s_B$. It is shown in §8 that (with a trivial exception) *each sojourn set A contains a sequence of equivalent sojourn sets A_ϵ which contract monotonically to the empty set as $\epsilon \rightarrow 0$* . We shall introduce them as

(*) Cf. the absurdly simple example IV, §17, where the meaning of a boundary homeomorphic to the interval $(0, 1)$ is intuitively clear. The construction can be modified so as to obtain almost any type of boundary.

(4) The normal derivatives occurring in the boundary conditions for harmonic functions are, of course, meaningless in a discrete space E . They will be replaced by expressions depending on the given basic operator and which in turn reduce to normal derivatives, etc. in classical cases.

neighborhoods of a set A of the boundary (which will be both open and closed). For the random walk governed by Π and starting at i the value $s_A(i) = s_{A_\epsilon}(i)$ is (for each ϵ) the probability that the path ultimately (after finitely many steps) enters and remains contained in A_ϵ . In other words, $s_A(i)$ is the *probability of an asymptotic approach to the boundary set A* . (See §5 and Theorem 9.4.) The simplest situation is described in §12: here the boundary consists of denumerably many isolated points, each being represented by one sojourn solution. In general, however, no sojourn solutions correspond to the individual *points* of the boundary. We have, in effect, a measure (or capacity) induced on the boundary, and only *sets of positive measure are represented by sojourn solutions*. To introduce *points* of the boundary we use a variation of the well-known method of *maximal ideals* (§13). The enlarged space $E + \mathfrak{B}$ is a Hausdorff space in which *all elements of \mathfrak{B} are continuous and possess continuous boundary values*.

The boundary defined in terms of the bounded solutions of (1.1) sets the stage for a study of the *unbounded positive solutions* of (1.1). They form a vector lattice^(*) \mathfrak{M} . The transformation mentioned above and studied in §14 defines a family \mathfrak{F} of stochastic matrices such that there exists an isomorphism between the corresponding vector lattices \mathfrak{M} . To each element $x \in \mathfrak{M}$ there corresponds a matrix of the family such that the image of x is the unit vector 1. In other words, the vectors dominated by x are mapped into a set corresponding to the cone \mathfrak{B} of bounded solutions. In this way *each vector $x \in \mathfrak{M}$ can be made to play the rôle of the unit vector*. These transformations have a simple probabilistic interpretation and also (a less obvious) counterpart in the transformation theory of differential equations of the Sturm-Liouville type.

Each matrix of the family \mathfrak{F} defines a boundary for E , and these boundaries do partly overlap and the topologies agree. It turns out that it is possible to *endow the space E with a boundary \mathfrak{B}^* common to all matrices of the family \mathfrak{F}* . The boundary \mathfrak{B} defined by the bounded solutions of (1.1) is a subset of \mathfrak{B}^* and \mathfrak{B}^* is the union of the boundaries defined in terms of bounded solutions of (1.1), as Π runs through the members of the family \mathfrak{F} . Both \mathfrak{B} and \mathfrak{B}^* have analytic and probabilistic significance. The construction of \mathfrak{B}^* is analogous to the procedure outlined for \mathfrak{B} , but in \mathfrak{M} we have no maximal ideals at our disposal. Instead we introduce *lattice ideals maximal relative to a fixed element*. They are characterized in §15.

At last (§16) we consider the *adjoint system* (1.2). Fortunately it requires no special theory since it can be reduced to (1.1) by a simple device which has been used for other purposes by Kolmogorov and Derman. Just as the solutions of (1.1) induce the boundary \mathfrak{B}^* so do the solutions of (1.2) induce an *adjoint boundary*. Example III (§17) shows that *the two topologies are in-*

(*) Unfortunately, \mathfrak{M} is in general not compact, and we have therefore neither extremals nor maximal ideals at our disposal. If each row of Π has only finitely many nonzero elements then \mathfrak{M} is compact and a boundary may be defined in terms of the extremals of \mathfrak{M} .

dependent of each other. This phenomenon is new because one usually deals with operators which are symmetric or nearly so. It leads to interesting relations and boundary conditions for stochastic processes in E .

§11 stands somewhat apart from the main part of the paper. In the ergodic theory of stochastic matrices the points and subsets of E are classified as *recurrent* or *transient*. A recurrent subset R in no way contributes to the boundary (and corresponds analytically to a *closed manifold*). For the matrix Π it has only nuisance value, since Π is partitioned in the form (11.3) and the theory effectively reduces to a study of a submatrix corresponding to the set $E - R$. For our purposes it was necessary to correlate these facts with properties of the solutions of (1.1). §11 contains a direct derivation of the basic properties of transient and recurrent sets⁽⁶⁾. A reader who is disturbed by this interruption of the theory may omit this section and simply assume that Π is not partitioned (of the form (11.3)).—Examples for the several phenomena are collected in §17.

HARMONIC FUNCTIONS—ASPECT

We conclude this introduction by indicating the *relationship of the present theory with classical harmonic functions*. To illustrate the meaning of the new boundaries, let G be a simply connected domain of the Euclidean plane with a complicated boundary Γ (containing prime ends, etc.). A conformal mapping reduces the theory of *bounded harmonic functions* in G to the theory of bounded harmonic functions in the circular disc. The inverse map from the disc into G induces for G an ideal boundary Γ^* which is not topologically equivalent to Γ , and which is more natural for the study of harmonic functions. In several dimensions the “natural” boundary is still less adequate, and a boundary appropriate for the study of (or induced by) the Laplacian operator has been defined in the classical paper [8] by R. S. Martin. The formal analogue for harmonic functions to our boundary is not the *Martin boundary* but a larger one: our topology would make the bounded harmonic functions continuous up to and including the ideal boundary. On the other hand, our boundary is smaller than the *Čech boundary* constructed by means of maximal ideals in the algebra of bounded real functions continuous in C . It constitutes a worth-while program to reduce our boundary by an appropriate identification of points to an analogue of the Martin boundary where points would stand in a one-to-one correspondence to the linearly independent positive solutions of (1.1).

A *direct treatment of harmonic functions by our methods* can proceed in two ways.

(a) As is well known, the theory of harmonic functions can be approached

⁽⁶⁾ It is now easy to go a step further and to derive the ergodic theory from our results. Incidentally, the uniqueness Theorem 14.4 properly belongs to §11, but is proved more naturally using the isomorphisms introduced in §14.

using the diffusion equation $u_t = \Delta u$. The latter corresponds to the Kolmogorov differential equations mentioned above, and (using Laplace transforms) can be treated in the same way.

(b) Completely within the framework of the present paper we can proceed as follows. Let Γ be an open circular disc and for each point $P \in \Gamma$ denote by C_P the *greatest* open circular disc with center at P and contained in Γ . Let $|C_P|$ be the area of C_P and define a kernel $K(P, Q)$ by

$$(1.4) \quad K(P, Q) = \begin{cases} 0 & \text{for } Q \notin C_P \\ \frac{1}{|C_P|} & \text{for } Q \in C_P \end{cases}$$

where $P, Q \in \Gamma$. Such a kernel plays the rôle of the matrix Π and the equation

$$(1.5) \quad x(P) = \int_{\Gamma} K(P, Q)x(Q)dQ$$

is analogous to (1.1). Each *harmonic* $x(P)$ is a solution of (1.5).

Probabilistically (1.4) describes a random walk where a step leads from P to a random point Q which has a uniform probability distribution in C_P . If A is a compact set in Γ , the probability that after n steps the moving point will be contained in A tends to zero as $n \rightarrow \infty$ (independently of the initial position). In other words, the moving point approaches in probability the (ordinary) boundary.

A bounded harmonic function is representable by means of the Poisson integral in terms of the boundary values $f(\theta)$ on the boundary of Γ . To our sojourn solutions (extremals) there correspond the solutions x where f assumes the value one on an arc α and 0 on the complement. Then $x(P)$ represents the probability that, starting from P , the random walk will asymptotically tend to the arc α . As explained above, a natural topological connection between the boundary and the interior is induced by K . For example, if the radii of C_P are chosen so as to decrease rapidly as P approaches the center of Γ and so that C_P excludes this center, then the center may become a boundary set in the topology induced by the kernel K .

2. Notations and conventions. We shall use *bold face* to denote *column vectors*. Thus \mathbf{x} stands for a vector with components $x(1), x(2), \dots$. Occasionally these components $x(i)$ will be defined only for $i \in C$, where C is a subset of the space E . For column vectors we use the conventional norm

$$(2.1) \quad \|\mathbf{x}\| = \sup_{i \in E} x(i).$$

Row vectors will not occur before §16. By 0 and 1 we denote the vectors all of whose components equal 0 or 1, respectively. An inequality such as $\mathbf{x} \leq \mathbf{y}$ has the obvious meaning: $x(i) \leq y(i)$ for all i . Thus the second inequality in (1.3) may be rewritten as $\Pi \mathbf{1} \leq \mathbf{1}$.

DEFINITION. We denote by \mathfrak{P} , \mathfrak{P}^* , and \mathfrak{P}_* the aggregates of vectors satisfying

$$(2.2) \quad 0 \leq z \leq 1$$

and, respectively, the conditions

$$(2.3) \quad \Pi z = z,$$

$$(2.4) \quad \Pi z \geq z,$$

$$(2.5) \quad \Pi z \leq z.$$

For any set $A \subset E$ we write

$$(2.6) \quad \Pi(i, A) = \sum_{j \in A} \Pi(i, j).$$

With this notation the second inequality in (1.1) reads $\Pi(i, E) \leq 1$. In the *random walk interpretation* $\Pi(i, A)$ is the transition probability from the point i to the set A .

The restriction Π_A of Π to a set A is the matrix defined by

$$(2.7) \quad \Pi_A(i, j) = \Pi(i, j) \quad \text{for } i, j \in A$$

and undefined for $i, j \in E - A$. Clearly, Π_A is again a sub-stochastic matrix and every theorem concerning Π applies equally to Π_A . For the probability interpretation cf. §5.

A matrix product $\Pi_A x$ makes sense only when x is a vector defined on A . For convenience we shall use the same notation even if x is defined on the full space E . Thus we write

$$(2.8) \quad y = \Pi_A x$$

to indicate that

$$(2.9) \quad y(i) = \sum_{j \in A} \Pi(i, j)x(j), \quad \text{for } i \in A.$$

The components $x(i)$ and $y(i)$ for $i \in E - A$ may, but need not, be defined; equation (2.8) contains no statement concerning them.

3. The basic lemmas. We begin with the observation that to each $z \in \mathfrak{P}^*$ there exists a *smallest element* $a \in \mathfrak{P}$ such that $a \geq z$. More precisely we have

LEMMA 3.1. For each $z \in \mathfrak{P}^*$ the limit

$$(3.1) \quad a = \lim_{n \rightarrow \infty} \Pi^n z$$

exists. One has $a \in \mathfrak{P}$ and $a \geq z$; moreover,

$$(3.2) \quad z \leq a \leq x,$$

for each $x \in \mathfrak{P}$ such that $x \geq z$.

Proof. Put

$$(3.3) \quad a_0 = z, \quad a_{n+1} = \Pi a_n = \Pi^n z.$$

Then $z \leq a \leq 1$ in consequence of (2.4) and (2.2). It follows by induction that $a_n \leq a_{n+1} \leq 1$ and therefore by bounded convergence $a_n \rightarrow a$; here $a = \Pi a \geq z$. To conclude the proof suppose that $x \in \mathfrak{P}$ and $x \geq z = a_0$. Then by induction $x \geq a_n$ so that (3.2) holds.

The same argument leads to

LEMMA 3.2. *For each $z \in \mathfrak{P}_*$ the limit (3.1) exists, and $a \in \mathfrak{P}$. One has $a \leq z$. Each $x \in \mathfrak{P}$ such that $x \leq z$ satisfies*

$$(3.4) \quad x \leq a \leq z.$$

The possibility $a = 0$ is not excluded.

THEOREM⁽⁷⁾ 3.1. *Each bounded solution of $\Pi x = x$ is a linear combination of two elements of \mathfrak{P} .*

Proof. Without loss of generality we assume $\|x\| \leq 1$. For each i put $z(i) = |x(i)|$. Then $z \in \mathfrak{P}^*$. With a defined by Lemma (3.1) we have

$$(3.5) \quad x = \frac{1}{2} (a + x) - \frac{1}{2} (a - x)$$

where $(a \pm x)/2 \in \mathfrak{P}$.

4. **Lattice properties of \mathfrak{P} .** It is hardly necessary to point out that \mathfrak{P} is a convex set; that is, if $u, v \in \mathfrak{P}$ and $p \geq 0, q \geq 0, p+q \leq 1$, then $pu+qv \in \mathfrak{P}$. We now prove that \mathfrak{P} is a vector lattice.

