GENERALIZED RANDOM VARIABLES(1)

BY

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I. Introduction and survey of the paper

1. If $P=[U, \mathfrak{U}, u]$ is a probability space(3) and $\mathfrak{B}$ is the σ-field(4) of all Borel subsets of the real line, S, and f is a real valued measurable function, then the map f^{-1} is a σ-algebra homomorphism of $\mathfrak{B}$ into $\mathfrak{U}$ and induces a σ-algebra homomorphism, F, of $\mathfrak{B}$ into the measure ring, $\mathfrak{M}$, of P, where $\mathfrak{M}$ is the quotient σ-algebra, $\mathfrak{U}$ modulo the ideal, $\mathfrak{N}$, of sets of measure zero. This map F from $\mathfrak{B}$ into the measure ring of P does not change if we vary f on a set of measure zero and is in some mathematical circumstances easier to deal with than the point function f, and is more basic from some conceptual statistical viewpoints. Consequently, following Segal [8] we make the following

Definition 1. Let $\mathfrak{B}$ be a Boolean σ-algebra and let $\mathfrak{M}$ be the measure ring of a probability space P. Then a $\mathfrak{B}$ *measurable generalized random variable* on P is a σ-algebra homomorphism, F, of $\mathfrak{B}$ into $\mathfrak{M}$. If $\mathfrak{B}$ is a σ-field of subsets of a set S then F is said to be *S-valued.*

In the event S is the dual of a topological vector space B, and $\mathfrak{B}$ is the σ-algebra of weak star measurable subsets of S (Definition 4 below) a notion of *integrable* generalized random variable is definable. A theory of integration for such random variables can be developed which is in some respects smoother than the theory for ordinary vector valued point functions. In particular, if B is metrizable, one can characterize the indefinite integrals of S-valued, $\mathfrak{B}$ measurable, generalized random variables; whereas, if B is not separable, it is not known how to characterize the indefinite integrals of S-valued, $\mathfrak{B}$ measurable, point functions. (A point function f is $\mathfrak{B}$ measurable means $f^{-1}(b)\in\mathfrak{U}$ for all $b\in\mathfrak{B}$.) If B is separable, then every S-valued, $\mathfrak{B}$ measurable generalized random variable is induced by a point function. Thus

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(1) This is substantially the same as the author's thesis submitted to the faculty of the University of Chicago in partial fulfillment of the requirements for the Ph.D. degree, September 2, 1955.

(2) The author is indebted to Professor I. E. Segal for suggesting this area of research as well as for his continued interest and encouragement during its completion.

(3) A probability space is an ordered triple $[U, \mathfrak{U}, u]$ where U is a set, $\mathfrak{U}$ is a σ-field of subsets of U, and u is a non-negative countably additive real valued function defined on $\mathfrak{U}$ such that $u(U)=1$.

(4) A field is a Boolean Algebra $\mathfrak{U}$ of subsets of a set U in which the Boolean operations are the usual set theoretic operations. If $\mathfrak{U}$ is a field, then it is a σ-field provided that whenever $A_n\in\mathfrak{U}$, $n=1, 2, \cdots$, then $A\in\mathfrak{U}$, where A is the set theoretic union of the A_n.

if B is separable we do have a characterization of the indefinite integrals of S-valued $\mathcal{B}$ measurable point functions. This latter result is related to earlier work of J. Dieudonné [2] on the Lebesgue-Nikodym theorem. A definitive Radon-Nikodym type theorem of a somewhat different nature than ours has been obtained by Rickart [7].

If B is a metrizable space, then our characterization of the indefinite integrals of generalized random variables permits a natural definition of conditional expectation appropriate to such variables. If B is a Banach space, this same characterization justifies natural definitions of the Cartesian product and sum of two generalized random variables.

We note here that every theorem in the sequel which applies to real topological vector spaces applies also to complex spaces. The reader will have no difficulty in making the necessary modifications in the proofs.

II. Generalized random variables induced by point functions

2. We first inquire: When is an S-valued $\mathcal{B}$ measurable generalized random variable, F, on a probability space P induced by a point function f as described above? Our knowledge concerning this question is quite incomplete. In this section we will state some relevant results of R. Sikorski and prove a theorem of our own.

We note that there is a distinction between the question which starts this section and the question: When is a σ-algebra homomorphism h of $\mathcal{B}$ into the σ-algebra $\mathfrak{U}$ of measurable subsets of U, induced by a point function? In the former case we are interested in knowing when a σ-algebra homomorphism into a certain *quotient σ-algebra* is induced by a point function (namely the quotient algebra $\mathfrak{U}/\mathfrak{N}$, where $\mathfrak{N}$ is the σ-ideal of sets of measure zero) whereas in the latter case we want to know when a σ-algebra homomorphism into a certain *σ-algebra of sets* is induced by a point function. To the latter question Sikorski supplies a satisfactory answer, but in order to state the answer we require a definition.

Definition 2. If $x \in S$, then the class of all sets $A \in \mathcal{B}$, which do not contain x, is a maximal σ-ideal in $\mathcal{B}$. Such a maximal σ-ideal in $\mathcal{B}$ will be said to be *induced by a point* in S.

We now state two results of Sikorski [9], namely: *Fact* 1. In order that every σ-algebra homomorphism h of $\mathcal{B}$ into an arbitrary σ-field $\mathfrak{U}$ of subsets of a set U be induced by a point function on U into S, it is necessary and sufficient that every maximal σ-ideal in $\mathcal{B}$ be induced by a point in S.

As a corollary he obtains *Fact* 2. If S is a metric space and $\mathcal{B}$ is the σ-field of all Borel subsets of S and the cardinality of S is not too great, namely if it is less than the first cardinal inaccessible in the strict sense(5), then every

(5) A cardinal $p = \aleph_\lambda > \aleph_0$ is called *inaccessible in the weak sense*, if λ is a limit ordinal, and if the condition $p_t < p$ implies $\sum p_t < p$ provided t runs over a set T of cardinality less than p. If moreover $m^n < p$ for every $m < p$ and $n < p$ then p is said to be *inaccessible in the strict sense*. See A. Tarski [11, p. 69].

σ-homomorphism h of $\mathcal{B}$ into an arbitrary σ-field of subsets of a set U, is induced by a point function of U into S.

We now return to the question with which we commenced this section. We know nothing for the case in which $\mathcal{B}$ is not countably generated. For the case in which $\mathcal{B}$ is countably generated we state a theorem of Sikorski's and then derive one of our own. It is convenient first to state a definition of Sikorski's.

DEFINITION 3. A topological space is called a *Borel space* if it is homeomorphic to a Borel subset of the Hilbert parallelotope(6).

We now state a result of Sikorski [9]. *Fact* 3. In order that every σ-homomorphism h of a countably generated σ-field $\mathcal{B}$ of subsets of a set S into an arbitrary σ-quotient algebra $\mathfrak{U}/\mathfrak{N}$ (of a set U) be induced by a point function on U into S it is necessary and sufficient that $\mathcal{B}$ be isomorphic to the σ-field of all Borel subsets of a Borel space.

Before applying Fact 3 to prove a theorem of our own it will be convenient to make a definition.

DEFINITION 4. Let S be the dual of a topological vector space B. For each $x \in B$ and $\phi \in S$, let (x, ϕ) be the value of ϕ at the point x and let $(x, \cdot)$ be the linear functional on S determined by x. By the *weak star measurable subsets of* S we mean the least σ-field, $\mathcal{B}$, of subsets of S, such that for all $x \in B$, $(x, \cdot)$ is measurable relative to $\mathcal{B}$.

THEOREM 1. *Let $\mathcal{B}$ be the weak star measurable subsets of the dual S of a real separable metrizable topological vector space B, and let h be a σ-algebra homomorphism of $\mathcal{B}$ into an arbitrary σ-quotient algebra $\mathfrak{U}/\mathfrak{N}$ (of a set U). Then h is induced by a point function on U into S.*

It is convenient to state as our first lemma, a well known theorem. We first require two well known definitions.

DEFINITION 5. Let $\mathcal{E}$ be any collection of subsets of a set Y. By $s(\mathcal{E})$ we mean the smallest σ-ring containing $\mathcal{E}$.

DEFINITION 6. For every class $\mathcal{E}$ of subsets of Y and every fixed subset Z of Y, we shall denote by $\mathcal{E} \cap Z$ the class of all sets of the form $E \cap Z$ with E in $\mathcal{E}$.

LEMMA 1. *If*(7) *$\mathcal{E}$ is any class of subsets of Y and if Z is any subset of Y,*

$$s(\mathcal{E}) \cap Z = s(\mathcal{E} \cap A).$$

Our next lemma is probably well known. Since we know of no reference we supply its proof.

LEMMA 2. *Let $\mathcal{B}(Y)$ be the Borel subsets of a topological space Y and let Z*

(6) The Hilbert parallelotope is the collection of all sequences $\alpha = \{\alpha_i\}$ of real numbers such that $|\alpha_i| \leq 1/i$.

(7) For a proof see Halmos [4, p. 25].

be a Borel subset of Y. Let $\mathcal{B}(Z)$ be the class of all Borel subsets of Z, i.e., the σ-field of subsets of Z generated by the relatively closed subsets of Z. Let $\mathcal{A}(Z)$ be the class of all subsets of Z which are elements of $\mathcal{B}(Y)$. Then $\mathcal{B}(Z)=\mathcal{A}(Z)$.

Proof of Lemma 2. Let $\mathcal{E}$ be the class of all closed subsets of Y. $\mathcal{E}\cap Z$ is the class of all relatively closed subsets of Z. Therefore $\mathcal{B}(Z)=s(\mathcal{E}\cap Z)$. An application of Lemma 1 yields

$$\mathcal{B}(Z) = s(\mathcal{E}) \cap Z.$$

Since $s(\mathcal{E})=\mathcal{B}(Y)$ it follows that

$$\mathcal{B}(Z) = \mathcal{B}(Y) \cap Z.$$

The lemma now follows from the observation that since Z is a Borel subset of Y,

$$\mathcal{B}(Y) \cap Z = \mathcal{A}(Z).$$

LEMMA 3. *Let Y be the Cartesian product of the real line with itself a countable number of times. Let $\mathcal{B}_1$ be the least σ-algebra of subsets of Y such that each projection mapping is measurable and let $\mathcal{C}_1$ be the collection of all Borel subsets of Y. Then $\mathcal{B}_1=\mathcal{C}_1$.*

Proof of Lemma 3. It is trivial that $\mathcal{B}_1\subset\mathcal{C}_1$. To show that $\mathcal{C}_1\subset\mathcal{B}_1$ it suffices to show that every open subset of Y is an element of $\mathcal{B}_1$. Since Y is second countable it suffices to show that every element of some base for the open subsets of Y is an element of $\mathcal{B}_1$. The inverse images of open sets under the projection mappings form a subbase and are certainly in $\mathcal{B}_1$. Since $\mathcal{B}_1$ is closed under finite intersections, the base generated by this subbase is a subcollection of $\mathcal{B}_1$.

LEMMA 4. *Let S be the dual of a real separable metrizable topological vector space B and let $\mathcal{C}$ be the σ-algebra of subsets of S generated by the weak star closed subsets. Then $\mathcal{C}$ is isomorphic to the σ-algebra of all Borel subsets of a Borel space.*

Proof of Lemma 4. Let x_i be a countable dense subset of B and let Y be as in Lemma 3. Let T be the mapping of S into Y defined by letting the ith coordinate of $T(\phi)$ be (x_i, ϕ) for all $\phi\in S$. Then T is easily seen to be a one-one continuous map of S into Y. There exists a countable base for the neighborhoods of the origin in B. The polars(8) of these neighborhoods are a countable set, K_j, of weak star compact subsets of S whose union is(9) S. Since $T(K_j)$ is compact and since

$$T(S) = T\left(\bigcup_{j=1}^{\infty} K_j\right) = \bigcup_{j=1}^{\infty} (T(K_j)),$$

(8) The polar of a subset W of B is the set of all $\phi\in S$ such that $|(x, \phi)|\leqq 1$.

