THE DIFFERENTIAL EQUATIONS OF BIRTH-AND-DEATH PROCESSES, AND THE STIELTJES MOMENT PROBLEM[1]

BY

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CHAPTER I

1. Introduction. A birth-and-death process is a stationary Markoff process whose path functions $X(t)$ assume non-negative integer values and whose transition probability function

$$P_{ij}(t) = Pr\left\{X(t+s) = j \mid X(s) = i\right\}$$

satisfies the conditions

$$\begin{aligned} P_{i,i+1}(t) &= \lambda_i t + o(t), \\ P_{i,i}(t) &= 1 - (\lambda_i + \mu_i)t + o(t), \\ P_{i,i-1}(t) &= \mu_i t + o(t), \end{aligned}$$

as $t \to 0$, where λ_i, μ_i are constants which may be thought of as the rates of absorption from state i into states $i+1$, $i-1$. As a guide to one's intuition it is useful to think of a material particle which moves from integer to neighboring integer, the path function $X(t)$ being the position of the particle at time t. An elegant description of these processes together with a survey of applications may be found in Feller's book [4, Chapter 17].

Using the above order conditions and the Markoffian nature of the process it is easy to show that the infinite matrix $P(t) = (P_{ij}(t))$, $i, j = 0, 1, 2, \cdots$ satisfies the equation

$$P'(t) = AP(t), \qquad t \geqq 0, \tag{1.1}$$

where A is the matrix

$$A = \begin{bmatrix} -(\lambda_0 + \mu_0), & \lambda_0 & 0 & 0 \\ \mu_1 & -(\lambda_1 + \mu_1), & \lambda_1 & 0 \\ 0 & \mu_2 & -(\lambda_2 + \mu_2), & \lambda_2 \\ & \vdots & \vdots & \ddots \end{bmatrix} = (a_{ij})$$

and $a_{ij} = \lambda_i$ if $j = i+1$, $-(\lambda_i + \mu_i)$ if $j = i$, μ_i if $j = i-1$, zero if $|j-i| > 1$. Equa-

Received by the editors March 28, 1956 and in revised form July 10, 1956.

(1) This work was supported in part by the Office of Naval Research under contract N onr-220(11).

tion (1.1) is called the *backward* equation. With the aid of additional assumptions about the process it can be shown that the equation

$$P'(t) = P(t)A, \qquad t \geqq 0, \tag{1.2}$$

called the *forward* equation, is also satisfied. In any case the initial condition

$$P(0) = I, \qquad \text{the identity matrix,} \tag{1.3}$$

is satisfied.

The main purpose of this paper is to study the existence, uniqueness, and the properties of the matrices $P(t)$ which satisfy (1.1), (1.2), (1.3) and certain auxiliary conditions. As will be shown later, for given A there are always infinitely many matrices which satisfy (1.1), (1.2) and (1.3). Consequently one is led to look for additional properties which may be used to pick out those matrices $P(t)$ which may serve as transition probability matrices. Two such properties are

$$P_{ij}(t) \geq 0, \tag{1.4}$$

$$\sum_{j=0}^{\infty} P_{ij}(t) \leqq 1. \tag{1.5}$$

The inequality in (1.5) expresses the possibility that the diffusing particle may disappear, either by going to infinity or by absorption at the zero state in case μ_0 is positive. Another property, here called the *semi-group property* is expressed by the Chapman-Kolmogoroff equation

$$P_{ij}(t + s) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(s). \tag{1.6}$$

It has recently been shown by Feller [2] that the forward equation is in general considerably more complicated than (1.2). The more complicated forward equations correspond to processes with a state at infinity from which a return to the finite states may occur with positive probability. No attempt is made here to construct the transition matrix of the most general such process, but it turns out that there are interesting families of such processes for which the forward equation, when the state at infinity is disregarded, is exactly (1.2). For these special processes the inequality

$$P_{ij}(t + s) \geq \sum_{0 \leq k \leq \infty} P_{ik}(t)P_{kj}(s), \qquad i, j < \infty$$

is satisfied.

In principle the method employed is to look for an integral representation of the matrix $P(t)$ in terms of the eigenvectors of A. This point of view leads to the revelation of a very intimate connection between the theory of birth-and-death processes and the theory of the Stieltjes moment problem. The study of the relationship between these two theories vastly enriches our knowledge of birth-and-death processes and at the same time produces important new properties of the moment problem and its associated system of orthogonal polynomials.

The time dependence of the transition probabilities is displayed in a particularly simple and lucid manner in the integral representation. This feature of the representation is crucial for the further study of birth-and-death processes. It is found that many questions of limiting behavior as $t \to \infty$ are reduced at once to trivialities or to relatively simple analytical problems.

The main results of the present paper were summarized in [8]. It has since been brought to our attention that integral representations for birth-and-death processes were also discovered by Reuter and Ledermann [9]. These authors used a method of passage to the limit from a system with a finite number of states, and obtained the integral representation of the minimal solution and of one other solution. By a similar limiting process an integral representation of the transition matrix of a random walk was found in a number of interesting cases by Kac [7]. The general representation formula for random walks was described by the authors in [8], and will be discussed in detail in a forthcoming paper. Integral representations for a special class of one-dimensional diffusion processes were found by Hille [6], and recently more general results have been obtained by McKean.

2. **Outline of the method and results.** It is assumed that the coefficients λ_i, $i \geqq 0$ and μ_i, $i > 0$ are *strictly positive* and that $\mu_0 \geqq 0$. If μ_0 is positive it may be interpreted as the rate of absorption from the zero state into a minus-one state which has the property that when the particle arrives in that state it remains there ever afterward. The recurrence relations

$$\begin{aligned} -xQ_0(x) &= -(\lambda_0 + \mu_0)Q_0(x) + \lambda_0 Q_1(x), \\ -xQ_n(x) &= \mu_n Q_{n-1}(x) - (\lambda_n + \mu_n)Q_n(x) + \lambda_n Q_{n+1}(x), \qquad n \geqq 1, \end{aligned}$$

or more compactly,

$$-xQ = AQ,$$

together with the normalizing condition

$$Q_0(x) \equiv 1$$

determine a sequence $\{Q_n(x)\}$ of polynomials. It is shown in Chapter II that these polynomials are the orthogonal polynomials of a solvable Stieltjes moment problem. That is, there is at least one positive regular measure ψ on $0 \leqq x < \infty$, of total mass one, with respect to which the polynomials are orthogonal.

The integral representation will now be derived in a *purely formal* way. One forms the sequence of functions

$$f_i(x, t) = \sum_{j=0}^{\infty} P_{ij}(t)Q_j(x)$$

or equivalently the vector

$$f(x, t) = P(t)Q(x).$$

This vector satisfies the equation

$$\frac{\partial f(x, t)}{\partial t} = P'(t)Q(x) = P(t)AQ(x) = -\, xf(x, t),$$

and the initial condition

$$f(x, 0) = Q(x).$$

Hence

$$f(x, t) = e^{-xt}Q(x)$$

or

$$f_i(x, t) = e^{-xt}Q_i(x).$$

Now $P_{ij}(t)$ is the jth Fourier coefficient of $f_i(x, t)$ and hence

$$P_{ij}(t) = \pi_j \int_0^{\infty} e^{-xt}Q_i(x)Q_j(x)d\psi(x) \tag{1.7}$$

where

$$\int_0^{\infty} Q_j^2(x)d\psi(x) = \frac{1}{\pi_j}\cdot$$

This is the integral representation formula.

No attempt is made to rigorize the above construction. Instead, after a preliminary investigation of the polynomials $Q_n(x)$, the restrictions which must be placed on ψ in order that the matrix $P(t)$ defined by (1.7) should have all the desirable properties are investigated. This is followed by a separate proof that a suitable matrix $P(t)$ is representable in the form (1.7).

In Chapter II the correspondence between the set of all matrices A belonging to birth-and-death processes and the set of all solvable Stieltjes moment problems is established. It transpires that the processes with $\mu_0=0$ generate all Stieltjes moment problems, and those with $\mu_0>0$ generate all Stieltjes moment problems which have a solution with a finite moment of order minus one. The remainder of Chapter II contains a summary of the elementary facts concerning the Hamburger moment problem and a survey of the important properties of the polynomials $Q_n(x)$ and several related systems of polynomials.

The first theorem of Chapter III is the assertion of a new positivity property of orthogonal polynomials. This theorem, which is of independent analytical interest, is the main tool for much of the later work. The relationship between the properties of ψ as a solution of the moment problem and the properties (1.1)–(1.6) for the corresponding matrix $P(t)$ determined by (1.7) are then investigated. Finally, by studying the properties of $P(t)$ as a semi-group of operators acting on a certain Hilbert space natural to the problem, it is shown that any matrix $P(t)$ with properties (1.1)–(1.6) has a representation of the form (1.7). The reader will find that these problems are treated in somewhat greater generality than is indicated here.

In Chapter IV the behavior under passage to the limit from a system with a finite number of states is considered. In this way some linearly ordered one-parameter families of solutions $P(t)$ are discovered. The results are used to obtain necessary and sufficient conditions for uniqueness of the matrix $P(t)$, and to prove a new theorem about completeness of the orthogonal polynomials. Even in the case when there is only one matrix which satisfies (1.1)–(1.5), the equality in (1.5) may fail to be satisfied. A necessary and sufficient condition for the equality to hold is given in this chapter.

In Chapter V the total positivity of the matrices $P(t)$ is studied. This is an analytical property of matrices (and continuous kernels) which is of fundamental importance for diffusion processes, but which has not previously been studied in this connection. The subdeterminants of the transition matrix $P(t)$ are positive when the path functions of the process are "continuous." Even when the path functions are not continuous the subdeterminants have an important probabilistic significance which is developed in a companion paper.

The value of the representation (1.7) lies in the facts that (i) the time dependence is contained entirely in the simple monotonic factor e^{-xt} of the integrand; and (ii) the dependence on i and j is also "factorized." The probabilistic consequences of the representation are investigated in the companion paper. The present paper is devoted to a purely analytical study of the basic properties of the representation. Many of the results were motivated by probabilistic considerations but on the other hand some of the main theorems were discovered first as analytical theorems and now lead to new ideas of probabilistic significance.

Chapter II. The related systems of orthogonal polynomials

1. The moment problem and the integral representation. The sequence of polynomials $\{Q_n(x)\}$ defined by the recurrence relations

$$
\tag{2.1}
\begin{cases}
Q_0(x) \equiv 1, \\
-xQ_0(x) = -(\lambda_0 + \mu_0)Q_0(x) + \lambda_0 Q_1(x), \\
-xQ_n(x) = \mu_n Q_{n-1}(x) - (\lambda_n + \mu_n)Q_n(x) + \lambda_n Q_{n+1}(x), \qquad n \geqq 1,
\end{cases}
$$

are called the polynomials belonging to the matrix A. Because each λ_n is positive, $Q_n(x)$ is a polynomial in x of exact degree n, the coefficient of x^n in $Q_n(x)$ being $(-1)^n/\lambda_0\lambda_1 \cdots \lambda_{n-1}$.

The quantities

$$\pi_0 = 1, \qquad \pi_n = \frac{\lambda_0\lambda_1 \cdots \lambda_{n-1}}{\mu_1\mu_2 \cdots \mu_n}, \qquad n \geqq 1, \tag{2.2}$$

play a very fundamental role in the theory of the differential equations (1.1) and (1.2). Since $\lambda_n\pi_n = \mu_{n+1}\pi_{n+1}$ the recurrence relation can be written in the form

$$\begin{aligned} -xQ_0(x)\pi_0 &= \lambda_0\pi_0[Q_1(x) - Q_0(x)] - \mu_0 \\ -xQ_n(x)\pi_n &= \lambda_n\pi_n[Q_{n+1}(x) - Q_n(x)] - \lambda_{n-1}\pi_{n-1}[Q_n(x) - Q_{n-1}(x)], \qquad n \geqq 1. \end{aligned}$$

Consequently

$$-x\sum_{j=0}^{n} Q_j(x)\pi_j = \lambda_n\pi_n[Q_{n+1}(x) - Q_n(x)] - \mu_0 \tag{2.3}$$

and it follows by an induction that for $x<0$

$$1 = Q_0(x) < Q_1(x) < \cdots < Q_n(x) < Q_{n+1}(x) < \cdots.$$

If $n \geqq 1$

$$Q_n(0) = 1 + \mu_0 \sum_{k=0}^{n-1} \frac{1}{\lambda_k\pi_k}, \tag{2.4}$$

and hence the above inequalities are also valid when $x=0$ if $\mu_0>0$. The following theorem is an extension of a result of Favard [1].

THEOREM 1. *There is at least one positive regular measure ψ on $0 \leqq x < \infty$, of total mass one, not supported by a finite set of points, such that*

$$\int_0^\infty Q_i(x)Q_j(x)d\psi(x) = \begin{cases} 0, & \text{if } i \neq j, \\ \dfrac{1}{\pi_i}, & \text{if } i = j. \end{cases} \tag{2.5}$$

Proof. The system of equations $\int Q_0 d\psi = 1$, $\int Q_n d\psi = 0$, $n>0$, can be solved recursively for the moments $c_n = \int x^n d\psi$. For example $\int Q_0 d\psi = 1$ gives $c_0 = 1$, and then

$$0 = \int Q_1 d\psi = \int \frac{\lambda_0 + \mu_0 - x}{\lambda_0} d\psi$$

gives $c_1 = \lambda_0 + \mu_0$. There are always [12] infinitely many regular, but not necessarily positive, measures on $0 \leqq x < \infty$ which have these moments. In the remainder of the proof ψ denotes one of these measures.

From the recurrence formula it follows that if $n \geqq 1$ then

$$\int_0^\infty x^k Q_n(x) d\psi(x) = 0, \qquad 0 \leqq k < n,$$

and

$$\int_0^\infty (-x)^n Q_n(x) d\psi(x) = \mu_n \int_0^\infty (-x)^{n-1} Q_{n-1}(x) d\psi(x).$$

Consequently

$$\int_0^\infty Q_m(x) Q_n(x) d\psi(x) = \frac{\delta_{m,n}}{\pi_n}, \qquad m, n \geqq 0.$$

To prove the theorem it is therefore sufficient [12, p. 6] to show that all of the determinants

$$(2.6) \qquad \Delta_n = \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ c_n & c_{n+1} & \cdots & c_{2n} \end{vmatrix}, \qquad \Delta_n^{(1)} = \begin{vmatrix} c_1 & c_2 & \cdots & c_{n+1} \\ c_2 & c_3 & \cdots & c_{n+2} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ c_{n+1} & c_{n+2} & \cdots & c_{2n+1} \end{vmatrix}, \qquad n \geq 0$$

are strictly positive.

For $n \geqq 1$ the determinant

$$P_n(x) = \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} & 1 \\ c_1 & c_2 & \cdots & c_n & x \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ c_n & c_{n+1} & \cdots & c_{2n-1} & x^n \end{vmatrix}$$

is a polynomial of degree $\leqq n$ and

$$\int_0^\infty x^k P_n(x) d\psi(x) = 0, \qquad 0 \leqq k < n.$$

Hence $P_n(x)$ is a constant multiple of $Q_n(x)$, and comparing coefficients of x^n

$$P_n(x) = (-1)^n (\lambda_0 \lambda_1 \cdots \lambda_{n-1}) \Delta_{n-1} Q_n(x).$$

It follows that

$$(2.7) \qquad \Delta_n = \int_0^\infty x^n P_n(x) d\psi(x) = \frac{(\lambda_0 \lambda_1 \cdots \lambda_{n-1})^2}{\pi_n} \Delta_{n-1},$$

and since $\Delta_0 = 1$, that $\Delta_n > 0$, $n = 1, 2, \cdots$. Moreover,

$$\Delta_n^{(1)} = (-1)^{n+1} P_{n+1}(0) = (\lambda_0\lambda_1 \cdots \lambda_n)\Delta_n Q_{n+1}(0) > 0. \tag{2.8}$$

This completes the proof.

From (2.7) and (2.8) it is found that for $n \geqq 1$

$$\frac{(\Delta_{n-1}^{(1)})^2}{\Delta_{n-1}\Delta_n} = \pi_n Q_n^2(0), \tag{2.9}$$

$$\frac{(\Delta_n)^2}{\Delta_{n-1}^{(1)}\Delta_n^{(1)}} = \frac{1}{\lambda_n \pi_n Q_n(0) Q_{n+1}(0)}. \tag{2.10}$$

Hereafter a *positive* measure with the properties required by the above theorem will be called a solution of the S moment problem. A not necessarily positive measure on $0 \leqq x < \infty$ with the same moments will be called a solution of the BVS moment problem. A positive (not necessarily positive) measure on $-\infty < x < \infty$ with the same moments will be called a solution of the H (BVH) moment problem. S and H are abbreviations of Stieltjes and Hamburger respectively while B.V. suggests bounded variation. A measure on an interval $a \leqq x < \infty$ where $a > -\infty$ will be said to have left-bounded support.

The above theorem establishes a correspondence between a given birth-and-death matrix A and a solvable Stieltjes moment problem. There is a converse theorem. Suppose $\{Q_n(x)\}$ is a sequence of real polynomials, the nth polynomial being of exact degree n, orthogonal on $0 \leqq x < \infty$ with respect to a positive measure ψ. The zeros of the polynomials are then interior to the interval $0 \leqq x < \infty$ and it can therefore be assumed that $Q_n(0) = 1$ for every n. The polynomials satisfy a recurrence formula [13, p. 41]

$$\begin{aligned} -xQ_0(x) &= B_0 Q_0(x) + C_0 Q_1(x), \\ -xQ_n(x) &= A_n Q_{n-1}(x) + B_n Q_n(x) + C_n Q_{n+1}(x), \qquad n \geqq 1 \end{aligned}$$

where A_n, B_n, C_n are real. Because of the normalization $B_0 + C_0 = 0$ and $A_n + B_n + C_n = 0$, $n \geqq 1$. The coefficient of x^n in $Q_n(x)$ is $(-1)^n / C_0 C_1 \cdots C_n$ and since $Q_n(0) = 1$ this is equal to $(-1)^n$ multiplied by the product of the reciprocals of the n positive roots of $Q_n(x)$. Consequently $C_n > 0$ for $n \geqq 1$. If $n \geqq 1$ then

$$\int_0^\infty (-x)^n Q_n(x) d\psi(x) = A_n / C_{n-1} \int_0^\infty (-x)^{n-1} Q_{n-1}(x) d\psi(x)$$

so $A_n > 0$. Hence the recurrence formula of the polynomials $Q_n(x)$ determines a birth-and-death matrix with $\mu_0 = 0$. The precise conditions under which it is possible to renormalize the polynomials so that the recurrence formula determines a birth-and-death matrix with $\mu_0 > 0$, are given by the following lemma.