THEOREM. *Let $x, y \in \mathfrak{P}$. Then \mathfrak{P} contains a uniquely determined least upper bound⁽⁸⁾ $x \cup y$ and a greatest lower bound $x \cap y$. (The latter may be zero.)*

Proof. For each i put $z(i) = \max \{x(i), y(i)\}$. Then $z \in \mathfrak{P}^*$, and the vector a defined by (3.1) has the properties required of $x \cup y$. Similarly, putting $z(i) = \min \{x(i), y(i)\}$ the construction of Lemma 3.2 leads to $x \cap y$.

The following two lemmas are known in more general contexts (cf. e.g. [1]).

LEMMA 4.1. *If*

$$(4.1) \quad x + u = y + v = w,$$

then

$$(4.2) \quad x \cup y + u \cap v = w.$$

⁽⁷⁾ An *unbounded* solution of $\Pi x = x$ is not necessarily the difference of two non-negative solutions.

⁽⁸⁾ This means, $a = x \cup y$ is defined as an element of \mathfrak{P} such that $a \geq x$ and $a \geq y$ and with the property that if $u \in \mathfrak{P}, u \geq x, u \geq y$ then $u \geq a$. A similar definition applies to $x \cap y$.

In particular, one has identically

$$(4.3) \quad x + y = x \cup y + x \cap y.$$

Proof. Clearly $w - u \cap v \geq w - u = x$ and equally $w - u \cap v \geq y$. Therefore $w - u \cap v \geq x \cup y$. On the other hand $w - x \cup y \leq w - x = u$ and so $w - x \cup y \leq u \cap v$.

LEMMA 4.2. *One has*

$$(4.4) \quad (x + y) \cap z \leq (x \cap z) + (y \cap z).$$

If $x \cap y = 0$ then the distributive law

$$(4.5) \quad (x + y) \cap z = (x \cap z) + (y \cap z)$$

holds.

Proof. For each i we have

$$\min \{x(i) + y(i), z(i)\} \leq \min \{x(i), z(i)\} + \min \{y(i), z(i)\}.$$

The assertion (4.4) is an immediate consequence of this and the definition of $x \cap y$ (cf. the construction of Lemma 3.2). Next suppose $x \cap y = 0$. From (4.3)

$$(4.6) \quad (x + y) \cap z = (x \cup y) \cap z \geq (x \cap z) \cup (y \cap z) = (x \cap z) + (y \cap z)$$

where the last step consists in a repeated application of (4.3). The two inequalities (4.4) and (4.6) together imply (4.5).

Note on unbounded solutions. The operations $x \cup y$ and $x \cap y$ are well defined for any two non-negative solutions of (1.2). In fact, the construction of $x \cup y$ depends only on the existence of some solution u of (1.2) such that $u \geq x$ and $u \geq y$. Now $u = x + y$ is such a solution and may replace 1 in our construction. Note however the footnote to §3 which shows that $x \cup y$ need not exist for arbitrary unbounded solutions. The theory of unbounded *non-negative* solutions is developed in §14.

5. **Sojourn sets.** Since $1 \in \mathfrak{P}_*$ we may apply Lemma 3.2 to $z = 1$. We conclude that

$$(5.1) \quad s_E(i) = \lim_{n \rightarrow \infty} \Pi^n(i, E)$$

exists for each i , and $s_E \in \mathfrak{P}$. For each $x \in \mathfrak{P}$ we have $x \leq 1$, and hence by induction $x \leq \Pi^n 1 = \Pi^n(i, E)$. This proves

THEOREM 5.1. *The vector s_E defined by (5.1) is the maximal element of \mathfrak{P} , that is, $s_E \in \mathfrak{P}$ and $x \leq s_E$ for each $x \in \mathfrak{P}$.*

Of course, it may happen that $s_E = 0$, in which case \mathfrak{P} contains 0 as the only element. If Π is strictly stochastic, then $s_E = 1$ (and conversely).

Now let $A \subset E$ be an arbitrary set and apply the above argument to the

restriction Π_A of Π to A . To s_E there corresponds the vector \mathfrak{d}_A which is defined on A only by

$$(5.2) \quad \sigma_A(i) = \lim_{n \rightarrow \infty} \Pi_A^n(i, A), \quad i \in A,$$

and which satisfies

$$(5.3) \quad \mathfrak{d}_A = \Pi_A \mathfrak{d}_A.$$

By Theorem 5.1 *this \mathfrak{d}_A may be characterized as the maximal solution of (5.3) subject to the condition $0 \leq \mathfrak{d}_A \leq 1$.*

For convenience we extend the definition of \mathfrak{d}_A throughout E by putting

$$(5.4) \quad \sigma_A(i) = 0 \quad \text{for } i \in E - A.$$

Clearly $\mathfrak{d}_A \in \mathfrak{P}^*$, and applying Lemma 3.1 to $z = \mathfrak{d}_A$ we see that

$$(5.5) \quad s_A(i) = \lim_{n \rightarrow \infty} \sum_j \Pi^n(i, j) \sigma_A(j)$$

exists for all i . The vector

$$(5.6) \quad s_A = \lim_{n \rightarrow \infty} \Pi^n \mathfrak{d}_A$$

is uniquely defined as the smallest vector satisfying

$$(5.7) \quad s_A \in \mathfrak{P}, \quad 0 \leq \mathfrak{d}_A \leq s_A \leq 1.$$

When $A = E$ the definitions (5.1) and (5.6) agree, and $\mathfrak{d}_E = s_E$.

DEFINITION 5.1. *The set A is called sojourn set if $s_A \neq 0$ (or, what amounts to the same, if $\mathfrak{d}_A \neq 0$).*

In view of the fact that the limit in (5.2) is attained monotonically, we have the following obvious

CRITERION. *For A to be a sojourn set it is necessary and sufficient that there exist an $i \in A$ and an $\eta < 1$ such that*

$$(5.8) \quad \Pi_A^n(i, A) > \eta$$

for all n .

PROBABILITY INTERPRETATION. Consider the random walk with stationary transition probabilities $\Pi(i, j)$ and interpret $1 - \Pi(i, E)$ as the probability that the random walk does not continue (terminates). Then $1 - \Pi^n(i, E)$ is the probability that the random walk with initial position i terminates at or before the n th step. From (5.1) we then conclude:

$s_E(i)$ is the probability that the random walk with initial position i continues indefinitely (= does not terminate after finitely many steps).

From the definition, $\Pi_A^n(i, A)$ is the probability that, starting from $i \in A$,

the random walk will continue for at least n steps without leaving the set A . Accordingly, for $i \in A$ we see that:

$\sigma_A(i)$ is the probability that starting from i the random walk will continue indefinitely without ever leaving the set A .

Finally, in (5.5) the sum on the right equals the probability that from an arbitrary starting point i the n th step in our random walk leads to a position $j \in A$, and that from then on the random walk continues indefinitely without ever leaving A . (During the first n steps the random walk may lead in and out of A .) Therefore:

$s_A(i)$ is the probability that from the starting point i the random walk will after finitely many steps lead into A and from then on continue indefinitely without ever leading out of A .

DEFINITION 5.2. Two sets A and B are equivalent if $s_A = s_B$.

DEFINITION 5.3. The aggregate of all vectors s_A and 0 will be denoted by \mathfrak{S} . The elements of \mathfrak{S} will be referred to as sojourn solutions.

LEMMA 5.1. If A and B are nonoverlapping, then

$$(5.9) \quad s_A \cap s_B = 0,$$

$$(5.10) \quad s_A \cup s_B = s_A + s_B \leq s_{A \cup B}.$$

Note. It will be shown in §9 that A and B can be replaced by equivalent sets such that the inequality sign in (5.10) becomes a strict equality sign.

Proof. Since, by definition, $\mathfrak{d}_A + \mathfrak{d}_B$ equals \mathfrak{d}_A in A and \mathfrak{d}_B in B we have

$$(5.11) \quad s_A \cup s_B \geq \mathfrak{d}_A + \mathfrak{d}_B.$$

Premultiplying by Π^n and letting $n \rightarrow \infty$ we get

$$(5.12) \quad s_A \cup s_B \geq s_A + s_B.$$

But the reversed inequality is trivially true, and therefore the equality sign holds. This proves the first half of (5.10). A comparison with (4.1) shows the truth of (5.9). Finally, the inequality $s_{A \cup B} \geq s_A \cup s_B$ follows directly from the definitions.

6. **Relativization.** Let A be a sojourn set, and $A \subset B \subset E$. In §5 we have considered A as a subset of E and Π_A as a submatrix of Π . Obviously the same considerations apply if we replace E and Π by B and Π_B , respectively. To the sojourn solution s_A there corresponds the vector s_A^B defined by

$$(6.1) \quad \begin{aligned} s_A^B(i) &= \lim_{n \rightarrow \infty} \sum_j \Pi_B^n(i, j) \sigma_A(j), & i \in B, \\ s_A^B(i) &= 0, & i \in E - B. \end{aligned}$$

With this notation we have $s_A = s_A^E$ and $\mathfrak{d}_A = s_A^A$. Clearly

$$(6.2) \quad \mathfrak{d}_A \leq s_A^B \leq s_A.$$

Premultiply (6.2) by Π^n . The right side remains unchanged, and (5.6) leads to

LEMMA 6.1. *We have*

$$(6.3) \quad s_A = \lim_{n \rightarrow \infty} \Pi^n s_A^B.$$

We shall repeatedly use the following corollary.

LEMMA 6.2. *Let $A \subset B$ and $\mathbf{A} \subset B$. If $s_A^B = s_{\mathbf{A}}^B$, then also $s_A = s_{\mathbf{A}}$.*

That is, if A and \mathbf{A} are equivalent relative to B , then they are equivalent.

7. Auxiliary lemmas.

LEMMA 7.1. *Let $z \in \mathfrak{P}_*$ and let A be an arbitrary set. For each n and each $i \in A$ we have*

$$(7.1) \quad z(i) \geq \sum_{j \in A} \Pi_A^n(i, j)z(j) + \sum_{r=0}^{n-1} \sum_{j \in A} \sum_{k \in B-A} \Pi_A^r(i, j)\Pi(j, k)z(k).$$

If $z \in \mathfrak{P}$, then the equality sign holds in (7.1).

Proof. For $n=1$ the relation (7.1) reduces to $z \geq \Pi z$. Assume (7.1) to hold for some n . Using $z \geq \Pi z$ we get

$$(7.2) \quad \sum_{j \in A} \Pi_A^n(i, j)z(j) \geq \sum_{j \in A} \Pi_A^{n+1}(i, j)z(j) + \sum_{j \in A} \sum_{k \in B-A} \Pi_A^n(i, j)\Pi(j, k)z(k).$$

Substituting this for the first term on the right in (7.1) we get the assertion (7.1) with n replaced by $n+1$. When $z = \Pi z$ then each of the above inequalities is replaced by an equality, and the lemma is proved.

Choosing in particular $z = 1$ we get the

COROLLARY 1. *For $i \in A$*

$$(7.3) \quad \Pi_A^n(i, A) + \sum_{r=0}^{n-1} \sum_{j \in A} \Pi_A(i, j)\Pi(j, E-A) \leq 1.$$

If Π is strictly stochastic, then the equality sign holds.

Letting $n \rightarrow \infty$ we get

COROLLARY 2. *For $i \in A$*

$$(7.4) \quad \sigma_A(i) + \sum_{r=0}^{\infty} \sum_{j \in A} \Pi_A(i, j)\Pi(j, E-A) \leq 1.$$

PROBABILITY INTERPRETATION. The n th term in the outer sum equals the probability that, starting from i , the random walk remains for n steps inside A whereas the step number $n+1$ leads into $E-A$ (i.e. the first passage into

$E-A$ occurs at the $(n+1)$ st step). The difference between the two sides in (7.4) is the probability that the random walk terminates after finitely many steps.

With each vector x there are associated its *contour sets*, that is, the sets of those i for which $x(i)$ exceeds a preassigned constant. Concerning these we prove

LEMMA 7.2. *Let x be a bounded solution of $\Pi x = x$, and $x \geq 0$, $\|x\| > 0$. For fixed $0 < \eta < \|x\|$ put*

$$(7.5) \quad X_\eta = \{i: x(i) > \|x\| - \eta\}.$$

Then X_η is a sojourn set. If

$$(7.6) \quad \frac{\delta}{\eta} < \epsilon, \quad \delta > 0,$$

then

$$(7.7) \quad \sigma_{X_\eta}(i) > 1 - \epsilon \quad \text{for } i \in X_\eta.$$

Proof. Applying Lemma 7.1 with $z = x$ and $A = X_\eta$ we get for $i \in X_\eta$

$$(7.8) \quad \|x\| - \delta < x(i) \leq \|x\| \cdot \Pi_{X_\eta}^n(i, X_\eta) + \{\|x\| - \eta\} \cdot \sum_{r=0}^{n-1} \sum_{j \in X_\eta} \Pi_{X_\eta}^r(i, j) \Pi(j, E - X_\eta).$$

Using (7.3) with $A = X_\eta$ this leads to

$$(7.9) \quad \|x\| - \delta \leq \|x\| \cdot \Pi_{X_\eta}^n(i, X_\eta) + \{\|x\| - \eta\} \cdot \{1 - \Pi_{X_\eta}^n(i, X_\eta),$$

or

$$(7.10) \quad \Pi_{X_\eta}^n(i, X_\eta) \geq 1 - \frac{\delta}{\eta}.$$

The assertion (7.7) now follows letting $n \rightarrow \infty$.