(9) For a proof see [1].

it follows that $T(S)$ is a σ-compact subset of Y and therefore certainly is a Borel subset of Y. Since T is continuous, every relatively closed subset of $T(S)$ is the image under T of a weak star closed subset of S. It follows by a standard argument that every Borel subset of $T(S)$ is the image under T of a weak star Borel subset of S. Conversely, given any $b \in \mathcal{C}$, we show that $T(b)$ is a Borel subset of $T(S)$. For, since $UK_j = S$ it easily follows that

$$T(b) = \bigcup_{j=1}^{\infty} (T(b \cap K_j)).$$

Furthermore, by Lemma 2, $b \cap K_j$ is a Borel subset of K_j. Since T cut down to K_j is a homeomorphism it follows that $T(b \cap K_j)$ is a Borel subset of $T(K_j)$. Then applying Lemma 2 again, we see that $T(b \cap K_j)$ is a Borel subset of $T(S)$. Therefore $T(b)$ is the union of a countable number of Borel subsets of $T(S)$, and is therefore also a Borel subset of $T(S)$. It is now trivial to complete the proof that the map

$$b \to T(b)$$

is an isomorphism of the σ-algebra, $\mathcal{C}$, with the σ-algebra of all Borel subsets of $T(S)$. The proof of Lemma 4 is completed by observing that Y is a Borel space and any Borel subset of a Borel space is likewise.

LEMMA 5. *The collection, $\mathcal{B}$, of weak star measurable subsets of the dual S of a real separable metrizable topological vector space B is the same as the σ-algebra, $\mathcal{C}$, generated by the weak star closed subsets of S.*

Proof of Lemma 5. It is trivial to verify that $\mathcal{B} \subset \mathcal{C}$. We proceed to show that $\mathcal{C} \subset \mathcal{B}$. Let T be the mapping of S into Y defined as in Lemma 4. Our lemma will be proved once we show that $T(\mathcal{C}) \subset T(\mathcal{B})$. In the course of proving Lemma 4, it was shown that $T(\mathcal{C})$ is the collection of all Borel subsets of $T(S)$. Therefore, applying Lemma 2, we see that $T(\mathcal{C}) = \mathcal{C}_1 \cap T(S)$, where $\mathcal{C}_1$ is as defined in Lemma 3. Then by Lemma 3, we get $T(\mathcal{C}) = \mathcal{B}_1 \cap T(S)$. Then by Lemma 1, $\mathcal{B}_1 \cap T(S)$ is the least σ-algebra of subsets of $T(S)$ such that each projection mapping $\{\alpha_i\} \to \alpha_i$ is measurable, when the mapping is restricted to $T(S)$. Since it is easy to check that each such projection mapping is measurable relative to $T(\mathcal{B})$ it follows that $\mathcal{B}_1 \cap T(S) \subset T(\mathcal{B})$. Therefore $T(\mathcal{C}) \subset T(\mathcal{B})$. This completes the proof.

LEMMA 6. *Let $\mathcal{B}$ be the weak star measurable subsets of the dual, S, of a real separable metrizable topological vector space B and let x_i be a countable dense subset of B. Then $\mathcal{B}$ is the least σ-algebra of subsets of S with respect to which all x_i are measurable.*

Proof of Lemma 6. It is only necessary to show that if $x \in B$ and all x_i are measurable relative to a σ-algebra of subsets of S then so is $(x, \cdot)$. Since x_i are dense in B, there exists a subsequence which converges to x. Therefore there

exists a subsequence of the measurable functions $(x_i, \cdot)$ which converges pointwise to the function $(x, \cdot)$. But the pointwise limit of a sequence of measurable functions is likewise measurable.

Proof of Theorem 1. From Lemma 6, it easily follows that $\mathcal{B}$ is countably generated. By Lemmas 4 and 5 we see that $\mathcal{B}$ is isomorphic to the σ-algebra of all Borel subsets of a Borel space. The theorem now follows from Fact 3 above.

III. Extending σ-algebra homomorphisms

3. It is not in general true that a σ-algebra homomorphism defined on a field of subsets of a set into an arbitrary σ-quotient algebra can be extended to the generated σ-field so as to be a σ-algebra homomorphism[10]. However, with suitable restrictions on the σ-quotient algebra, the result is true. In particular we have the following:

THEOREM 2. *Every σ-algebra homomorphism of a field $\mathcal{R}$ of subsets of a set S into the measure ring of a probability space possesses a unique extension to a σ-algebra homomorphism of the σ-field $\mathcal{B}$ generated by $\mathcal{R}$.*

Proof of Theorem 2. The uniqueness of the extension is trivial. We proceed with the proof of existence. Let h be a σ-algebra homomorphism of $\mathcal{R}$ into the measure algebra $\mathfrak{M}$ of a probability space and let m be the measure on $\mathfrak{M}$. Since the measure algebra $\mathfrak{M}$ of a probability space is complete as a partially ordered set we can first extend h to every subset A of S as follows.

$$\bar{h}(A) = \bigwedge \left[\bigvee_{i=1}^{\infty} h(R_i) \right] \tag{1}$$

where the inf, $\bigwedge$, is taken over all countable sequences $\{R_i\}$ such that $R_i \in \mathcal{R}$ and such that

$$A \subset \bigcup_{i=1}^{\infty} R_i. \tag{2}$$

We first verify in a straightforward way that $\bar{h}$ is indeed an extension of h. Let $R \in \mathcal{R}$. We show that $\bar{h}(R) \leqq h(R)$. Let R_i equal R for all $i = 1, 2, \cdots$. Therefore $\bigvee h(R_i) = h(R)$. Thus $h(R)$ is one of the terms over which the infimum on the right side of (1) is taken. Therefore $\bar{h}(R) \leqq h(R)$. We now show that $\bar{h}(R) \geqq h(R)$. It suffices to show that for every sequence $\{R_i\}$ over which the infimum in (1) is taken, $h(R) \leqq \bigvee h(R_i)$. If R_i is such a sequence then $R \subset \bigcup R_i$. In this event $R = \bigcup (R \cap R_i)$. Since h is countably additive on $\mathcal{R}$, $h(R) = \bigvee h(R \cap R_i)$, which in turn is clearly $\leqq \bigvee h(R_i)$. We have now shown that $\bar{h}$ equals h on $\mathcal{R}$.

We next show that $\bar{h}$ on $\mathcal{B}$ preserves complements and countable unions. Let A be any subset of S. We will show that

[10] For a counterexample see Sikorski [10, p. 13].

$$\bar{h}(A) \vee \bar{h}(A') = e \tag{3}$$

where e is the unit in the measure ring $\mathfrak{M}$.

By definition of $\bar{h}$

$$\bar{h}(A) \vee \bar{h}(A') = \left\{ \wedge \left[\bigvee_{i=1}^{\infty} h(R_i) \right] \right\} \vee \left\{ \wedge \left[\bigvee_{j=1}^{\infty} h(S_j) \right] \right\} \tag{4}$$

where the first inf, $\wedge$, is taken over all sequences $\{R_i\}$ such that $R_i \in \mathfrak{R}$, $A \subset \bigcup_{i=1}^{\infty} R_i$ and the second inf, $\wedge$, is taken over all sequences $\{S_j\}$ such that $S_j \in \mathfrak{R}$, $A' \subset \bigcup_{j=1}^{\infty} S_j$.

Now since any complete Boolean algebra is infinitely distributive we have from (4)

$$\bar{h}(A) \vee \bar{h}(A') = \wedge \left\{ \left[\bigvee_{i=1}^{\infty} h(R_i) \right] \vee \left[\bigvee_{j=1}^{\infty} h(S_j) \right] \right\}, \tag{5}$$

where $\wedge$ is taken over all pairs of sequences $\{R_i\}$, $\{S_j\}$ such that

$$A \subset \bigcup R_i \text{ and } A' \subset \bigcup S_j.$$

The right side of (5) is easily seen to be equal to

$$\wedge \left\{ \bigvee_{i,j} \left[h(R_i) \vee h(S_j) \right] \right\}. \tag{6}$$

Since h is a homomorphism (6) equals

$$\wedge \left\{ \bigvee_{i,j} h(R_i \cup S_j) \right\}. \tag{7}$$

Further, since, h is a σ-homomorphism on $\mathfrak{R}$ and since the unit S of $\mathfrak{R}$ equals $\bigcup_{i,j} (R_i \cup S_j)$ we have $e = h(S) = \bigvee_{i,j} h(R_i \cup S_j)$. Therefore (3) is established.

We now wish to show that for all $A \in \mathfrak{B}$

$$\bar{h}(A) \cap \bar{h}(A') = 0. \tag{8}$$

In order to prove (8) and the fact that $\bar{h}$ on $\mathfrak{B}$ preserves countable unions we define a probability measure on S as follows:

First for each $R \in \mathfrak{R}$ we define

$$v(R) = m(h(R)) \tag{9}$$

where m is the measure on the measure ring $\mathfrak{M}$. It is straightforward to verify that v is countably additive on $\mathfrak{R}$. Thus as is well known v may be extended in a unique manner to be a probability measure on $\mathfrak{B}$. We now wish to show that

$$m \circ \bar{h}(A) \leqq v(A) \qquad \text{for every } A \in \mathfrak{B}. \tag{10}$$

Before proving (10) we first show that (10) is sufficient to complete the proof of the theorem. We return to the proof of (8). By (10) we get

$$m \circ \bar{h}(A) + m \circ \bar{h}(A') \leqq v(A) + v(A') = v(A \cup A') = 1. \tag{11}$$

From (3) and the fact that m is a probability measure we easily see that

$$m[\bar{h}(A) \vee \bar{h}(A')] = 1. \tag{12}$$

Since m is additive (11) and (12) are known to imply (8).

We continue to show that (10) is sufficient to complete the proof of the theorem. We need to show that if A_i is a monotone decreasing sequence of sets in $\mathcal{B}$ with an empty intersection then $\bar{h}(A_i) \searrow 0$. From the definition of $\bar{h}$ it easily follows that $\bar{h}$ is monotone. Therefore $\bar{h}(A_i)$ is a monotone decreasing sequence. To show it decreases to zero it is sufficient to show $m[\bar{h}(A_i)] \searrow 0$. But this latter follows from (10) since $v(A_i) \searrow 0$ for v is a countably additive measure on $\mathcal{B}$. The following sequence of inequalities establishes (10)

$$\begin{aligned} m \circ \bar{h}(A) &= m\left[\bigwedge \left\{ \bigvee_{i=1}^{\infty} h(R_i) \right\}\right] \leqq \bigwedge m\left\{ \bigvee_{i=1}^{\infty} h(R_i) \right\} \\ &\leqq \bigwedge \sum_{i=1}^{\infty} m \circ h(R_i) = v(A) \end{aligned}$$

where the infimum is taken over all countable sequences $\{R_i\}$, of elements $R_i \in \mathcal{R}$, such that $A \subset \bigcup R_i$. The first equality follows from the definition of $\bar{h}$. The two inequalities are immediate from the fact that m is a measure. The last equality is demonstrated in the well known proof of the existence of an extension of a σ-measure defined on a ring $\mathcal{R}$ of subsets of a set to the σ-ring $\mathcal{B}$ generated by $\mathcal{R}$. This completes the proof of Theorem 2.

IV. Integration of generalized random variables

4. Discussion. In this chapter we define a notion of integral for generalized random variables which agrees with the weak integral of Pettis([11]) in the event that the generalized random variable is induced by an ordinary point function into the dual of a Banach space. We then prove a Radon-Nikodym type theorem, Theorem 5 below, which is fundamental for the theory of generalized random variables.

Throughout this chapter, S will be the dual of a real topological vector space B, and $\mathcal{B}$ will be the collection of weak star measurable subsets of S, and F will be an S-valued, $\mathcal{B}$ measurable, generalized random variable on a probability space $P = [U, \mathfrak{U}, u]$ whose measure ring is $\mathfrak{M}$. We will occasionally impose additional restrictions on B and F. Let f be an ordinary function defined on U into S such that for every $x \in B$ the real valued function

([11]) For a discussion of various notions of integral including the weak integral of Pettis see Hille [5].

$(x, f(\cdot))$ is measurable relative to $\mathfrak{U}$. If for every $x \in B$

$$\int (x, f(w))du(w)$$

exists, it determines a linear functional on B. If furthermore it is a continuous linear functional of B it determines a unique element of S. It seems natural to say that in this event, f is integrable, and to define $\int f$ to be the element of S thereby determined. If one accepts this definition, then it seems natural to say that for any measurable subset $M \in \mathfrak{U}$, if for every $x \in B$, $\int_M (x, f(w))du(w)$ exists, and determines a continuous linear functional of B then f is integrable over M, and to define $\int_M f$ to be the unique element of S thereby determined. It would be desirable if the set of M such that f is integrable over M formed an ideal. If we assume that B is complete and metrizable then one can prove that the collection of such M is indeed an ideal; however, the collection need not, in general, form an ideal. It seems desirable therefore to modify our proposed definition somewhat, so that this collection is always an ideal, and so that the definition is not really different in the event B is both complete and metrizable. We are thereby led to consider f to be integrable over a measurable set $M_0 \in \mathfrak{U}$ provided that for all measurable $M \leqq M_0$, f is integrable over M in the sense previously defined and in this event to define $\int_M f$ as before. If one examines what this concept means in terms of the mapping f^{-1} of $\mathfrak{B}$ into $\mathfrak{U}$ one is led to definitions appropriate to generalized random variables.