LEMMA 1. *Suppose $\mu_0=0$ and let $\{Q_n(x)\}$ be the sequence of polynomials determined by* (2.1). *Let μ be a given positive number. Then there is a sequence of positive constants $\{\alpha_n\}$ such that the polynomials $R_n(x)=\alpha_n Q_n(x)$ satisfy a recurrence relation of the form*

$$\begin{aligned} R_0(x) &\equiv 1, \\ -xR_0(x) &= -(\lambda_0' + \mu_0')R_0(x) + \lambda_0' R_1(x), \\ -xR_n(x) &= \mu_n' R_{n-1}(x) - (\lambda_n' + \mu_n')R_n(x) + \lambda_n' R_{n+1}(x), \qquad n \geqq 1, \\ &\qquad\qquad \mu_n' > 0,\ \lambda_n' > 0, \end{aligned}$$

with $\mu_0'=\mu$, if and only if the series $\sum_0^\infty (1/\lambda_n\pi_n)$ converges and μ satisfies

$$\mu \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n} \leqq 1.$$

Proof. Let $\{\alpha_n\}$ be any sequence of positive constants and $R_n(x)=\alpha_n Q_n(x)$. Then

$$\begin{aligned} R_0(x) &\equiv \alpha_0, \\ -xR_0 &= -\lambda_0 R_0 + \lambda_0 \frac{\alpha_0}{\alpha_1} R_1, \\ -xR_n &= \mu_n \frac{\alpha_n}{\alpha_{n-1}} R_{n-1} - (\lambda_n + \mu_n)R_n + \lambda_n \frac{\alpha_n}{\alpha_{n+1}} R_{n+1}. \end{aligned}$$

This recurrence formula is of the required type if and only if

$$\begin{aligned} \alpha_0 &= 1, \\ \lambda_0 \frac{\alpha_0}{\alpha_1} + \mu &= \lambda_0, \\ \lambda_n \frac{\alpha_n}{\alpha_{n+1}} + \mu_n \frac{\alpha_n}{\alpha_{n-1}} &= \lambda_n + \mu_n, \qquad n \geqq 1. \end{aligned}$$

Let

$$r_0 = \frac{\mu}{\lambda_0}, \qquad r_n = \frac{\mu_n}{\lambda_n}, \qquad s_n = \frac{\alpha_n}{\alpha_{n-1}}$$

and

$$\begin{aligned} t_n &= r_0 + r_0 r_1 + r_0 r_1 r_2 + \cdots + r_0 r_1 \cdots r_n \\ &= \mu \sum_{i=1}^{n} \frac{1}{\lambda_i \pi_i} \cdot \end{aligned}$$

Then the above relations may be written

$$(*)\qquad \begin{aligned} 1 - \frac{1}{s_1} &= r_0, \\ 1 - \frac{1}{s_{n+1}} &= r_n(s_n - 1), \qquad n \geqq 1. \end{aligned}$$

The s_n are positive and $s_1 = 1/(1-r_0) > 1$, so it follows by induction that $s_n > 1$ for all n. Expressing s_{n+1} in terms of the r_i gives

$$s_{n+1} = 1 + \frac{r_0 r_1 \cdots r_n}{1 - t_n}.$$

Hence * has a solution with $s_n > 1$ for all n if and only if $t_n < 1$ for every n, that is, if and only if

$$\mu \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n} \leqq 1.$$

REMARK. Using (2.10) the condition that $\sum(1/\lambda_n\pi_n)$ converge can be expressed in terms of the moments. It will be seen later (Chapter IV, especially Lemma 6) that the minimum value of $\int_0^\infty d\psi/x$ as ψ ranges over all solutions of the S moment problem is attained for a certain solution $\psi_{\min}$ and $\int_0^\infty d\psi_{\min}/x = \sum_0^\infty (1/\lambda_n\pi_n)$.

THEOREM 2. *Let ψ be any solution of the BVH moment problem with left bounded support and suppose the integrals*

$$(2.11)\qquad \int_{-\infty}^{\infty} x^n d\psi(x), \qquad n = 0, 1, 2, \cdots$$

are all absolutely convergent. Then the matrix $P(t) = P(t; \psi)$ defined by

$$P_{ij}(t) = \pi_j \int_{-\infty}^{\infty} e^{-xt} Q_i(x) Q_j(x) d\psi(x)$$

is (componentwise) analytic in the half-plane Re $t > 0$, *continuous in the half-plane* Re $t \geqq 0$, *and satisfies* (1.1), (1.2), (1.3). *If ψ, ψ_1 are two such measures and for some i, j and all t in an interval $a \leqq t \leqq b$ $(0 \leqq a < b)$, $P_{ij}(t; \psi) = P_{ij}(t; \psi_1)$ then $\psi = \psi_1$.*

Proof. The analyticity and continuity properties are apparent, and the orthogonality condition (2.5) shows that $P(t; \psi)$ satisfies (1.3). The derivatives $P'_{ij}(t)$ can be computed by differentiation under the integral sign:

$$P'_{ij}(t) = \pi_j \int_{-\infty}^{\infty} e^{-xt}(-x) Q_i(x) Q_j(x) d\psi(x).$$

Using the recurrence formula for $-xQ_i(x)$ shows that P satisfies (1.1), and using the recurrence formula for $-xQ_j(x)$ and the relations $\lambda_n\pi_n=\mu_{n+1}\pi_{n+1}$ shows that P satisfies (1.2).

To prove the uniqueness statement, observe that for the special value of i, j the analytic function $P_{ij}(t;\psi)-P_{ij}(t;\psi_1)$ is identically zero, and hence by the uniqueness theory of Laplace transforms the measure $\psi-\psi_1$ is supported by the roots of the polynomial $Q_i(x)Q_j(x)$. Since all the moments of $\psi-\psi_1$ are zero it follows that $\psi=\psi_1$.

Since there are always infinitely many solutions of the BVH moment problem which have left bounded support and for which the integrals (2.11) are absolutely convergent, the above theorem shows that there are always infinitely many matrices $P(t)$ which satisfy (1.1), (1.2) and (1.3). Conditions on ψ under which $P(t;\psi)$ will also satisfy (1.4), (1.5) and (1.6) are discussed in Chapter III. The remainder of this chapter is devoted to a review of the elementary facts concerning the Hamburger moment problem, and to an analysis of some of the properties of the polynomials $Q_n(x)$ and of certain related systems of polynomials.

2. **The H moment problem.** We first summarize a number of results which are proved in [12, pp. 23–76].

DEFINITIONS. A measure ψ_n is called a *distribution of order* $n+1$ associated with the H moment problem if it consists of $n+1$ masses located on the real axis and has the correct moments of orders $\leqq 2n$; that is

$$\int_{-\infty}^{\infty} x^k d\psi_n = c_k, \qquad k = 0, 1, \cdots, 2n.$$

A polynomial q is called a (real) *quasi-orthogonal polynomial of degree* $n+1$ associated with the H moment problem if it is of the form

$$q(x) = AQ_{n+1}(x) + BQ_n(x)$$

where A and B are real and $A\neq 0$. (This definition differs slightly from that of [12, p. 35].)

(i) A quasi-orthogonal polynomial of degree $n+1$ has $n+1$ real simple roots; two such polynomials have a common root if and only if one polynomial is a multiple of the other. For any real x_0 which is not a root of $Q_n(x)$ there is a quasi-orthogonal polynomial of degree $n+1$ which vanishes at x_0.

(ii) If ψ_n is a distribution of order $n+1$ then its support is the set of zeros of a quasi-orthogonal polynomial of degree $n+1$. Conversely, if q is a quasi-orthogonal polynomial of degree $n+1$, then there is a distribution of order $n+1$ supported by the set of zeros of q.

(iii) Let

$$\rho(x) = 1 \Big/ \sum_{k=0}^{\infty} Q_k^2(x)\pi_k \tag{2.12}$$

if the series in the denominator converges, and let $\rho(x)=0$ otherwise. For each real x_0 the maximal mass which a solution of the H moment problem may have at x_0 is $\rho(x_0)$. If $\rho(x_0)>0$ there is one and only one solution of the H moment problem with mass $\rho(x_0)$ at x_0.

(iv) Let ψ_n be the distribution of order $n+1$ which has mass at a fixed point x_0. Then as $n\to\infty$, ψ_n converges to the solution ψ of the H moment problem which has the maximal mass $\rho(x_0)$ at x_0, in the sense that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)d\psi_n(x) = \int_{-\infty}^{\infty} f(x)d\psi(x)$$

for every continuous function f which vanishes at infinity.

The first zero $\xi_{1,n}$ of $Q_n(x)$ is a decreasing function of n and hence tends to a limit $\xi_1\geqq 0$ as $n\to\infty$. The distribution of order $n+1$ supported by the zeros of Q_{n+1} converges as $n\to\infty$ to the solution of the H moment problem with the maximal mass $\rho(\xi_1)$ at ξ_1. This solution is of course also a solution of the S moment problem.

(v) Let q be any quasi-orthogonal polynomial of degree $n+1$ and let $0\leqq i<n+1$. Let $\xi_{1,i}<\xi_{2,i}<\cdots<\xi_{i,i}$ be the roots of $Q_i(x)$. Then each of the $i+1$ open intervals

$$\begin{aligned} -\infty &< x < \xi_{1,i}, \\ \xi_{k,i} &< x < \xi_{k+1,i}, \qquad 1 \leqq k < i, \\ \xi_{i,i} &< x < +\infty \end{aligned}$$

contains at least one zero of q. This precise statement will not be found in [12]. The proof goes as follows. Let ψ_n be the distribution of order $n+1$ supported by the zeros of q. If ψ_n has no mass in one of the above open intervals, then there is a polynomial $f(x)$ of degree $<i$ which vanishes only at zeros of $Q_i(x)$, such that $f(x)Q_i(x)\geqq 0$ at each root of q.

Since

$$\int f(x)Q_i(x)d\psi_n(x) = 0, \qquad \int Q_i^2(x)d\psi_n(x) \neq 0,$$

this is impossible.

(vi) Let q be any quasi-orthogonal polynomial of degree $n+1$, and $\eta_1,\cdots,\eta_{n+1}$ be its zeros. If ψ_n is the corresponding distribution of order $n+1$ with mass γ_i at η_i, then the polynomial

$$p(x) = q(x)\sum_{i=1}^{n+1}\frac{\gamma_i}{x-\eta_i} \tag{2.13}$$

is representable in the form

$$p(x) = \int_{-\infty}^{\infty}\frac{q(x)-q(t)}{x-t}\,d\psi(t) \tag{2.14}$$

where ψ is any solution of the H moment problem or even any measure with the correct moments of orders $\leqq n$. $p(x)$ is called the numerator of the quasi-orthogonal polynomial $q(x)$, and in the sequel will be denoted by $q^{(0)}(x)$. The mapping $q(x)\rightarrow q^{(0)}(x)$ is induced by a linear transformation in the space of polynomials.

(vii) A solution ψ of the H moment problem is called *extremal* if the Parseval equation

$$\int_{-\infty}^{\infty} |f(x)|^2 d\psi(x) = \sum_{n=0}^{\infty} \left| \int_{-\infty}^{\infty} f(x)Q_n(x)d\psi(x)\right|^2 \pi_n$$

is valid for every function f in $L_2(\psi)$. If the solution of the H moment problem is unique then it is an extremal solution; if it is not unique then there is a one-parameter family of extremal solutions. In the latter case each extremal solution is a "step function," i.e., it consists of a countable number of positive masses located at the points of a discrete set on the real axis. The extremal solution with mass at a point x_0 has the maximal possible mass $\rho(x_0)$ at that point. In the case under consideration here, the S moment problem has a solution, and every extremal solution of the H moment problem has left bounded support. If the solution of the S moment problem is unique, it is an extremal solution (of the H moment problem); if it is not unique then there is a one-parameter family of solutions of the S moment problem which are extremal. In fact, if the solution is not unique then the first zero $\xi_{1,n}$ of $Q_n(x)$ converges to a positive limit ξ as $n\rightarrow\infty$. For each x_0 in the closed interval $0\leqq x_0\leqq \xi$ there is an extremal solution of the S moment problem with mass at x_0, and no extremal solution has mass at two points of this interval. Furthermore this accounts for all the extremal solutions of the S moment problem.

The final three sections of this chapter list a series of formulas of fundamental use throughout the paper. Aside from Lemmas 2 and 3, the results are essentially known.

3. **Properties of the polynomials.** The polynomials

$$\left\{\begin{aligned} H_0(x) &\equiv \mu_0, \\ H_{n+1}(x) &= \lambda_n\pi_n[Q_{n+1}(x) - Q_n(x)], \qquad n \geqq 0, \end{aligned}\right. \tag{2.15}$$

satisfy the recurrence relation

$$\left\{\begin{aligned} -xH_0(x) &= -\mu_0H_0(x) + \mu_0H_1(x), \\ -xH_{n+1}(x) &= \lambda_nH_n(x) - (\lambda_n + \mu_{n+1})H_{n+1}(x) + \mu_{n+1}H_{n+2}(x). \end{aligned}\right. \tag{2.16}$$

The polynomials

$$Q_n^{(0)}(x) = \int \frac{Q_n(x) - Q_n(y)}{x - y}\, d\psi(y), \qquad n = 0, 1, \cdots \tag{2.17}$$

(ψ is any solution of the moment problem) satisfy

$$Q_0^{(0)}(x) = 0,$$
$$Q_1^{(0)}(x) = -\frac{1}{\lambda_0}$$

$$(2.18)\qquad \begin{cases} -xQ_0^{(0)}(x) = -(\lambda_0+\mu_0)Q_0^{(0)}(x) + \lambda_0 Q_1^{(0)}(x) + 1, \\ -xQ_n^{(0)}(x) = \mu_n Q_{n-1}^{(0)}(x) - (\lambda_n+\mu_n)Q_n^{(0)}(x) + \lambda_n Q_{n+1}^{(0)}(x). \end{cases}$$

The polynomials

$$(2.19)\qquad H_{n+1}^{(0)}(x) = \int \frac{H_{n+1}(x) - H_{n+1}(y)}{x-y}\, d\psi(y), \qquad n = 0, 1, \cdots$$

satisfy

$$H_{n+1}^{(0)}(x) = \lambda_n\pi_n[Q_{n+1}^{(0)}(x) - Q_n^{(0)}(x)],$$

and

$$H_0^{(0)}(x) \equiv 0,$$
$$H_1^{(0)}(x) \equiv -1,$$

$$(2.20)\qquad \begin{cases} -xH_0^{(0)}(x) = -\mu_0 H_0^{(0)}(x) + \mu_0 H_1^{(0)}(x) + \mu_0, \\ -xH_1^{(0)}(x) = \lambda_0 H_0^{(0)}(x) - (\lambda_0+\mu_1)H_1^{(0)} + \mu_1 H_2^{(0)}(x) - \lambda_0, \\ -xH_{n+1}^{(0)}(x) = \lambda_n H_n^{(0)}(x) - (\lambda_n+\mu_{n+1})H_{n+1}^{(0)}(x) + \mu_{n+1}H_{n+2}^{(0)}(x), \quad n \geqq 1. \end{cases}$$

The following identities are easy consequences of the definitions and the recurrence formulas

$$(2.21)\qquad Q_n(x) = 1 + \sum_{k=0}^{n-1} H_{k+1}(x)\frac{1}{\lambda_k\pi_k},$$

$$(2.22)\qquad Q_n^{(0)}(x) = \sum_{k=0}^{n-1} H_{k+1}^{(0)}(x)\frac{1}{\lambda_k\pi_k},$$

$$(2.23)\qquad H_{n+1}(x) = \mu_0 - x\sum_{k=0}^{n} Q_k(x)\pi_k,$$

$$(2.24)\qquad H_{n+1}^{(0)}(x) = -1 - x\sum_{k=0}^{n} Q_k^{(0)}(x)\pi_k,$$

$$(2.25)\qquad \begin{aligned} (x-y)\sum_{k=0}^{n} Q_k(x)Q_k(y)\pi_k &= \lambda_n\pi_n[Q_{n+1}(y)Q_n(x) - Q_{n+1}(x)Q_n(y)] \\ &= H_{n+1}(y)Q_n(x) - H_{n+1}(x)Q_n(y), \end{aligned}$$

$$(2.26)\qquad \begin{aligned} H_{n+1}(x)Q_n^{(0)}(x) - H_{n+1}^{(0)}(x)Q_n(x) &= \lambda_n\pi_n[Q_{n+1}(x)Q_n^{(0)}(x) - Q_{n+1}^{(0)}(x)Q_n(x)] \\ &\equiv 1. \end{aligned}$$

Inspection of the recurrence formulas shows that the two systems of polynomials $\{Q_n^{(0)}(x)\}$, $n \geqq 1$ and $\{H_{n+1}^{(0)}(x)\}$, $n \geqq 0$, are orthogonal systems on $0 \leqq x < \infty$. If $\mu_0 > 0$ the system $\{H_k(x)\}$, $k \geqq 0$ is an orthogonal system on $0 \leqq x < \infty$, while if $\mu_0 = 0$ the system $\{H_{k+1}(x)/-x\}$ is a system of polynomials orthogonal on $0 \leqq x < \infty$. In either case there is a natural correspondence between solutions ψ of the original S moment problem and measures θ relative to which the H system is orthogonal. This correspondence is expressed in the next two lemmas.

LEMMA 2. *Suppose $\mu_0 > 0$. Then there is at least one positive regular measure θ on $0 \leqq x < \infty$ such that*

$$\int_0^\infty \frac{H_m(x)}{\mu_0} \frac{H_n(x)}{\mu_0} d\theta(x) = \begin{cases} 0 & \text{if } m \neq n, \\ \dfrac{1}{\pi_m'}, & \text{if } m = n \end{cases} \tag{2.27}$$

where $\pi_0' = 1$, $\pi_n' = \mu_0/\lambda_{n-1}\pi_{n-1}$, $n \geqq 1$. If θ is such a measure then

$$d\psi(x) = \frac{x d\theta(x)}{\mu_0}$$

defines a solution ψ of the original S moment problem for which

$$\mu_0 \int_0^\infty \frac{d\psi(x)}{x} \leqq 1. \tag{2.28a}$$

Conversely if ψ is a solution of the original S moment problem such that (2.28a) *is valid, then the measure θ, which consists of a mass $1 - \mu_0 \int_0^\infty d\psi/x$ located at $x = 0$ and is defined on $0 < x < \infty$ by $d\theta(x) = \mu_0 d\psi(x)/x$, satisfies* (2.27).