LEMMA 7.3. *For each sojourn set $\|\mathfrak{d}_A\| = 1$ and $\|s_A\| = 1$.*

Proof. Since s_E is the maximal sojourn solution we conclude from (7.7) directly that $\|s_E\| = 1$. Applying this conclusion to the matrix Π_A instead of Π we see that $\|\mathfrak{d}_A\| = 1$. Finally, $\mathfrak{d}_A \leq s_A \leq 1$ and the lemma is proved.

LEMMA 7.4. *With the notations of Lemma 7.2 one has*

$$(7.11) \quad x \geq \{\|x\| - \eta\} \cdot s_{X_\eta}.$$

Proof. By definition $x \geq \{\|x\| - \eta\} \cdot \mathfrak{d}_{X_\eta}$. The assertion follows on pre-multiplying this inequality by Π^n and letting $n \rightarrow \infty$.

8. Nests of equivalent sojourn sets. The next theorem supplements

Lemma 7.2 for the special case of sojourn solutions and is of prime importance for the sequel.

THEOREM^(*) 8. For any sojourn set A and $0 < \eta < 1$ put

$$(8.1) \quad A_\eta = \{i \in A: s_A(i) > 1 - \eta\}$$

and

$$(8.2) \quad \mathbf{A}_\eta = \{i: \sigma_A(i) > 1 - \eta\}.$$

Then

$$(8.3) \quad s_A = s_{A_\eta} = s_{\mathbf{A}_\eta},$$

that is, A , A_η , \mathbf{A}_η are equivalent.

Proof. When $A = E$ the sets defined by (8.1) and (8.2) are identical and the lemma reduces to the proposition

$$(8.4) \quad s_E = s_{E_\eta}.$$

We begin now by showing that (8.3) is more general than (8.4) in appearance only. First note that $\mathbf{A}_\eta \subset A_\eta \subset A$, so that (8.3) really reduces to $s_A = s_{\mathbf{A}_\eta}$. Now recall Lemma 6.2 according to which A and \mathbf{A}_η are equivalent whenever they are equivalent relative to A (that is, if $s_{\mathbf{A}_\eta}^A = \mathfrak{d}_A$). Now this statement differs from (8.4) only notationally, and so it suffices to prove (8.4).

For that purpose put

$$(8.5) \quad s_E - s_{E_\eta} = u$$

and suppose that $\|u\| > \alpha > 0$. The set

$$(8.6) \quad B = \{i: u(i) > \alpha\}$$

is a sojourn set, by Lemma 7.2. We now apply that Lemma to $x = s_E$. Then $X_\eta = E_\eta$ and by (7.7) we have

$$(8.7) \quad s_{E_\eta}(i) > 1 - \alpha \quad \text{for } i \in E_\delta$$

provided $\delta < \alpha\eta$. For $i \in E_\delta \cap B$ we have both (8.7) and $u(i) > \alpha$, and therefore $s_E(i) > 1$ which is impossible. Thus $B \subset E - E_\delta$, and hence by the definition of E_δ

$$(8.8) \quad s_E(i) \leq 1 - \delta \quad \text{for } i \in B.$$

But $s_E \geq s_B \geq \mathfrak{d}_B$ and from (7.7) we see that $\sigma_B(i) > 1 - \delta$ for some $i \in B$. This contradicts (8.8), and so $u = 0$ as asserted.

An important implication of the last theorem is that each sojourn set A contains an *equivalent* sojourn set \mathbf{A} such that

(*) Compare with Lemma 9.2 at the end of §9.

$$(8.9) \quad s_A(i) > 1 - \eta \quad \text{for } i \in A.$$

When dealing with a particular sojourn solution $\mathbf{x} = \mathbf{s}_A$ we may therefore always choose the representative set so that (8.9) holds. In fact, it is necessary to do this if set theoretical operations on sojourn sets are to correspond to lattice operations on sojourn solutions. For example, (8.10) is *false* without the assumption that A satisfies (8.9). We therefore introduce the

DEFINITION. *A sojourn set A is called representative if (8.9) holds for some $\eta > 0$.*

As was already observed, we have

LEMMA 8.1. *Each sojourn set A contains an equivalent subset A which is representative.*

LEMMA 8.2. *Let A be representative, and $A \subset B$. Then*

$$(8.10) \quad \mathbf{s}_B = \mathbf{s}_A + \mathbf{s}_{B-A}.$$

Proof. We begin with the special case $B = E$. Put

$$(8.11) \quad \mathbf{s}_E - \mathbf{s}_A - \mathbf{s}_{E-A} = \mathbf{u}.$$

We propose to show that $\mathbf{u} = \mathbf{0}$. From (5.10) we have $\mathbf{u} \geq \mathbf{0}$. Suppose $\|\mathbf{u}\| > 0$ and define three families of sets as follows:

$$(8.12) \quad \begin{aligned} A_\delta &= \{i \in A : s_A(i) > 1 - \delta\}, \\ C_\delta &= \{i \in E - A : s_{E-A}(i) > 1 - \delta\}, \\ U_\delta &= \{i : u(i) \geq \|\mathbf{u}\| - \delta\} \end{aligned}$$

(where C_δ may be empty). Choose $2\delta < \|\mathbf{u}\|$. At any point i common to A_δ and C_δ we would have $s_E(i) > 2 - 2\delta > 1$ which is impossible. Therefore A_δ and C_δ are nonoverlapping. The same argument shows that

$$(8.13) \quad A_\delta \cap C_\delta = 0, \quad A_\delta \cap U_\delta = 0, \quad C_\delta \cap U_\delta = 0.$$

It follows then from (5.10) that

$$(8.14) \quad \mathbf{s}_E \geq \mathbf{s}_{A_\delta} + \mathbf{s}_{C_\delta} + \mathbf{s}_{U_\delta}.$$

In view of Theorem 8 this is the same as

$$(8.15) \quad \mathbf{s}_E \geq \mathbf{s}_A + \mathbf{s}_{E-A} + \mathbf{s}_{U_\delta}.$$

Thus $\mathbf{u} \geq \mathbf{s}_{U_\delta}$ and therefore $\|\mathbf{u}\| = 1$. This means that for $i \in A \cap U_\delta$ we have $s_E(i) \geq 2 - \eta - \delta$ and therefore $A \cap U_\delta = 0$ provided $\delta < 1 - \eta$. With this choice of δ we see that $U_\delta \subset E - A$ and hence

$$(8.16) \quad \mathbf{s}_E \geq \mathbf{s}_{E-A} + \mathbf{u} \geq \mathbf{s}_{E-A} + \mathbf{s}_{U_\delta} \geq 2\mathbf{s}_{U_\delta}.$$

But this is absurd since $\|\mathbf{s}_{U_\delta}\| = 1$ by Lemma 7.3. We conclude therefore finally that $\mathbf{u} = \mathbf{0}$ or $\mathbf{s}_E = \mathbf{s}_A + \mathbf{s}_{E-A}$.

Now consider the case of an arbitrary set $B \supset A$. Applying the last result to the matrix Π_B (instead of Π) we see that there exists a subset $A \subset A$ such that

$$(8.17) \quad \mathfrak{d}_B = \mathfrak{s}_A^B + \mathfrak{s}_{B-A}^B, \quad \mathfrak{s}_A^B = \mathfrak{s}_A^B$$

(see (6.1) for the notation). Lemma 6.1 shows at once that (8.17) is equivalent to

$$(8.18) \quad \mathfrak{s}_B = \mathfrak{s}_A + \mathfrak{s}_{B-A}, \quad \mathfrak{s}_A = \mathfrak{s}_A.$$

Now $B-A \supset B-A$, and the difference is contained in A . But for $i \in A$ we have $\mathfrak{s}_{B-A}(i) \leq 1 - \mathfrak{s}_A(i) \leq 1 - \eta$ and hence $B-A$ is equivalent to $B-A$.

9. Properties of sojourn solutions.

THEOREM 9.1 (CRITERION). *For an element $x \in \mathfrak{P}$ to be a sojourn solution it is necessary and sufficient that*

$$(9.1) \quad x \cap (\mathfrak{s}_E - x) = 0.$$

Equivalently it is necessary and sufficient for any sojourn set C

$$(9.2) \quad x > t\mathfrak{s}_C, \quad t > 0 \text{ implies } x \geq \mathfrak{s}_C.$$

Proof. (1) Let $x = \mathfrak{s}_A$. Then A may be supposed to be representative. By Lemma 8.2 we have in this case $\mathfrak{s}_E - x = \mathfrak{s}_{E-A}$, and (9.1) is contained in (5.9). Thus (9.1) is necessary.

(2) Proof that (9.1) implies (9.2). In consequence of (9.1) we may apply (4.3) to obtain

$$(9.3) \quad \mathfrak{s}_C = \mathfrak{s}_C \cap \{x + (\mathfrak{s}_E - x)\} = \mathfrak{s}_C \cap x + \mathfrak{s}_C \cap (\mathfrak{s}_E - x).$$

Suppose now that $t\mathfrak{s}_C \leq x$ where $t > 0$. By (9.1) the last term in (9.3) vanishes, and thus (9.3) reduces to $\mathfrak{s}_C = \mathfrak{s}_C \cap x$ or $\mathfrak{s}_C \leq x$.

(3) Suppose now that (9.2) holds. We have to show that $x = \mathfrak{s}_X$ for some sojourn set X .

Let $x \in \mathfrak{P}$ and consider the set X_η defined in (7.5). Then (7.11) holds, and therefore $x \geq \mathfrak{s}_{X_\eta}$ by (9.2). Thus

$$(9.4) \quad x - \mathfrak{s}_{X_\eta} = u, \quad u \in \mathfrak{P}.$$

Applying the preceding remark to u instead of x we conclude that if $u \neq 0$ then there exists a sojourn set A such that $u \geq t\mathfrak{s}_A$ with $t > 0$. It follows then from (9.4) and (9.2) that $x \geq \mathfrak{s}_A$.

Let A_η be defined as in (8.1). For $i \in A_\eta$ we have $x(i) \geq \mathfrak{s}_A(i) \geq 1 - \eta$ and thus $A_\eta \subset X_\eta$. But then from (9.4)

$$(9.5) \quad x \geq \mathfrak{s}_{A_\eta} + u \geq (1 + t)\mathfrak{s}_{A_\eta}$$

or $\|x\| \geq 1 + t$ against assumption. Therefore $u = 0$ and $x = \mathfrak{s}_{X_\eta}$.

THEOREM 9.2. *If A and B are sojourn sets, then*

$$(9.6) \quad s_{A \cap B} = s_A \cap s_B.$$

Warning. The formal analogue for $A \cup B$ is false.

Proof. Since obviously $s_{A \cap B} \leq s_A \cap s_B$ it suffices to prove that there exists a set $C \subset A \cap B$ such that $s_C = s_A \cap s_B$. We are therefore permitted to replace A and B by equivalent subsets and to prove the relation (9.6) for them. This means that there is no loss of generality in supposing that both A and B are representative. Accordingly, we shall assume that

$$(9.7) \quad s_A(i) > 3/4 \text{ for } i \in A, \quad s_B(i) > 3/4 \text{ for } i \in B.$$

From the probability interpretation⁽¹⁰⁾ of sojourn probabilities one sees that

$$(9.8) \quad s_{A \cup B} \leq s_{A \cap B} + \{s_E - s_A\} + \{s_E - s_B\}.$$

We see from this and (9.7) that $s_{A \cap B}(i) > 1/4$ when $i \in A \cap B$. Thus $A \cap B$ satisfies an inequality of the form (8.9) and we may apply Lemma 8.2 to obtain

$$(9.9) \quad s_A = s_{A \cap B} + s_{A-A \cap B}, \quad s_B = s_{A \cap B} + s_{B-A \cap B}.$$

From this we get

$$(9.10) \quad s_A \cap s_B = s_{A \cap B} + s_{A-A \cap B} \cap s_{A \cap B} + s_{A \cap B} \cap s_{B-A \cap B} + s_{A-A \cap B} \cap s_{B-A \cap B}$$

(the distributive law (4.5) is applicable in consequence of (5.9)). The last three terms in (9.10) vanish since the two sets involved are in each case non-overlapping (cf. (5.9)). Thus (9.10) reduces to (9.6).