5. Some elementary properties of indefinite integrals of generalized random variables.

DEFINITION 7. For every S-valued $\mathfrak{B}$ measurable generalized random variable F, and every M in the measure ring of P we define a linear transformation F_M as follows.

The *domain of* F_M is the set of all $x \in B$ such that the linear functional on S, $(x, \cdot)$, is integrable relative to the measure $u_M \circ F$ where $(u_M \circ F)(A) = u_M(F(A)) = u(M \cap F(A))$ for all $A \in \mathfrak{B}$.

If x is in the domain of F_M then we define $F_M(x)$ to be that element of $L_1(u_M \circ F)$ determined by $(x, \cdot)$. For brevity we write

$$F_M(x) = (x, \cdot).$$

PROPOSITION 1. *F_M is a linear transformation defined on a subspace of B.*

Proof is immediate from definition of F_M.

PROPOSITION 2. *If the domain of F_M is a complete metrizable subspace of B then F_M is continuous.*

Proof of Proposition 2. By the closed graph theorem it is sufficient to show that F_M is closed. Since metrizability implies first countability it suffices to show that if $x_n \in$ domain of F_M and $x_n \to x$ and $F_M(x_n) \to y$ then $x \in$ domain of F_M and $F_M(x) = y$. Since the domain of F_M is closed it follows that x is in the

domain of F_M. Since x_n converges to x, $(x_n, \cdot)$ converges everywhere to $(x, \cdot)$. Since $F_M(x_n) = (x_n, \cdot)$ converges to y in $L_1(u_M \circ F)$ it follows that $y = (x, \cdot)$ almost everywhere $(u_M \circ F)$. Therefore $F_M(x) = (x, \cdot) = y$ in $L_1(u_M \circ F)$. This completes the proof.

DEFINITION 8. F is said to be *weak star semi-integrable over* M provided that for all $x \in B$, $(x, \cdot)$ is integrable relative to the measure $u_M \circ F$ and the mapping

$$x \to \int (x, \phi) d(u_M \circ F)(\phi)$$

is a continuous linear functional defined on B.

DEFINITION 9. F is said to be *weak star integrable* over M_0 provided that for all $M \leqq M_0$, F is weak star semi-integrable over M.

In the event B is complete and metrizable there is no need to distinguish between weak star semi-integrability and weak star integrability as the following proposition implies.

PROPOSITION 3. *Suppose B is complete and metrizable. If for all $x \in B$, $(x, \cdot)$ is integrable relative to the measure $u_{M_0} \circ F$ then F is weak star integrable over M_0.*

Proof of Proposition 3. Let $M \leqq M_0$. The measure $u_M \circ F$ is less than or equal to the measure $u_{M_0} \circ F$ on every measurable set $b \in \mathcal{B}$. Therefore for all $x \in B$, $(x, \cdot)$ is integrable relative to $u_M \circ F$. That is, every $x \in B$ is in the domain of F_M. Therefore by Proposition 2, F_M is continuous. So the mapping $x \to (x, \cdot)$ is a continuous mapping of B into $L_1(u_M \circ F)$. It easily follows that the mapping $x \to \int (x, \cdot) d(u_M \circ F)$ is a continuous linear functional on B. Thus F is weak star semi-integrable over M. By Definition 9 it follows that F is weak star integrable over M_0.

DEFINITION 10. If F is weak star semi-integrable over M then we define $\int_M F$ to be that unique element in S determined by

$$\left(x, \int_M F\right) = \int (x, \phi) d(u_M \circ F) \phi.$$

DEFINITION 11. Let F be any S-valued, $\mathcal{B}$ measurable generalized random variable. We define the *indefinite integral* of F to be the vector valued function v with domain the set of all M in the measure ring of P such that F is weak star integrable over M, and where

$$v(M) = \int_M F.$$

The following proposition follows immediately from the definitions and is stated here for later reference:

PROPOSITION 4. *Let F be an S-valued, $\mathcal{B}$ measurable, generalized random*

variable and M_0 an element of the measure ring of P. Then any one of the following hold if and only if they all hold.

1. *M_0 is in the domain of the indefinite integral of F.*
2. *F is weak star integrable over M_0.*
3. *For all $M \leqq M_0$, F is weak star semi-integrable over M.*
4. *For all $M \leqq M_0$, F is weak star integrable over M.*
5. *For all $M \leqq M_0$, M is in the domain of the indefinite integral of F.*
6. *For all $M \leqq M_0$, and for all $x \in B$, $(x, \cdot)$ is integrable relative to the measure $u_M \circ F$ and the mapping $x \to \int (x, \phi) d(u_M \circ F)\phi$ is a continuous linear functional defined on B.*

DEFINITION 12. F is said to be *weak star integrable* provided that for all M in the measure ring of P, F is weak star integrable over M.

COROLLARY TO PROPOSITION 4. *An S-valued, $\mathcal{B}$ measurable generalized random variable is weak star integrable if and only if it is weak star integrable over the unit of the measure ring of P.*

Our primary objective is to characterize the indefinite integral of a generalized random variable. An obvious property of indefinite integrals is their countable additivity, a concept which we now define explicitly.

DEFINITION 13. A mapping v of a subset of a σ-ring into S will be called *weak star countably additive* provided that for every disjoint sequence M_i in the domain of v such that $\bigvee_{i=1}^{\infty} M_i$ is also in the domain of v, we have

$$v\left(\bigvee_{i=1}^{\infty} M_i\right) = \sum_{i=1}^{\infty} v(M_i),$$

where the sum, $\sum$, is meant in the sense of the weak star topology on S.

PROPOSITION 5. *The indefinite integral of a generalized random variable F is weak star countably additive.*

Proof of Proposition 5. Let v be the indefinite integral of F and let M_i be a disjoint sequence in the domain of v such that $M = \bigvee M_i$ is also in the domain of v. We need to show that $v(M) = \sum v(M_i)$ where the sum, $\sum$, is taken in the sense of the weak star topology on S. It suffices to show that for all $x \in B$

$$(x, v(M)) = \sum (x, v(M_i)).$$

That is we wish to show that for all $x \in B$

$$\left(x, \int_M F\right) = \sum \left(x, \int_{M_i} F\right).$$

Equivalently, we need only show that

$$\int F_M(x) d(u_M \circ F) = \sum \int_{F_{M_i}} (x) d(u_{M_i} \circ F).$$

That is we must only show that

$$\int (x, \cdot)d(u_M \circ F) = \sum \int (x, \cdot)d(u_{M_i} \circ F).$$

This equality does indeed hold since the measure $u_M \circ F$ is the sum of the measures $u_{M_i} \circ F$. Therefore the proposition is established.

6. **A preliminary Radon-Nikodym theorem for generalized random variables.** We are now ready to prove a preliminary theorem of Radon-Nikodym type.

THEOREM 3. *Let v be a weak star countably additive function whose domain is the measure ring of a probability space $P=[U, \mathfrak{U}, u]$ and with values in S. Suppose that for all $M\neq 0$ in the measure ring of P, $v(M)/u(M)\in K$, where K is a convex subset of S which is compact in the weak star topology. Then v is the indefinite integral of an S-valued, $\mathfrak{B}$ measurable, generalized random variable h.*

Proof of Theorem 3. Our proof will be accomplished as follows. First we will define h on sets of the form $x^{-1}(J)$ where J is a left closed right open interval of real numbers and where $x^{-1}(J)$ is the set of $\phi\in S$ such that $(x, \phi)\in J$. We will then prove that h can be extended to be a σ-algebra homomorphism of the Boolean subalgebra of $\mathfrak{B}$ generated by sets of the form $x^{-1}(J)$. Then by Theorem 2 there exists a further extension of h to $\mathfrak{B}$. We will then show that v is the indefinite integral of h by showing, that for all M in the measure ring of P, M is in the domain of the indefinite integral of h and $\int_M h=v(M)$.

We now proceed to the details of the proof. We will use the letter x to represent either a vector in B or the continuous linear functional on S to which it gives rise. Thus by $x\circ v$ we mean the mapping which assigns to each M in the measure ring $\mathfrak{M}$ of P, the real number $\langle x, v(M)\rangle$. Clearly $x\circ v$ is a countably additive real valued measure defined on the measure ring $\mathfrak{M}$. Therefore by the usual Radon-Nikodym theorem, there exists an integrable function f_x defined on U into the reals, unique up to a set of u measure zero such that

$$\int_D f_x du = x\circ v\circ \pi(D), \tag{1}$$

where π is the canonical projection mapping of $\mathfrak{U}$ onto the measure ring $\mathfrak{M}$, and $D\in\mathfrak{U}$.

Let J be any left closed, right open, interval of real numbers, infinite intervals not being excluded. Then $\pi f_x^{-1}(J)$ is an element of the measure ring and is uniquely determined by x and J. We now define h on sets of the form $x^{-1}(J)$ as follows.

$$h[x^{-1}(J)] = \pi f_x^{-1}(J). \tag{2}$$

We omit the easy proof that h is well defined on such sets. Let g be a set of the form

$$g = x_1^{-1}(J_1) \cap \cdots \cap x_n^{-1}(J_n) \tag{3}$$

and let b be a finite union of sets of the form (3)

$$b = \bigcup_{j=1}^{l} g_j. \tag{4}$$

It is straightforward to check that the class of all sets of the form (4) is the Boolean algebra generated by sets of the form $x^{-1}(J)$. It then follows([12]) that a necessary and sufficient condition for h to be uniquely extended to a Boolean algebra homomorphism of this Boolean algebra is:

$$\begin{aligned} &\text{If } x_1^{-1}(J_1) \cap \cdots \cap x_n^{-1}(J_n) \text{ is empty then} \\ &\qquad h[x_1^{-1}(J_1)] \wedge \cdots \wedge h[x_n^{-1}(J_n)] = 0. \end{aligned} \tag{5}$$

We proceed to prove (5).

LEMMA 1. *Let E be a convex set of reals and x a vector in B and f_x as defined above. Suppose $0 < M \leqq \pi f_x^{-1}(E)$. Then $v(M)/u(M) \in x^{-1}(E)$.*

Proof of Lemma 1. From (1) we get

$$\frac{1}{u(M)} \int_M f_x du = \frac{x \circ v(M)}{u(M)} \tag{6}$$

where $\int_M f_x du$ means the integral of f_x over any set in $\mathfrak{U}$ which represents M. It is well known that the left side of (6) is in the least convex set containing the image under f_x of any representative of M. By hypotheses E is such a set. Therefore

$$\frac{x \circ v(M)}{u(M)} \in E$$

or equivalently

$$\frac{v(M)}{u(M)} \in x^{-1}(E).$$

LEMMA 2. *Let $E_1, \cdots, E_n$ be convex sets of reals and $x_1, \cdots, x_n$ vectors in B. Let $f_{x_1}, \cdots, f_{x_n}$ be as defined above and suppose*

$$0 < M = \pi f_{x_1}^{-1}(E_1) \wedge \cdots \wedge \pi f_{x_n}^{-1}(E_n)$$

then

([12]) For a proof see Sikorski [10].

$$\frac{v(M)}{u(M)} \in x_1^{-1}(E_1) \cap \cdots \cap x_n^{-1}(E_n).$$

Proof is immediate from Lemma 1.

It is easy to see that (5) above is immediate from Lemma 2. Thus h can be extended to be a Boolean algebra homomorphism of the Boolean algebra generated by sets of the form $x^{-1}(J)$.

We next wish to show that h as thus extended is a σ-homomorphism. It is sufficient to show that if b_n is any monotone decreasing sequence of subsets of S whose intersection is empty, (where each b_n is of the form (4) above) then $\bigwedge h(b_n) = 0$. We now proceed to show that this sufficient condition holds.

LEMMA 3. *If $\pi f_x^{-1}(J) \neq 0$, then for any $\epsilon > 0$, there exists a closed interval, Z, such that*

$$Z \subset J \text{ and } u[\pi f_x^{-1}(Z)] > u[\pi f_x^{-1}(J)] - \epsilon.$$

Proof of Lemma 3. $u\pi f_x^{-1}$ is a regular measure on Borel subsets of reals.