Proof. From (2.23)

$$x Q_n(x) \pi_n = H_n(x) - H_{n+1}(x)$$

and hence if θ satisfies (2.27) then

$$\begin{aligned} \pi_n \int_0^\infty x^k Q_n(x) \frac{x d\theta(x)}{\mu_0} &= \int_0^\infty x^k \left[\frac{H_n(x)}{\mu_0} - \frac{H_{n+1}(x)}{\mu_0} \right] d\theta(x) \\ &= 0 \qquad \text{for } k = 0, 1, \cdots, n-1, \end{aligned}$$

and the integral is not zero if $k = n$. It follows at once that $d\psi = x d\theta/\mu_0$ is a solution of the original S moment problem, and $\mu_0 \int_0^\infty d\psi/x \leqq 1$.

Now suppose ψ is a solution of the original moment problem which satisfies (2.28a) and let θ be the measure described in the second part of the lemma. Then θ is a positive measure and $\int_0^\infty d\theta(x) = 1$. Since $H_n(0) = \mu_0$ for every n

$$\int_0^\infty \frac{H_n(x)}{\mu_0}\,d\theta(x) = 1 - \mu_0\int_0^\infty \frac{d\psi}{x} + \int_0^\infty \left[\mu_0 - x\sum_{k=0}^{n-1} Q_k(x)\pi_k\right]\frac{d\psi(x)}{x},$$

and this is zero if $n \geqq 1$. Moreover if $1 \leqq k < n$

$$\int_0^\infty x^k\,\frac{H_n(x)}{\mu_0}\,d\theta(x) = \lambda_{n-1}\pi_{n-1}\int_0^\infty x^{k-1}[Q_n(x) - Q_{n-1}(x)]d\psi(x)$$
$$= 0,$$

while the integral is not zero if $k = n$. This completes the proof.

LEMMA 3. *If $\mu_0 = 0$ there is at least one positive regular measure θ on $0 \leqq x < \infty$ such that*

$$\int_0^\infty \frac{H_{m+1}(x)}{-x}\cdot\frac{H_{n+1}(x)}{-x}\,d\theta(x) = \frac{\delta_{m,n}}{\pi_n''}, \qquad m, n \geqq 0,$$

where $\pi_0'' = 1$, $\pi_n'' = \lambda_0/\lambda_n\pi_n$ for $n \geqq 1$, and such that

$$\lambda_0\int_0^\infty \frac{d\theta(x)}{x} \leqq 1.$$

If θ is such a measure then the measure ψ which has a mass $1 - \lambda_0\int_0^\infty d\theta/x$ located at $x = 0$, and is defined on $0 < x < \infty$ by $d\psi(x) = \lambda_0 d\theta(x)/x$, is a solution of the original S moment problem. Conversely if ψ is any solution of the original S moment problem then $d\theta(x) = xd\psi(x)/\lambda_0$ defines a measure θ which satisfies the above conditions.

The proof may be made to depend on the preceding lemma, and is omitted.

The convergence properties of the four systems of polynomials Q_n, $Q_n^{(0)}$, H_{n+1}, $H_{n+1}^{(0)}$ were investigated by Stieltjes, who used the following lemma.

LEMMA 4. *The following statements are equivalent:*

(1) *As $n \to \infty$, $Q_n(x)$ converges for every complex x, uniformly in every circle $|x| \leqq R$;*

(2) *$Q_n(x)$ is bounded as $n \to \infty$ for at least one $x < 0$;*

(3) *The series*

$$\sum_{n=0}^\infty \frac{1}{\lambda_n\pi_n}\sum_{i=0}^n \pi_i \tag{2.28}$$

is convergent.

Proof. Let ξ_i, $i = 1, 2, \cdots, n$ be the zeros of $Q_n(x)$. From

$$Q_n(x) = Q_n(0)\prod_{i=1}^n\left(1 - \frac{x}{\xi_i}\right)$$

it is seen that for every complex x

$$(2.29)\qquad \begin{aligned} |Q_n(x)| &\leqq Q_n(-|x|) \\ &\leqq Q_n(0)\left[1+\frac{|x|}{n}\sum\frac{1}{\xi_i}\right]^n \\ &\leqq Q_n(0)e^{|x|\sigma_n} \end{aligned}$$

where

$$\sigma_n = \sum_{i=1}^{n}\frac{1}{\xi_i} = -\frac{Q_n'(0)}{Q_n(0)},$$

and that for $s>0$,

$$(2.30)\qquad Q_n(-s) \geqq Q_n(0)[1+\sigma_n s].$$

Hence in order that $Q_n(x)$ be bounded for some $x<0$ it is necessary that $Q_n(0)$ and σ_n are both bounded, in which case $Q_n(x)$ is bounded for every complex x. Now $Q_n(0)$ is given by (2.4), and from (2.21), (2.23)

$$-Q_n'(0) = \sum_{k=0}^{n-1}\frac{1}{\lambda_k\pi_k}\sum_{i=0}^{k}\pi_i Q_i(0).$$

It follows that (2) and (3) are equivalent. Clearly (1) implies (2). If (2) is valid then $Q_n(x)$ is uniformly bounded in every circle $|x| \leqq R$, and since $Q_n(x)$ is a monotone sequence for each $x<0$, $Q_n(x)$ converges for each $x<0$, and by the Stieltjes-Vitali theorem $Q_n(x)$ converges uniformly in every circle $|x| \leqq R$. Thus (1) and (2) are equivalent.

Applying the above lemma to the other systems of polynomials, the following results are obtained.

(a) If the series (2.28) converges then $Q_n^{(0)}(x)$ converges uniformly in every circle $|x| \leqq R$.

(b) If the series

$$(2.31)\qquad \sum_{n=0}^{\infty}\pi_n\sum_{i=0}^{n-1}\frac{1}{\lambda_i\pi_i}$$

converges then $H_{n+1}(x)$ and $H_{n+1}^{(0)}(x)$ both converge uniformly in every circle $|x| \leqq R$.

4. **The limiting functions.** In this section it will be assumed that both of the series (2.28) and (2.31) converge, or what is the same thing, that both of the series

$$\sum_{n=0}^{\infty}\pi_n,\qquad \sum_{n=0}^{\infty}\frac{1}{\lambda_n\pi_n}$$

converge. Under these circumstances the limits

$$Q_\infty(x) = \lim_{n\to\infty} Q_n(x), \qquad Q_\infty^{(0)}(x) = \lim_{n\to\infty} Q_n^{(0)}(x),$$

$$H_\infty(x) = \lim_{n\to\infty} H_{n+1}(x), \qquad H_\infty^{(0)}(x) = \lim_{n\to\infty} H_{n+1}^{(0)}(x)$$

all exist and are representable by the series

$$Q_\infty(x) = 1 + \sum_{n=0}^{\infty} H_{n+1}(x) \frac{1}{\lambda_n \pi_n}, \tag{2.32}$$

$$Q_\infty^{(0)}(x) = \sum_{n=0}^{\infty} H_{n+1}^{(0)}(x) \frac{1}{\lambda_n \pi_n}, \tag{2.34}$$

$$H_\infty(x) = \mu_0 - x \sum_{n=0}^{\infty} Q_n(x)\pi_n, \tag{2.35}$$

$$H_\infty^{(0)}(x) = -1 - x \sum_{n=0}^{\infty} Q_n^{(0)}(x)\pi_n, \tag{2.36}$$

the convergence in each case being uniform in every circle $|x| \leqq R$. The limiting functions are entire functions and satisfy the identities

$$(x - y) \sum_{n=0}^{\infty} Q_n(x)Q_n(y)\pi_n = H_\infty(y)Q_\infty(x) - H_\infty(x)Q_\infty(y), \tag{2.37}$$

$$H_\infty(x)Q_\infty^{(0)}(x) - H_\infty^{(0)}(x)Q_\infty(x) = 1 \tag{2.38}$$

where the series in (2.37) converges uniformly in every circle.

Since the only zeros of the polynomials are on $0 \leqq x < \infty$ it follows from Hurwitz' theorem that the only zeros of the limiting functions are on $0 \leqq x < \infty$. Moreover any zero of $Q_\infty(x)$ of order $k \geqq 2$ is a limit of k zeros of $Q_n(x)$, and because of the interlacing of zeros, is also a limit of zeros of $H_{n+1}(x)$, so it is also a zero of $H_\infty(x)$. On the other hand (2.38) shows that Q_∞ and H_∞ have no common zeros. Hence all the zeros of Q_∞ are simple zeros. Similar arguments show that the functions $Q_\infty^{(0)}$, H_∞, $H_\infty^{(0)}$ have only simple zeros.

Between each pair of successive zeros of $H_{n+1}(x)$ there is exactly one zero of $Q_n(x)$. Hence each pair of successive zeros of $H_\infty(x)$ are separated by exactly one zero of $Q_\infty(x)$.

An entire function $f(x)$ is said to be at most of order one and of minimal type if

$$f(x) = e^{|x| \cdot \delta(|x|)}$$

where $\delta(|x|)$ is a bounded function which is $o(1)$ as $|x| \to \infty$. It is shown in [12] that the functions Q_∞, $Q_\infty^{(0)}$, H_∞, $H_\infty^{(0)}$ are at most of order one and of minimal type. The referee has observed that in the present case these functions are of order $\leqq 1/2$. For example, suppose $\mu_0 = 0$, and let $Q_n(x)$

$=\sum_{k=0}^{n} a_{n,k}(-x)^k$. Then $a_{n,0}=1$ for all n and (2.21), (2.23) give for $n \geqq k \geqq 1$

$$a_{n,k} = \sum_{i=0}^{n-1} \frac{1}{\lambda_i \pi_i} \sum_{j=0}^{i} \pi_j a_{j,k-1}.$$

A simple computation shows that $0 \leqq a_{n,k} \leqq C^k/(k!)^2$, where $C=\left(\sum_{i=0}^{\infty}(1/\lambda_i\pi_i)\right) \cdot \left(\sum_{j=0}^{\infty}\pi_j\right)$, and it follows that Q_∞ is of order $\leqq 1/2$.

Chapter III. The auxiliary properties of $P(t, \psi)$

1. The fundamental positivity theorem.

Definition. A set of $m \geqq 1$ real numbers $\eta_1 \leqq \eta_2 \leqq \cdots \leqq \eta_m$ is said to be *separated* by a set of $m+1$ distinct real numbers $\xi_1 < \xi_2 < \cdots < \xi_{m+1}$ if

$$\xi_i \leqq \eta_i \leqq \xi_{i+1}, \qquad i = 1, 2, \cdots, m.$$

Theorem 3. *Let ψ be any solution of the H-moment problem with left bounded support. Then the integrals*

$$\int_{-\infty}^{\infty} e^{-xt} Q_n(x) d\psi(x), \qquad n = 0, 1, 2, \cdots$$

are all strictly positive for $t>0$. Moreover if $n \geqq 2$ and if $p(x)$ is a polynomial of degree r, $1 \leqq r < n$ whose roots are all real and separated by the roots of $Q_n(x)$, and if $p(0)>0$, then the integral

$$\int_{-\infty}^{\infty} e^{-xt} p(x) Q_n(x) d\psi(x),$$

is strictly positive for $t>0$.

In the course of the proof of this theorem the following simple lemma is required.

Lemma 5. *Let $\eta_1 \leqq \eta_2 \leqq \cdots \leqq \eta_n$ be real numbers separated by the numbers $\xi_1 < \xi_2 < \cdots < \xi_{n+1}$. Let*

$$Q(x) = \prod_{i=1}^{n+1} (\xi_i - x),$$

$$p(x) = \prod_{i=1}^{n} (\eta_i - x),$$

$$A_i(x) = \frac{Q(x)}{\xi_i - x}, \qquad i = 1, 2, \cdots, n+1.$$

Then

$$p(x) = \sum_{i=1}^{n+1} \alpha_i A_i(x)$$

where the coefficients α_i are non-negative.

Proof. Since the ξ_i are distinct

$$\frac{p(x)}{Q(x)} = \sum_{i=1}^{n+1} \frac{\alpha_i}{\xi_i - x}$$

where

$$\alpha_i = \frac{p(\xi_i)}{A_i(\xi_i)} \cdot$$

Since $A_i(\xi_i)$ is not zero and has the sign of $(-1)^{i-1}$ and since $p(\xi_i)$ is either zero or else has the sign of $(-1)^{i-1}$, the lemma follows.

Proof of the theorem. It is trivial that $\int_{-\infty}^{\infty} e^{-xt} d\psi(x) > 0$ for $t \geqq 0$. Let

$$F_n(t) = \int_{-\infty}^{\infty} e^{-xt} Q_n(x) d\psi(x).$$

Then

$$\frac{d}{dt}\left[e^{(\lambda_0+\mu_0)t} F_1(t)\right] = \lambda_0 e^{(\lambda_0+\mu_0)t} \int_{-\infty}^{\infty} e^{-xt} Q_1^2(x) d\psi(x) > 0 \qquad \text{for } t \geqq 0.$$

Since $F_1(0) = 0$ it follows that $F_1(t) > 0$ for $t > 0$.

Now suppose $n \geqq 2$. Then $Q_n(x) = k_n(\xi_1 - x)(\xi_2 - x) \cdots (\xi_n - x)$ where $k_n > 0$ and $0 < \xi_1 < \xi_2 < \cdots < \xi_n$. A polynomial of the form

$$A_{i_1 i_2 \cdots i_r}(x) = \prod_{k=1}^{r} (\xi_{i_k} - x)$$

where $1 \leqq i_1 < i_2 < \cdots < i_r \leqq n$ will be called a factor of $Q_n(x)$ of degree r. If $p(x)$ is a polynomial of the type described in the theorem then by the lemma p has a representation

$$p(x) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} \alpha_{i_1 i_2 \cdots i_r} A_{i_1 i_2 \cdots i_r}(x)$$

where the coefficients $\alpha_{i_1 i_2 \cdots i_r}$ are non-negative and not all zero. To each factor of $Q_n(x)$ corresponds a function

$$f_{i_1 i_2 \cdots i_r}(t) = \int_{-\infty}^{\infty} e^{-xt} A_{i_1 i_2 \cdots i_r}(x) Q_n(x) d\psi(x)$$

which vanishes at $t = 0$ if $1 \leqq r < n$. Assume $1 \leqq r < n$ and let $j_1, j_2, \cdots, j_{n-r}$ be the complement of the set $i_1, i_2, \cdots, i_r$ in the set $1, 2, \cdots, n$. Let D_i be the operator such that

$$D_i g(t) = e^{-\xi_i t}\frac{d}{dt}\left[e^{\xi_i t} g(t)\right].$$

Then

$$(*)\qquad D_{j_1}D_{j_2}\cdots D_{j_{n-r}} f_{i_1 i_2\cdots i_r}(t) = \frac{1}{k_n}\int_{-\infty}^{\infty} e^{-xt} Q_n^2(x)\,d\psi(x) > 0 \qquad \text{for } t \geqq 0,$$

and if $k<n-r$

$$(**)\qquad D_{j_1}D_{j_2}\cdots D_{j_k} f_{i_1 i_2\cdots i_r}(t)\big]_{t=0} = f_{i_1 i_2\cdots i_r\, j_1 j_2\cdots j_k}(0) = 0.$$

Thus $f_{i_1 i_2\cdots i_r}(t)$ is the solution of the differential equation (*) which satisfies the initial conditions (**). The differential equation can be solved by successive integrations. Hence $f_{i_1 i_2\cdots i_r}(t)>0$ for $t>0$, and

$$\int_{-\infty}^{\infty} e^{-xt} p(x) Q_n(x)\,d\psi(x) = \sum \alpha_{i_1 i_2\cdots i_r} f_{i_1 i_2\cdots i_r}(t) > 0 \qquad \text{for } t > 0.$$

Finally since $F_n(0)=0$ and

$$e^{-\xi_1 t}\frac{d}{dt}\left[e^{\xi_1 t} F_n(t)\right] = f_1(t)$$

it follows that $F_n(t)>0$ for $t>0$.

Many interesting results can be obtained as corollaries of the above theorem. Its most immediate consequence is the following theorem:

THEOREM 4. *Let ψ be any solution of the H moment problem with left bounded support, and $P(t;\psi)$ the corresponding matrix (defined in Theorem 2). Then*

$$P_{ij}(t;\psi) > 0$$

for every i and j, and for $t>0$.

Proof. It is trivial that

$$\int_{-\infty}^{\infty} e^{-xt} Q_m(x) Q_n(x)\,d\psi(x)$$

is positive for $t\geqq 0$ if $m=n$. If $m\neq n$, say $m<n$ then the roots of $Q_n(x)$ separate the roots of $Q_m(x)$, $Q_m(0)>0$, and hence the integral is positive for $t>0$ by Theorem 3.