THEOREM 9.3. *Let x , y , and x_n be sojourn solutions (elements of \mathfrak{S}). Then*

- (a) $x \cap y \in \mathfrak{S}$,
- (b) $y - x \in \mathfrak{S}$ provided $x \leq y$.
- (c) $x \cup y \in \mathfrak{S}$,
- (d) $x + y \in \mathfrak{S}$ provided $x \cap y = 0$.
- (e) *If either $x_n \downarrow u$ or $x_n \uparrow u$ then $u \in \mathfrak{S}$.*

Proof. Theorem 9.2 contains a statement which is stronger (and deeper) than (a). Also, (a) is an immediate consequence of the criterion (9.2). Similarly, (b) is contained in Lemma 8.2. To prove (c) note that by (4.2) we have

$$(9.11) \quad x \cup y = s_E - (s_E - x) \cap (s_E - y),$$

and the right side is a sojourn solution in consequence of (b) and (a). Next,

⁽¹⁰⁾ For the reader unacquainted with (or distrusting) probability arguments we give a direct proof of (9.8). For the purpose of this proof we set $\Pi_A(i, j) = 0$ when $i \notin A$. A simple induction shows that $\Pi_{A \cup B}^*(i, A \cup B) \leq \Pi_{A \cap B}^*(i, A \cap B) + \{1 - \Pi_A^*(i, A)\} + \{1 - \Pi_B^*(i, B)\}$ (note that the right side is ≥ 1 when $i \notin A \cap B$). Letting $n \rightarrow \infty$ we get $s_{A \cup B} \leq s_{A \cap B} + (1 - s_A) + (1 - s_B)$. Premultiplying by Π^* and letting $n \rightarrow \infty$ we obtain (9.8).

(d) is but a special case of (a). Finally, if $x_n \downarrow u$ then the criterion (9.2) shows trivially that $u \in \mathfrak{S}$. For the case $x_n \uparrow u$ the proof goes by complementation as under (c).

The following theorem is listed for its intrinsic interest and will not be used in the sequel.

THEOREM 9.4. *Let A be representative. Then*

$$(9.12) \quad s_A(i) = \lim_{n \rightarrow \infty} \Pi^n(i, A).$$

Warning. For an arbitrary sojourn set (9.12) need not be true.

For the *random walk* with initial position i the relation (9.12) states that *the probability of finding the moving point at time n in the set A approaches $s_A(i)$ as $n \rightarrow \infty$.*

Proof. From the definition (5.5) we have

$$(9.13) \quad s_A(i) \leq \liminf_{n \rightarrow \infty} \Pi^n(i, A),$$

and equally

$$(9.14) \quad s_{E-A}(i) \leq \liminf_{n \rightarrow \infty} \Pi^n(i, E - A).$$

Now (8.10) applies, and so the left sides add to $s_E(i)$. Thus

$$(9.15) \quad s_A(i) + s_{E-A}(i) = \lim_{n \rightarrow \infty} \{ \Pi^n(i, A) + \Pi^n(i, E - A) \},$$

and the theorem follows trivially.

We conclude with a lemma which supplements Theorem 8 inasmuch as it associates with each sojourn solution s_A a *sequence of sets which is independent of the representative set A* and thus intrinsically connected with s_A . However, this lemma will not be used in the sequel.

LEMMA 9.1. *Let A be a sojourn set and*

$$(9.16) \quad S_\eta = \{ i : s_A(i) > 1 - \eta \}, \quad 0 < \eta < 1.$$

Then S_η is equivalent to A .

Proof. Once more we are permitted to replace A by any equivalent set, in particular by the set A_η defined in (8.1). Now $A_\eta \subset S_\eta$ and Lemma 8.2 applies. Thus

$$(9.17) \quad s_{S_\eta} = s_{A_\eta} + s_{S_\eta - A_\eta}.$$

However, for $i \in S_\eta$ we have

$$(9.18) \quad s_{S_\eta - A_\eta}(i) \leq s_{E - A_\eta}(i) = s_E(i) - s_{A_\eta}(i) \leq 1 - (1 - \eta)$$

and therefore $\|\delta_{S_\eta - A_\eta}\| \leq \eta$. Accordingly $s_{S_\eta - A_\eta} = 0$ in consequence of Lemma 7.3.

10. Sojourn solutions as extremals of \mathfrak{P} .

THEOREM 10. *In order that an element $x \in \mathfrak{P}$ be a sojourn solution it is necessary and sufficient that the relations*

$$(10.1) \quad x = tu + (1 - t)v, \quad 0 < t < 1, \quad u, v \in \mathfrak{P},$$

imply $u = v = x$.

This amounts to saying that the sojourn solutions coincide with the extremals of \mathfrak{P} (cf. [2, Livre V, Ch. 2]).

Proof. (1) Suppose that $x \in \mathfrak{S}$ and that (10.1) holds. We have

$$(10.2) \quad s_E - x = t \cdot (s_E - u) + (1 - t) \cdot (s_E - v).$$

Since $tu \leq x$ and $t(s_E - u) \leq s_E - x$ we conclude from (9.1) that $u \cap (s_E - u) = 0$ and so $u \in \mathfrak{S}$ by Theorem 9.1. For the same reason $v \in \mathfrak{S}$. Accordingly, (9.2) assures us that $x \geq u$ and also $x \geq v$. Therefore

$$(1 - t)v = x - tu \geq (1 - t)u.$$

In this way $v \geq u$ and, by the same argument, $u \geq v$ and so $u = v$.

(2) If x is not a sojourn probability then by Theorem 9.1

$$(10.3) \quad x \cap (s_E - x) = z \quad z \neq 0.$$

We show that

$$(10.4) \quad x = \frac{1}{2}(x + z) + \frac{1}{2}(x - z)$$

is a decomposition of the form (10.1) with $t = 1/2$. Clearly $0 \leq x - z \leq 1$, and so $x - z \in \mathfrak{P}$. Also, for each i we have $z(i) \leq s_E(i) - x(i) \leq 1 - x(i)$, so that $0 \leq x + z \leq 1$ and $x + z \in \mathfrak{P}$.

Note. The Krein-Milman theorem leads to an alternative proof of the uniform approximation Theorem 13.4.

11. Recurrent and transient sets. For the further development we require a few elementary facts about *partitioned matrices*. As was stated in the introduction, part of the results could be obtained from the ergodic theory of stochastic matrices. It is simpler and more natural to derive all required facts in one sweep purely analytically. The present method opens a new access to the ergodic theory. The following terminology will, perhaps, appear artificial, but it comes closest to established usage in probability theory.

DEFINITION 11. *A set $R \subseteq E$ is called indecomposable-recurrent if it is a sojourn set, but no proper subset of R is a sojourn set. A set is called recurrent if it is the union of indecomposable-recurrent sets. A point i is called recurrent if there exists a recurrent set R such that $i \in R$. A point which is not recurrent is transient.*

Warning. The set C_i which consists of the single point i obviously is recurrent if, and only if, $\Pi(i, i) = 1$. Thus i can be recurrent without C_i being recurrent. It is hoped that this will cause less confusion than would the introduction of two new terms.

THEOREM 11.1. *In order that r be recurrent, it is necessary and sufficient that⁽¹¹⁾*

$$(11.1) \quad s_E(r) = 1, \quad s_{E-r}(r) = 0.$$

PROBABILITY INTERPRETATION. For a random walk starting at r the relation $s_E(r) = 1$ attributes probability zero to the event that the random walk terminates after finitely many steps. On the other hand, $s_{E-r}(r) = 0$ means that the probability of only finitely many returns to r is zero. Thus, according to Theorem 11.1, *the point r is recurrent if and only if there is probability one that, starting from r , the random walk leads infinitely often back to r .* Our definition therefore agrees with the definition used in probability theory.

Proof. (1) **Necessity.** Consider first the special case where E is indecomposable-recurrent. By Theorem 8 the set where $s_E(i) > 1 - \eta$ is a sojourn set, and since no proper subset of E is a sojourn set we have $s_E(i) = 1$ for all i . Furthermore, $E - i$ is not a sojourn set, and therefore $s_{E-i} = 0$ for each i . Thus (11.1) holds for all points. Moreover, since $s_E = 1$ we see that Π is *strictly stochastic*.

Turning to the general case, let r be a fixed recurrent point. By definition there exists an *indecomposable-recurrent* set R such that $r \in R$. Applying our last conclusion to Π_R instead of Π we see first of all that Π_R is *strictly stochastic*. Therefore

$$(11.2) \quad \Pi_R(i, R) = 1, \quad \Pi_R(i, E - R) = 0 \quad \text{for } i \in R.$$

This means that Π is of the form of a *partitioned* matrix

$$(11.3) \quad \Pi = \begin{pmatrix} \Pi_R & 0 \\ M & \Pi_{E-R} \end{pmatrix}.$$

When the whole space is indecomposable-recurrent we have shown that $s_E(j) = 1$, $s_{E-i}(j) = 0$ for all i, j . Using the notation of §6 we can write the corresponding equations for our subspace R in the form

$$(11.4) \quad s_R(i) = 1, \quad s_{R-i}^R(j) = 0, \quad i \in R, j \in E.$$

Applying (6.3) we get at once

$$(11.5) \quad s_R(i) = 1, \quad s_{R-i}(j) = 0, \quad i \in R, j \in E.$$

Thus $s_E(i) = 1$. For $r \in R$ we have from (11.3) clearly $s_{E-R}(r) = 0$, and thus

⁽¹¹⁾ We denote the complement of r by $E - r$ rather than by the more correct $E - \{r\}$.

$s_{E-i}(r) \leq s_{R-i}(r) + s_{E-R}(r) = 0$ for each $i \in R$. This statement includes (11.1).

(2) **Sufficiency.** To each $r \in E$ define a set $R \subset E$ as follows: $k \in R$ if and only if

$$(11.6) \quad \Pi^n(r, k) > 0 \quad \text{for some } n \geq 0.$$

For $k \in R$ and $j \in E - R$ we have obviously $\Pi(k, j) = 0$ and so $\Pi(k, E - R) = 0$ for each $k \in R$. This means that Π is again a partitioned matrix of the form (11.3), but in general Π_R will not be strictly stochastic.

We begin with the following simple remark (which will be used also in the proof of Theorem 11.3). *If $s_E(r) = 1$ for one $r \in R$, then Π_R is strictly stochastic* (so that (11.2) holds). In fact, for each n we have

$$(11.7) \quad 1 = s_E(r) = \sum_{j \in R} \Pi^n(r, j) s_E(j);$$

the right side can equal unity only if each positive $\Pi^n(r, j)$ is multiplied by unity. From the definition (11.6) then $s_E(k) = 1$ for each $k \in R$ and thus $\Pi_R \mathbf{1} = \mathbf{1}$.

Now to the proof that (11.1) is sufficient. If (11.1) holds for some r , then Π_R is strictly stochastic and so R is a sojourn set. We have to prove: *if $A \subset R$ is a sojourn set, then $A = R$* . Assume the contrary and choose $k \in R - A$. Then $R - k \supset A$ is a sojourn set. Choose n so that (11.6) holds. Then

$$(11.8) \quad \sigma_{R-k}(r) = \sum_j \Pi_{R-k}^n(r, j) \sigma_{R-k}(j) \leq \Pi_{R-k}^n(r, R - k) \leq 1 - \Pi^n(r, k) < 1.$$

By Theorem 8 the set B of all j such that $\sigma_{R-k}(j) > \sigma_{R-k}(r)$ is a sojourn set, and r is not in B . Therefore $R - r \supset B$ is a sojourn set, and hence there exists an $i \in R - r$ such that $s_{R-r}(i) > 0$. Now

$$(11.9) \quad s_{R-r}(r) = \sum_j \Pi^m(r, j) s_{R-r}(j) \geq \Pi^m(r, i) s_{R-r}(i).$$

By an appropriate choice of m the last term can be made positive. Thus $s_{R-r}(r) > 0$ against assumption, and the proof is completed.

THEOREM 11.2. *The space E is partitioned into mutually nonoverlapping sets T, R_1, R_2, \dots as follows: T is the set of all transient points; each R_n is indecomposable-recurrent. Each submatrix R_n satisfies (11.2). One has*

$$(11.10) \quad s_{R_n}(i) = \delta_{n,m} \quad \text{for } i \in R_m$$

where $\delta_{n,m}$ is the Kronecker symbol. For each sojourn set A

$$(11.11) \quad s_A = s_{A \cap T} + s_{A \cap R}$$

where $R = \bigcup R_n$ is the set of recurrent points. Furthermore

$$(11.12) \quad s_{A \cap T}(i) = 0 \quad \text{for } i \in R;$$

$$(11.13) \quad s_{A \cap T} = \Pi_T s_{A \cap T}, \quad s_{A \cap R} = \sum_{R_n \subset A} s_{R_n}.$$

Proof. In the sufficiency proof to Theorem 11.1 we have constructed the set R which is obviously the *unique* indecomposable-recurrent set containing the recurrent point r . It follows that no two indecomposable-recurrent sets can overlap. It is then clear that Π is again of the form (11.3), where $E - R = T$ and Π_R is completely partitioned

$$(11.14) \quad \Pi_R = \begin{pmatrix} R_1 & 0 & \cdots \\ 0 & R_2 & \cdots \\ \cdot & \cdot & \cdot \end{pmatrix}.$$

The remaining assertions are immediate consequences of this and the definitions.