LEMMA 4. *If $\pi f_{x_1}^{-1}(J_1) \wedge \cdots \wedge \pi f_{x_n}^{-1}(J_n) \neq 0$ then for any $\epsilon > 0$, there exist closed intervals Z_i such that $Z_i \subset J_i$ and*

$$u[\pi f_{x_1}^{-1}(Z_1) \wedge \cdots \wedge \pi f_{x_n}^{-1}(Z_n)] > u[\pi f_{x_1}^{-1}(J_1) \wedge \cdots \wedge \pi f_{x_n}^{-1}(J_n)] - \epsilon.$$

Proof of Lemma 4. This lemma follows from Lemma 3.

LEMMA 5. *If $\gamma = \bigvee_{j=1}^{l} [\pi f_{x_{j,1}}^{-1}(J_{j,1}) \wedge \cdots \wedge \pi f_{x_{j,n_j}}^{-1}(J_{j,n_j})] \neq 0$ then for any $\epsilon > 0$, there exist closed intervals $Z_{j,i}$ such that $Z_{j,i} \subset J_{j,i}$ and such that*

$$u\left\{\bigvee_{j=1}^{l}\left[\bigwedge_{i=1}^{n_j} \pi f_{x_{j,i}}^{-1}(Z_{j,i})\right]\right\} > u(\gamma) - \epsilon.$$

Proof of Lemma 5. This lemma follows easily from Lemma 4.

LEMMA 6. *Let $\gamma_m = \bigvee_{j=1}^{l_m} \bigwedge_{i=1}^{n_{m,j}} \pi f_{x_{m,j,i}}^{-1}(J_{m,j,i})$ and suppose γ_m is monotone decreasing. Let $M = \bigwedge \gamma_m$ and suppose $M \neq 0$. Then there exist closed intervals $Z_{m,j,i} \subset J_{m,j,i}$ such that*

$$u\left\{\bigwedge_{m=1}^{\infty} \bigvee_{j=1}^{l_m} \bigwedge_{i=1}^{n_{m,j}} \pi f_{x_{m,j,i}}^{-1}(Z_{m,j,i})\right\} \geqq \frac{u(M)}{2}.$$

Proof of Lemma 6. Let $\epsilon_m = u(M)/4^m$. We apply Lemma 5 to γ_m and ϵ_m to get the existence of closed intervals $Z_{m,j,i} \subset J_{m,j,i}$, and such that

$$u\left\{\bigvee_{j=1}^{l_m} \bigwedge_{i=1}^{n_{m,j}} \pi f_{x_{m,j,i}}^{-1}(Z_{m,j,i})\right\} > u(\gamma_m) - \epsilon_m.$$

We easily confirm that

$$u\left\{\bigwedge_{m=1}^{\infty}\bigvee_{j=1}^{l_m}\bigwedge_{i=1}^{n_{m,j}}\pi f^{-1}_{x_{m,j,i}}(Z_{m,j,i})\right\} \geqq u[\wedge\gamma_m] - \sum\epsilon_m = u(M) - \frac{u(M)}{2} = \frac{u(M)}{2}.$$

LEMMA 7. *If b_n is any monotone decreasing sequence of subsets of S of form* (4) *above, whose intersection is empty, then $\wedge h(b_n)=0$.*

Proof of Lemma 7. Let $M=\wedge h(b_n)$ and assume $M\neq 0$,

$$b_m = \bigcup_{j=1}^{l_m}\bigcap_{i=1}^{n_{m,j}} x^{-1}_{m,j,i}(J_{m,j,i}).$$

Let $\gamma_m=h(b_m)$. By Lemma 6, there exist closed intervals $Z_{m,j,i}\subset J_{m,j,i}$ such that

$$u\left\{\bigwedge_{m=1}^{\infty}\bigvee_{j=1}^{l_m}\bigwedge_{i=1}^{n_{m,j}}\pi f^{-1}_{x_{m,j,i}}(Z_{m,j,i})\right\} \geqq \frac{u(M)}{2}.$$

In particular

$$\delta_N = \bigwedge_{m=1}^{N}\bigvee_{j=1}^{l_m}\bigwedge_{i=1}^{n_{m,j}}\pi f^{-1}_{x_{m,j,i}}(Z_{m,j,i}) \neq 0 \text{ for every } N.$$

Then I claim

$$\frac{v(\delta_N)}{u(\delta_N)} \in \bigcap_{m=1}^{N}\bigcup_{j=1}^{l_m}\bigcap_{i=1}^{n_{m,j}} x^{-1}_{m,j,i}(Z_{m,j,i}).$$

To see this we use the distributive law and apply Lemma 2. Furthermore $v(\delta_N)/u(\delta_N)\in K$. Thus we have proved that for every N

$$K\cap\bigcap_{m=1}^{N}\bigcup_{j=1}^{l_m}\bigcap_{i=1}^{n_{m,j}} x^{-1}_{m,j,i}(Z_{m,j,i})$$

is nonempty.

For every N this set is compact in the weak star topology. These sets obviously possess the finite intersection property. Therefore

$$K\cap\bigcap_{m=1}^{\infty}\bigcup_{j=1}^{l_m}\bigcap_{i=1}^{n_{m,j}} x^{-1}_{m,j,i}(Z_{m,j,i})$$

is nonempty. It easily follows that

$$\bigcap_{m=1}^{\infty}\bigcup_{j=1}^{l_m}\bigcap_{i=1}^{n_{m,j}} x^{-1}_{m,j,i}(Z_{m,j,i})$$

is nonempty. And since $Z_{m,j,i}\subset J_{m,j,i}$ we see that

$$\bigcap_{m=1}^{\infty}\bigcup_{j=1}^{l_m}\bigcap_{i=1}^{n_{m,j}} x^{-1}_{m,j,i}(J_{m,j,i})$$

is nonempty, i.e, $\bigcap_{m=1}^{\infty} b_m$ is nonempty. This contradiction completes the proof of the lemma.

Lemma 7 implies that h is a σ-homomorphism of the Boolean algebra generated by sets of the form $x^{-1}(J)$. Now by Theorem 2, h can be extended to be a σ-homomorphism of the σ-algebra generated by sets of the form $x^{-1}(J)$. To complete the proof of the theorem we need only show that for all M in the measure ring of P, M is in the domain of the indefinite integral of h and $\int_M h = v(M)$. By Proposition 4, it suffices to show that for all M in the measure ring, h is weak star semi-integrable over M and $\int_M h = v(M)$. That is we need to show that for all $x \in B$, $(x, \cdot)$ is integrable relative to the measure $u_M \circ h$ and:

$$(x, v(M)) = \int (x, \phi) d(u_M \circ h)(\phi).$$

This is easily established via the following sequence of equalities

$$\begin{aligned}(x, v(M)) &= \int_M f_x du = \int y d(u_M \circ f_x^{-1})(y) \\ &= \int y d(u_M \circ h \circ x^{-1})(y) = \int (x, \phi) d(u_M \circ h)(\phi).\end{aligned}$$

The first equality follows from the definition of f_x. The second equality is standard measure theory. The third equality is by definition of h. The last equality is again standard measure theory. The proof of Theorem 3 is now complete.

7. **A sufficient condition for absolute continuity of the indefinite integral of generalized random variables.** It is easy to see that if v is weak* countably additive on the measure ring $\mathfrak{M}$ of a probability space, then $v(0) = 0$; that is, v necessarily possesses a kind of absolute continuity property with respect to the probability measure u on the measure ring. In case S is the real numbers the usual Radon-Nikodym theorem is therefore easily seen to imply that every weak* countably additive function defined on the measure ring of a probability space into S is the indefinite integral of an integrable random variable. It is therefore natural to inquire whether this result is true if S is the dual of an arbitrary Banach space, B. This however is not the case. Errett Bishop called to my attention the following counterexample. Let the measure space be the unit interval under Lebesgue measure. Let B be L_2 of the unit interval and S its dual. Let $v(M)$ be the element of S corresponding to the characteristic function of any representative of M where M is any element of the measure ring. Then v is weak* countably additive. However v is not the indefinite integral of a generalized random variable F. For assume that it were. Then there exists a sphere V, say the sphere of radius k in S, such that $F(V) \neq 0$.

Let $0<M<F(V)$. Then for all $x\in L_2$,

$$\begin{aligned} |(x, v(M))| &= \left|\left(x, \int_M F\right)\right| = \left|\int (x, \phi) du_M \circ F(\phi)\right| \\ &\leqq \int |(x,\phi)|\, du_M \circ F(\phi) \leqq \int \|x\|\,\|\phi\| du_M \circ F(\phi) \\ &= \|x\| \int_V \|\phi\| du_M \circ F(\phi) \leqq \|x\|\, k\cdot u(M). \end{aligned}$$

Now let x be $v(M)$. Therefore $u(M)=(v(M), v(M))\leqq \|v(M)\|\cdot ku(M)$. Equivalently $1\leqq \|v(M)\|\cdot k$. Therefore $1\leqq (v(M), v(M))k^2=u(M)k^2$. We arrive at a contradiction by letting M have sufficiently small positive measure.

Therefore in order that v be the indefinite integral of a generalized random variable we need an additional restriction on v besides weak star countable additivity. In order to state the additional property v must satisfy, it is convenient to make a definition.

DEFINITION 14. Let $\mathfrak{M}$ be the measure ring of a probability space $P=[U, \mathfrak{U}, u]$ and let S be the dual of a topological vector space B. Let v be a function defined on a subset of $\mathfrak{M}$ into S. We say that v is *absolutely continuous* with respect to u provided that there exists a collection M_α of elements of $\mathfrak{M}$ whose sup is the unit of $\mathfrak{M}$ and such that for each α and each $M\in\mathfrak{M}$ with $M\leqq M_\alpha$, v is defined on M and $v(M)\in u(M)\cdot K_\alpha$, where K_α is a convex weak star compact subset of S corresponding to M_α.

In the event v were a countably additive real valued function defined on $\mathfrak{U}$, it is well known that v is absolutely continuous in the usual sense if and only if the function $\bar{v}$ defined on the measure ring of P by

$$\bar{v}(\{A\}) = v(A)$$

is well defined, where $\{A\}$ is the element in the measure ring corresponding to $A\in\mathfrak{U}$. In order to show that for real valued countably additive functions, our use of the words absolutely continuous is consistent with previous usage we observe that the usual Radon-Nikodym theorem can be used to show that if v is a countably additive real valued function defined on the measure ring of a probability space $P=[U, \mathfrak{U}, u]$ then v is absolutely continuous with respect to u in the sense of Definition 14.

We do not know if the indefinite integral of an S-valued, $\mathfrak{B}$ measurable, generalized random variable is necessarily absolutely continuous with respect to u. We do however have the following:

THEOREM 4. *Let h be an S-valued, $\mathfrak{B}$ measurable generalized random variable on a probability space $P=[U, \mathfrak{U}, u]$ where S is the dual of a real metrizable*

topological vector space B. Then the indefinite integral of h is absolutely continuous with respect to u.

LEMMA. *Let S be the dual of any real topological vector space B and let $[S, \mathcal{B}, v]$ be a probability space where $\mathcal{B}$ is the collection of all weak star measurable subsets of S. Suppose further that S is a measurable cover of the polar, K, of a neighborhood, W, of the origin in B. Then the mapping*

$$x \to (x, \cdot)$$

is a continuous linear transformation of B into $L_\infty(v)$, the Banach algebra of essentially bounded measurable functions on S. Furthermore it is a continuous linear transformation of B into $L_1(v)$ and the mapping

$$x \to \int (x, \phi) dv(\phi)$$

is a continuous linear functional, γ, on B. Also $\gamma \in K$.