2. **Convergence when $\mu_0=0$.**

THEOREM 5. *Suppose $\mu_0=0$ and let ψ be any solution of the S moment problem. Then for each i and n and every $t>0$*

$$0 < \sum_{j=0}^{n} P_{i+1,j}(t;\psi) < \sum_{j=0}^{n} P_{ij}(t;\psi) < 1. \tag{3.1}$$

Each of the series

$$\sum_{j=0}^{\infty} P_{ij}(t;\psi) = f_i(t;\psi) \qquad i = 0, 1, 2, \cdots \tag{3.2}$$

converges uniformly on every finite interval $0 \leqq t \leqq t_0 < \infty$, *and can be differentiated termwise any number of times, the resulting series being uniformly convergent on every finite interval. The sums satisfy the inequalities*

$$0 < f_{i+1}(t;\psi) \leqq f_i(t;\psi) \leqq 1 \tag{3.3}$$

and the sequence $\{f_i(t;\psi)\}$ *is a solution of*

$$\frac{df}{dt} = Af. \tag{3.4}$$

Proof. From (2.23)

$$\sum_{j=0}^{n} P_{ij}(t;\psi) = \int_0^\infty e^{-xt} Q_i(x) \frac{H_{n+1}(x)}{-x} d\psi(x). \tag{3.5}$$

The polynomials $H_{n+1}(x)/-x$ are orthogonal with respect to the positive measure $d\theta(x) = xd\psi(x)/\lambda_0$. From Theorem 4 and

$$\sum_{j=0}^{n} P'_{0j}(t;\psi) = -\lambda_0 \int_0^\infty e^{-xt} \frac{H_{n+1}(x)}{-x} d\theta(x)$$

it follows that

$$\sum_{j=0}^{n} P'_{0j}(t;\psi) < 0, \qquad t > 0.$$

Hence the sum $\sum_{j=0}^{n} P_{0j}(t;\psi)$ is strictly decreasing on $0 \leqq t < \infty$. Since this sum is non-negative and has the value one at $t=0$, the inequality

$$0 < \sum_{j=0}^{n} P_{0j}(t;\psi) < 1, \qquad t > 0,$$

follows. Theorem 4 implies that the function

$$\sum_{j=0}^{n} [P_{ij}(t;\psi) - P_{i+1,j}(t;\psi)] = \frac{\lambda_0}{\lambda_i \pi_i} \int_0^\infty e^{-xt} \frac{H_{i+1}(x)}{-x} \cdot \frac{H_{n+1}(x)}{-x} d\theta(x)$$

is strictly positive for $t>0$. This proves (3.1) and consequently that the series (3.2) are convergent, with sums satisfying (3.3). Using (1.1)

$$(3.6)\quad \begin{aligned}\sum_{j=0}^{n}\frac{d}{dt}P_{ij}(t;\psi) &= \mu_i\sum_{j=0}^{n}P_{i-1,j}(t;\psi) - (\lambda_i+\mu_i)\sum_{j=0}^{n}P_{ij}(t;\psi)\\ &\quad + \lambda_i\sum_{j=0}^{n}P_{i+1,j}(t;\psi)\end{aligned}$$

with an obvious modification if $i=0$. It follows that

$$(3.7)\qquad \sum_{j=0}^{\infty}\frac{d}{dt}P_{ij}(t;\psi)$$

converges on $0\leqq t<\infty$ and has uniformly bounded partial sums. Consequently the partial sums of the series

$$\sum_{j=0}^{\infty}P_{ij}(t;\psi)$$

are uniformly bounded and uniformly equicontinuous. Therefore this latter series converges uniformly on every finite interval, and (3.6) shows that (3.7) also converges uniformly on every finite interval. By differentiating (3.6) k times and making an induction argument on k, it is seen that the series

$$\sum_{j=0}^{\infty}\frac{d^k}{dt^k}P_{ij}(t;\psi),\qquad k\geqq 0$$

all converge uniformly on every finite interval. Letting $n\to\infty$ in (3.6) gives (3.4).

THEOREM 6. *Suppose $\mu_0=0$ and let ψ be any solution of the S moment problem. Then the series*

$$(3.8)\qquad \sum_{j=0}^{\infty}P_{ij}(t)Q_j(x),\qquad P_{ij}(t)=P_{ij}(t;\psi),$$

converges absolutely for $t\geqq 0$ and all complex x, the convergence being uniform over every bounded set $0\leqq t\leqq T$, $|x|\leqq a$. Moreover

$$(3.9)\qquad \sum_{j=0}^{\infty}\left|P_{ij}(t)Q_j(x)\right|\leqq e^{|x|t}Q_i(-|x|).$$

Proof. For any $a>0$ the polynomials

$$Q_n^*(x)=\frac{Q_n(x-a)}{Q_n(-a)},\qquad n=0,1,2,\cdots$$

satisfy the recurrence formula

$$-xQ_0^*(x) = \lambda_0^* Q_0^*(x) + \lambda_0^* Q_1^*(x),$$
$$-xQ_n^*(x) = \mu_n^* Q_{n-1}^*(x) - (\lambda_n^* + \mu_n^*)Q_n^*(x) + \lambda_n^* Q_{n+1}^*(x),$$

where

$$\lambda_n^* = \lambda_n \frac{Q_{n+1}(-a)}{Q_n(-a)}, \qquad \mu_n^* = \mu_n \frac{Q_{n-1}(-a)}{Q_n(-a)}.$$

These polynomials are orthogonal with respect to the measure ψ^* defined by

$$\int_{-\infty}^{x} d\psi^*(y) = \int_{-\infty}^{x-a} d\psi(y).$$

Let

$$P_{ij}^*(t) = \pi_j^* \int_0^\infty e^{-xt} Q_i^*(x) Q_j^*(x) d\psi^*(x)$$

where $\pi_j^* = \pi_j Q_j^2(-a)$. Then

$$P_{ij}(t)Q_j(-a) = e^{at}Q_i(-a)P_{ij}^*(t).$$

If $|x| \leqq a$ then $|Q_j(x)| \leqq Q_j(-a)$ and hence the series (3.8) is dominated by the uniformly converging series

$$\sum_{j=0}^{\infty} e^{at}Q_i(-a)P_{ij}^*(t).$$

Since $\sum_j P_{ij}^*(t) \leqq 1$ by Theorem 5, (3.9) follows.

3. **Convergence when** $\mu_0 > 0$. The case $\mu_0 > 0$ seems to be more difficult. The theorem below gives a sufficient condition on a solution ψ of the S moment problem in order that the corresponding matrix satisfy (1.5). This sufficient condition has a natural probabilistic interpretation, discussed in the companion paper. The remark after the theorem below shows that when the solution of the S moment problem is not unique there is a solution whose corresponding matrix does not satisfy (1.5).

It was shown in Lemma 2 that when $\mu_0 > 0$ there is at least one solution ψ of the S moment problem for which

$$\mu_0 \int_0^\infty \frac{d\psi(x)}{x} \leqq 1. \tag{3.10}$$

THEOREM 7. *Suppose $\mu_0 > 0$ and let ψ be a solution of the S moment problem which satisfies* (3.10). *Then for each i*

$$\sum_{j=0}^{\infty} P_{ij}(t;\psi) = f_i(t;\psi) \tag{3.11}$$

converges uniformly on every finite interval $0 \leqq t \leqq T$, and

$$0 < f_i(t; \psi) < 1 \qquad \textit{for } t > 0.$$

All of the series obtained by differentiating (3.11) *termwise a finite number of times converge uniformly on every finite interval.*

Proof. The proof employs the polynomials $H_n(x)/\mu_0$, which are orthogonal on $0 \leqq x < \infty$ with respect to the measure θ obtained from ψ by the method described in Lemma 2. From (2.23)

$$\sum_{j=0}^{n} P_{ij}(t; \psi) = \int_0^\infty e^{-xt} Q_i(x) \frac{[H_{n+1}(x) - \mu_0]}{-x} d\psi(x),$$

and since $H_{n+1}(0) - \mu_0 = 0$,

$$\sum_{j=0}^{n} P_{ij}(t; \psi) = \int_0^\infty e^{-xt} Q_i(x) d\theta(x) - \int_0^\infty e^{-xt} Q_i(x) \frac{H_{n+1}(x)}{\mu_0} d\theta(x). \tag{3.12}$$

Assume that $n \geqq i$, and let

$$g_i(t) = \int_0^\infty e^{-xt} Q_i(x) d\theta(x).$$

Setting $t = 0$ in (3.12) gives $g_i(0) = 1$. By Theorem 3

$$g_i'(t) = -\mu_0 \int_0^\infty e^{-xt} Q_i(x) d\psi(x) < 0 \qquad \text{for } t > 0,$$

so

$$g_i(t) < 1 \qquad \text{for } t > 0.$$

The roots of $Q_i(x)$ are separated by the roots of $H_{n+1}(x)$, and θ is a positive measure because of (3.10). Hence by Theorem 3

$$\int_0^\infty e^{-xt} Q_i(x) \frac{H_{n+1}(x)}{\mu_0} d\theta(x) > 0 \qquad \text{for } t > 0$$

and

$$\sum_{j=0}^{n} P_{ij}(t; \psi) < g_i(t), \qquad t > 0.$$

Letting $n \to \infty$ gives

$$0 < f_i(t; \psi) \leqq g_i(t) < 1, \qquad t > 0.$$

The statements about uniform convergence follow by the argument that was used in Theorem 5.

REMARK. Suppose $\mu_0 > 0$ and the solution of the S moment problem is not unique. Let $\psi_{\max}$ be the extremal solution with mass at $x = 0$, and $P(t)$

$=P(t;\psi_{\max})$. Then $P(t)$ does not satisfy (1.5) for all $t>0$.

Proof. The mass concentrated at $x=0$ by $\psi_{\max}$ is

$$\rho = \frac{1}{\sum_{n=0}^{\infty} \pi_n Q_n^2(0)} > 0,$$

and

$$\lim_{t\to\infty} P_{ij}(t) = \rho\pi_j Q_i(0)Q_j(0) = \frac{Q_i(0)}{Q_j(0)} a_j$$

where $\sum_0^\infty a_j=1$. If $Q_i(0)\to\infty$ as $i\to\infty$ then

$$\lim_{i\to\infty} \lim_{t\to\infty} P_{ij}(t) = \infty$$

and hence $P_{ij}(t)>1$ if i and t are both large enough (depending on j). On the other hand if $Q_i(0)\to M<\infty$ as $i\to\infty$ then

$$\lim_{n\to\infty} \lim_{i\to\infty} \lim_{t\to\infty} \sum_{j=0}^{n} P_{ij}(t) = M \sum_{j=0}^{\infty} \frac{a_j}{Q_j(0)} > 1$$

and hence

$$\sum_{j=0}^{n} P_{ij}(t) > 1$$

if n, i and t are all sufficiently large.

4. **Convergence when ψ is a solution of the H moment problem.** If ψ is a solution of the H moment problem with mass to the left of $x=0$ then $P_{00}(t;\psi)\to+\infty$ as $t\to\infty$. However the matrix $P(t;\psi)$ still has very strong convergence properties, as is shown by the following.

THEOREM 8. *Suppose $\mu_0\geqq 0$. Let ψ be any solution of the H moment problem with support in the interval $-a\leqq x<\infty$ where $a\geqq 0$. Then the series*

$$\sum_{j=0}^{\infty} P_{ij}(t;\psi)Q_j(x) = f_i(t, x)$$

converges uniformly on every bounded region $0\leqq t\leqq T$, $0\leqq |x| \leqq R$, and the sum satisfies

$$|f_i(t, x)| \leqq e^{bt}Q_i(-b)$$

where $b=\max\{a, |x|\}$.

The proof is very similar to the proof of Theorem 6.

5. **The semi-group property.** In the next theorem μ_0 may be either zero or positive.

THEOREM 9. *Let ψ be any solution of the H moment problem with left bounded support. Let $P(t) = P(t; \psi)$. Then for $s \geqq 0$, $t \geqq 0$ and every i, j the series*

$$\sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s)$$

converges. $P(t)$ has the semi-group property (1.6) *if and only if ψ is extremal.*

Proof. The functions

$$f(x) = e^{-xt} Q_i(x), \qquad g(x) = e^{-xs} Q_j(x)$$

are in $L_2(\psi)$, and their Fourier coefficients relative to the orthonormal system $\{(\pi_k)^{1/2} Q_k(x)\}$ are

$$a_k = \int_{-\infty}^{\infty} e^{-xt} Q_i(x) (\pi_k)^{1/2} Q_k(x) d\psi(x),$$

$$b_k = \int_{-\infty}^{\infty} e^{-xs} Q_j(x) (\pi_k)^{1/2} Q_k(x) d\psi(x).$$

Hence the series

$$\sum_{k=0}^{\infty} a_k b_k = \sum_{k=0}^{\infty} \frac{P_{ik}(t) P_{kj}(s)}{\pi_j}$$

is convergent. If ψ is extremal the sum of this series is

$$\int_{-\infty}^{\infty} f(x) g(x) d\psi(x) = \int_{-\infty}^{\infty} e^{-xt} Q_i(x) e^{-xs} Q_j(x) d\psi(x) = \frac{P_{ij}(t+s)}{\pi_j},$$

that is, $P(t)$ satisfies (1.6).

Conversely suppose $P(t)$ satisfies (1.6). Let $f(x) = e^{-xt} R(x)$ where $t > 0$ is fixed and R is a polynomial. Then R is a finite linear combination of the Q_i,

$$R(x) = \sum_i \alpha_i Q_i(x)$$

and

$$\int_{-\infty}^{\infty} |f(x)|^2 d\psi(x) = \sum_{i,j} \alpha_i \bar{\alpha}_j \int_{-\infty}^{\infty} e^{-2xt} Q_i(x) Q_j(x) d\psi(x)$$

$$= \sum_{i,j} \alpha_i \bar{\alpha}_j \frac{P_{ij}(2t)}{\pi_j} = \sum_{i,j} \alpha_i \bar{\alpha}_j \sum_{k=0}^{\infty} \frac{P_{ik}(t) P_{jk}(t)}{\pi_k}$$

$$= \sum_{k=0}^{\infty} \left| \sum_i \alpha_i \frac{P_{ik}(t)}{(\pi_k)^{1/2}} \right|^2$$

$$= \sum_{k=0}^{\infty} \left| \int_{-\infty}^{\infty} f(x) Q_k(x) (\pi_k)^{1/2} d\psi(x) \right|^2.$$

Thus the Parseval equation holds for f. But functions of the type $f(x)=e^{-xt}R(x)$ are dense in $L_2(\psi)$ and consequently the Parseval equation is valid for every f in $L_2(\psi)$, so ψ is extremal.

6. **The representation theorem.** In this section it is shown that the interesting solutions of (1.1) and (1.2) have a representation of the form $P(t) = P(t;\psi)$. Throughout this section μ_0 may be either zero or positive.

The sequence $\{\pi_n\}$ may be regarded as a positive measure on the space of the non-negative integers. To within a constant factor it is the only positive measure on this space such that the matrix A acts as a symmetric operator on a suitable dense subspace of the Hilbert space $L_2(\pi)$ consisting of all sequences $f=\{f(n)\}$ of complex numbers for which

$$\|f\|^2 = \sum_{n=0}^{\infty} |f(n)|^2\pi_n < \infty.$$

The inner product of this space is(2)

$$(f, g) = \sum_{n=0}^{\infty} f(n)[g(n)]^* \pi_n$$

and Af is defined by

$$(Af)(n) = \sum_{k=0}^{\infty} a_{n,k}f(k)$$

where

$$a_{n,k} = \begin{cases} \mu_n & \text{if } k = n-1, \\ -(\lambda_n+\mu_n) & \text{if } k = n, \\ \lambda_n & \text{if } k = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the relations $\lambda_n\pi_n = \mu_{n+1}\pi_{n+1}$, or what is the same thing, $a_{ij}\pi_i = a_{ji}\pi_j$, that

$$(Af, g) = (f, Ag)$$

whenever f and g terminate ($f(n)=g(n)=0$ for all large n).

DEFINITION. A matrix $C=(c_{ij})$ is said to have the *symmetry property* if $c_{ij}\pi_i = c_{ji}\pi_j$ for every i and j.

Since A is both row finite and column finite the matrix products AC and CA are well defined for every matrix C.

LEMMA 6. *Every matrix which commutes with A has the symmetry property. In particular every matrix which satisfies both* (1.1) *and* (1.2) *has the symmetry property.*

(2) * and — represent symbols for conjugate complex.

Proof. Suppose $C=(c_{ij})$ commutes with A. Then for every i and j, $\pi_i(CA)_{ij} = \pi_i(AC)_{ij}$ and $\pi_j(AC)_{ji} = \pi_j(CA)_{ji}$ so $\pi_i(CA)_{ij} - \pi_j(AC)_{ji} = \pi_i(AC)_{ij} - \pi_j(CA)_{ji}$. Using the symmetry property of A this may be written as

$$(*) \qquad \sum_{k=0}^{\infty} (c_{ik}\pi_i - c_{ki}\pi_k)a_{kj} = \sum_{k=0}^{\infty} (c_{kj}\pi_k - c_{jk}\pi_j)a_{ki}.$$

When $j=i$ the two members of (*) differ only in sign, so each member is zero:

$$\sum_k (c_{ik}\pi_i - c_{ki}\pi_k)a_{ki} = 0, \qquad i = 0, 1, 2, \cdots .$$

This equation is of the form

$$(c_{01}\pi_0 - c_{10}\pi_1)a_{10} = 0$$

if $i=0$, and

$$(c_{i,i+1}\pi_i - c_{i+1,i}\pi_{i+1})a_{i+1,i} + (c_{i,i-1}\pi_i - c_{i-1,i}\pi_{i-1})a_{i-1,i} = 0$$

if $i>0$, from which it follows that

$$c_{i,i+1}\pi_i - c_{i+1,i}\pi_{i+1} = 0, \qquad i = 0, 1, 2, \cdots .$$

Now assume that for $i=0, 1, 2, \cdots$

$$c_{i,i+n}\pi_i - c_{i+n,i}\pi_{i+n} = 0$$

whenever $1 \leqq n \leqq p$. Setting $j=i+p$ in (*) gives

$$\sum_k (c_{ik}\pi_i - c_{ki}\pi_k)a_{k,i+p} = \sum_k (c_{k,i+p}\pi_k - c_{i+p,k}\pi_{i+p})a_{ki}.$$

By the above assumption the left member reduces to

$$(c_{i,i+p+1}\pi_i - c_{i+p+1,i}\pi_{i+p+1})a_{i+p+1,i+p},$$

and the right member reduces to zero if $i=0$ and to

$$(c_{i-1,i+p}\pi_{i-1} - c_{i+p,i-1}\pi_{i+p})a_{i-1,i}$$

if $i>0$. It follows that

$$c_{i,i+p+1}\pi_i - c_{i+p+1,i}\pi_{i+p+1} = 0, \qquad i = 0, 1, 2, \cdots$$

and hence by induction on p

$$c_{ij}\pi_i - c_{ji}\pi_j = 0, \qquad i, j = 0, 1, 2, \cdots .$$

THEOREM 10. *Let $P(t)$, $t \geqq 0$, be a matrix which satisfies* (1.1), (1.2), (1.3), (1.4) *and such that*

$$(3.13) \qquad \sum_{j=0}^{\infty} P_{ij}(t) \leqq Me^{at}$$

for every i and all $t \geqq 0$, where $M > 0$, $a \geqq 0$ are constants. Then

$$(T_t f)(i) = \sum_{j=0}^{\infty} P_{ij}(t) f(j) \tag{3.14}$$

converges absolutely for every f in $L_2(\pi)$ and defines a bounded linear self-adjoint operator of $L_2(\pi)$ into itself. The mapping $t \to T_t$ is continuous on $0 \leqq t < \infty$ relative to the strong operator topology.