The main implication of Theorem 11.2 is that in the study of a sojourn set A we need only worry about the transient part $A \cap T$. Since $s_{A \cap T}$ vanishes on R we are in substance dealing with the submatrix Π_T only. In other words, *we have reduced our problem to the case where all points are transient*. (Each indecomposable-recurrent class contributes one single element to \mathfrak{P} .)

We proceed to the most important

THEOREM 11.3. *Let the sojourn set A be transient. Then there exists a sequence of sets A_n equivalent to A such that*

$$(11.15) \quad s_{A_n} = s_A, \quad A \supset A_1 \supset A_2 \cdots, \quad A_n \rightarrow 0.$$

COROLLARY. *The complement of any finite set in A is equivalent to A .*

Proof. It suffices to prove that A and $A - r$ are equivalent for each r . If $\sigma_A(r) < 1$ then this statement is amply contained in Theorem 8. Accordingly, assume $\sigma_A(r) = 1$. Define the set R as above by the property (11.6). As has been remarked in the sufficiency proof of Theorem 11.1, the relation $\sigma_A(r) = 1$ implies that Π_R is *strictly stochastic*, so that Π is of the form (11.3). Also $A \supset R$ and we have only to prove that $R - r$ is equivalent to R .

Since Π_R is strictly stochastic, R is a sojourn set. It contains a sojourn set B as a proper subset, for otherwise r would be recurrent. By Lemma 8.1 we can choose B so that $s_R = s_B + s_{R-B}$. If $r \in R - B$ then $R - r \supset B$ is a sojourn set and the statement is proved. If $r \in B$ then $\sigma_B(r) < 1$. In fact, choose $k \in R - B$ and choose n so that (11.6) holds. Then $\sigma_B(r) \leq 1 - \Pi^n(r, k) < 1$. By Theorem 8 the set B' of all $i \in B$ such that $\sigma_B(i) > \sigma_B(r)$ is a sojourn set equivalent to B . Therefore $B - r$ is equivalent to B . But then

$$(11.16) \quad s_R = s_{B-r} + s_{R-B} \leq s_{R-r} \leq s_R$$

and so R and $R - r$ are equivalent.

12. The discrete part of the boundary.

DEFINITION 12.1. *A sojourn solution $s_A \neq 0$ corresponding to a transient⁽¹²⁾*

⁽¹²⁾ The restriction to transient sets is introduced only to avoid clumsy formulations and trivialities.

set A is called minimal if $s_B \leq s_A$ implies that $s_A = s_B$ or $s_B = 0$. A sojourn solution s_A is continuous if there exists no minimal sojourn solution $s_B \leq s_A$.

Clearly if s_A is minimal and $x \leq s_A$ (where $x \in \mathfrak{P}$), then $x = ts_A$.

LEMMA 12.1. *There exist at most denumerably many minimal sojourn solutions $s^{(1)}, s^{(2)}, \dots$.*

Proof. For two distinct minimal solutions one has $s^{(i)} \cap s^{(k)} = 0$, by the very definition. Theorem 8 shows that we may choose sojourn sets S_n so that

$$(12.1) \quad s_{S_n} = s^{(n)}$$

and that the S_n are mutually nonoverlapping, in fact so that $s_{S_n}(i) > 1 - \epsilon_i$ for $i \in S_n$.

DEFINITION 12.2. *We enlarge the space E by adding for each $s^{(n)}$ a new point γ_n . In the set $E + \Gamma$ thus obtained we introduce a topology as follows. The set Ω is open if for each $\gamma_n \in \Omega$ there exists a sojourn set $S_n \subset \Omega$ such that (12.1) holds.*

According to this definition each subset of E is open.

THEOREM 12.1. *With this topology $E + \Gamma$ is a Hausdorff space. Each of the subsets E and Γ in itself has the discrete topology⁽¹³⁾.*

Proof. If $s_A = s_B = s^{(n)}$ then also $s_{A \cap B} = s^{(n)}$ by Theorem 9.2. Thus the union of open sets and the intersection of finitely many open sets are again open. That γ_n and γ_m ($n \neq m$) have nonoverlapping neighborhoods was shown above in the proof to Lemma 12.1. Clearly each point $i \in E$ represents an open set, and each γ is relatively open in Γ .

THEOREM 12.2. *In order that*

$$(12.2) \quad x = \sum \alpha_n s^{(n)}$$

be an element of \mathfrak{P} it is necessary and sufficient that $0 \leq \alpha_n \leq 1$. Then

$$(12.3) \quad x(i) \rightarrow \alpha_n \quad \text{as} \quad i \rightarrow \gamma_n, \quad i \in E.$$

If \mathfrak{P} contains no continuous sojourn solutions⁽¹⁴⁾ and all points of E are transient then each $x \in \mathfrak{P}$ is of the form (12.2).

Proof. If $0 \leq \alpha_n \leq 1$ then $x \in \mathfrak{P}$. When all the α_n are bounded, (12.2) represents a bounded solution of $\Pi x = x$. Relation (12.3) follows then from the fact that there exists a neighborhood of γ_n in which $s^{(n)}(i) \geq 1 - \epsilon$, and hence

⁽¹³⁾ It will be noticed that the recurrent part of E in no way influences the boundary. For a better understanding consider the case where E is indecomposable-recurrent, that is, where E is the only sojourn set. Then \mathfrak{S} contains the unique element $\mathbf{1}$. If one added a boundary point γ , it would have no neighborhood except the whole space, and thus we would not have a Hausdorff space.

⁽¹⁴⁾ By Theorem 11.2 each indecomposable-recurrent set in E contributes exactly one trivial sojourn solution, and these should be added to (12.2) in the general case.

$s_E(i) - s^{(n)}(i) \leq \epsilon$. The necessity of the condition $0 \leq \alpha_n \leq 1$ is a trivial consequence of this. Finally, given an arbitrary $x \in \mathfrak{B}$, we have

$$(12.4) \quad \alpha_n s^{(n)} = s^{(n)} \cap x.$$

With this definition of α_n the right side of (12.2) is $\leq x$, and clearly the difference is a continuous solution.

Once the boundary is defined we may say that $s^{(n)}(i)$ is the probability that, starting from i , the random walk will asymptotically approach the boundary point γ_n .

13. The maximal ideal space. We wish to define the boundary so that the sojourn solutions $s \in \mathfrak{S}$ will stand in one-to-one correspondence with certain sets of boundary points. In anticipation of the final outcome these sets may be described as sets of positive measure or capacity⁽¹⁵⁾. For each continuous sojourn solution (Definition 12.1) $s \in \mathfrak{S}$ there exist null-sequences such that $s \geq s_1 \geq \dots$, $s_n \rightarrow 0$, and such a sequence corresponds to a monotone sequence of sets of the boundary. The intersection may be an arbitrary set of measure zero, and not necessarily a point. It seems therefore hopeless to define points of the boundary directly in terms of sequences of sojourn solutions. Instead, we adapt to our purposes ideas widely used in representation theory.

We recall (cf. [1]) that a set J of sojourn solutions is a *lattice ideal*⁽¹⁶⁾ in \mathfrak{S} if the following is true: whenever $x \in J$ and $y \in J$ then also $x \cup y \in J$ and $z \in J$ where $z \in \mathfrak{S}$ and $z \leq x$. An ideal is *maximal* if \mathfrak{S} is the only ideal containing J as a proper subset, and $J \neq \mathfrak{S}$. There exist maximal ideals whenever \mathfrak{S} contains more than one element.

DEFINITION 13. Suppose that all points of E are transient⁽¹⁷⁾. Let \mathfrak{B} be the set of maximal ideals. In the space $E + \mathfrak{B}$ we introduce a topology as follows. A set $\Omega \subset E + \mathfrak{B}$ is open if to each maximal ideal $\omega \in \Omega$ there corresponds an element $s_A \in \mathfrak{S}$ such that $s_A \notin \omega$ and $A \subset \Omega$, and moreover Ω contains each maximal ideal β such that s_{E-A} .

Before proceeding it is necessary to show that the discrete boundary introduced in §12 is really part of the boundary \mathfrak{B} as defined in the last definition and that there is no contradiction between the two definitions of open sets.

⁽¹⁵⁾ Formula (12.2) expresses the value $x(i)$ as an integral over the boundary values α_n with respect to the measure which attaches weight $s^{(n)}(i)$ to the point γ_n . This is a representation of the Green-function type. An abstract representation of this type is possible also in the most general case, using the Krein-Milman theorem. For this purpose it is not necessary to introduce a boundary, but only a measure defined on \mathfrak{S} . See, for example, the excellent representation in Choquet [3, Chapter 7].

⁽¹⁶⁾ For our purposes it would be more natural to use dual ideals instead of ideals.

⁽¹⁷⁾ Alternatively, replace in the sequel \mathfrak{S} by the sublattice of sojourn solutions corresponding to transient sets, that is, of sojourn solutions such that $s_A = s_{A_n}$ where $A_n \downarrow 0$. Cf. the footnote to Theorem 12.1.

LEMMA 13.1. *The two definitions 13 and 12.2 are mutually consistent.*

Proof. Let s_A be a *minimal* sojourn solution and define J as the set of all elements $x \in \mathfrak{S}$ such that $x \cap s_A = 0$. It is obvious that $s_A \notin J$, that J is a maximal ideal, and that J but no other maximal ideal contains $s_{E-A} = s_E - s_A$. It follows then that the set $A + J$ is open according to either of the definitions.

For the proof that $E + \mathfrak{B}$ is a Hausdorff space we require two lemmas.

LEMMA 13.2. *For any maximal ideal either $s_A \in J$ or $s_{E-A} \in J$.*

Proof. Suppose that $s_A \notin J$ and define I as the set of all elements of the form $x \cup y$, where $x \in J$ and $y \leq s_A$. It is clear that I is an ideal and that $J \subset I$. As J is maximal we have $I = \mathfrak{S}$, and therefore $s_{E-A} \in I$. This means that $s_{E-A} = x \cup y$ with $x \in J$ and $y \leq s_A$. Now $y \cap s_{E-A} = (x \cup y) \cap y = y$ or $y \leq s_{E-A}$. Therefore $y \leq s_A \cap s_{E-A} = 0$ or $s_{E-A} = x \in J$ as asserted.

LEMMA 13.3. *Let J be a maximal ideal and suppose that $s_A \notin J$ and $s_B \notin J$. Then $s_{A \cap B} \notin J$.*

Proof. In accordance with Lemma 8.2 we may assume without loss of generality that the sets A and B have been chosen so that

$$s_E = s_A + s_{E-A} = s_B + s_{E-B}.$$

Then, by Lemma 4.1

$$(13.1) \quad s_A \cap s_B = s_E - s_{E-A} \cup s_{E-B}.$$

Now in consequence of the preceding lemma both s_{E-A} and s_{E-B} are in J , and therefore $u = s_{E-A} \cup s_{E-B} \in J$. Again, $s_E = u \cup (s_E - u)$, and therefore $u \in J$ implies that $s_E - u \notin J$. A recall of Theorem 9.2 now completes the proof.

THEOREM 13.1. *The space $E + \mathfrak{B}$ of Definition 13 is a Hausdorff space.*

Proof. The union of open sets is obviously open. We prove that the intersection of two open sets Ω_1 and Ω_2 is open. Every subset of E is open and therefore there is nothing to be proved in case $\Omega_1 \cap \Omega_2 \subset E$. Suppose then that there exists a maximal ideal $\omega \in \Omega_1 \cap \Omega_2$, and let $A_i \subset \Omega_i$ be a set as described in the Definition 13. Then $s_{A_1} \notin \omega$ and $s_{A_2} \notin \omega$, and so $s_{A_1 \cap A_2} \notin \omega$ by Lemma 13.3. Therefore $s_{A_1 \cap A_2} \neq 0$, and $A_1 \cap A_2$ is a sojourn set contained in $\Omega_1 \cap \Omega_2$. Finally, if β is a maximal ideal such that $s_{E-A_1 \cap A_2} \in \beta$, then a fortiori $s_{E-A_1} \in \beta$ and $s_{E-A_2} \in \beta$, so that $\beta \in \Omega_1 \cap \Omega_2$. This proves that $\Omega_1 \cap \Omega_2$ is open.

To verify the separation postulate, let α and β be two distinct maximal ideals. Then there exists a sojourn solution $s_A \in \alpha$ such that $s_A \notin \beta$. By Lemma 13.2 we have in this case $s_{E-A} \in \beta$. Again, the set A is determined only up to an equivalence class and may be chosen so that $s_E = s_A + s_{E-A} = s_A \cup s_{E-A}$ (Lemma 8.2). Then no maximal ideal can contain both s_A and s_{E-A} . Thus if we put $\Omega_1 = (E - A) + \text{all maximal ideals containing } s_A$, and similarly $\Omega_2 = A + \text{all maximal ideals containing } s_{E-A}$, then Ω_1 and Ω_2 are nonoverlapping open sets such that $\alpha \in \Omega_1$ and $\beta \in \Omega_2$. If α is maximal ideal and $i \in E$, then i is its

own open neighborhood and α has a neighborhood excluding i by virtue of Theorem 11.3⁽¹⁸⁾.