Proof of Lemma. It is clear that for each $x \in B$, $(x, \cdot)$ is measurable $(\mathcal{B})$. We show that it is essentially bounded. K is a weak star compact subset of S, and therefore the image of K under the continuous function $(x, \cdot)$ is a compact interval I of real numbers which depends upon x. Let b be the set of $\phi \in S$ such that $(x, \phi) \in' I$. Clearly $b \in \mathcal{B}$ and $b \cap K = 0$. Therefore $v(b) = 0$ since S is a measurable cover of K. Therefore $(x, \cdot)$ is essentially bounded by its bound on K. The mapping $x \to (x, \cdot)$ is clearly a linear transformation of B into the essentially bounded measurable functions on S. To show continuity it is sufficient to show continuity at the origin. In fact it is sufficient to show that some neighborhood of the origin in B maps into the open unit sphere. For all $x \in W$ and all $\phi \in K$ $|(x, \phi)| \leqq 1$. Since $(x, \cdot)$ is essentially bounded by its bound on K, it follows that $|(x, \phi)| \leqq 1$ almost everywhere (v). Therefore continuity into $L_\infty(v)$ is established. Since the injection map of $L_\infty(v)$ into $L_1(v)$ is continuous it follows that $x \to (x, \cdot)$ is continuous on B into $L_1(v)$ and the mapping $x \to \int (x, \phi) dv(\phi)$ is clearly a continuous linear functional γ on B. We now show $\gamma \in K$. Assume $\gamma \in' K$. Then since S is locally convex in the weak star topology there exists a continuous linear functional separating γ from K. Since the only continuous linear functionals on S in the weak star topology are in B it follows that there exists an $x \in B$ such that

$$(x, \phi) \neq (x, \gamma) \qquad \text{for all } \phi \in K.$$

Since v is a probability measure, $\int (x, \phi) dv(\phi)$ is an element of the closed convex hull of the essential range of $(x, \cdot)$. It was shown above that the essential range of $(x, \cdot)$ is contained in I, the image of K under the continuous function $(x, \cdot)$. Therefore $\int (x, \phi) dv(\phi)$ equals (x, ϕ) for some $\phi \in K$. Therefore $(x, \phi) = \int (x, \phi) dv(\phi) = (x, \gamma)$. This contradiction completes the proof of the lemma.

Proof of Theorem 4. Let W_n be a countable base for the neighborhoods of the origin in B, and let D_n be a measurable cover for K_n, the polar of W_n. Since S equals the union of the K_n it follows that the same is true for the union of the D_n. Also, since h is a σ-algebra homomorphism it follows that the unit of $\mathfrak{M}$ is the sup of $M_n = h(D_n)$. Let $M \in \mathfrak{M}$ with $0 \neq M \leqq M_\alpha$ for some fixed α. We now apply the lemma where $v = (1/u(M)(u_M \circ h)$. Therefore the mapping

$$x \to \int (x, \phi) dv(\phi)$$

is a continuous linear functional, γ, on B; and $\gamma \in K$. That is, the mapping $x \to (1/u(M)) \int (x, \phi) d(u_M \circ h)(\phi)$ determines an element of K. Therefore the mapping $x \to \int (x, \phi) d(u_M \circ h)(\phi)$ is a continuous linear functional. Therefore h is weak star semi-integrable over M. This is true for every $M \leqq M_\alpha$ and therefore by Proposition 4, M is in the domain of the indefinite integral of h. Furthermore $(1/u(M))(x, \int_M h) = (1/u(M)) \int (x, \phi) d(u_M \circ h)(\phi)$. Therefore $(1/u(M)) \int_M h \in K$. This completes the proof.

8. A Radon-Nikodym theorem.

THEOREM 5. *Let $\mathfrak{M}$ be the measure ring of a probability space $P = [U, \mathfrak{U}, u]$ and let S be the dual of a topological vector space B. Let v be a function defined on $\mathfrak{M}$ with values in S which is both weak star countably additive and absolutely continuous with respect to u. Then v is the indefinite integral of a unique S-valued, $\mathfrak{B}$ measurable, weak star integrable, generalized random variable, h, on P.*

Proof of Theorem 5. We first prove existence. Since v is absolutely continuous with respect to u, there exists a collection $M_k \in \mathfrak{M}$, where k ranges over some set A, such that sup $M_k = 1 \in \mathfrak{M}$ and such that for each $k \in A$ and each $M \leqq M_k$, $v(M) \in u(M) \cdot K_k$ where K_k is a convex weak star compact subset of S.

Let $v_k = v$ cut down to M_k. We observe that Theorem 3 is valid not only for probability spaces but for any finite measure space. Therefore there exists a collection h_k of σ-algebra homomorphisms of $\mathfrak{B}$, into the subring of $\mathfrak{M}$ consisting of the principal ideal generated by M_k with the property

$$\int_M h_k = v(M) \text{ for every } M \leqq M_k.$$

Now define $h(b) = \bigvee h_k(b)$ for each $b \in \mathfrak{B}$. We now check that h is a σ-homomorphism.

The fact that h preserves countable unions is trivial. We proceed to show h preserves complements. We need to show that

$$\bigvee h_j(b) \quad \text{and} \quad \bigvee (M_k - h_k(b))$$

are complements. The sup of these two is easily seen to be the identity in $\mathfrak{M}$.

We need only prove that their inf is zero. Their inf is

$$\bigvee_{j,k} h_j(b) \wedge (M_k - h_k(b)).$$

Thus we wish to show that for each j and k

$$h_j(b) \wedge (M_k - h_k(b)) = 0.$$

We observe that the collection of $b \in \mathcal{B}$ for which this latter equation holds forms a σ-subalgebra of $\mathcal{B}$ and that therefore we need only show that it holds for a generating subset of $\mathcal{B}$. In particular we need only show that it holds for sets of the form $x^{-1}(J)$ where $x \in B$ and J is a left closed right open interval of real numbers. By the usual Radon-Nikodym theorem, there exists an essentially unique real valued integrable function f_x defined on U such that

$$\int_M f_x = (x, v(M) \text{ for all } M \in \mathfrak{M}.$$

By definition of h_k and h_j (in Theorem 3)

$$h_k(x^{-1}(J)) = \pi f_x^{-1}(J) \wedge M_k$$

and

$$h_j(x^{-1}(J)) = \pi f_x^{-1}(J) \wedge M_j.$$

These two equations imply that

$$h_j(x^{-1}(J)) \wedge (M_k - h_k(x^{-1}(J))) = 0.$$

Therefore h preserves complements. Thus we have shown that h is a σ-algebra homomorphism.

It is convenient to note here for later use the fact that

$$h \circ x^{-1} = \pi f_x^{-1}.$$

For: $h(x^{-1}(J)) = \sup_k h_k(x^{-1}(J)) = \sup_k \pi f_x^{-1}(J) \wedge M_k = \pi f_x^{-1}(J)$. Therefore for every Borel set C of real numbers,

$$h(x^{-1}(C)) = \pi f_x^{-1}(C).$$

Equivalently

$$h \circ x^{-1} = \pi f_x^{-1}.$$

We now show that h is weak star integrable and that v is the indefinite integral of h. That is, we will show that for all $M \in \mathfrak{M}$ and all $x \in B$

$$(x, v(M)) = \int (x, \phi) du_M \circ h(\phi)$$

in the sense that the integral on the right exists and equals $(x, v(M))$. This follows from the following sequence of equalities

$$(x, v(M)) = \int_M f_x = \int y du_M \circ f_x^{-1}(y)$$

$$= \int y du_M \circ h \circ x^{-1}(y) = \int (x, \phi) du_M \circ h(\phi).$$

The first equality follows from the definition of f_x. The second is standard measure theory. The third follows from the fact already noted; namely $h \circ x^{-1} = \pi f_x^{-1}$, and the fourth is standard measure theory.

We now prove uniqueness. We observe that the function f_x was determined as that essentially unique function such that $\int_M f_x = (x, v(M))$ for all $M \in \mathfrak{M}$. Therefore πf_x^{-1} is uniquely determined by x and v. Therefore $h \circ x^{-1}$ is uniquely determined by x and v. Thus h is uniquely determined on sets of the form $x^{-1}(C)$ for C a Borel subset of the reals. Therefore h is uniquely determined on a generating subcollection of $\mathfrak{B}$. By a standard argument h is uniquely determined on $\mathfrak{B}$. This completes the proof.

DEFINITION 15. The unique h determined by v we call the *Radon-Nikodym derivative* of v.

9. **The main result: a characterization of the indefinite integral of S-valued generalized random variables for S the dual of a metrizable space.** As an immediate consequence of Proposition 5 and Theorems 4 and 5 we deduce our main result:

COROLLARY 1. *Let $\mathfrak{M}$ be the measure ring of a probability space $P = [U, \mathfrak{U}, u]$ and let S be the dual of a real metrizable topological vector space B. Let $\mathfrak{B}$ be the collection of weak star measurable subsets of S and let v be a function defined on $\mathfrak{M}$ into S. Then v is the indefinite integral of a unique S-valued $\mathfrak{B}$ measurable, weak star integrable, generalized random variable on P if and only if v is both weak star countably additive and absolutely continuous with respect to u.*

10. **A characterization of the indefinite integral of S-valued weak star-integrable ordinary random variables for S the dual of a separable metrizable space.** Theorems 1 and 5 together imply a generalization of a theorem of Dieudonné [2, p. 132] which we state as:

COROLLARY 2. *Let $\mathfrak{M}$ be the measure ring of a probability space $P = [U, \mathfrak{U}, u]$ and let S be the dual of a real separable metrizable topological vector space B. Let $\mathfrak{B}$ be the collection of weak star measurable subsets of S and let v be a function defined on $\mathfrak{M}$ into S. Assume v is both absolutely continuous with respect to u and weak star countably additive. Then there exists a point function f on U into S such that $f^{-1}(b) \in \mathfrak{U}$ for all $b \in \mathfrak{B}$ and such that for each $A \in \mathfrak{U}$ and each $x \in B$*

$$(x, X_A(\cdot) \cdot f(\cdot))$$

is integrable with respect to u and

$$\int (x, X_A(w)\cdot f(w))du(w) = (x, v \circ \pi(A))$$

where π is the canonical mapping of $\mathfrak{U}$ onto $\mathfrak{M}$ and X_A is the characteristic function of A.

Proof of Corollary 2. By Theorem 5, v is the indefinite integral of a generalized random variable h. By Theorem 1, h is induced by a point function f. It is straightforward to check that f has the desired properties.

We observe that Corollary 1 implies the converse to Corollary 2. Thus if S is the dual of a separable metrizable space, we have a necessary and sufficient condition that an S-valued function v defined on the measure ring of a probability space be the indefinite integral of an S-valued, weak star integrable point function.

11. **Conditional expectation and the Fubini theorem for generalized random variables.** Corollary 1 enables us to define conditional expectation. Let g be a weak star integrable S-valued $\mathfrak{B}$ measurable generalized random variable on a probability space $P=[U, \mathfrak{U}, u]$ with measure ring $\mathfrak{M}$ where S is assumed to be the dual of a metrizable space B and $\mathfrak{B}$ as usual is the collection of weak star measurable subsets of S. Let $\mathfrak{M}_0$ be a σ-subalgebra of $\mathfrak{M}$. It is easy to see that there exists a σ-subalgebra of $\mathfrak{U}$, $\mathfrak{U}_0$, such that $\mathfrak{M}_0$ is the measure ring of the probability space $[U, \mathfrak{U}_0, u_0]=P_0$ where u_0 equals u cut down to $\mathfrak{U}_0$. Let v be the indefinite integral of g

$$v(M) = \int_M g du \qquad \text{for all } M \in \mathfrak{M}.$$

Then by Corollary 1 to Theorem 4, v is both weak star countably additive and absolutely continuous with respect to u. Let v_0 equal v cut down to $\mathfrak{M}_0$. It is easy to see that v_0 is also weak star countably additive and absolutely continuous with respect to u_0. Then we apply Corollary 1 again, to find a unique weak star integrable, S-valued, $\mathfrak{B}$ measurable generalized random variable, g_0, on P_0 which is the indefinite integral of v_0. We call g_0 the *conditional expectation* of g with respect to $\mathfrak{M}_0$ and designate it by $E[g|\mathfrak{M}_0]$. Then, if h is any S-valued $\mathfrak{B}$ measurable generalized random variable on P we define as usual the conditional expectation of g with respect to h to be $E[g|h(\mathfrak{B})]$. The uniqueness of the Radon-Nikodym derivative permits one easily to show that $E\{E[g|\mathfrak{M}_0]|\mathfrak{M}_1\}=E[g|\mathfrak{M}_1]$ whenever $\mathfrak{M}_1\subset\mathfrak{M}_0$. It is convenient here to remark, that the usual Fubini theorem can be interpreted to say two things. Firstly, if a function $f(x, y)$ is integrable with respect to a product measure then the conditional expectation of f with respect to the x coordinate exists and is integrable, with the same integral as f, and secondly, the conditional expectation, f_0, of f with respect to the x coordinate can be obtained as follows:

$$f_0(x) = \int f(x, y)du(y).$$

It is easy to see that whenever we have a conditional expectation for generalized random variables, the analogue of the first part of the usual Fubini theorem is valid. However there seems to be no analogue for the second part of the usual Fubini theorem.