Proof. Let $f \in L_2(\pi)$, $t \geqq 0$. Then

$$\begin{aligned} \|T_t f\|^2 &= \sum_{i=0}^{\infty} \left| \sum_{j=0}^{\infty} P_{ij}(t) f(j) \right|^2 \pi_i \\ &\leqq \sum_{i=0}^{\infty} \left(\sum_{k=0}^{\infty} P_{ik}(t) \right) \left(\sum_{j=0}^{\infty} P_{ij}(t) \, | f(j) |^2 \right) \pi_i \\ &\leqq M e^{at} \sum_{ij} P_{ji}(t) \pi_j \, | f(j) |^2 \\ &\leqq (M e^{at})^2 \|f\|^2. \end{aligned}$$

Hence T_t is a bounded operator with $\|T_t\| \leqq M e^{at}$. That T_t is self-adjoint follows from the fact that $P(t)$ is real and has the symmetry property.

It follows by the argument used in Theorem 6 that for each i the series (3.13) converges uniformly on every finite interval. Let g be a terminating element of $L_2(\pi)$, let $\epsilon > 0$ and $t \geqq 0$ be given. If $0 \leqq s \leqq t+1 = A$ then

$$\|T_t g - T_s g\|^2 \leqq 2 M e^{aA} \sum_j | g(j) |^2 \pi_j \sum_{i=0}^{\infty} | P_{ji}(t) - P_{ji}(s) | .$$

For each fixed j there is an index n_j such that

$$2 M e^{aA} \|g\|^2 \sum_{i=n_j}^{\infty} P_{ji}(t) + P_{ji}(s) < \epsilon$$

provided $0 \leqq s \leqq A$, and hence

$$\|T_t g - T_s g\|^2 < \epsilon + 2 M e^{aA} \|g\|^2 \max_{j \in E} \sum_{i=0}^{n_j - 1} | P_{ji}(t) - P_{ji}(s) |$$

where E is the set of j for which $g(j) \neq 0$. The right hand side $\to \epsilon$ as $s \to t$. Since ϵ is arbitrary it follows that $T_s g \to T_t g$ as $s \to t$. But terminating functions are dense in $L_2(\pi)$ and hence the strong continuity is established.

A bounded operator T on $L_2(\pi)$ is called positive definite if $(Tf, f) \geqq 0$ for every f in $L_2(\pi)$. If T, S are bounded operators the inequality $T \geqq S$ means that $T - S$ is positive definite.

THEOREM 11. *Let ψ be a solution of the II moment problem with left bounded support and $\{T_t\}$, $t \geqq 0$, be the family operators determined by $P(t; \psi)$. Then the* *operators* $\{T_t\}$ *are positive definite and*

$$T_{2t} \geqq T_t^2 \qquad \text{for } t \geqq 0. \tag{3.15}$$

If the equality in (3.15) *holds for some* $t>0$ *then* ψ *is extremal, and in this case the operators* $\{T_t\}$, $t\geqq 0$, *form a one parameter semigroup. The inequality*

$$T_s \geqq T_t \qquad \text{for } 0 \leqq s \leqq t \tag{3.16}$$

is valid if and only if ψ *is a solution of the S moment problem.*

Proof. Let f be a terminating element of $L_2(\pi)$ and

$$\hat{f}(x) = \sum_i f(i)Q_i(x)\pi_i.$$

Then

$$\begin{aligned}(T_tf, f) &= \sum_{ij} P_{ij}(t)f(j)[f(i)]^*\pi_i \\ &= \sum_{ij} f(j)[f(i)]^*\pi_i\pi_j \int_{-\infty}^{\infty} e^{-xt}Q_i(x)Q_j(x)d\psi(x) \\ &= \int_{-\infty}^{\infty} e^{-xt}|\hat{f}(x)|^2 d\psi(x) \\ &\geqq 0.\end{aligned}$$

Since terminating functions are dense in $L_2(\pi)$ it follows that $(T_tf, f)\geqq 0$ for all f in $L_2(\pi)$. It is clear also that (T_tf, f) is a decreasing function of t for every terminating f and hence for all f, if and only if ψ has no mass to the left of $x=0$. Again if f is terminating

$$\begin{aligned}(T_t^2f, f) &= \sum_{ijk} [f(i)]^*f(k)\pi_i\pi_j\pi_k \int_{-\infty}^{\infty} e^{-xt}Q_i(x)Q_j(x)d\psi(x) \int_{-\infty}^{\infty} e^{-yt}Q_j(y)Q_k(y)d\psi(y) \\ &= \sum_{j=0}^{\infty} \left| \int_{-\infty}^{\infty} e^{-xt}\hat{f}(x)Q_j(x)d\psi(x) \right|^2 \pi_j,\end{aligned}$$

and by Bessel's inequality

$$\begin{aligned}(T_t^2f, f) &\leqq \int_{-\infty}^{\infty} e^{-2xt}|\hat{f}(x)|^2 d\psi(x) \\ &= (T_{2t}f, f).\end{aligned}$$

As before this inequality remains valid for any f in $L_2(\pi)$. Now suppose $T_{t_0}^2 = T_{2t_0}$ for some $t_0>0$. Then the Parseval equation

$$\int_{-\infty}^{\infty} |e^{-xt_0}\hat{f}(x)|^2 d\psi(x) = \sum_{j=0}^{\infty} \left| \int_{-\infty}^{\infty} e^{-xt_0}\hat{f}(x)Q_j(x)d\psi(x) \right|^2 \pi_j$$

is valid whenever $\hat{f}(x)$ is a polynomial, and therefore for every function in $L_2(\psi)$, showing that ψ is extremal. On the other hand if ψ is extremal, then by Theorem 9

$$\begin{aligned}(T_tT_sf)(i) &= \sum_j \sum_k P_{ij}(t)P_{jk}(s)f(k) \\ &= \sum_k P_{ik}(t+s)f(k)\end{aligned}$$

for all f in $L_2(\pi)$, and therefore $T_tT_s = T_{t+s}$.

THEOREM 12. *Let $P(t)$ be a solution of* (1.1), (1.2), (1.3), (1.4) *such that*

(a) *for every i and for $t \geqq 0$*

$$\sum_{j=0}^{\infty} P_{ij}(t) \leqq Me^{\alpha t}$$

where M is a positive constant and α is a real constant;

(b) *The operators $\{T_t\}$, $t \geqq 0$, determined by $P(t)$ are positive definite.*

Then there is a unique solution ψ of the H moment problem with left bounded support, such that $P(t) = P(t;\psi)$. The support of ψ is contained in the interval $-\alpha \leqq x < \infty$.

Proof. Theorem 2 implies that ψ, if it exists, is unique.

The operators $K_t = e^{-\alpha t}T_t$, $t \geqq 0$ are positive definite and $\|K_t\| \leqq M$. Let f be any terminating sequence and form the function

$$\phi(t;f) = \phi(t) = (K_tf, f).$$

Then $\phi(t) \geqq 0$ for $t \geqq 0$. Using (1.1) and (1.2) it is easily shown that

$$\phi''(t;f) = \phi(t;(A - \alpha I)f).$$

Thus $\phi''(t) \geqq 0$, and since $(A-\alpha I)f$ is terminating, the argument can be repeated. Evidently all the even order derivatives of ϕ are non-negative on $0 \leqq t < \infty$. Suppose $\phi'(t_0) > 0$ for some $t_0 \geqq 0$. Then $\phi''(t) \geqq 0$ implies $\phi'(t) \geqq \phi'(t_0)$ for $t \geqq t_0$ and

$$\phi(t) = \phi(t_0) + \int_{t_0}^{t} \phi'(\tau)d\tau \geqq (t - t_0)\phi'(t_0), \qquad t \geqq t_0.$$

Hence $\phi(t) \to \infty$ as $t \to \infty$. But $\phi(t) = (K_tf, f) \leqq M \cdot \|f\|^2$. This is a contradiction, and therefore $\phi'(t) \leqq 0$. Again the argument can be repeated, and it is seen that all the odd order derivatives of ϕ are non-positive on $0 \leqq t < \infty$. Consequently $\phi(t)$ is completely monotonic on $0 \leqq t < \infty$, and has a representation

$$\phi(t) = \int_0^{\infty} e^{-\xi t} d\beta(\xi)$$

where β is a positive measure on $0 \leqq \xi < \infty$. Since ϕ and all of its derivatives are continuous at $t=0$, the integrals

$$\int_0^\infty \xi^n d\beta(\xi), \qquad n = 0, 1, 2, \cdots$$

are all convergent. Choosing f to be the function

$$f(i) = \begin{cases} 1 & \text{if} \quad i = 0, \\ 0 & \text{if} \quad i \neq 0 \end{cases}$$

gives

$$P_{00}(t) = e^{\alpha t}\phi(t) = \int_0^\infty e^{-(\xi-\alpha)t} d\beta(\xi)$$

or

$$P_{00}(t) = \int_{-\alpha}^\infty e^{-xt} d\psi(x), \tag{3.17}$$

where ψ is a positive measure on $-\alpha \leqq x < \infty$ with finite moments of all orders. The representation for the other elements of $P(t)$ now follows by differentiating (3.17) under the integral sign and using (1.1), (1.2). That ψ is a solution of the moment problem is a consequence of (1.3).

In the probabilistic applications hypothesis (b) of the above theorem is somewhat unnatural. It should be observed however that when $P(t)$ has the semi-group property the operators $\{T_t\}$, $t \geqq 0$ form a semi-group of self adjoint operators and are automatically positive definite;

$$(T_t f, f) = (T_{t/2} f, T_{t/2} f) \geqq 0.$$

In view of this remark, Theorem 12, and Theorems 2, 4, 5, 9 the following assertion can be made.

When $\mu_0 = 0$ formula (2.12) establishes a bi-unique correspondence between the set of all extremal solutions of the S moment problem and the set of all matrices $P(t)$ which satisfy (1.1), (1.2), (1.3), (1.4), (1.5), (1.6).

7. **Laplace transforms.** The Laplace transform of a matrix $P(t)$ which satisfies (1.1), (1.2), (1.3) and

$$|P_{ij}(t)| \leqq Me^{\alpha t}, \qquad 0 \leqq t < \infty,$$

where M and α are constants, will be expressed in terms of polynomials. Let

$$R(s) = \int_0^\infty e^{-st} P(t) dt, \qquad s > \alpha. \tag{3.18}$$

Then

$$-I + sR(s) = AR(s) = R(s)A. \tag{3.19}$$

The solution of (3.19) leads to the introduction of the sequences of polynomials $Q^{(k)}(x) = \{Q_n^{(k)}(x)\}$, n, $k \geqq 0$, defined by

$$\begin{cases} Q_0^{(k)}(x) \equiv 0, & k = 0, 1, 2, \cdots, \\ -\delta^{(k)} - xQ^{(k)}(x) = AQ^{(k)}(x), \end{cases} \tag{3.20}$$

where $\delta^{(k)} = \{\delta_n^k\}$ and δ_n^k is the Kronecker symbol. For $k=0$ the polynomials $\{Q_n^{(k)}(x)\}$ are the polynomials defined in (2.17). It is easily seen that

$$\begin{aligned} Q_n^{(k)}(x) &\equiv 0 & \text{for } n \leqq k, \\ Q_{k+1}^{(k)}(x) &= -1/\lambda_k \end{aligned}$$

and that for $n \geqq k+1$, $Q_n^{(k)}(x)$ is of exact degree $n-k-1$. A particular solution of $-I+sR(s)=AR(s)$ is

$$R_{ij}(s) = Q_i^{(j)}(-s),$$

and the most general solution of $sR(s)=AR(s)$ is of the form

$$R_{ij}(s) = g_j(s)Q_i(-s).$$

Hence the most general solution of (3.19) is of the form

$$R_{ij}(s) = Q_i^{(j)}(s) + g_j(s)Q_i(-s)$$

where

$$g_j(s) = R_{0j}(s).$$

Since $R(s)$ has the symmetry property

$$\begin{aligned} R_{0j}(s) &= R_{j0}(s)\pi_j \\ &= Q_j^{(0)}(-s)\pi_j + g_0(s)Q_j(-s)\pi \end{aligned}$$

and it follows that

$$R_{ij}(s) = Q_i^{(j)}(-s) + Q_i(-s)Q_j^{(0)}(-s)\pi_j + Q_i(-s)Q_j(-s)\pi_j R_{0,0}(s). \tag{3.21}$$

Similar considerations show that the most general solution of

$$sR(s) = AR(s) = R(s)A \tag{3.22}$$

is of the form

$$R_{ij}(s) = Q_i(-s)Q_j(-s)\pi_j R_{00}(s). \tag{3.23}$$

From (3.21) and the symmetry property of $R(s)$

$$Q_i^{(j)}(-s)\pi_i + Q_i(-s)Q_j^{(0)}(-s)\pi_i\pi_j = Q_j^{(i)}(-s)\pi_j + Q_j(-s)Q_i^{(0)}(-s)\pi_i\pi_j.$$

If $j>i$, say $j=i+n$ then $Q_i^{(j)}(-s)=0$ and

$$(3.24) \qquad Q_{i+n}(x)Q_i^{(0)}(x) - Q_i(x)Q_{i+n}^{(0)}(x) = -\frac{Q_{i+n}^{(i)}(x)}{\pi_i}.$$

From the special case of (3.24) with $n=1$ we obtain the useful formula

$$(3.25) \qquad \frac{Q_{n+1}^{(0)}(x)}{Q_{n+1}(x)} = -\sum_{k=0}^{n} \frac{1}{\lambda_k\pi_k Q_{k+1}(x)Q_k(x)}.$$

Now let ψ be a solution of the S moment problem and $R(s)$ the Laplace transform of $P(t;\psi)$. Then for $s>0$

$$\begin{aligned}\pi_j\int \frac{Q_i(x)Q_j(x)}{x+s}\,d\psi(x) &= R_{ij}(s)\\ &= Q_i^{(j)}(-s) + Q_i(-s)R_{0j}(s)\\ &= Q_i^{(j)}(s) + Q_i(-s)\pi_j\int\frac{Q_j(x)}{x+s}\,d\psi(x)\end{aligned}$$

and therefore

$$Q_i^{(j)}(s) = \pi_j\int Q_j(x)\frac{Q_i(x)-Q_i(s)}{x-s}\,d\psi(x).$$

This equation is clearly valid for all s, and remains valid if ψ is replaced by any measure with the correct moments of orders $\leqq i+j-1$.

Chapter IV. Truncated problems, ordered families of solutions, uniqueness theorems, honesty of the minimal solution

1. **Linear ordering of the extremal solutions.** As $n\to\infty$ the first zero of $Q_n(x)$ converges to a finite limit $\xi\geqq 0$. If the solution of the H moment problem is not unique, then for each x_0 in the interval $-\infty<x_0\leqq\xi$ there is an extremal solution of the H moment problem with positive mass at x_0. This solution, ψ_{x_0}, has no other mass to the left of ξ. The following theorem will be proved.

Theorem 13. *Suppose the solution of the H moment problem is not unique, and let $-\infty<x_0<x_1\leqq\xi$. Then*

$$P_{ij}(t;\psi_{x_1}) \leqq P_{ij}(t;\psi_{x_0}), \qquad t\geqq 0,\ i,j,=0,1,\cdots.$$

Before proving this theorem the order relations between solutions of certain truncated problems will be studied. For $n=1,2,3,\cdots$ and $-\infty<\lambda<\infty$, let

$$(4.1)\quad A(n,\lambda) = \begin{bmatrix} -(\lambda_0+\mu_0) & \lambda_0 & & & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \mu_{n-1} & -(\lambda_{n-1}+\mu_{n-1}) & \lambda_{n-1} \\ & & & & \mu_n & -(\lambda+\mu_n) \end{bmatrix}$$

and

$$(4.2)\qquad P(t; n, \lambda) = e^{tA(n,\lambda)}.$$

$P(t;\ n, \lambda)$ is the unique solution of

$$(4.3)\qquad P'(t; n, \lambda) = A(n, \lambda)P(t; n, \lambda),$$

$$(4.4)\qquad P(0; n, \lambda) = I$$

where here I is the $(n+1)$-square identity matrix. The behavior of $P(t; n, 0)$ and $P(t; n, \lambda_n)$ as $n\to\infty$ was investigated by Reuter and Lederman (11]. They showed that $P(t; n, \lambda_n)$ converges to the minimal (see below) solution, and that a subsequence of $P(t; n, 0)$ converges to a solution of (1.1), (1.2), (1.3), (1.4), (1.5) which is in some cases different from the minimal solution. For each of these two solutions they established an integral representation.

Let $\psi_{n,\lambda}$ be the distribution of order $n+1$ supported by the zeros of the quasi-orthogonal polynomial

$$(4.5)\qquad \begin{aligned} F(x, n, \lambda) &= -\ xQ_n(x) - [\mu_n Q_{n-1}(x) - (\lambda+\mu_n)Q_n(x)] \\ &= \lambda_n Q_{n+1}(x) + (\lambda - \lambda_n)Q_n(x). \end{aligned}$$

Then

$$P_{ij}(t) = \pi_j \int e^{-xt}Q_i(x)Q_j(x)d\psi_{n,\lambda}(x), \qquad 0 \leqq i, j \leqq n$$

defines a matrix $P(t)$ which satisfies (4.3), (4.4) and hence $P(t) = P(t; n, \lambda)$.

LEMMA 5. *If* $\lambda_* < \lambda$ *then*

$$(4.6)\qquad P_{ij}(t; n, \lambda) < P_{ij}(t; n, \lambda_*)$$

for every $t > 0$ *and* $0 \leq i, j \leq n$.