THEOREM 13.2. *Let $s_A \in \mathfrak{S}$, and let ω be a boundary point (maximal ideal). Then, as $i \rightarrow \omega$ (where $i \in E$)*

$$(13.2) \quad s_A(i) \rightarrow \begin{cases} 1 & \text{if } s_A \notin \omega \\ 0 & \text{if } s_A \in \omega. \end{cases}$$

Proof. Among the equivalent sets defining s_A choose A so that (8.9) holds. If $s_A \notin \omega$ choose as a neighborhood of ω the set Ω_2 defined in the preceding proof. Then $s_A(i) > 1 - \eta$ for $i \in \Omega_2 \cap E$, and this proves the first relation in (13.2). The second follows on replacing A by $E - A$.

We see thus that each sojourn solution s_A can be extended to a *continuous function* on the whole space $E + \mathfrak{B}$. On the boundary such a function assumes only the values 0 and 1. Finite linear combinations of sojourn solutions induce on the boundary \mathfrak{B} continuous functions assuming only finitely many values. We call such functions *step-functions*. The last theorem is now supplemented by

THEOREM 13.3. *Each $x \in \mathfrak{B}$ has continuous boundary values.*

This is an immediate consequence of:

THEOREM 13.4. *Each $x \in \mathfrak{B}$ can be approximated uniformly in E by finite linear combinations of sojourn solutions.*

In other words: in order that a function $f(\beta)$ defined on \mathfrak{B} represent boundary values of some $x \in \mathfrak{B}$ it is necessary and sufficient that $0 \leq f \leq 1$ and that f be the uniform limit of step functions.

The proof of Theorem 13.4 will be based on the following lemma which will be used again for the proof of Theorem 15.1. With a view to this latter application the lemma is formulated so as to cover also the case of *unbounded solutions*.

LEMMA 13.4. *Let $x \geq 0$ be a solution of $\Pi x = x$ and put $X = \{i: x(i) > \eta\}$ where η is a positive constant. Then*

$$(13.3) \quad x \geq \eta s_X.$$

Moreover, letting $s_A = s_E - s_X$, one has

$$(13.4) \quad x \cap s_A \leq \eta s_A.$$

NOTE. If x is bounded and $\eta < \|x\|$, then X is a sojourn set and $s_X \neq 0$. For an unbounded x it may happen that $s_X = 0$ for all η ,

⁽¹⁸⁾ Note that sojourn solutions corresponding to recurrent sets have been excluded only to establish this point.

Proof. Clearly $x \geq \eta \delta_X$. Premultiplying by Π^n and letting $n \rightarrow \infty$ we obtain (13.3).

Next put $x \cap s_A = y$ and suppose for the moment that $\|y\| > \eta$. By Lemma 7.2 the set $Y = \{i: y(i) > \eta\}$ is a sojourn set and applying (13.3) we see that $y \geq \eta s_Y$. Now $y \leq s_A$ and therefore $s_Y \leq s_A$. This, however, is impossible, for $y \leq x$ implies that $Y \subset X$, whence $s_Y \leq s_X$ and therefore $s_Y \leq s_A \cap s_X = 0$. It follows that $\|y\| \leq \eta$ and hence $y \leq \eta \cdot 1$. Premultiplying by Π^n and letting $n \rightarrow \infty$ we get $y \leq \eta s_E$, that is, $x \cap s_A \leq \eta \cdot s_E$. This is the same as (13.4), and the lemma is proved.

Proof of Theorem 13.4. Without loss of generality assume that $\|x\| = 1$. Choose $\eta = 1/2$ and define the sets X and A as in the proof of Lemma 13.4. By this lemma

$$(13.5) \quad \frac{1}{2}s_X \leq x \leq s_X + \frac{1}{2}s_A$$

or

$$(13.6) \quad 0 \leq x - \frac{1}{2}s_X \leq \frac{1}{2}s_E.$$

Thus $x - 2^{-1}s_X \in \mathfrak{P}$ and $\|x - 2^{-1}s_X\| \leq 1/2$. Applying the same procedure to $x - 2^{-1}s_X$ we get a linear combination $L = 2^{-1}s_X + \alpha s_Y$ such that $x - L \in \mathfrak{P}$, and $\|x - L\| \leq 1/4$, etc.

14. Unbounded solutions. Isomorphisms. It has been remarked at the end of §4 that *non-negative* unbounded solutions of $\Pi x = x$ enjoy lattice properties similar to those of bounded solutions. We shall now outline a new approach to the theory of unbounded solutions which has analytical advantages and important probabilistic implications. It has a close analogue in a familiar transformation of the Sturm-Liouville differential equations, although this analogy is hidden by the altogether different formalism. For simplicity of formulations we shall consider only strictly stochastic matrices, that is, we assume $\Pi 1 = 1$.

Let z be a strictly positive⁽¹⁹⁾ solution of $\Pi z = z$ and define a matrix Π' by

$$(14.1) \quad \Pi'(i, j) = \Pi(i, j) \frac{z(j)}{z(i)}.$$

Clearly Π' is again strictly stochastic and for its powers one has

$$(14.2) \quad \Pi'^n(i, j) = \Pi^n(i, j) \frac{z(j)}{z(i)}.$$

⁽¹⁹⁾ The restriction to strictly positive solutions is introduced only to simplify formulations and represents no serious loss of generality. For, if $z \geq 0$ is a solution which is not strictly positive, let R be the set of all i such that $z(i) = 0$. It is obvious that Π is *partitioned* in the form (11.3). All the matrices similar to Π in the sense of Definition 14 will be of the same partitioned form and there will be a one-to-one correspondence between the solutions of $\Pi x = x$ and $\Pi' x' = x'$ which vanish on R .

If \mathbf{x} is any non-negative solution of $\Pi\mathbf{x} = \mathbf{x}$, then the vector \mathbf{x}' defined by

$$(14.3) \quad x'(i) = \frac{x(i)}{z(i)}$$

is a non-negative solution of $\Pi'\mathbf{x}' = \mathbf{x}'$.

NOTATION. We denote by \mathfrak{M} the aggregate of all strictly positive solutions of $\Pi\mathbf{x} = \mathbf{x}$, and by \mathfrak{M}' the corresponding set for Π' .

DEFINITION 14. Two strictly stochastic matrices Π and Π' will be called similar if they stand in the relationship (14.1), where $\mathbf{z} \in \mathfrak{M}$. The vector \mathbf{x}' defined in (14.3) will be called the (canonical) image of \mathbf{x} .

We have now the obvious but important

THEOREM 14.1. The similarity is a symmetric and transitive relationship. The transformation (14.3) is a one-to-one mapping of \mathfrak{M} onto \mathfrak{M}' which is a lattice isomorphism, that is, for $\mathbf{x}, \mathbf{y} \in \mathfrak{M}$

$$(14.4) \quad (\mathbf{x} \cap \mathbf{y})' = \mathbf{x}' \cap \mathbf{y}', \quad (\mathbf{x} \cup \mathbf{y})' = \mathbf{x}' \cup \mathbf{y}'.$$

In particular, the vector \mathbf{z} itself is mapped into $\mathbf{z}' = \mathbf{1}$. In this way an arbitrary positive solution of $\Pi\mathbf{z} = \mathbf{z}$ can be made to play the rôle of the unit solution. It is possible to introduce notions of relative boundedness, relative sojourn solutions, etc. However, we can proceed in a more direct manner.

THEOREM 14.2. Let both \mathbf{z} and \mathbf{x} be sojourn solutions. Then the image \mathbf{x}' is a sojourn solution if $\mathbf{x} \leq \mathbf{z}$, and is unbounded otherwise.

Proof. To begin with, let $\mathbf{x} \leq \mathbf{z}$. From (9.1) we have $\mathbf{x} \cap (\mathbf{z} - \mathbf{x}) = \mathbf{0}$, and therefore $\mathbf{x}' \cap (\mathbf{1} - \mathbf{x}') = \mathbf{0}$. By Theorem 9.1 this is equivalent to the assertion that \mathbf{x} is a sojourn solution. Next, suppose that $\mathbf{x} \leq \mathbf{z}$ does not hold. Then there exists a nonzero sojourn $\mathbf{y} \leq \mathbf{1} - \mathbf{z}$ such that $\mathbf{x} \geq \mathbf{y}$. By Theorem 8 there exists a sojourn set Y such that $y(i) > 1 - \epsilon$ for $i \in Y$. Now $x'(i) > \epsilon^{-1}(1 - \epsilon)$ for $i \in Y$ and \mathbf{x}' is therefore unbounded.

We see thus that when \mathbf{z} is a sojourn solution, the boundary of E defined by the matrix Π' consists of a part of the boundary defined by Π . On the other hand, if \mathbf{z} is not a sojourn solution then the sojourn solutions of $\Pi'\mathbf{x}' = \mathbf{x}'$ need not be images of the sojourn solutions of $\Pi\mathbf{x} = \mathbf{x}$. In the next section it will be shown that nevertheless there exists a simple connection between the boundaries of all similar stochastic matrices.

Probability interpretation. For simplicity let us begin with the case where \mathbf{z} is a minimal sojourn solution (Definition 12.1). To \mathbf{z} there corresponds an isolated boundary point γ and, for the random walk defined by Π , we know that $z(i)$ represents the probability of an ultimate asymptotic approach to γ if the initial position is i . It is then clear that $\Pi'(i, j)$, as defined in (14.1) represents the conditional one-step transition probability from i to j evaluated

on the hypothesis of an ultimate asymptotic approach to γ . In statistical terminology: out of a sample of mutually independent random walks obeying Π we consider the subpopulation of those which ultimately land at γ . The process defined in this way is the same as a random walk with transition probabilities Π' . It is now clear that in this new process the boundary point γ will be approached with probability one, so that γ must be the unique boundary point for Π' .

Next consider the case where $\mathbf{z} = p_1 \mathbf{z}_1 + p_2 \mathbf{z}_2$, where the \mathbf{z}_i are minimal sojourn solutions and $p_i > 0$. If $p_1 = p_2 = 1$, then \mathbf{z} is again a sojourn solution, and the above probabilistic interpretation requires only a slight rephrasing. In particular, the boundary of Π' will now consist of the two boundary points γ_1 and γ_2 corresponding to \mathbf{z}_1 and \mathbf{z}_2 . If $p_1 \neq p_2$ different weights are attached to the two boundary points: if a particle ultimately approaches γ_i it has probability $p_i/(p_1 + p_2)$ to belong to our subpopulation defining Π' .

Obviously a similar probabilistic interpretation can be given in the most general case. Using the notions of boundary and real valued functions on this boundary our description requires only a trivial rephrasing.

We conclude this section by two theorems which, though of considerable interest will not be used in the sequel. The second illustrates the power of the present method.

THEOREM 14.3. *If the point i is transient [recurrent] for the matrix Π , then it is transient [recurrent] for each similar matrix Π' .*

Proof. It is known (see [6, Chapter 15, §5]), that i is transient if, and only if, $\sum_n \Pi^n(i, i) < \infty$. A glance at (14.2) completes the proof.

THEOREM 14.4. *Suppose that E is indecomposable-recurrent⁽²⁰⁾. Then $x(i) = \text{const.}$ is the only non-negative solution of $\Pi \mathbf{x} = \mathbf{x}$.*

Proof. By assumption no proper subset of E is a sojourn set, and therefore by Lemma 7.2 there can exist no nonconstant bounded solution $\mathbf{x} \geq 0$. (That $\mathbf{x} = 1$ is a solution follows from Theorem 11.1.) Suppose now that $\mathbf{x} > 0$ is an unbounded solution of $\Pi \mathbf{x} = \mathbf{x}$, and consider the matrix Π' of (14.1) with $\mathbf{z} = 1 + \mathbf{x}$. Let \mathbf{u}'_1 and \mathbf{u}'_2 be the vectors with components

$$u'_1(i) = \frac{1}{1 + x(i)}, \quad u'_2(i) = \frac{x(i)}{1 + x(i)}.$$

Then \mathbf{u}'_1 and \mathbf{u}'_2 are independent bounded solutions of $\Pi' \mathbf{u}' = \mathbf{u}'$. But this is impossible, since E is indecomposable-recurrent not only relative to Π , but relative to Π' as well.

15. Relatively maximal ideals. The total boundary. The preceding considerations show that, probabilistically and analytically, similar matrices

⁽²⁰⁾ Cf. Definition 11. The character of E remains unchanged when Π is replaced by a similar matrix.

are closely related. Theorem 14.2 establishes in some cases an obvious connection between parts of the boundaries \mathfrak{B} and \mathfrak{B}' defined by two similar matrices Π and Π' . It is easily seen by the same method (and will be shown in a different way) that in general the boundaries Π and Π' have a common part (defined by solutions such that both \mathbf{x} and \mathbf{x}' are bounded) but that each boundary contains points which do not have an image on the other.