12. **A determination of the weak star distributions on the dual of a Banach space induced by generalized random variables.** If F is an integrable generalized random variable and $x \in B$, then the real valued, Borel measurable generalized random variable, $F \circ x^{-1}$ is induced by a real valued point function defined on U. This point function is necessarily in $L_1(u)$ and we designate it by $\hat{F}(x)$. $\hat{F}(x)$ can easily be seen to be the usual Radon-Nikodym derivative of the signed measure

$$M \to (x, v(M))$$

where v is the indefinite integral of F. From this observation one easily deduces that $\hat{F}$ is a linear transformation of B into $L_1(u)$. The question naturally arises as to which linear transformations from B into $L_1(u)$ arise in this manner.

The answer is essentially a reformulation of Corollary 1 which for simplicity we state only for the case B is a Banach space. It is convenient to have some terminology. We modify slightly the notion of weak distribution due to Segal [8].

DEFINITION 16. Let S be the dual of a topological vector space B and let $P = [U, \mathfrak{U}, u]$ be a probability space. A *weak star distribution on S associated with P* is a linear transformation T from B to the random variables on P. T is said to be $L_1(u)$ *valued* provided that for all $x \in B$, $T(x) \in L_1(u)$.

DEFINITION 17. Let $\mathfrak{B}$ be the collection of all weak star measurable subsets of the dual S of a topological vector space B and let F be an S-valued $\mathfrak{B}$ measurable weak star integrable generalized random variable on a probability space $P = [U, \mathfrak{U}, u]$ with measure ring $\mathfrak{M}$. Let v be the indefinite integral of F. For each $x \in B$, we define $\hat{F}(x)$ to be the Radon-Nikodym derivative of the signed countably additive measure

$$M \to (x, v(M))$$

defined for all $M \in \mathfrak{M}$. We say that a linear transformation T from B into $L_1(u)$ is *induced by a generalized random variable* provided there exists an F such that $T = \hat{F}$.

COROLLARY 3. *Let $\mathfrak{M}$ be the measure ring of a probability space $P = [U, \mathfrak{U}, u]$ and let S be the dual of a real Banach space B. Let $\mathfrak{B}$ be the collection of weak star measurable subsets of S. Then a continuous, $L_1(u)$ valued, weak star distribution T on S is induced by an S-valued $\mathfrak{B}$ measurable weak star integrable generalized*

random variable, if and only if, there exists a monotone increasing sequence $M_k \in \mathfrak{M}$ *whose sup is the unit of* $\mathfrak{M}$ *and such that for each* k *and each* $M \in \mathfrak{M}$ *with* $M \leqq M_k$

$$\int_M T(x) \leqq ku(M)\|x\|.$$

Proof of Corollary 3. Suppose T is induced by F. Then let $M_k = F(D_k)$ where D_k is a measurable cover for the closed sphere of radius k in S for the measure $u \circ F$. $T(x)$ is the Radon-Nikodym derivative of the signed measure

$$M \to \left(x, \int_M F\right)$$

for M ranging over the measure ring $\mathfrak{M}$ of P. Therefore $\int_M T(x) = (x, \int_M F) = \int (x, \phi) du_M \circ F(\phi)$ for all $M \in \mathfrak{M}$. Now let $0 \neq M \leqq M_k$. We use the lemma for Theorem 4 where v is $(1/u(M))(u_M \circ h)$, and K is the sphere of radius k in S. By this lemma, the map $x \to (1/u(M)) \int (x, \phi) du_M \circ F(\phi)$ is a bounded linear functional on B of norm $\leqq k$. Therefore

$$\int_M T(x) \leqq ku(M)\|x\|.$$

It is easy to verify that the M_k are monotone increasing with sup equal to the unit of $\mathfrak{M}$.

Now suppose T has the property stated in the corollary. For each M, let $v(M) \in S$ be defined by

$$(x, v(M)) = \int_M T(x).$$

Then v satisfies the hypotheses of Theorem 5, and therefore v is the indefinite integral of a unique F. Furthermore it is clear that $T(x)$ is the Radon-Nikodym derivative of the signed measure $M \to (x, v(M))$. Therefore $T(x) = \hat{F}(x)$. This completes the proof.

V. Special classes of generalized random variables

13. **A characterization of the indefinite integrals of bounded generalized random variables.** A particularly important class of integrable point functions defined on a probability space are the essentially bounded measurable functions. We investigate the appropriate analogous classes of generalized random variables. A point function f is bounded if and only if for some bounded set K of real numbers $f^{-1}(K)$ is the unit of the probability space. This suggests proposing that an S-valued, $\mathfrak{B}$ measurable generalized random variable F be called bounded provided that $F(K)$ is the unit of the measure ring for some bounded set K, or equivalently provided $u \circ F(K) = 1$. However bounded

subsets of S will in general not be in $\mathcal{B}$, and therefore F will not be defined on them.

We will therefore say that F is bounded provided some bounded subset K of S is a thick([13]) subset of S for the measure $u \circ F$. This latter condition is equivalent to the condition that S be a measurable cover for some bounded set. This discussion motivates our next definition.

DEFINITION 18. Let $\mathcal{B}$ be the weak star measurable subsets of the dual S of a topological vector space B and let F be an S-valued $\mathcal{B}$ measurable generalized random variable on a probability space $P = [U, \mathfrak{U}, u]$. We say that F is *weak star bounded* if S is a measurable cover for some weak star bounded subset K of S (relative to the measure $u \circ F$). We say that F is *weak star bounded in the restricted sense* if S is a measurable cover for the polar of a neighborhood of the origin in B.

The lemma to Theorem 4 implies immediately that if F is weak star bounded in the restricted sense then it is weak star integrable. An easy consequence of this observation is

PROPOSITION 6. *If B is a Banach space and F is weak star bounded then it is weak star integrable.*

Proof of Proposition 6. Suppose S is a measurable cover for a weak star bounded subset K of S. By the uniform boundedness theorem K is bounded in norm. Therefore S is a measurable cover of some closed sphere in S. Every closed sphere in S is the polar of a neighborhood of the origin in B. Therefore F is weak star bounded in the restricted sense. The observation above completes the proof.

We wish to characterize the indefinite integrals of generalized random variables which are weak star bounded in the restricted sense. For this purpose we need a preliminary result:

PROPOSITION 7. *Let $\mathcal{B}$ be the weak star measurable subsets of the dual S of a topological vector space B, and suppose that K is a convex weak star compact subset of S. Let $[S, \mathcal{B}, v]$ be a probability space and suppose $K \subset b \in \mathcal{B}$. Then there exists a countable family $x_i \in B$ such that for all i, $K \subset [\phi : (x_i, \phi) \leqq 1]$ and, except possibly for a set of v measure zero,*

$$b \supset \bigcap_{i=1}^{\infty} [\phi : (x_i, \phi) \leqq 1].$$

LEMMA 1. *The proposition holds in the event B is finite dimensional.*

LEMMA 2. *The proposition holds for the situation in which b is in the σ-ring generated by finitely many $x_i \in B$.*

([13]) For a definition of *thick* see Halmos [4].

Proof of Lemma 2. Let b be in the σ-ring generated by $x_1, \cdots, x_n$ and let V be the subspace of B generated by $x_1, \cdots, x_n$. Then there exists a natural projection mapping, p, of S onto V', the dual of V. The image of K, $p(K)$ is a compact convex subset of V' since p is continuous for the weak star topology on S and the unique topological vector space topology on V'. Let vp^{-1} be the measure induced on the Borel subsets of V'. Since b is in the σ-ring generated by $x_1, \cdots, x_n$ it follows that there exists a Borel subset $\bar{b}$ of V' such that $b = p^{-1}(\bar{b})$. Since $K \subset b$ it follows that $p(K) \subset \bar{b}$. Therefore by Lemma 1 there exists a countable family $x_i \in V$ such that for all i

$$p(K) \subset [\theta : (x_i, \theta) \leqq 1]$$

and except possibly for a set of measure zero (vp^{-1})

$$\bar{b} \supset \bigcap_{i=1}^{\infty} [\theta : (x_i, \theta) \leqq 1].$$

Therefore for all i, $K \subset [\phi : (x_i, \phi) \leqq 1]$ and

$$b \supset \bigcap_{i=1}^{\infty} [\phi : (x_i, \phi) \leqq 1].$$

This completes the proof of the lemma.

For fixed K, the collection b for which the proposition holds is closed under monotone limits and by Lemma 2 contains a ring whose generated σ-ring is $\mathfrak{B}$. This completes the proof of the proposition.

If we apply this proposition to any measurable cover b of K we get the following:

COROLLARY. *K possesses a measurable cover which is the intersection of a countable number of closed half spaces.*

We are now ready to characterize the indefinite integrals of generalized random variables which are weak star bounded in the restricted sense.

PROPOSITION 8. *Let $\mathfrak{B}$ be the weak star measurable subsets of the dual S of a topological vector space B and let F be an S-valued $\mathfrak{B}$ measurable generalized random variable on a probability space $P = [U, \mathfrak{U}, u]$ with measure ring $\mathfrak{M}$. Let v be the indefinite integral of F. Then F is weak star bounded in the restricted sense if and only if, for the polar K of a neighborhood of the origin in B,*

$$v(M)/u(M) \in K,$$

for all nonzero $M \in \mathfrak{M}$.

Proof of Proposition 8. Assume first that F is weak star bounded in the restricted sense. Therefore S is a measurable cover for the polar K of some neighborhood of the origin in B for the measure $u \circ F$. Let M be nonzero in

$\mathfrak{M}$. Then $[S, B, (1/u(M))u_M \circ F]$ is a probability space. S is clearly a measurable cover for K relative to the measure $u_M \circ F$.

The definition of the indefinite integral implies that for all x

$$\frac{1}{u(M)}(x, v(M)) = \frac{1}{u(M)}\int (x, \phi)d(u_M \circ F)(\phi).$$

Thus, by the lemma to Theorem 4, $v(M)/u(M) \in K$.

Now assume that $v(M)/u(M) \in K$ for all nonzero $M \in \mathfrak{M}$, where K is the polar of a neighborhood of the origin of B. We will show that F is weak star bounded in the restricted sense by showing that S is a measurable cover of K (relative to the measure $u \circ F$). We need to show that if $b \in \mathfrak{B}$ and $K \subset b$ then $u \circ F(b) = 1$. Assume $b = [\phi : (x, \phi) \leqq 1]$ and $K \subset b$. Therefore for all $\phi \in K$, $(x, \phi) \leqq 1$. In particular,

$$(x, v(M)/u(M)) \leqq 1$$

for all $M \in \mathfrak{M}$. We now consider the real valued signed measure which assigns to M the real number $(x, v(M))$. It is elementary to verify that since $(x, v(M)/u(M)) \leqq 1$ for all $M \in \mathfrak{M}$, then the Radon-Nikodym derivative f_x is almost everywhere $\leqq 1$. Therefore $\pi \circ f_x^{-1}(-\infty, 1]$ is the unit of $\mathfrak{M}$ where π is the canonical projection mapping of $\mathfrak{U}$ onto $\mathfrak{M}$. In the proof of Theorem 5 it was shown that

$$F(b) = \pi f_x^{-1}(-\infty, 1].$$

Therefore $F(b)$ equals the unit of $\mathfrak{M}$. From this it easily follows that if b is the intersection of a countable number of closed half spaces then $u \circ F(b) = 1$. The proposition now follows from the corollary to the previous proposition.

14. **Absolutely and strongly integrable generalized random variables.** If f is a real valued measurable function defined on a probability space, then f is integrable if and only if $|f|$ is integrable. If f has values in Euclidean n space then the same assertion holds provided $|f(w)|$ is the Euclidean norm of the vector $f(w)$. In the event f has values in the dual of a Banach space and possesses the requisite measurability, then the integrability of $\|f\|$ is a sufficient condition for the weak star integrability(14) of f. If f has the requisite measurability and $\|f\|$ is integrable then one might say that such a function f is strongly integrable. In the event f is only weak star measurable it is not immediately clear how to define strong integrability. However, the concept of measurable cover permits us to define a related notion. We are thereby led to

DEFINITION 19. If F is an S-valued, $\mathfrak{B}$ measurable generalized random variable on a probability space $P = [U, \mathfrak{U}, u]$ where S is the dual of a Banach

(14) See Hille [5] for a discussion of various notions of integrability.

space B we define $|F|$ to be the real valued function of a real variable determined as follows:

$$|F|(y) = u \circ F(C\{\phi : \|\phi\| \leqq y\})$$

where for any subset γ of S, $C(\gamma)$ is any measurable cover of γ for the measure $u \circ F$.