Proof. Let

$$c_k(\lambda) = \int x^k d\psi_{n,\lambda}(x), \qquad k = 0, 1, 2, \cdots.$$

The moments $c_k(\lambda)$, $k \leqq 2n$ are independent of λ, in fact they are the correct moments for the original S moment problem. The moment $c_{2n+1}(\lambda)$ can be computed from the equation

$$0 = \int (-x)^n F(x, n, \lambda) d\psi_{n,\lambda}(x)$$

which is of the form

$$0 = \frac{-1}{\lambda_0\lambda_1 \cdots \lambda_{n-1}} c_{2n+1}(\lambda) + \frac{\lambda}{\lambda_0\lambda_1 \cdots \lambda_{n-1}} c_{2n} + \text{terms independent of } \lambda.$$

Hence $c_{2n+1}(\lambda)$ is a strictly increasing linear function of λ. Using the Taylor expansion of e^{-xt}

$$P_{ij}(t; n, \lambda) - P_{ij}(t; n, \lambda_*) = \pi_j \int e^{-xt} Q_i(x) Q_j(x) [d\psi_{n,\lambda}(x) - d\psi_{n,\lambda_*}(x)]$$

$$= \pi_j \frac{(-1)^i}{\lambda_0\lambda_1 \cdots \lambda_{i-1}} \frac{(-1)^j}{\lambda_0\lambda_1 \cdots \lambda_{j-1}} \frac{(-t)^{2n+1-i-j}}{(2n+1-i-j)!} [c_{2n+1}(\lambda) - c_{2n+1}(\lambda_*)]$$
$$+ O(t^{2n+2-i-j}),$$

with a slight modification if $i=0$ or $j=0$. Hence

$$P_{ij}(t; n, \lambda) < P_{ij}(t; n, \lambda_*)$$

for all sufficiently small positive t. If t_1, t_2 are two positive values of t for which the inequalities hold then

$$\begin{aligned} P_{ij}(t_1 + t_2; n, \lambda) &= \sum_k P_{ik}(t_1; n, \lambda) P_{kj}(t_2; n, \lambda) \\ &< \sum_k P_{ik}(t_1; n, \lambda_*) P_{kj}(t_2; n, \lambda_*) \\ &= P_{ij}(t_1 + t_2; n, \lambda_*), \end{aligned}$$

and the lemma follows.

The matrix $P(t; n, \lambda)$ is also a solution of

(4.7) $$P'(t; n, \lambda) = P(t; n, \lambda) A(n, \lambda), \text{ and}$$

its elements are strictly positive on $0 < t < \infty$. A convergent sequence of matrices may be obtained by making λ depend on n, $\lambda = \lambda(n)$ in such a way that the distributions $\psi_{n,\lambda(n)}$ converge to a solution ψ of the H moment problem, and hence

$$\lim_{n\to\infty} P(t; n, \lambda(n)) = P(t; \psi).$$

It was pointed out in Chapter 2 that the sequence of distributions of order $n+1$ supported by the zeros of $Q_{n+1}(x)$ converges to an extremal solution. This solution will henceforth be denoted by $\psi_{\min}$. Thus

(4.8) $$\lim_{n\to\infty} P_{ij}(t; n, \lambda_n) = P_{ij}(t; \psi_{\min})$$

for every i, j, and for Re $t \geqq 0$. Reuter and Lederman [9; 11], showed that if $\xi(t)=\{\xi_j(t)\}$ is any solution of

$$(1)\qquad \xi'(t) = \xi(t)A, \qquad t \geqq 0,$$

$$(2)\qquad \xi_j(t) \geqq 0, \qquad \text{for } t \geqq 0 \text{ and each } j$$

$$(3)\qquad \xi_j(0) = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i \end{cases}$$

then

$$\xi_j(t) \geqq P_{ij}(t;\psi_{\min}) \qquad \text{for } t \geqq 0 \text{ and each } j.$$

For this reason $P(t;\psi_{\min})$ is called *the minimal solution*. When (2.28) converges $\psi_{\min}$ is supported by the zeros of $Q_\infty(x)$.

The distributions $\psi_{n,\lambda(n)}$ also converge if they all have mass at some fixed point x_0, and if the solution of the H moment problem is not unique the limiting distribution is the extremal solution with mass at x_0. If

$$x_0 = \lim_{n\to\infty} \xi_{1,n} \equiv \xi$$

where $\xi_{1,n}$ is the first zero of $Q_n(x)$, then the limit is again $\psi_{\min}$.

Proof of Theorem 13. The first zero of $F(x, n, \lambda)$ is the smallest solution x_0 of

$$\frac{Q_{n+1}(x)}{Q_n(x)} = 1 - \frac{\lambda}{\lambda_n}\cdot$$

Dividing (2.25) through by $x-y$ and letting $y\to x$ it is found that

$$\frac{d}{dx}\,\frac{Q_{n+1}(x)}{Q_n(x)} < 0$$

if x is real and $Q_n(x)\neq 0$. Since $Q_{n+1}(x)/Q_n(x)\to+\infty$ as $x\to-\infty$ and $\to-\infty$ as $x\to\xi_{1,n}^-$, it follows that for each real λ, $F(x, n, \lambda)$ has exactly one zero in the interval $-\infty<x<\xi_{1,n}$, and this zero $x_0(\lambda)$ is a strictly increasing continuous function of λ, with

$$\lim_{\lambda\to+\infty} x_0(\lambda) = \xi_{1,n},$$

$$\lim_{\lambda\to-\infty} x_0(\lambda) = -\infty.$$

Now suppose $-\infty<x_0\leqq\xi=\lim_{n\to\infty}\xi_{1,n}$. The value $\lambda(x_0)$ of λ for which $\psi_{n,\lambda}$ has mass at x_0 is the inverse function of $x_0(\lambda)$. By the lemma $P_{ij}(t;n,\lambda(x_0))$ is a decreasing function of x_0. The theorem follows by letting $n\to\infty$.

When the solution of the S moment problem is not unique it is convenient to have a name for the solution with maximal mass at $x=0$. It will henceforth be called $\psi_{\max}$, although $P(t;\psi_{\max})$ is not maximal in the same sense that $P(t;\psi_{\min})$ is minimal.

As $\lambda \to +\infty$, n of the zeros of $F(x, n, \lambda)$ converge to the n zeros of $Q_n(x)$ and the other zero $\eta_{n+1}(\lambda)$ tends to $+\infty$. Since the moments of orders $\leqq 2n$ are independent of λ, the mass $\gamma_{n+1}(\lambda)$ located at $\eta_{n+1}(\lambda)$ vanishes, in fact

$$\gamma_{n+1}(\lambda)(\eta_{n+1}(\lambda))^{2n-1} \to 0 \text{ as } \lambda \to \infty.$$

Consequently $\psi_{n,\lambda}$ converges to the distribution $\psi_{n-1,\lambda_{n-1}}$ supported by the zeros of $Q_n(x)$. The functions $e^{-xt}Q_i(x)Q_j(x)$ are all bounded on $0 \leqq x < \infty$ and therefore

$$\lim_{\lambda \to \infty} P_{ij}(t; n, \lambda) = P_{ij}(t; n-1, \lambda_{n-1})$$

for $t > 0$, $i, j, \leqq n-1$. One corollary of this is that $P_{ij}(t; n, \lambda_n)$ *increases as* n *increases.*

2. **Uniqueness theorems.** Necessary and sufficient conditions in order that there be one and only one solution of (1.1), (1.2), (1.3), (1.4), (1.5) will now be obtained.

LEMMA 6. *For* $s \geqq 0$

$$\int_0^\infty \frac{d\psi_{\min}}{x+s} = \sum_{n=0}^\infty \frac{1}{\lambda_n \pi_n Q_n(-s)Q_{n+1}(-s)}, \tag{4.9}$$

where both members may be infinite for $s = 0$. *If* $\mu_0 > 0$ *then*

$$\mu_0 \int_0^\infty \frac{d\psi_{\min}}{x} = \mu_0 \sum_{n=0}^\infty \frac{1}{\lambda_n \pi_n Q_n(0)Q_{n+1}(0)} = 1 - \lim_{n\to\infty} \frac{1}{Q_n(0)}.$$

Proof. Let ψ_n be the distribution of order $n+1$ supported by the zeros of $Q_{n+1}(x)$. From (2.17) and (3.25)

$$\int_0^\infty \frac{d\psi_n(x)}{x+s} = -\frac{Q_{n+1}^{(0)}(-s)}{Q_{n+1}(-s)} = \sum_{k=0}^n \frac{1}{\lambda_k \pi_k Q_k(-s)Q_{k+1}(-s)}$$

and since

$$\int_0^\infty \frac{d\psi_n(x)}{x+s} = \int_0^\infty e^{-st} P_{0,0}(t; n, \lambda_n) dt$$

(4.9) follows by monotone convergence as $n \to \infty$. If $\mu_0 > 0$ then $H_{n+1}(0) = \mu_0$ and hence

$$\mu_0 \int_0^\infty \frac{d\psi_{\min}}{x} = \mu_0 \sum_{n=0}^\infty \frac{1}{H_{n+1}(0)} \left[\frac{1}{Q_n(0)} - \frac{1}{Q_{n+1}(0)}\right] = 1 - \lim_{n\to\infty} \frac{1}{Q_n(0)}.$$

THEOREM 14. *If* $\mu_0 = 0$ *the following statements are either all true or all false.*

(1) *There is only one matrix* $P(t)$ *which satisfies* (1.1), (1.2), (1.3) *and which for some* $M \geqq 1$ *satisfies*

$$\left|\sum_{j=0}^{n} P_{ij}(t)\right| \leqq M \qquad \textit{for all } t \geqq 0 \textit{ and every } i, n.$$

(2) *The solution of the S moment problem is unique.*

(3) *The series*

$$\sum_{n=0}^{\infty}\left(\pi_n + \frac{1}{\lambda_n \pi_n}\right)$$

is divergent.

Proof. If (3) is false then $\psi_{\max}$ has the mass $(\sum_{n=0}^{\infty}\pi_n)^{-1}$ located at the origin, while $\int_0^\infty (d\psi_{\min}/x)$ converges by Lemma 6. Hence $\psi_{\max}$ and $\psi_{\min}$ are different and (2) is false.

If (2) is false then Theorems 2, 4, 5 show that (1) is false.

Suppose (1) is false. Let $P(t)$, $P^{(1)}(t)$ be two distinct matrices with the stated properties and let

$$R_{ij}(s) = \int_0^\infty e^{-st}[P_{ij}(t) - P_{ij}^{(1)}(t)]dt, \qquad s > 0.$$

Then

$$\left|\sum_{j=0}^{n} R_{ij}(s)\right| \leqq \frac{2M}{s} \qquad \text{for all } s > 0, \text{ and every } i, n.$$

Now from (3.23)

$$R_{ij}(s) = Q_i(-s)Q_j(-s)\pi_j R_{00}(s)$$

and $R_{00}(s_0) \neq 0$ for some $s_0 > 0$. Hence $Q_i(-s_0)$ is bounded and $\sum_{j=0}^{\infty} Q_j(-s_0)\pi_j$ converges. The first of these conditions implies by Lemma 4 that $\sum(1/\lambda_n\pi_n)$ converges and the second implies that $\sum \pi_n$ converges. Thus (3) is false, and this proves the theorem.

COROLLARY. *When $\mu_0 > 0$ the solution of the S moment problem is unique if and only if*

$$\sum_{n=0}^{\infty} \pi_n Q_n^2(0) \text{ diverges.}$$

Proof. Applied to the recurrence formula for the polynomials $Q_n(x)/Q_n(0)$, Theorem 14 shows that for the solution of the moment problem to be unique it is necessary and sufficient that one of the series

$$\sum_{n=0}^{\infty} \pi_n Q_n^2(0), \qquad \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n Q_n(0) Q_{n+1}(0)}$$

is divergent. Lemma 4 shows that the second of these two series is convergent.

THEOREM 15. *If* $\mu_0>0$ *then in order that there be only one matrix* $P(t)$ *which satisfies* (1.1), (1.2), (1.3), (1.4), (1.5), *it is necessary and sufficient that at least one of the two conditions*

(1) *The solution of the S moment problem is unique,*

(2) $$\mu_0 \int_0^\infty \frac{d\psi_{\min}(x)}{x} = 1,$$

be satisfied; or equivalently that the single condition

(3) $$\sum_0^\infty \left(\pi_n + \frac{1}{\lambda_n\pi_n}\right) \textit{ diverges}$$

be satisfied.

Proof. Denote by $R(s)$ the Laplace transform of the difference of any two matrices which satisfy (1.1), (1.2), (1.3), (1.4). Then

$$R_{ij}(s) = Q_i(-s)Q_j(-s)\pi_j R_{00}(s).$$

If (2) is satisfied then, by Lemma 6 it is seen that $Q_i(-s)\to\infty$ as $i\to\infty$. But $R_{ij}(s)$ is bounded as $i\to\infty$ and hence $R_{0,0}(s)\equiv 0$. On the other hand if (2) is not satisfied but (1) is satisfied then $Q_j(-s)$ is bounded as $j\to\infty$ and

$$\sum_{j=0}^\infty \pi_j Q_j^2(0) \text{ diverges.}$$

Hence since

$$\sum_{j=0}^\infty R_{ij}(s) < \infty$$

it again follows that $R_{00}(s)\equiv 0$. Thus if either of the two conditions hold the matrix $P(t)$ is unique.

To show that (3) holds if and only if at least one of (1) and (2) hold, observe first that by Lemma 6, (2) holds if and only if $Q_n(0)\to\infty$, and hence by (2.4), if and only if $\sum(1/\lambda_n\pi_n)$ diverges. Thus if (3) is satisfied then either (2) holds or $\sum\pi_n$ diverges and (1) is satisfied. Conversely if (3) is not satisfied then neither is (2), and moreover $Q_n(0)$ is bounded so $\sum\pi_n Q_n^2(0)$ converges and (1) is not satisfied.

Now suppose neither (1) nor (2) is satisfied. Then since (3) is false the limiting functions Q_∞, H_∞, $Q_\infty^{(0)}$, $H_\infty^{(0)}$ exist. The first zero ξ of $Q_\infty(x)$ is also the first point in the support of $\psi_{\min}$. For $0<x_0\leq\xi$ let ψ_{x_0} denote the extremal solution whose first jump is at x_0, and let

$$\phi(x_0) = \mu_0 \int_0^\infty \frac{d\psi_{x_0}(x)}{x}.$$

It will be shown that ϕ is continuous, and since $\phi(\xi)<1$, that $\phi(x_0)<1$ for all x_0 sufficiently close to ξ. Theorem 7 then guarantees that $P(t)$ is not unique.

If $\psi_{x_0,n}$ is the distribution of order $n+1$ whose first jump is at x_0 then

$$\int_0^\infty \frac{d\psi_{x_0,n}(y)}{x-y} = \frac{H_{n+1}^{(0)}(x)Q_n(x_0) - H_{n+1}(x_0)Q_n^{(0)}(x)}{H_{n+1}(x)Q_n(x_0) - H_{n+1}(x_0)Q_n(x)}.$$

Letting $n\to\infty$ and then setting $x=0$ gives

$$\phi(x_0) = -\mu_0 \frac{H_\infty^{(0)}(0)Q_\infty(x_0) - H_\infty(x_0)Q_\infty^{(0)}(0)}{H_\infty(0)Q_\infty(x_0) - H_\infty(x_0)Q_\infty(0)}.$$

The quasi-orthogonal polynomial $H_{n+1}(0)Q_n(x)-H_{n+1}(x)Q_n(0)$ has a zero at $x=0$, a zero in each of the $n-1$ open intervals formed by the successive zeros of $Q_n(x)$, and a zero beyond the last zero of $Q_n(x)$. Hence the entire function $H_\infty(0)Q_\infty(x)-H_\infty(x)Q_\infty(0)$ has no zero in $0<x<\xi$, and in fact it has no zero at $x=\xi$ because H_∞ and Q_∞ have no common zero. Thus ϕ is continuous on $0<x\leqq\xi$, and the theorem is proved.

3. **Honesty of the minimal solution.** A matrix $P(t)$ which satisfies (1.1)–(1.5) is sometimes called honest if

$$\sum_{j=0}^\infty P_{ij}(t) = 1, \qquad t \geqq 0,\ i = 0, 1, \cdots. \tag{4.10}$$

Using (1.6) and the kind of argument found in the remark after Theorem 7, it is easily shown that if there is a solution of the S moment problem with mass at $x=0$, then the solution with maximal mass at $x=0$ generates an honest matrix provided $\mu_0=0$. On the other hand, when $\mu_0=0$ and $\psi_{\min}$ is the only solution of the S moment problem, the corresponding matrix $P(t;\psi_{\min})$ may fail to be honest. The analytical aspect of this situation will be examined and a necessary and sufficient condition for the honesty of $P(t;\psi_{\min})$ will be given.

LEMMA 7. *If $P(t)$ is a solution of* (1.1), (1.2), (1.3), (1.4), (1.5) *such that for some i*

$$\sum_{j=0}^\infty P_{ij}(t) = 1 \qquad \textit{for all } t \geqq 0, \tag{4.11}$$

then $\mu_0=0$ and (4.11) *is valid for every i.*

Proof. Let

$$f_i(s) = \int_0^\infty e^{-st}\left[1 - \sum_{j=0}^\infty P_{ij}(t)\right]dt, \qquad s > 0.$$

Then $f(s) = \{f_i(s)\}$ is a solution of $Af(s)=sf(s)$ and hence $f_i(s)=\text{Const. } Q_i(-s)$.

Since $f_i(s)=0$ for some i, it follows that $f_i(s)=0$ for each i and hence (4.11) is valid for every i. The argument used in Theorem 6 shows that (4.11) can be differentiated term by term, and for $i=0$ this gives

$$0 = -(\lambda_0+\mu_0)\sum_{j=0}^{\infty} P_{0j}(t) + \lambda_0 \sum_{j=0}^{\infty} P_{ij}(t),$$

so that $\mu_0=0$.

THEOREM 16. *In order that*

$$\sum_{j=0}^{\infty} P_{ij}(t;\psi_{\min}) = 1, \qquad t \geqq 0,\ i = 0, 1, 2, \cdots \tag{4.12}$$

it is necessary and sufficient that $\mu_0=0$ and the series

$$\sum_{j=0}^{\infty} \frac{1}{\lambda_j \pi_j} \sum_{i=0}^{j} \pi_i \tag{4.13}$$

be divergent.

Proof. Lemma 7 shows $\mu_0=0$ is necessary. Suppose $\mu_0=0$ and (4.13) converges. By Lemma 4, $Q_n(x)$ converges as $n\to\infty$ to an entire function $Q_\infty(x)$ with $Q_\infty(0)=1$. The first zero of $Q_\infty(x)$ is positive, say at $\xi=2\alpha>0$. Consequently $\psi_{\min}$ has no mass in the interval $0\leqq x<2\alpha$. The polynomials

$$U_n(x) = \frac{Q_n(x+\alpha)}{Q_n(\alpha)}$$

are orthogonal on $0\leqq x<\infty$ with respect to the measure β defined by

$$\int_0^x d\beta(x) = \int_0^{x+\alpha} d\psi(x).$$

Let

$$P^*_{ij}(t) = \pi^*_j \int_0^\infty e^{-xt} U_i(x) U_j(x) d\beta(x)$$

where $\pi^*_n=\pi_n Q_n^2(\alpha)$. Then

$$P_{ij}(t;\psi_{\min}) = e^{-\alpha t}\frac{Q_i(\alpha)}{Q_j(\alpha)} P^*_{ij}(t),$$

and since $0<Q_\infty(\alpha)<Q_n(\alpha)<1$ for every n and $\sum_j P^*_{ij}(t)\leqq 1$,

$$\sum_j P_{ij}(t;\psi_{\min}) \leqq \frac{e^{-\alpha t}}{Q_\infty(\alpha)}$$

which is less than one for large t. Thus the divergence of (4.13) is necessary.