It will now be shown that it is possible, and natural, using the totality \mathfrak{M} of all positive solutions of $\Pi\mathbf{x}=\mathbf{x}$, to introduce a boundary \mathfrak{B}^* which is the same for all matrices similar to Π ; the boundary \mathfrak{B} as introduced in §13 by means of bounded solutions will be a subset of \mathfrak{B}^* , and $E+\mathfrak{B}$ embedded in $E+\mathfrak{B}^*$.

The notion of sojourn set can be formulated invariantly for the whole family of similar matrices (Definition 15.2), but the notion of sojourn solution has no intrinsic meaning. The procedure of §13 requires therefore two modifications. Instead of ideals in \mathfrak{S} we have to use ideals in \mathfrak{M} , but here we change the definition so as not to distinguish between \mathbf{x} and the scalar multiples $p\mathbf{x}$. Moreover, in \mathfrak{S} we had maximal ideals because \mathfrak{S} has a maximal element (lattice unit) \mathbf{s}_E . Maximal ideals in \mathfrak{M} need not exist, and are in any case not usable for our purposes. We use a *relative maximality*.

DEFINITION 15.1. A subset $I \subset \mathfrak{M}$ will be called an ideal in \mathfrak{M} if

- (1) $\mathbf{x} \cup \mathbf{y} \in I$, whenever $\mathbf{x}, \mathbf{y} \in I$;
- (2) $\mathbf{z} \in I$, whenever $\mathbf{z} \leq \mathbf{x}$, $\mathbf{x} \in I$, $\mathbf{z} \in \mathfrak{M}$;
- (3) $p\mathbf{x} \in I$, whenever $p \geq 0$, $\mathbf{x} \in I$.

The ideal I will be called maximal relative to $\mathbf{u} \in \mathfrak{M}$ if $\mathbf{u} \notin I$ but $\mathbf{u} \in I_1$ for each ideal I_1 properly containing I .

It will be noticed that the transformation (14.3) takes an ideal which is maximal relative to \mathbf{u} into an ideal maximal relative to \mathbf{u}' . In this sense the notion of relatively maximal ideals refers to the family of similar matrices rather than to an individual matrix. Note that if I is maximal relative to both \mathbf{u} and \mathbf{v} then it is maximal also relative to $\mathbf{u} \cup \mathbf{v}$ and to $\mathbf{u} \cap \mathbf{v}$.

We proceed to prove the existence of relatively maximal ideals and to describe them. If Π is strictly stochastic, we may simplify the language by taking $\mathbf{u} = \mathbf{1}$. In fact, the transformation (14.3) permits us to reduce the general case to this apparently special case.

THEOREM 15.1. Let Π be strictly stochastic, and I an ideal in \mathfrak{M} maximal relative to $\mathbf{1}$. Let J be the set of all sojourn solutions in I , that is, $J = I \cap \mathfrak{S}$. Then: (a) J is a maximal ideal in \mathfrak{S} . (b) in order that an element $\mathbf{x} \in \mathfrak{M}$ belong to I it is necessary and sufficient that to each $\epsilon > 0$ there exist a sojourn solution \mathbf{s} such that

$$(15.1) \quad \mathbf{x} \cap \mathbf{s} < \epsilon \mathbf{s} \quad \mathbf{s} \in J.$$

(c) Conversely, if J is an arbitrary maximal ideal in \mathfrak{S} and I is defined as the

set of all $x \in \mathfrak{M}$ which satisfy condition (b), then I is an ideal in \mathfrak{M} , maximal relative to 1.

Proof. (1) Obviously J is an ideal in \mathfrak{S} and we have to prove that it is maximal. Let J_1 be any ideal in \mathfrak{S} such that $J \subset J_1$. If J is a proper subset of J_1 there exists an element s such that $s \in J_1$ but $s \notin J$. Denote by I_1 the set of all elements of \mathfrak{M} of the form $x \cup y$, where $x \in I$ and $y \leq ps$. Clearly I_1 is an ideal, and $I \subset I_1$. Now I is maximal relative to 1, and therefore x and y can be chosen so that $x \cup y = 1$. We have then⁽²¹⁾

$$(15.2) \quad 1 - s = (x \cup y) \cap (1 - s) = \{(1 - s) \cap x\} \cup \{(1 - s) \cap y\}.$$

But $y \leq ps$ and so $(1 - s) \cap y = 0$ by Theorem 9.2. Accordingly, (15.2) reduces to $1 - s = (1 - s) \cap x$ or $1 - s \leq x$. Now $x \in I$ and since I is an ideal, we have $1 - s \in I$ and therefore $1 - s \in J$. Again, $J_1 \subset J$, so that $1 - s \in J_1$. It is seen that J_1 contains both s and $1 - s$, and therefore also $s \cup (1 - s) = 1$. Thus $J_1 = \mathfrak{S}$, and J is maximal as asserted.

(2) Next let J be an arbitrary maximal ideal in \mathfrak{S} and define I by the property (b). We show that I is an ideal. Requirements (2) and (3) of Definition 15.1 are trivially fulfilled. To verify that (1) holds, choose $x, y \in I$. By (15.1) there exist two sojourn solutions s_A and s_B such that

$$(15.3) \quad x \cap s_A \leq \epsilon \cdot s_A, \quad y \cap s_B \leq \epsilon \cdot s_B$$

and $s_A \notin J$, $s_B \notin J$. By Lemma 13.3 this implies that also $s_A \cap s_B \notin J$. On the other hand, using Theorem 9.2 we get from (15.3)

$$(15.4) \quad x \cap s_A \cap s_B \leq \epsilon \cdot s_A \cap s_B, \quad y \cap s_A \cap s_B \leq \epsilon \cdot s_A \cap s_B$$

and therefore (using the distributive law as in (15.2))

$$(15.5) \quad (x \cup y) \cap s_A \cap s_B = (x \cap s_A \cap s_B) \cup (y \cap s_A \cap s_B) \leq \epsilon \cdot s_A \cap s_B.$$

Thus $x \cup y \in I$ and so I is an ideal in \mathfrak{M} .

(3) We show that I is maximal relative to 1. Let $x \in \mathfrak{M}$ be an arbitrary element such that $x \notin I$. By definition this means that there exists an $\eta > 0$ such that for each sojourn solution s_A

$$(15.6) \quad \text{if } x \cap s_A \leq \eta \cdot s_A \text{ then } s_A \in J.$$

Now define s_x and s_A as in Lemma 13.4. A comparison of (15.6) and (13.4) shows that $s_A \in J$ (possibly $s_A = 0$). On the other hand, $1 = s_A + s_x = s_A \cup s_x$, and therefore $s_x \notin J$. Moreover, s_x satisfies (13.3). In other words, if $x \notin I$, then there exist sojourn solutions s_A and s_x such that

$$(15.7) \quad x \geq \eta s_x, \quad s_x \notin I, \quad s_A \in I, \quad s_x + s_A = 1.$$

⁽²¹⁾ The distributive law used in (15.2) holds in every vector-lattice; see [1, Theorem 7.6] or [2].

(This result will be used in the proof of Theorem 15.2.)

To complete the proof, consider an arbitrary ideal I_1 in \mathfrak{M} such that $I_1 \supset J$ and $1 \notin I_1$. Then $s_A \in I_1$ and consequently $s_x \in I_1$. It follows now from the first inequality in (15.7) that $x \in I_1$. We see thus that $x \in I$ implies $x \in I_1$, and hence $I_1 \subset I$. This proves the asserted maximality of I relative to 1, and also that if I is maximal relative to 1 it is necessarily of the form described in the theorem.

We propose to define a boundary \mathfrak{B}^* by a procedure analogous to that of §13, except that ideals in \mathfrak{M} are to be used instead of ideals in \mathfrak{S} . For that purpose it is necessary to define an intrinsic analogue to sojourn sets.

DEFINITION 15.2. *Let Π be strictly stochastic and let z be an arbitrary element of \mathfrak{M} . A set $A \subset E$ will be called carrier of z if for the matrix Π' of (14.1) the sets A and E are equivalent.*

Spelled out in detail and without reference to the transformation (14.1) our definition amounts to the following. As in §3 it is seen that

$$(15.8) \quad \zeta(i) = \lim_{n \rightarrow \infty} \Pi_A^n(i, j)z(j), \quad i \in A;$$

$$\zeta(i) = 0, \quad i \notin A,$$

always defines a vector $\zeta \leq z$ (the limit being attained monotonically). Similarly

$$(15.9) \quad z^* = \lim_{n \rightarrow \infty} \Pi^n \zeta$$

exists, and $z^* \leq z$. The set A is carrier of z if and only if $z^* = z$.

It will be noticed that if A is a carrier for z , then it is also carrier of the image z' of z for any similar matrix Π' . Moreover, if E is transient, then by Theorem 11.3 there exists a sequence $A_1 \supset A_2 \supset \cdots$, $\cap A_n = 0$, of carriers of z . Each sojourn set A is, of course, carrier of s_A .

We let \mathfrak{B}^* stand for the set of all ideals in \mathfrak{M} which are maximal relative to some $x \in \mathfrak{M}$.

DEFINITION 15.3. *For a strictly stochastic Π and a transient⁽²²⁾ E we introduce in $E + \mathfrak{B}^*$ a topology as follows.*

A set $\Omega \subset E + \mathfrak{B}^$ is open if to each point $\omega \in \Omega \cap \mathfrak{B}^*$ there corresponds a vector $x \in \mathfrak{M}$ such that*

- (a) *The ideal ω is maximal relative to x .*
- (b) *The set Ω contains a subset of E which is carrier of x .*
- (c) *The set $\Omega \cap \mathfrak{B}^*$ contains each $\beta \in \mathfrak{B}^*$ which is maximal relative to some $u \leq x$.*

THEOREM 15.2. *The space $E + \mathfrak{B}^*$ of Definition 15.3 is a Hausdorff space.*

⁽²²⁾ As in Definition 13, the assumption that all points be transient is introduced for convenience of formulations only. Note that E is transient with respect to any matrix similar to Π (Theorem 14.3).

Proof. The union of open sets is trivially open. Also, every subset of E is open. To prove that the intersection of two open sets Ω_1 and Ω_2 is open, consider an element $\omega \in \mathfrak{B}^*$ common to Ω_1 and Ω_2 . Let \mathbf{x} and \mathbf{y} be the corresponding elements described in the definition. In view of the isomorphisms described in Theorem 14.1 there is no loss of generality (but only change of notation) in supposing that $\mathbf{x} \cup \mathbf{y} = 1$. Then ω is maximal relative to 1. We know from (15.7) that there exist two sojourn sets X and Y such that

$$(15.10) \quad \mathbf{x} \geq \eta s_X, \quad \mathbf{y} \geq \eta s_Y, \quad s_X \notin \omega, \quad s_Y \notin \omega.$$

By the definition there exist two sets $A \subset \Omega_1$ and $B \subset \Omega_2$ which are carriers of \mathbf{x} and \mathbf{y} , respectively. Now $\|\mathbf{x}\| \leq 1$ and it is seen from the definition of a carrier set (see (15.8)–(15.9)) that $\mathbf{x} \leq s_A$. Hence $s_X \leq s_A$ by (15.10). Furthermore, the set X is defined only up to an equivalence, and in view of Theorem 9.2 we may choose $X \subset A$. A similar argument holds for Y . We have therefore

$$(15.11) \quad X \subset \Omega_1, \quad Y \subset \Omega_2,$$

and obviously X and Y are carriers of s_X and s_Y , respectively. As before, let $J = \omega \cap \mathfrak{S}$ be the set of all sojourn solutions contained in ω . Theorem 15.1 shows that J is a maximal ideal. Then $s_X \notin J$ and $s_Y \notin J$ by (15.10), and hence $s_{X \cap Y} \notin J$ by Lemma 13.3. Now $J \subset \omega$, and thus $s_{X \cap Y} \notin J$. The set $X \cap Y$ is a carrier of $s_{X \cap Y}$, and is contained in the intersection of Ω_1 and Ω_2 (see (15.11)). Thus $\Omega_1 \cap \Omega_2$ satisfies the conditions (a) and (b) of Definition 15.3 with $s_{X \cap Y}$ playing the rôle of \mathbf{x} . Condition (c) is trivially satisfied since if β is maximal relative to some $\mathbf{u} \leq s_{X \cap Y}$ then by definition $\beta \in \Omega_1$ and $\beta \in \Omega_2$ and so $\beta \in \Omega_1 \cap \Omega_2$.

It remains to prove that to any two points of $E + \mathfrak{B}^*$ one may find two nonoverlapping neighborhoods. By Theorem 11.3 this is trivial unless both points are in \mathfrak{B}^* . Let $\omega_1 \neq \omega_2$ be two points of \mathfrak{B}^* . If ω_i is maximal relative to \mathbf{x}_i , then automatically ω_i is maximal relative to $\mathbf{x}_1 \cup \mathbf{x}_2$. As before there is no loss of generality in assuming that $\mathbf{x}_1 \cup \mathbf{x}_2 = 1$. In other words, it suffices to verify the separation property for two elements ω_i both of which are maximal with respect to 1.