It is clear that $|F|$ is a monotone nondecreasing non-negative real valued function of a real variable and $|F|(-1)=0$ and $|F|(y)$ converges to 1 as y goes to $+\infty$. It follows that $|F|$ determines a unique probability measure on the real line. We may therefore make

DEFINITION 20. If F is as described in Definition 19 we say that F is *absolutely integrable* provided the probability measure determined by $|F|$ possesses a mean.

We remark that it is straightforward to verify if B is a Banach space and F is weak star bounded then it is absolutely integrable.

Our purpose in introducing the concept of absolute integrability is the following:

PROPOSITION 9. *If F is absolutely integrable then F is integrable.*

Proof of Proposition 9. By Proposition 3 we need only show the existence of

$$\int (x, \phi) du \circ F(\phi) \tag{1}$$

for all $x \in B$ of norm 1. Equivalently we need only show that

$$\sum_{k=1}^{\infty} u \circ F(b_k) < \infty \tag{2}$$

where b_k is the set of $\phi \in S$ such that $|(x, \phi)| > k$. Clearly b_k is a subset of c_k, the set of $\phi \in S$ of norm $> k$. If γ_k is a measurable cover of the complement of c_k, then $u \circ F(b_k \cap \gamma_k) = 0$. It follows that $u \circ F(b_k) \leqq 1 - u \circ F(\gamma_k) = 1 - |F|(k)$. Therefore

$$\sum_{k=1}^{\infty} u \circ F(b_k) \leqq \sum_{k=1}^{\infty} (1 - |F|(k)). \tag{3}$$

Since $|F|$ possesses a mean, the right side of (3) is finite, and thus (2) is established. This completes the proof.

We now inquire as to the relationship of absolute integrability and strong integrability. That is, an ordinary point function f on a probability space $[U, \mathfrak{U}, u]$ into the dual S of a Banach space B is said to be strongly integrable provided (1) $f^{-1}(b) \in \mathfrak{U}$ for every subset b of S which is open in the norm topology and (2) $\int \|f(w)\| du(w) < \infty$.

It is easy to see that if f is strongly integrable then the generalized random variable $F=\pi\circ f^{-1}$ induced by f is S-valued and $\mathcal{B}$ measurable where $\mathcal{B}$ as usual are the weak star measurable subsets of S. Further F is necessarily absolutely integrable. For since $\|f\|$ is integrable, the probability distribution on the real line induced by $\|f\|$ possesses a mean. That is the real measure determined by $\|F\|$ where $\|F\|(y)=u\circ f^{-1}[\phi:\|\phi\|\leqq y]$ possesses a mean. It is trivial to verify that $\|F\|(y)\leqq |F|(y)$. Furthermore any probability distribution which lives on the positive reals and whose distribution function is everywhere greater than or equal to the distribution function of a probability distribution with a mean also possesses a mean. Therefore F is absolutely integrable.

We do not possess an example of a strongly measurable f which is not strongly integrable and which induces an absolutely integrable generalized random variable. However in the event B is separable it is easy to show that strong integrability is also a necessary condition to absolute integrability. For one can then show that $\|F\|=|F|$. In order to show $\|F\|=|F|$ it is certainly sufficient to show that the set of $\phi\in S$ such that $\|\phi\|\leqq y$ is an element of $\mathcal{B}$. That is it is sufficient to show that the closed unit sphere is an element of $\mathcal{B}$. To see this latter fact we observe that the closed unit sphere is weak star closed and apply Lemma 5 of Theorem 1. That which we have just proved can be stated in terms of generalized random variables. First we need two definitions.

DEFINITION 21. Let S be the dual of a Banach space B and let $\mathcal{S}$ be the σ-algebra generated by the subsets of S which are open in the norm topology. Suppose F is an S-valued, $\mathcal{S}$ measurable generalized random variable on a probability space $P=[U, \mathfrak{U}, u]$. We define $\|F\|$ to be the real valued function of a real variable determined as follows:

$$\|F\|(y) = u\circ F(\phi:\|\phi\| \leqq y).$$

It is easy to see that $\|F\|$ is the distribution function of a unique probability measure on the real line.

DEFINITION 22. If F is as described in Definition 21 we say that F is *strongly integrable* provided the probability measure determined by $\|F\|$ possesses a mean.

If F is an S-valued, $\mathcal{S}$ measurable generalized random variable on a probability space then it determines a unique S-valued $\mathcal{B}$ measurable one, by restricting the domain of F to $\mathcal{B}$. There will be no confusion in the sequel if we also designate this random variable by F.

That which we proved above can now be summarized by

PROPOSITION 10. *If F is strongly integrable then F is absolutely integrable. Furthermore if B is separable, then F is $\mathcal{B}$ measurable if and only if F is $\mathcal{S}$ measurable, and (consequently) F is strongly integrable if and only if F is absolutely integrable.*

VI. The composition of a generalized random variable with a linear transformation

15. If f is an S-valued integrable point function defined on a probability space $[U, \mathfrak{U}, u]$ and if T is a reasonable linear transformation on S then $T \circ f$ is also integrable and $T(\int f) = \int (T \circ f)$. Our next proposition shows that an analogous result holds for integrable generalized random variables.

Proposition 11. *Let $\mathfrak{B}_i$ be the collection of all weak star measurable subsets of the dual S_i of a topological vector space B_i for $i=1$ and $i=2$. Let F be an S_1-valued, $\mathfrak{B}_1$ measurable generalized random variable on a probability space P. Suppose T is a continuous linear transformation on B_2 into B_1 and let T^* be its adjoint. Then G is an S_2-valued, $\mathfrak{B}_2$ measurable generalized random variable on P where*

$$G(b_2) = F[\phi_1 : T^*(\phi_1) \in b_2]$$

for all $b_2 \in \mathfrak{B}_2$. Furthermore if F is integrable then so is G, and

$$\int_M G = T^*\left(\int_M F\right)$$

for all M in the measure ring of P.

Proof of Proposition 11. We first show that for all $b_2 \in \mathfrak{B}_2$

$$[\phi_1 : T^*(\phi_1) \in b_2]$$

is an element of $\mathfrak{B}_1$. For suppose b_2 is of the form

$$[\phi_2 : (x_2, \phi_2) \leqq \alpha].$$

Then

$$[\phi_1 : T^*(\phi_1) \in b_2] = [\phi_1 : (x_2, T(\phi_1)) \leqq \alpha] = [\phi_1 : (T(x_2), \phi_1) \leqq \alpha]$$

which is an element of $\mathfrak{B}_1$ since $T(x_2) \in B_1$. We now make the usual observation that the set of $b_2 \in \mathfrak{B}_2$ for which the assertion holds is a σ-algebra and includes a generating collection of $b_2 \in \mathfrak{B}_2$. It is now clear that G is an S_2-valued, $\mathfrak{B}_2$ measurable generalized random variable on P. Now suppose F to be integrable. To complete the proof we need only show that for all M in the measure ring of P, G is weak star semi-integrable over M, and

$$\int_M G = T^*\left(\int_M F\right).$$

Equivalently we need to show that for all $x_2 \in B_2$,

$$\int (x_2, \phi_2) du_M \circ G(\phi_2) \text{ exists and equals } \left(x_2, T^*\left(\int_M F\right)\right).$$

The following sequence of identities completes the proof.

$$\int (x_2, \phi_2) du_M \circ G(\phi_2) = \int (x_2, \phi_2) du_M \circ F \circ T^{*-1}(\phi_2)$$

$$= \int (x_2, T^*(\phi_1)) du_M \circ F(\phi_1) = \int (T(x_2), \phi_1) du_M \circ F(\phi_1)$$

$$= \left(T(x_2), \int_M F\right) = \left(x_2, T^*\left(\int_M F\right)\right).$$

This completes the proof.

VII. Operations on generalized random variables

16. **The existence of the cartesian product of two generalized random variables.** If f_i is an ordinary real valued random variable on a probability space $P = [U, \mathfrak{U}, u]$ for $1 \leqq i \leqq 2$ then there is an obvious notion of the cartesian product $f_1 \times f_2$ of f_1 with f_2. Namely $f_1 \times f_2$ is the function defined on P into the plane by

$$(f_1 \times f_2)(w) = (f_1(w), f_2(w)).$$

The inverse mapping $g = (f_1 \times f_2)^{-1}$ is determined uniquely (up to sets of measure zero) by the requirement

$$g(A_1 \times A_2) = f_1^{-1}(A_1) \cap f_2^{-1}(A_2)$$

for A_i a Borel subset of the reals $i = 1$ and $i = 2$.

Therefore, if F_i is an S_i-valued, $\mathfrak{B}_i$ measurable generalized random variable on a probability space P where S_i is an arbitrary set and $\mathfrak{B}_i$ is an arbitrary σ-field of subsets of S_i it is natural to attempt to define the cartesian product of F_1 and F_2 to be a σ-algebra homomorphism F of $\mathfrak{B} = \mathfrak{B}_1 \times \mathfrak{B}_2$ into the measure ring of P, determined by the condition

$$F(A_1 \times A_2) = F_1(A_1) \wedge F_2(A_2).$$

It is easy to see that there exists at most one such generalized random variable. However there may not exist any such F. For Kakutani has constructed a bimeasure (where a bimeasure is a non-negative finite valued function γ defined on all measurable rectangles $A_1 \times A_2$ in the cartesian product of two measurable spaces such that for each fixed A_2, $\gamma(A_1 \times A_2)$ is a countably additive measure of the first variable and for each fixed A_1, $\gamma(A_1 \times A_2)$ is a countably additive measure of the second variable) which cannot be extended to be a measure. A modification by Halmos([15]) of Kakutani's construction can be used to exhibit generalized random variables F_i such that for no generalized random variable F does

([15]) The construction of P. Halmos has not been published.

$$F(A_1 \times A_2) = F_1(A_1) \times F_2(A_2)$$

for all $A_1 \in B_1$ and all $A_2 \in B_2$.

In at least one important special case, however, the cartesian product can be defined. This is the case in which S_i is the dual of a Banach space B_i, and $\mathcal{B}_i$ is the σ-algebra of all weak star measurable subsets of S_i. For in this case we let v_i be the indefinite integral of F_i and define for each M in the measure ring of P which is in the domain of both v_1 and v_2, an element $v(M)$ in $S_1 \times S_2$ by

$$v(M) = (v_1(M), v_2(M)).$$

There exists a natural isomorphism of $S_1 \times S_2$ onto S, the dual of $B_1 \times B_2$. Because of this isomorphism we can view v as a mapping into S. Then by Theorem 6 below, which is a slight strengthening of Theorem 5, so as to characterize the indefinite integrals of all generalized random variables, it follows that this mapping into S is the indefinite integral of a unique S-valued, $\mathcal{B}$ measurable generalized random variable on P, where $\mathcal{B}$ is the σ-algebra of all weak* measurable subsets of S. Then we again use the natural isomorphism of $S_1 \times S_2$ onto S to obtain a unique σ-algebra homomorphism of $\mathcal{B}_1 \times \mathcal{B}_2$ into $\mathfrak{M}$. This latter σ-homomorphism we define as the *cartesian* product of F_1 with F_2 and designate it by $F_1 \times F_2$.

Thus, in order that our definition of cartesian product make sense, it only remains to establish:

THEOREM 6. *Let $\mathfrak{M}$ be the measure ring of a probability space $P = [U, \mathfrak{U}, u]$ and let S be the dual of a Banach space B. Let $\mathcal{B}$ be the σ-algebra of weak star measurable subsets of S and let v be a function whose domain is an ideal in $\mathfrak{M}$ and whose range is contained in S. Then v is the indefinite integral of a unique S-valued, $\mathcal{B}$ measurable, generalized random variable on P if and only if* (1) *v is weak star countably additive and* (2) *there exists a monotone increasing sequence of elements $M_k \in \mathfrak{M} 1 \leqq k < \infty$ such that* (a) *the sup of the M_k is the unit of $\mathfrak{M}$ and* (b) *for every $M \in \mathfrak{M}$ with $M \leqq M_k$, v is defined on M and $\|v(M)\| \leqq ku(M)$, and* (c) *if for every $M \leqq M_0$, $\lim_{k\to\infty} v(M \cap M_k)$ exists in the weak star topology, then M_0 is in the domain of v.*

Proof of Theorem 6. If v is an indefinite integral then, by Proposition 5, v is weak star countably additive, and by Theorem 4 there exists a sequence $M_k \in \mathfrak{M}$ such that $\bigvee M_k$ is the unit in $\mathfrak{M}$ and such that for every $M \in \mathfrak{M}$ with $M \leqq M_k$, v is defined on M and $\|v(M)\| \leqq ku(M)$. Suppose that, for every $M \leqq M_0$, $\lim_{k\to\infty} v(M \cap M_k)$ exists in w^* topology. We wish to show M_0 is in the domain of v. That is, we wish to show that F is weak star integrable over M_0, where v is the indefinite integral of F. It is sufficient to show that F is weak star semi-integrable over M_0. Let $x \in B$.