Now suppose $\mu_0=0$ and (4.13) diverges. Defining $f(s)$ as in Lemma 7 we again have $f_i(s)=\text{const.}\ Q_i(-s)$. Since (4.13) diverges, $Q_i(-s)\to\infty$, while $f_i(s)$ is bounded as $i\to\infty$. Hence $f_i(s)=0$ for all i and (4.12) follows.

It is interesting to observe that there may be matrices $P(t)$ which satisfy (1.1), (1.2), (1.3), (1.4) and for which the sums

$$\sum_{j=0}^{\infty} P_{ij}(t)$$

are bounded uniformly in i and t, but not bounded by one. For example if $\mu_0=0$, and the solution of the S moment problem is not unique then for any $c>1$

$$P(t) = P(t;\psi_{\min}) + c[P(t;\psi_{\max}) - P(t;\psi_{\min})]$$

defines a matrix which satisfies (1.1)–(1.4) but

$$\sum_{j=0}^{\infty} P_{ij}(t) \leqq c = \lim_{t\to\infty} \sum_{j=0}^{\infty} P_{ij}(t).$$

4. **Extremal solutions of the second kind.** Throughout this section it is assumed that $\sum \pi_n + 1/\lambda_n\pi_n < \infty$. A linearly ordered family of solutions different from the extremal solutions will now be studied. The solutions in question are obtained by truncating the matrix A in a different way than was done in §1, and then passing to the limit. The matrix (4.1) corresponds to a process in which the particle may disappear permanently after it reaches the final (nth) state. Variation of the parameter λ amounts to variation of the rate at which this absorption occurs. The method of truncation now considered corresponds to allowing the particle to return into the system after it has been absorbed. The parameter which is varied is the length of time the particle remains in the absorbed condition. By passing to the limit one obtains the transition probabilities of a process which is of the type called "elementary return process" by Feller [2; 3]. The study of these processes reveals a remarkable new fact about the Stieltjes moment problem, namely that when the solution is not unique there is a natural family of solutions ψ such that the polynomials $\{Q_n(x)\}$ together with the limiting function $Q_\infty(x)$ form a *complete orthogonal* system in $L_2(\psi)$.

For the sake of simplicity only the case $\mu_0=0$ is treated. Let

$$(4.14)\quad B(n,\mu) = \begin{bmatrix} -\lambda_0 & \lambda_0 & & & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & \mu_{n-1} & -(\lambda_{n-1}+\mu_{n-1}) & \lambda_{n-1} \\ & & & & \mu & -\mu \end{bmatrix}$$

where $0<\mu<\infty$. The state n corresponds to the absorbed condition and the limiting values $\mu=0$, $\mu=\infty$ correspond respectively to permanent absorption and instantaneous return. The transition probability matrix of the corresponding process is

$$P_n(t) = e^{tB(n,\mu)}.$$

Although the polynomial

$$G_n(x) = -xQ_n(x) + \mu\{Q_n(x) - Q_{n-1}(x)\} \tag{4.15}$$

is not a quasi-orthogonal polynomial belonging to the original S moment problem, it may be viewed as a quasi-orthogonal polynomial belonging to a new S moment problem generated by a new infinite matrix A^* which agrees with A in the first n rows only. Consequently there is a distribution ψ_n with jumps at the $n+1$ zeros of $G_n(x)$ such that

$$P_{n;i,j}(t) = \overset{*}{\pi}_j \int_0^\infty e^{-xt}Q_i(x)Q_j(x)d\psi_n(x)$$

where

$$\overset{*}{\pi}_j = \pi_j \qquad \text{for } 0 \leqq j \leqq n-1,$$

$$\overset{*}{\pi}_n = \frac{\mu_n}{\mu}\pi_n.$$

This distribution has the first $2n-1$ moments prescribed for the original S moment problem, but if $\mu\neq\mu_n$ the moment $\overset{*}{c}_{2n}$ of order $2n$ will be different. This moment can be computed from the equation

$$0 = \int (-x)^{n-1}G_n(x)d\psi_n(x)$$

which is of the form

$$\begin{aligned} 0 &= \int (-x)^nQ_n(x)d\psi_n(x) + \mu\int(-x)^{n-1}[Q_n(x) - Q_{n-1}(x)]d\psi_n(x) \\ &= \frac{\overset{*}{c}_{2n}}{\lambda_0\lambda_1\cdots\lambda_{n-1}} - \mu\int(-x)^{n-1}Q_{n-1}(x)d\psi_n(x) \\ &\quad + \text{terms independent of } \mu. \end{aligned}$$

Hence $\overset{*}{c}_{2n}$ is a strictly increasing linear function of μ.

LEMMA 8. *If* $0\leqq i\leqq n$, $0\leqq j\leqq n-1$ *then for each fixed* $t>0$, $P_{n;i,j}(t)$ *is a strictly increasing function of* μ.

The proof, which is similar to the proof of Lemma 5 is omitted. The main difference is that here the result is not true for $j=n$ because π_n has been replaced by π_n^*.

The mass ρ_n^* of the distribution ψ_n at $x=0$ is

$$\rho_n^* = 1 \Big/ \sum_{k=0}^{n} \pi_k^* = \mu \Big/ \left(\mu_n\pi_n + \mu \sum_{k=0}^{n-1} \pi_k\right).$$

In order that this have the value $1/(\pi_\infty + \sum_{k=0}^{\infty} \pi_k)$ where $\pi_\infty > 0$ is a prescribed constant, it is necessary and sufficient that

$$\mu = \frac{\mu_n\pi_n}{\pi_\infty + \sum_{k=n}^{\infty} \pi_k}. \tag{4.16}$$

With this choice of μ as a function of n, it will be shown that the sequence of distributions ψ_n converges to a solution of the S moment problem. Replacing μ by its value as given above,

$$G_n(x) = -xQ_n(x) + H_n(x) \Big/ \left(\pi_\infty + \sum_{k=n}^{\infty} \pi_k\right) \tag{4.17}$$

which converges uniformly in every circle to

$$G_\infty(x) = -xQ_\infty(x) + \frac{H_\infty(x)}{\pi_\infty} \tag{4.18}$$

when $n \to \infty$. The numerator $G_n^0(x)$ of $G_n(x)$ (relative to the new moment problem) is given by

$$G_n^{(0)}(x) = \int_0^\infty \frac{G_n(x) - G_n(y)}{x - y} \, d\psi_n(y)$$

and since the integrand is a polynomial in y of degree n it is permissible to replace ψ_n by any solution ψ of the original S moment problem. Hence

$$G_n^{(0)}(x) = -xQ_n^{(0)}(x) + H_n^{(0)}(x) \Big/ \left(\pi_\infty + \sum_{k=n}^{\infty} \pi_k\right) \tag{4.19}$$

and when $n \to \infty$ this converges uniformly in every circle to

$$G_\infty^{(0)}(x) = -xQ_\infty^{(0)}(x) + \frac{H_\infty^{(0)}(x)}{\pi_\infty}. \tag{4.20}$$

Since $G_n(x)$ has a zero at $x=0$, a zero in each of the $n-1$ open intervals formed by successive zeros of $Q_n(x)$, a zero beyond the last zero of $Q_n(x)$, and no other zeros, it follows that $G_\infty(x)$ has a simple zero at $x=0$, a simple zero in each of the intervals formed by successive zeros of $Q_\infty(x)$, and no other zeros. The identity

$$G_\infty(x)Q_\infty^{(0)}(x) - G_\infty^{(0)}(x)Q_\infty(x) = \frac{1}{\pi_\infty} \tag{4.21}$$

shows that $G_\infty(x)$ and $G_\infty^{(0)}(x)$ have no common zeros.

Since for each $x<0$

$$\int_0^\infty \frac{d\psi_n(y)}{x-y} = \frac{G_n^{(0)}(x)}{G_n(x)} \to \frac{G_\infty^{(0)}(x)}{G_\infty(x)} \text{ as } n \to \infty,$$

the distributions ψ_n converge to a distribution ψ and

$$\int_0^\infty \frac{d\psi(y)}{x-y} = \frac{G_\infty^{(0)}(x)}{G_\infty(x)}.$$

For $A>0$, $n \geqq k$,

$$\int_0^A x^k d\psi_n(x) \leqq c_k, \qquad \int_A^\infty x^k d\psi_n(x) \leqq \frac{c_{k+1}}{A},$$

from which it follows that ψ is a solution of the original S moment problem. Clearly ψ is a discrete distribution supported by the zeros of G_∞.

The measure ψ will now be written $\psi(x; \pi_\infty)$ to display its dependence on π_∞. When the parameter μ in the truncated matrix $B(n, \mu)$ is determined by the formula (4.16) the corresponding matrix $P_n(t) = e^{tB(n,\mu)}$ becomes a function $P_n(t; \pi_\infty)$ of π_∞. In view of the convergence of ψ_n to ψ

$$P_{ij}(t; \psi(x; \pi_\infty)) = \lim_{n\to\infty} P_{n;i,j}(t)$$

for each i, j and every $t>0$. Since $P_{n;i,j}(t)$ is an increasing function of μ and μ is a decreasing function of π_∞ the following theorem is immediate.

Theorem 17. *For each i, j and every $t \geqq 0$*

$$P_{i,j}(t; \psi(x; \pi_\infty))$$

is a decreasing function of π_∞ on $0<\pi_\infty<\infty$.

As remarked at the end of Chapter II, the entire functions $Q_\infty(x)$, $H_\infty(x)$ are of order one and of minimal type, i.e.,

$$|Q_\infty(x)|, \qquad |H_\infty(x)| = e^{|x|\delta(|x|)}$$

where $\delta(|x|)$ is a generic symbol for a function which is bounded and $o(1)$ as $|x| \to \infty$. It follows that $e^{-xt}Q_\infty(x)$, $e^{-xt}H_\infty(x)$, considered as functions on $0 \leqq x < \infty$, vanish at infinity for each $t>0$. Moreover the sequence of functions $e^{-xt}Q_n(x)$ is dominated on $0 \leqq x < \infty$ by the function $e^{-xt}Q_\infty(-x)$ which itself vanishes at infinity. Hence for $t>0$ and each j

$$\lim_{n\to\infty} \pi_j \int_0^\infty e^{-xt} Q_n(x) Q_j(x) d\psi(x) = \pi_j \int_0^\infty e^{-xt} Q_\infty(x) Q_j(x) d\psi(x).$$

Here ψ can be taken to be any solution of the moment problem. By a similar argument

$$\lim_{n\to\infty} \int_0^\infty e^{-xt} Q_i(x) \frac{H_{n+1}(x)}{-x} d\psi(x) = \int_0^\infty e^{-xt} Q_i(x) \frac{H_\infty(x)}{-x} d\psi(x), \qquad t > 0.$$

If x is a point in the support of $\psi(x; \pi_\infty)$ then

$$0 = \pi_\infty G_\infty(x) = - xQ_\infty(x)\pi_\infty + H_\infty(x)$$

so

$$\frac{H_\infty(x)}{-x} = \begin{cases} -Q_\infty(x)\pi_\infty & \text{if } x \neq 0, \\ -Q_\infty(x)\pi_\infty + \left(\pi_\infty + \sum_{k=0}^\infty \pi_k\right) & \text{if } x = 0. \end{cases} \tag{4.22}$$

Consequently for $t>0$

$$\begin{aligned} \sum_{j=0}^\infty P_{ij}(t; \psi(x; \pi_\infty)) &= \lim_{n\to\infty} \int_0^\infty e^{-xt} Q_i(x) \frac{H_{n+1}(x)}{-x} d\psi(x; \pi_\infty) \\ &= \int_0^\infty e^{-xt} Q_i(x) \frac{H_\infty(x)}{-x} d\psi(x; \pi_\infty) \\ &= 1 - \pi_\infty \int_0^\infty e^{-xt} Q_i(x) Q_\infty(x) d\psi(x; \pi_\infty). \end{aligned} \tag{4.23}$$

By a similar calculation, for $t>0$,

$$\sum_{j=0}^\infty \pi_j \int_0^\infty e^{-xt} Q_\infty(x) Q_j(x) d\psi(x; \pi_\infty) = 1 - \pi_\infty \int_0^\infty e^{-xt} Q_\infty^2(x) d\psi(x; \pi_\infty). \tag{4.24}$$

Since the left member of (4.24) is non-negative

$$\pi_\infty \int_0^\infty e^{-xt} Q_\infty^2(x) d\psi(x; \pi_\infty) \leqq 1$$

and it follows by monotone convergence as $t \to 0$ that $Q_\infty(x) \in L_2(\psi(x; \pi_\infty))$ and

$$\pi_\infty \int_0^\infty Q_\infty^2(x) d\psi(x; \pi_\infty) \leqq 1.$$

This integral is not zero because $Q_\infty(0) = 1$. The left member of (4.23) is continuous and has the value 1 at $t = 0$. Hence

$$\pi_\infty \int_0^\infty Q_i(x) Q_\infty(x) d\psi(x; \pi_\infty) = 0, \qquad i = 0, 1, 2, \cdots. \tag{4.25}$$

Thus $Q_\infty, Q_0, Q_1, Q_2, \cdots$ is an orthogonal system in $L_2(\psi(x;\pi_\infty))$. It will be shown that this system is complete.

Let

$$K(x, y) = Q_\infty(x)Q_\infty(y)\pi_\infty + \sum_{k=0}^{\infty} Q_k(x)Q_k(y)\pi_k. \tag{4.26}$$

This series converges uniformly in every finite square $0 \leqq x, y \leqq A < \infty$. If $x \neq y$ (2.37) gives

$$K(x, y) = Q_\infty(x)Q_\infty(y)\pi_\infty + \frac{Q_\infty(x)H_\infty(y) - Q_\infty(y)H_\infty(x)}{x - y}, \qquad x \neq y, \tag{4.28}$$

and letting $x \to y$

$$K(y, y) = Q_\infty^2(y)\pi_\infty + Q_\infty'(x)H_\infty(y) - Q_\infty(y)H_\infty'(y). \tag{4.29}$$

Now suppose x, y are in the support of $\psi(y;\pi_\infty)$ and $x \neq y$. Then $H_\infty(z) = zQ_\infty(z)\pi_\infty$ for $z = x, y$ and hence $K(x, y) = 0$. If y is in the support of $\psi(x;\pi_\infty)$ and $\gamma(y)$ is the mass at y then

$$\gamma(y) = \frac{G_\infty^{(0)}(y)}{G_\infty'(y)} = \frac{H_\infty^{(0)}(y) - yQ_\infty^{(0)}(y)}{H_\infty'(y) - yQ_\infty'(y)\pi_\infty - Q_\infty(y)\pi_\infty}. \tag{4.30}$$

Setting $H_\infty(y) = yQ_\infty(y)\pi_\infty$ in the identity (2.38) gives

$$H_\infty^{(0)}(y) - yQ_\infty^{(0)}(y)\pi_\infty \equiv \frac{-1}{Q_\infty(y)}$$

and hence

$$\gamma(y) = \frac{1}{Q_\infty^2(y)\pi_\infty + yQ_\infty(y)Q_\infty'(y)\pi_\infty - Q_\infty(y)H_\infty'(y)} = \frac{1}{K(y, y)}. \tag{4.31}$$

Thus if x, y are in the support of $\psi(x;\pi_\infty)$

$$K(x, y) = \begin{cases} 0 & \text{if} \quad x \neq y, \\ \dfrac{1}{\gamma(y)} & \text{if} \quad x = y. \end{cases} \tag{4.32}$$

For fixed x in the support, $K(x, y)$ considered as a function of y is therefore in $L_2(\psi(x;\pi_\infty))$ and for any g in $L_2(\psi(x;\pi_\infty))$

$$g(x) = \int_0^\infty g(y)K(x, y)d\psi(y;\pi_\infty) = (g, K(x, \cdot)).$$

Moreover the partial sums

$$K_n(x, y) = Q_\infty(x)Q_\infty(y)\pi_\infty + \sum_{k=0}^{n} Q_k(x)Q_k(y)\pi_k$$

considered as functions of y converge in $L_2(\psi(x; \pi_\infty))$ because

$$\|K_n(x, \cdot) - K_{n+p}(x, \cdot)\|^2 = \sum_{k=n+1}^{n+p} Q_k^2(x)\pi_k \leqq \sum_{k=n+1}^{\infty} Q_k^2(x)\pi_k$$

which $\to 0$ as $n \to \infty$.

Now let g be any continuous function with compact support. Then

$$\lim_{n\to\infty} (g, K_n(x, \cdot) = \lim_{n\to\infty} \int_0^\infty g(y)K_n(x, y)d\psi(y; \pi_\infty) = \int_0^\infty g(y)\, K(x, y)d\psi(y; \pi_\infty),$$

which shows that $K_n(x, \cdot) \to K(x, \cdot)$ in $L_2(\psi(x; \pi_\infty))$. It follows that if g is orthogonal to Q_∞ and every Q_k, then for each x in the support of $L_2(\psi(x; \pi_\infty))$

$$g(x) = (g, K(x, \cdot)) = \lim_{n\to\infty} (g, K_n(x, \cdot)) = 0,$$

that is $g = 0$. This completes the proof of the following theorem.

THEOREM 18. *The functions Q_∞, Q_k, $k = 0, 1, 2, \cdots$ form a complete orthogonal system in $L_2(\psi(x; \pi_\infty))$.*

The norm of Q_∞ is obtained from

$$\begin{aligned} 1 = Q_\infty(0) &= \int_0^\infty K(0, y)Q_\infty(y)d\psi(y; \pi_\infty) = \lim_{n\to\infty} \int_0^\infty K_n(0, y)Q_\infty(y)d\psi(y; \pi_\infty) \\ &= \pi_\infty \int_0^\infty Q_\infty^2(y)d\psi(y; \pi_\infty). \end{aligned} \tag{4.33}$$

The entire functions $\pi_\infty G_\infty(x)$, $\pi_\infty G_\infty^0(x)$ converge to $H_\infty(x)$ and $H_\infty^0(x)$ as $\pi_\infty \to 0$, and it is easily shown that $\psi(x; \pi_\infty) \to \psi_{\max}$ when $\pi_\infty \to 0$. When $\pi_\infty \to 0$ the functions $G_\infty(x)$, $G_\infty^0(x)$ converge to $-xQ_\infty(x)$ and $-xQ_\infty^0(x)$, and it can be shown that $\psi(x; \pi_\infty) \to \psi_{\min}$. These facts will be assumed without giving the details of the proof.