According to Theorem 15.1 the intersections $J_i = \omega_i \cap \mathfrak{S}$ are two distinct maximal ideals in \mathfrak{S} . As in the proof of Theorem 13.1 there exist two sojourn sets A_1 and $A_2 = E - A_1$, such that $s_{A_i} \in \omega_i$, but $s_{A_1} \notin \omega_2$ and $s_{A_2} \notin \omega_1$. Moreover, $s_{A_1} + s_{A_2} = 1$. Let $\Omega_i = A_i +$ all elements of \mathfrak{B}^* which are maximal relative to some $\mathbf{x} \leq s_{A_i}$. Then Ω_i is an open set containing ω_i . If $\omega \in \mathfrak{B}^*$ is maximal with respect to some $\mathbf{x} \leq s_{A_1}$, then there exists an element $s_A \leq s_{A_1}$ such that ω is maximal relative to s_A (see (15.7)). Now the ideal $J = \omega \cap \mathfrak{S}$ does not contain s_A and hence it contains s_{A_2} by Lemma 13.2. Accordingly ω can not be maximal relative to any element $\mathbf{y} \leq s_{A_1}$, and so $\omega \in \Omega_1$ implies $\omega \notin \Omega_2$. Then Ω_1 and Ω_2 are nonoverlapping, and the theorem is proved.

The points of \mathfrak{B}^* are ideals in \mathfrak{M} and therefore \mathfrak{B}^* formally depends on

the matrix Π . However, if Π is replaced by a similar matrix Π' then the image I' of an ideal I is again an ideal and it is clear that the boundary as such remains unchanged. Therefore we have

THEOREM 15.3. *Each matrix of the family \mathfrak{F} induces the same boundary \mathfrak{B}^* and the same topology.*

The space $E + \mathfrak{B}$ of Definition 13 is an open set in $E + \mathfrak{B}^$.*

In other words, \mathfrak{B}^* may be considered as the union of the boundaries, constructed by means of bounded solutions, for all matrices similar to the given matrix Π .

16. The adjoint boundary. Instead of considering Π as an operator on column vectors, we now consider the dual operator. That is, if $\alpha = \{\alpha(1), \alpha(2), \dots\}$ is a *row vector*, we consider the new row vector $\alpha\Pi$, provided it is meaningful. This is certainly the case whenever $\sum |\alpha(j)| < \infty$.

We are interested in particular in the *eigenvectors* $\alpha = \alpha\Pi$, that is, the solutions of (1.3). To the set \mathfrak{B} there correspond the eigenvectors satisfying the conditions

$$(16.1) \quad \alpha(j) \geq 0, \quad \sum \alpha(j) \leq 1,$$

to \mathfrak{M} all eigenvectors such that $\alpha(j) \geq 0$. It is well known (see, for example [6, p. 329]) that in the case of a matrix Π which is not partitioned (i.e., not of the form (11.3)) there exists at most one eigenvector α satisfying (16.1). However, Derman [5] has shown that there may exist many unbounded positive eigenvectors.

If α is any strictly positive⁽²⁾ solution of $\alpha = \alpha\Pi$ we define a new matrix

$$(16.2) \quad \Pi^*(i, j) = \frac{\alpha(j)}{\alpha(i)} \Pi(j, i)$$

which is strictly stochastic. In the case where α is a probability vector, $\Pi^*(i, j)$ is simply the transition probability of the *time reversed random walk*, or the inverse probability to Π . This notion has been introduced by Kolmogorov [7] (see also [6, p. 321]). To Derman [5] is due the simple, but important, observation that the transformation (16.2) remains probabilistically meaningful and useful even if $\sum \alpha(j) = \infty$.

Suppose now that there exists a second strictly positive solution β of $\beta = \beta\Pi$. Put

$$(16.3) \quad x^*(i) = \frac{\beta(i)}{\alpha(i)}.$$

Then

$$(16.4) \quad \Pi^* x^* = x^*, \quad x^* > 0.$$

(²) Cf. the footnote to §14.

Conversely, if (16.4) has a solution $x^* > 0$, then the vector \mathfrak{g} defined by (16.3) is a solution of $\mathfrak{g} = \mathfrak{g}\Pi$. Also

$$(16.5) \quad \frac{\beta(j)}{\beta(i)} \Pi(j, i) = \Pi^*(i, j) \frac{x^*(j)}{x^*(i)}.$$

We have thus

THEOREM 16. *If α runs through the set of all positive solutions of $\alpha = \alpha\Pi$, then the matrix Π^* of (16.2) runs through a family of similar matrices. Conversely, each matrix similar to Π^* is of the form (16.5) where \mathfrak{g} is a positive solution of $\mathfrak{g} = \mathfrak{g}\Pi$.*

It is thus seen that the study of positive solutions of $\alpha = \alpha\Pi$ is reduced to that of solutions $x = \Pi x$.

DEFINITION 16. *The boundary of E as defined in §15 by the similar matrices Π^* will be called the adjoint boundary defined by Π .*

The probabilistic meaning is given inasmuch as (16.2) corresponds to a time-reversal for the random walk.

17. Examples.

I. *Unbounded solutions:* the unsymmetric one-dimensional random walk. Changing the notation slightly, we let i run through all integers, and put

$$(17.1) \quad \Pi(i, i+1) = p, \quad \Pi(i, i-1) = q, \quad -\infty < i < \infty$$

where $p > q > 0$, $p+q=1$. Clearly $\Pi x = x$ has the unique bounded solution $x=1$, and the unique unbounded solution defined by $x(i) = (q/p)^i$. Each set $i \geq a$ is a carrier set for the former, and $i \leq a$ for the latter. The boundary \mathfrak{B} consists of a single point, \mathfrak{B}^* of two. The equation $\alpha = \alpha\Pi$ has the two unbounded solutions $\alpha(i) \equiv 1$, and $\alpha(i) = (p/q)^i$. *The two topologies coincide in this case.*

II. *Symmetric random walk in $N \geq 3$ dimensions.* It has been proved by Murdoch [9] that in this case $x=1$ is the only non-negative solution of $\Pi x = x$. Moreover, $\alpha = \alpha\Pi$ has the solution $\alpha(i) = 1$, and with this solution the matrices Π and Π^* are identical. Both the boundary and the adjoint boundary reduce to a single point each and again the two topologies are identical.

III. We show that *the boundary and the adjoint boundary need have no relationship to each other.*

Relabeling the points, let E consist of the points A_n , C_n ($n \geq 0$), and B_n ($n \geq 1$). The reader is asked to draw a diagram with the points ordered on three parallel lines A , B , C . We define

$$(17.2) \quad \begin{aligned} \Pi(A_n, A_{n+1}) &= \Pi(C_n, C_{n+1}) = 1 - \epsilon_n, & n \geq 0, \\ \Pi(A_n, B_n) &= \Pi(C_n, B_n) = \epsilon_n, & n \geq 1, \\ \Pi(B_n, A_{n-1}) &= \Pi(B_n, C_{n-1}) = 1/2, & n \geq 1 \end{aligned}$$

where

$$(17.3) \quad \epsilon_0 = 0, \quad \epsilon_i > 0 \ (i \geq 1), \quad \sum \epsilon_i < \infty.$$

The equations $\Pi \mathbf{x} = \mathbf{x}$ admit of the two linearly independent solutions

$$(17.4) \quad \begin{aligned} x_{1,2}(A_n) &= \frac{1}{2} \pm \frac{1}{2} \sum_{k=n}^{\infty} (1 - \epsilon_k), & x_{1,2}(B_n) &= \frac{1}{2}, \\ x_{1,2}(C_n) &= \frac{1}{2} \mp \frac{1}{2} \sum_{k=n}^{\infty} (1 - \epsilon_k). \end{aligned}$$

There exists no solution, bounded or unbounded, of $\Pi \mathbf{x} = \mathbf{x}$ linearly independent of \mathbf{x}_1 and \mathbf{x}_2 .

The boundary $\mathfrak{B} = \mathfrak{B}^*$ consists of two points α and γ ; a set is a neighborhood of α if it contains α and all points A_n with $n \geq N$. Intuitively α and γ represent the "points at infinity" of the lines A and C , respectively, and \mathbf{x}_1 gives the probability that the random walk will be ultimately constrained to the line A . The set $E - B_1 - B_2 - \dots$ is equivalent to E .

On writing down the adjoint equations $\alpha = \alpha \Pi$ one notices immediately that $\alpha(C_n) = \alpha(A_n)$ and hence that $\alpha = \alpha \Pi$ has a solution which is unique up to a scalar multiplier. This solution is

$$(17.5) \quad \begin{aligned} \alpha(A_0) &= \alpha(C_0) = 1, \\ \alpha(A_n) &= \alpha(C_n) = \frac{1}{2\epsilon_n} \alpha(B_n) = \frac{(1 - \epsilon_1) \cdots (1 - \epsilon_{n-1})}{\epsilon_1 \cdots \epsilon_n}. \end{aligned}$$

Now the inverse matrix is

$$(17.6) \quad \begin{aligned} \Pi^*(A_n, B_{n+1}) &= \Pi^*(C_n, B_{n+1}) = 1 - \epsilon_n, \\ \Pi^*(A_n, A_{n-1}) &= \Pi^*(C_n, C_{n-1}) = \epsilon_n, \\ \Pi^*(B_n, A_n) &= \Pi^*(B_n, C_n) = 1/2. \end{aligned}$$

The uniqueness of the solution of $\alpha = \alpha \Pi$ shows that the adjoint boundary consists of a single point. From (17.6) it is clear that with probability one the random walk will pass infinitely often through each of the lines A, B, C . Thus the adjoint boundary consists of a unique point: to each neighborhood of it there exists an integer N such that the neighborhood contains all points A_n, B_n, C_n with $n \geq N$. That is to say, each neighborhood contains the complete neighborhood of two boundary points of \mathfrak{B} plus infinitely many other points.

IV. *Continuous solutions.* Imagine the points of E ordered according to a dyadic branching scheme and labelled accordingly as 0, 1, 00, 01, 10, \dots . Generally, if δ stands for any finite sequence of zeros and ones, then δ represents a point of E and we define

$$(17.7) \quad \Pi(\delta, \delta 0) = \Pi(\delta, \delta 1) = 1/2.$$

Obviously the set of all points starting with δ is a sojourn set, and the

corresponding sojourn solution is easily written down. For example, if $\delta = 0110$ then s equals 1 at all points δ , $\delta 0$, $\delta 1$, etc. It equals $1/2$, $1/4$, $1/8$ respectively at the points 011 , 01 , and 0 and s equals 0 at all other points.

Clearly there are no minimal sojourn solutions, so *all sojourn solutions are continuous*. We have seen that to each dyadic interval of $(0, 1)$ there corresponds a sojourn solution. Taking linear combinations it is readily seen that *the sojourn solutions are in one-to-one correspondence with sets of positive Lebesgue measure in $(0, 1)$* so that to sets on $(0, 1)$ which are equivalent in the sense of Lebesgue there corresponds the same element $s \in \mathfrak{S}$. Likewise, *the elements $x \in \mathfrak{P}$ are in one-to-one correspondence with the measurable functions on $(0, 1)$ which are non-negative and whose essential upper bound does not exceed unity*.

In this particular case it is natural to define a boundary isomorphic to the interval $(0, 1)$. The boundary which we have defined in terms of maximal ideals is larger: on it the boundary values of $x \in \mathfrak{P}$ are continuous instead of merely being bounded.

REFERENCES

1. G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, vol. 25, New York; 1940.
2. N. Bourbaki, *Éléments de mathématique*. Première partie. *Les structures fondamentales de l'analyse*, Paris, Hermann, 1939–1955.
3. G. Choquet, *Theory of capacities*, Annales de l'Institut Fourier vol. 5 (1955) pp. 131–295.
4. C. Derman, *A solution to a set of fundamental equations in Markov chains*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 332–334.
5. ———, *Some contributions to the theory of Markov chains*, Thesis, and Technical Report No. 7, Columbia University, Department of Mathematical Statistics, 1955.
6. W. Feller, *An introduction to probability theory and its applications*, New York, Wiley, 1950.
7. A. Kolmogoroff, *Zur Theorie der Markoffschen Ketten*, Math. Ann. vol. 112 (1935) pp. 155–160.
8. R. S. Martin, *Minimal positive harmonic functions*, Trans. Amer. Math. Soc. vol. 49 (1941) pp. 137–172.
9. B. H. Murdoch, *Preharmonic functions*, Princeton thesis, 1954.
Added in proof.
10. D. Blackwell, *On transient Markov processes with a countable number of states and stationary transition probabilities*, Ann. Math. Statist. vol. 26 (1955) pp. 654–658.

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