It is sufficient to show that (x, ϕ) is integrable with respect to $u_{M_0} \circ F$ and that

$$\int (x, \phi) du_{M_0} \circ F(\phi) = \left(x, \lim_{k\to\infty} \int_{M_0 \cap M_k} F du\right).$$

Let Q_x be the set of $\phi \in S$ such that $(x, \phi) \leqq 0$ and let Q_x' be its complement. It is well known that (x, ϕ) is integrable with respect to $u_{M_0} \circ F$ if and only if both $\int_{Q_x}(x, \phi) du_{M_0} \circ F(\phi)$ exists and $\int_{Q_x'}(x, \phi) du_{M_0} \circ F(\phi)$ exists and in this event its integral is the sum of these two integrals. Also we have $\int_{Q_x}(x, \phi) du_{M_0} \circ F(\phi)$ exists if and only if $\int (x, \phi) dr_x \circ F(\phi)$ exists and then they are equal, where $r_x = u_{M_0 \cap F(Q_x)}$. Since (x, ϕ) is a non-negative function of ϕ almost everywhere with respect to the measure $r_x \circ F$ it follows that

$$\int (x, \phi) dr_x \circ F(\phi) = \lim_{k\to\infty} \int (x, \phi) dr_{x,k} \circ F(\phi) \tag{1}$$

where $r_{x,k} = u_{M_0 \cap F(Q_x) \cap M_k}$. The right side of equation (1) is easily seen to be equal to

$$\lim_{k\to\infty} \left(x, \int_{M_0 \cap F(Q_x) \cap M_k} F\right). \tag{2}$$

The expression (2) clearly equals

$$\lim_{k\to\infty} (x, v(M_0 \cap F(Q_x) \cap M_k)) = \left(x, \lim_{k\to\infty} v(M_0 \cap F(Q_x) \cap M_k)\right).$$

This latter quantity exists by hypothesis. Therefore

$$\int_{Q_x} (x, \phi) du_{M_0} \circ F(\phi) = \left(x, \lim_{k\to\infty} v(M_0 \cap F(Q_x) \cap M_k)\right).$$

Likewise

$$\int_{Q_x'} (x, \phi) du_{M_0} \circ F(\phi) = \left(x, \lim_{k\to\infty} v(M_0 \cap F(Q_x') \cap M_k)\right).$$

From these latter two equalities it easily follows that

$$\int (x, \phi) du_{M_0} \circ F(\phi) \text{ exists and equals } \int (x, \lim_{k\to\infty} v(M_0 \cap M_k)).$$

Thus indefinite integrals necessarily possess the properties mentioned in Theorem 6.

For the converse we merely mention that if v possesses the properties stated in Theorem 6, then one defines an S-valued $\mathcal{B}$ measurable generalized random variable, F, as in Theorem 5. One needs then to show that v is the indefinite integral of F. As in Theorem 5 one shows that if M is in domain of v then F is weakly semi-integrable over M and $\int_M F du = v(M)$. Since the

F11)

domain of v is an ideal it follows that F is weakly integrable over M. We need now only show that if F is weakly integrable over M_0, then M_0 is in the domain of v. To show this we use property $2c$ of v. If F is weakly integrable over M_0 then for every $M \leqq M_0$, F is weakly semi integrable over M. For $k=1, \cdots, n, \cdots$ let F_k be the generalized random variable defined in the proof of Theorem 5. For every x in B we have the following equalities

$$(3) \qquad \left(x, \int_M F du\right) - \left(\int_{M \cap M_k} F_k du\right) = \left(x, \int_M F du\right) - \left(x, \int_{M \cap M_k} F_k du\right)$$

$$(4) \qquad = \int (x, \phi) du_M \circ F(\phi) - \int (x, \phi) dp_k \circ F_k(\phi)$$

where $p_k = u_{M \cap M_k}$. It is easy to see that (4) equals

$$(5) \qquad \int (x, \phi) du_M \circ F(\phi) - \int (x, \phi) dp_k \circ F(\phi).$$

Since M_k converges to the unit of $\mathfrak{M}$, one sees that (5) converges to zero. That is

$$(6) \qquad \int_{M \cap M_k} F_k du \text{ converges to } \int_M F du \text{ in the weak star}$$

topology. It is easy to see that

$$(7) \qquad \int_{M \cap M_k} F_k du = v_k(M \cap M_k) = v(M \cap M_k).$$

From (6) and (7) we conclude that $v(M \cap M_k)$ converges in the weak star topology. Therefore by hypothesis M_0 is in the domain of v. We conclude the proof by remarking that the uniqueness of F is proved in a manner similar to that of the proof of uniqueness in Theorem 5.

17. The existence of the sum of two generalized random variables. The set of all ordinary real valued random variables on a probability space forms a vector space under the usual definitions of sum and scalar product. Theorem 6 enables us to define the sum and scalar product for certain generalized random variables so as to form a vector space. For, if v_1 and v_2 are any two vector valued functions defined on subsets R_1 and R_2 respectively of the measure ring of a probability space then we define $v_1 + v_2$ on $R_1 \cap R_2$ in the natural way

$$(v_1 + v_2)(M) = v_1(M) + v_2(M)$$

for all $M \in R_1 \cap R_2$.

And if α is a scalar we define

$$(\alpha v_1)(M) = \alpha v_1(M)$$

for all $M \in R_1$. It is straightforward to verify

COROLLARY 1 TO THEOREM 6. *The collection of all indefinite integrals of S-valued, $\mathfrak{B}$ measurable generalized random variables on a probability space $P = [U, \mathfrak{U}, u]$ form a vector space.*

Now we can define scalar multiplication and addition for generalized random variables. Namely if F_1, F_2 are generalized random variables and α is a scalar, we define $F_1 + F_2$ to be the Radon-Nikodym derivative of $v_1 + v_2$ where v_i is the indefinite integral of F_i. Likewise αF_1 is the Radon-Nikodym derivative of αv_1.

COROLLARY 2. *The collection of S-valued, $\mathfrak{B}$ measurable generalized random variables on P form a vector space.*

Theorem 5 is easily seen to imply that the sum of two integrable generalized random variables is likewise integrable. Similarly the scalar product of an integrable generalized random variable is integrable. Therefore we have:

COROLLARY 3. *The collection of S-valued, $\mathfrak{B}$ measurable integrable generalized random variables on P is a subspace of the S-valued, $\mathfrak{B}$ measurable generalized random variables on P.*

We mention without proof that the notion of cartesian product enables us to give an alternative but equivalent definition of the sum of two generalized random variables. Namely let $F_1 \times F_2$ be the cartesian product of F_1 and F_2. For any two vectors x and y in S there is a unique vector $x+y$ in S.

This operation of addition in S induces a σ-homomorphism s of $\mathfrak{B}$ into $\mathfrak{B} \times \mathfrak{B}$. We then define $F_1 + F_2$ to be the composition σ-homomorphism $(F_1 \times F_2) \circ s$. In a similar manner one can define scalar multiplication. These definitions however leave one with the problem of establishing the associative and distributive laws in order to prove that the generalized random variables form a vector space. These laws are however trivial to verify in the case of the indefinite integrals. We remark in conclusion that the existence of the sum of two generalized random variables, for either definition, appears to require the Radon-Nikodym theorem.

VIII. A REPRESENTATION OF BOUNDED LINEAR TRANSFORMATIONS OF L_1 INTO THE DUAL OF A BANACH SPACE

In this section S will be the dual of a real Banach space B, $\mathfrak{B}$ will be the collection of weak star measurable subsets of S and $\mathfrak{M}$ will be the measure ring of a probability space $P = [U, \mathfrak{U}, u]$. L_1 is the Banach space of classes of real valued summable functions defined on U. A function g defined on U into S is said to be weak star measurable provided $g^{-1}(b) \in \mathfrak{U}$ for all $b \in \mathfrak{B}$. It is strongly bounded provided $\|g(w)\|$ is a bounded function of w. The validity of the following theorem is open.

THEOREM A. *Every continuous linear transformation of the Banach space L_1 into S is of the form*

$$f \to \int f(w)g(w)du \tag{1}$$

where g is a weak star measurable and strongly bounded function defined on U into S. (The second element of (1) *being the unique element in S defined by*

$$\left(x, \int f(w)g(w)du\right) = \int f(w)(x, g(w))du(w)$$

for all $x \in B$.)

This theorem is known if B is a *separable* Banach space (theorem of Dunford-Pettis, see [12]) and has been analyzed by Dieudonné, see [13] and [14]. We will show that Theorem A is equivalent to Theorem C below.

Let H be the real Banach space of all finite real valued signed measures which are absolutely continuous with respect to u. Each $m \in H$ determines a unique signed measure on the measure ring. There will be no confusion if we also designate this induced measure by m. For any bounded $\mathcal{B}$-measurable generalized random variable G, and any $m \in H$, $m \circ G$ is a finite signed measure so that $[S, \mathcal{B}, m \circ G]$ is a signed measure space. By $\int m \circ G$ we mean the unique element in S determined by

$$\left(x, \int m \circ G\right) = \int (x, \phi)dm \circ G(\phi)$$

for all $x \in B$.

Theorem A is equivalent to the conjunction of the following two theorems, the first of which is a representation theorem for bounded linear operators on L_1 of a probability space into the dual of a Banach space.

THEOREM B. *Every continuous linear transformation of the Banach space H into S is of the form*

$$m \to \int m \circ G \tag{2}$$

when G is a bounded weak star measurable generalized random variable.

THEOREM C. *Every bounded S-valued, $\mathcal{B}$-measurable generalized random variable G defined on $\mathcal{B}$ into $\mathfrak{M}$ is induced by a point function.*

Theorem C is open. Theorem B is an easy consequence of Theorem 3.

Proof of Theorem B. Let T be a bounded linear transformation of H into S. For each $M \in \mathfrak{M}$ let u_M be the measure u cut down to M. Define $v(M) = T(u_M)$. It is easy to see that v satisfies the hypotheses of Theorem 3. There-

fore v is the indefinite integral of a unique generalized random variable. It is now elementary to verify that this generalized random variable G is bounded and that $T(m) = \int m \circ G$. This completes the proof.

References

1. J. Dieudonné and L. Schwartz, *La dualité dans les espaces (F) et (LF)*, Annales de L'Institute Fourier vol. 1 (1949) pp. 61–101.

2. ———, *Sur le théorème de Lebesgue Nikodym* (V), Canadian Journal of Mathematics (1951) p. 132.

3. N. Dunford, *Uniformity in linear spaces*, Trans. Amer. Math. Soc. vol. 44 (1938) pp. 305–356.

4. P. Halmos, *Measure theory*, Toronto, New York, London, van Nostrand, 1950.

5. E. Hille, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publications, vol. 31, 1948.

6. B. J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc. vol. 44 (1938) pp. 277–304.

7. C. E. Rickart, *An abstract Radon-Nikodym theorem*, Trans. Amer. Math. Soc. vol. 56, (1944) pp. 50–66.

8. I. E. Segal, *Abstract probability spaces and a theorem of Kolmogoroff*, Amer. J. Math. vol. 76 (1954) pp. 721–732.

9. R. Sikorski, *On the inducing of homomorphisms by mappings*, Fund. Math. vol. 36 (1949).

10. ———, *On an analogy between measures and homomorphisms*, Annales de la Société Polonaise de Mathématique vol. 23 (1950).

11. A. Tarski, *Über unereichbare Kardinalzahlen*, Fund. Math. vol. 30 (1938).

12. N. Dunford and B. J. Pettis, *Linear operations on summable functions*, Trans. Amer. Math. Soc. vol. 47 (1940) pp. 323–392.

13. J. Dieudonné, *On the theorem of Lebesgue-Nikodym* (III), Annales de l'Université de Grenoble vol. 23 (1948) pp. 25–53.

14. ———, *On the theorem of Lebesgue-Nikodym* (IV), J. Indian Math. Soc. vol. 15 (1951) pp. 77–86.

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