Define the matrix $P(t; \pi_\infty)$ for $t \geqq 0$, $0 < \pi_\infty < \infty$ by

$$P_{ij}(t; \pi_\infty) = \pi_j \int_0^\infty e^{-xt}Q_i(x)Q_j(x)d\psi(x; \pi_\infty), \qquad i, j = \infty, 0, 1, \cdots. \tag{4.34}$$

For $i, j < \infty$ it has been shown that $P_{ij}(t; \pi_\infty)$ is a nonincreasing function of π_∞ for $t \geqq 0$. Moreover if $t > 0$ and $i, j < \infty$ then

$$1 = \sum_{j<\infty} P_{ij}(t; \psi_{\max}) > \sum_{j<\infty} P_{ij}(t; \pi_\infty) > \sum_{j<\infty} P_{ij}(t; \psi_{\min}), \tag{4.35}$$

the inequalities being strict because since the sums are nonincreasing functions of t, failure of strict inequality for some $t_0>0$ would imply, for example, that $P_{i0}(t;\psi_{\max})=P_{i0}(t;\pi_\infty)$ for $0\leqq t\leqq t_0$ and hence $\psi_{\max}=\psi(x;\pi_\infty)$ which is false. Consequently

$$(4.36)\qquad 0 < P_{i\infty}(t;\pi_\infty) < 1 \qquad \text{for } t > 0,\ i = 0, 1, 2, \cdots.$$

Finally $P_{\infty\infty}(t;\pi_\infty)$ is clearly a strictly decreasing positive function of t. Thus $P(t;\pi_\infty)$ is elementwise strictly positive for $t>0$. It is clear that $P_{ij}(0;\pi_\infty)=\delta_{ij}$, $i, j=\infty, 0, 1, \cdots$, and from (4.23), (4.24),

$$(4.37)\qquad \sum_{0\leq j\leq\infty} P_{ij}(t;\pi_\infty) = 1 \qquad \text{for } t \geqq 0,\ 0 \leqq i \leqq \infty.$$

From the above completeness theorem it follows that $P(t;\pi_\infty)$ has the semi-group property. This completes the proof of the following:

THEOREM 19. *The matrix $P(t;\pi_\infty)$ defined by* (4.34) *is elementwise strictly positive for $t>0$, reduces to the identity matrix for $t=0$, its rows sum to* 1 *for all $t\geqq 0$, and it has the semi-group property*

$$P_{ij}(t+s;\pi_\infty) = \sum_{0\leq k\leq\infty} P_{ik}(t;\pi_\infty)P_{kj}(s;\pi_\infty).$$

5. **Other methods of passage to the limit.** We end the chapter with a brief discussion of a more general truncation procedure.

Consider the matrix

$$C(n,\lambda,\mu) = \begin{bmatrix} -\lambda_0 & \lambda_0 & & & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \mu_{n-1} & -(\lambda_{n-1}+\mu_{n-1}) & \lambda_{n-1} \\ & & & & \mu & -(\lambda+\mu) \end{bmatrix}.$$

Assume the solution of the S moment problem is not unique and let

$$0 \leqq x_0 < \xi = \lim_{n\to\infty} \xi_{1,n}.$$

Let $\pi_\infty>0$ be a given constant. If μ is determined by the equation $\mu_n\pi_n=\mu\pi_\infty$ and then λ is chosen so that

$$\lambda = x_0 - \frac{H_n(x_0)}{Q_n(x_0)\pi_\infty},$$

which gives $\lambda>x_0\geqq 0$ then the unique measure ψ_n of order $n+1$ used in the representation of $e^{tC(n,\lambda,\mu)}$ is supported by the zeros of the polynomial

$$G_n(x) = (x_0 - x)Q_n(x_0)Q_n(x)\pi_\infty + [H_n(x)Q_n(x_0) - H_n(x_0)Q_n(x)],$$

and ψ_n has mass

$$\rho_n(x_0) = \frac{1}{\sum_{k=0}^{n-1} Q_k^2(x_0)\pi_k + Q_n^2(x_0)\pi_\infty}$$

at x_0. When $n \to \infty$ the polynomials $G_n(x)$ converge to an entire function $G_\infty(x)$, the distributions ψ_n converge to a distribution ψ supported by the zeros of G_∞, and ψ has mass

$$\rho_\infty(x_0) = \frac{1}{\sum_0^{\infty} Q_k^2(x_0)\pi_k + Q_\infty^2(x_0)\pi_\infty}$$

at x_0. As before the polynomials $Q_n(x)$ together with the limiting function $Q_\infty(x)$ form a complete orthogonal system in $L_2(\psi)$. Defining an augmented matrix $P(t;\, x_0,\, \pi_\infty)$ in the obvious way one obtains a semi-group of element-wise positive matrices for which

$$\sum_{0 \leq j \leq \infty} P_{ij}(t;\, x_0,\, \pi_\infty) \leqq 1.$$

Here however, the equality fails for $t>0$ if $x_0>0$.

Chapter V. Total positivity of the semi-group solutions

1. The positivity of the matrices $P(t;\,\psi)$ for all solutions ψ of the S moment problem played a fundamental role in studying the structure of these matrices. In this chapter a more general kind of positivity theorem is established. It is shown that if $P(t)$ is a matrix belonging to either of the two linearly ordered families of solutions studied in Chapter IV then the determinants

$$\det\,(P_{i_\mu j_\nu}(t)),$$

for $t>0$; $i_1<i_2< \cdots <i_n$, $j_1<j_2< \cdots <j_n$ are strictly positive.

In a separate paper the probabilistic meaning of these determinants is given (for stationary Markoff processes in general). Here we are concerned with establishing the positivity of the subdeterminants for the special processes considered.

If $C=(c_{ij})$ is any finite or infinite matrix the k-square determinant

$$\begin{vmatrix} c_{i_1 j_1} & c_{i_1 j_2} & \cdots & c_{i_1 j_k} \\ c_{i_2 j_1} & c_{i_2 j_2} & \cdots & c_{i_2 j_k} \\ \cdot & & & \\ c_{i_k j_1} & c_{i_k j_2} & \cdots & c_{i_k j_k} \end{vmatrix}$$

will be denoted by

$$C\begin{pmatrix} i_1 i_2 \cdots i_k \\ j_1 j_2 \cdots j_k \end{pmatrix}.$$

It will always be assumed that $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$. The determinant is called a *subdeterminant* of C and is called a *principal subdeterminant* if $i_1 = j_1, i_2 = j_2, \cdots, i_k = j_k$.

C is called *totally positive* (*strictly totally positive*) if its subdeterminants of all orders are non-negative (strictly positive). C is called a *Jacobi* matrix if $c_{ij} = 0$ for $|i-j| > 1$.

LEMMA 9. *Any Jacobi matrix whose elements are all non-negative and whose principal subdeterminants are all non-negative is totally positive.*

This is proved in [5, p. 457].

LEMMA 10. *If C is a finite real Jacobi matrix with off-diagonal elements all non-negative then e^{tC} is totally positive for $t \geqq 0$.*

Proof. For given $t \geqq 0$, $I + tC/n$ is totally positive when n is large, by Lemma 9. The multiplication rule for subdeterminants shows that a product of totally positive matrices is totally positive. Hence

$$e^{tC} = \lim_{n \to \infty} \left(I + \frac{tC}{n} \right)^n$$

is totally positive.

If ψ is an extremal solution of the S moment problem, or an extremal solutions of the second kind, then $P(t, \psi)$ is an elementwise limit of matrices of the form e^{tA_n} where the A_n are Jacobi matrices with off-diagonal elements non-negative. It follows that for any such ψ and for $t \geqq 0$ the matrix $P(t, \psi)$ is totally positive. It will be shown that these matrices are strictly totally positive for $t > 0$.

LEMMA 11. *Let ψ be any solution of the S moment problem and let $t \geqq 0$. Then*

$$P(t; \psi)\begin{pmatrix} i_1 \cdots i_k \\ i_1 \cdots i_k \end{pmatrix} > 0,$$

i.e., the principal subdeterminants are strictly positive.

Proof. The formula

$$P(t; \psi)\begin{pmatrix} i_1 \cdots i_k \\ i_1 \cdots i_k \end{pmatrix}$$

$$= \pi_{i_1} \cdots \pi_{i_k} \int \cdots \int_{0 \leqq x_1 < \cdots x_k < \infty} e^{-(x_1 + \cdots + x_k)t} \begin{vmatrix} Q_{i_1}(x_1) & \cdots & Q_{i_1}(x_k) \\ \vdots & & \\ Q_{i_k}(x_1) & \cdots & Q_{i_k}(x_k) \end{vmatrix}^2 d\psi(x_1) \cdots d\psi(x_k)$$

shows that if the left member is zero for some $t \geqq 0$ then the determinant in the integrand must be identically zero on the support of $d\psi(x_1) \cdots d\psi(x_k)$ and hence the left member is zero for $t=0$, which is a contradiction. The result is also valid for $P(t; \pi_\infty)$ with $i_k = \infty$.

The next two lemmas are due to Gantmacher and Krein [5, pp. 453–454].

LEMMA 12. *Let $B=(b_{ij})$ be a totally positive matrix with m rows and n columns, and let $1=i_1<i_2< \cdots <i_p=m$ be a set of p row indices beginning with 1 and ending with m. If the rows $i_1, \cdots, i_p$ are linearly dependent while the rows $i_1, \cdots, i_{p-1}$ and the rows $i_2, \cdots, i_p$ are linearly independent, then the rank of B is $p-1$.*

LEMMA 13. *Let B be a totally positive $m \times n$ matrix and $1=i_1<i_2< \cdots <i_p =m$: $1=k_1<k_2< \cdots <k_p=n$. If*

$$B\begin{pmatrix} i_1 \cdots i_p \\ k_1 \cdots k_p \end{pmatrix} = 0$$

while

$$B\begin{pmatrix} i_1 \cdots i_{p-1} \\ k_1 \cdots k_{p-1} \end{pmatrix} \neq 0, \qquad B\begin{pmatrix} i_2 \cdots i_p \\ k_2 \cdots k_p \end{pmatrix} \neq 0$$

then B has rank $p-1$.

DEFINITION.

$$C\begin{pmatrix} i_1 \cdots i_k \\ j_1 \cdots j_k \end{pmatrix}$$

is called *quasi-principal* if $\sum_{\nu=1}^{k} |i_\nu - j_\nu| \leqq 1$.

LEMMA 14. *If B is a totally positive matrix whose elements are all positive and whose principal subdeterminants are all positive, then all the quasi-principal subdeterminants of B are positive.*

Proof. An induction on the order p of the subdeterminant is made. The hypothesis guarantees the case $p=1$. Assume the result proved for all orders $<p$, and suppose if possible that

$$B\begin{pmatrix} i_1 \cdots i_p \\ k_1 \cdots k_p \end{pmatrix} = 0$$

where the subdeterminant is quasi-principal. Let B^* be the matrix obtained from B by deleting all rows of index $<i_1$ or $>i_p$ and all columns of index $<k_1$ or $>k_p$. Since

$$B\begin{pmatrix} i_1 \cdots i_{p-1} \\ k_1 \cdots k_{p-1} \end{pmatrix} \neq 0 \neq B\begin{pmatrix} i_2 \cdots i_p \\ k_2 \cdots k_p \end{pmatrix},$$

Lemma 13 shows that B^* has rank $p-1$. Let $h=\max(i_1, k_1)$. Because of the quasi-principal property

$$i_1,\ k_1 \leqq h \leqq h + p - 1 \leqq i_p,\ k_p.$$

Consequently the principal subdeterminant

$$B\begin{pmatrix} h, & h+1, & \cdots, & h+p-1 \\ k, & k+1, & \cdots, & k+p-1 \end{pmatrix}$$

of order p is zero, which is a contradiction.

THEOREM 20. *If ψ is an extremal solution of the S moment problem, or an extremal solution of the second kind, then $P(t;\psi)$ is strictly totally positive for each $t>0$.*

Proof. It follows from Lemma 14, Lemma 11, and the fact that $P_{ij}(t)>0$ for $t>0$, that all the quasi-principal subdeterminants of $P(t,\psi)$ are strictly positive. Let ψ be an extremal solution and

$$B(t) = P(t;\psi)\begin{pmatrix} i_1, & \cdots, & i_p \\ j_1, & \cdots, & j_p \end{pmatrix}, \qquad (t > 0),$$

be a subdeterminant of $P(t,\psi)$ and call

$$M = \sum_{\nu=1}^{p} | i_\nu - j_\nu |$$

the index sum of $B(t)$. If $M\leqq 1$ then $B(t)>0$. The proof will be by induction on M. We assume every subdeterminant of $P(t;\psi)$ with index sum $<M$ is positive.

Suppose $m>\max(i_p, j_p)$ and let $C(t)$ be the m-square matrix formed by the first $m+1$ rows and columns of $B(t)$. If $t>0$, $s>0$, then

$$B(t+s) = \begin{vmatrix} \sum P_{i_1k}(t;\psi)P_{kj_1}(s;\psi), & \cdots, & \sum P_{i_1k}(t;\psi)P_{kj_p}(s;\psi) \\ \cdot & & \\ \sum P_{i_pk}(t;\psi)P_{kj_1}(s;\psi), & \cdots & \end{vmatrix}.$$

Consequently

$$\begin{aligned} B(t+s) &= \sum_{0\leq k_1<k_2<\cdots<k_p} P(t;\psi)\begin{pmatrix} i_1 & \cdots & i_p \\ k_1 & \cdots & k_p \end{pmatrix} P(s;\psi)\begin{pmatrix} k_1 & \cdots & k_p \\ j_1 & \cdots & j_p \end{pmatrix} \\ &\geqq \sum_{0\leq k_1<\cdots<k_p\leq m} C(t)\begin{pmatrix} i_1 & \cdots & i_p \\ k_1 & \cdots & k_p \end{pmatrix} C(s)\begin{pmatrix} k_1 & \cdots & k_p \\ j_1 & \cdots & j_p \end{pmatrix}. \end{aligned}$$

Now when the index sum of $B(t)$ is $M>1$, there is at least one term in the above sum for which both factors have an index sum $<M$. This term is posi-

tive, all other terms are non-negative, and hence $B(t+s)>0$. With minor modifications the same method can be used when ψ is an extremal solution of the second kind.

The next theorem is a slight extension of a theorem of Loewner [10] on totally positive matrices.

THEOREM 21. *Let $P(t)$, $0\leqq t<t_0$ be a family of totally positive matrices which is elementwise differentiable with respect to t at $t=0$, and for which $P(0)=I$. If all subdeterminants of $P(t)$ of orders 1 and 2 are non-negative then the infinitesimal matrix $P'(0)$ is a Jacobi matrix.*

Proof. Let

$$a_{ij} = \lim_{t\to 0} \frac{P_{ij}(t)}{t}, \qquad i \neq j,$$

$$a_{ii} = \lim_{t\to 0} \frac{P_{ii}(t) - 1}{t} \cdot$$

If $j>i+1$,

$$0 \leqq P(t)\begin{pmatrix} i & i+1 \\ i+1 & j \end{pmatrix} = -a_{ij}t + o(t),$$

and from $a_{ij}\geqq 0$ it follows that $a_{ij}=0$. If $j<i-1$ a similar argument shows that $a_{ij}=0$.

2. **Variation diminishing properties.** A sequence $x=\{x_j\}$ is said to have a change of sign at k if $x_kx_j<0$, where j is the first index $>k$ for which $x_j\neq 0$. It will be shown that when P is a strictly totally positive matrix and $x=\{x_j\}$ has exactly n changes of sign, then $y=Px$ has at most n changes of sign.

The matrix $P=(P_{ij})$ is first extended so that i becomes a continuous variable. $P=(P_{\alpha,j})$ is defined by

$$P_{\alpha j} = \theta P_{ij} + (1-\theta)P_{i+1,j}$$

where

$$\alpha = \theta i + (1-\theta)(i+1), \qquad 0 \leqq \theta \leqq 1.$$

The extended matrix is still strictly totally positive in the sense that if $\alpha_1<\alpha_2<\cdots<\alpha_n$, $j_1<j_2<\cdots<j_n$ and if no half-open interval $i\leqq\alpha<i+1$ formed by two consecutive integers contains more than one of the α_i, then

$$P\begin{pmatrix} \alpha_1, & \cdots, & \alpha_n \\ j_1, & \cdots, & j_n \end{pmatrix} > 0.$$

This can be seen by expressing the determinant as a convex combination of the 2^n determinants

$$P\begin{pmatrix} i_1', & \cdots, & i_n' \\ j_1, & \cdots, & j_n \end{pmatrix}$$

where i_ν' is either i_ν or $i_\nu+1$.

Suppose $x=\{x_j\}$ has exactly k changes of sign. Let

$$y(\alpha) = \sum_j P_{\alpha j}x_j.$$

It is assumed that either P is elementwise bounded and $\sum_j |x_j| < \infty$ or that x is a bounded sequence and $\sum_j P_{ij} < \infty$ for each i. The numbers x_j can be divided into groups

$$x_1, x_2, \cdots, x_{\nu_1}; \qquad x_{\nu_1+1}, x_{\nu_1+2}, \cdots, x_{\nu_2}; \qquad \cdots; \qquad x_{\nu_k+1}, x_{\nu_k+2}, \cdots$$

so that the elements of the first group are, say, all $\leqq 0$, the second group all $\geqq 0$, the third group all $\leqq 0$, etc., and so that each group contains a nonzero element.

The function $y(\alpha)$ can be expressed in the form

$$y(\alpha) = \sum_{r=1}^{k+1} \epsilon_r C_{\alpha r}$$

where

$$C_{\alpha r} = \sum_{j=\nu_{r-1}+1}^{\nu_r} P_{\alpha j} | x_j |$$

and $\epsilon_1 = -1$, $\epsilon_r = -\epsilon_{r-1}$. Now if $\alpha_1 < \alpha_2 < \cdots < \alpha_{k+1}$ and no two of the α_ν are in the same half open interval $i \leqq \alpha < i+1$, then det $(C_{\alpha_\nu r})$ can be expressed as a positive linear combination of the determinants

$$P\begin{pmatrix} \alpha_1 & \cdots & \alpha_{k+1} \\ j_1 & \cdots & j_{k+1} \end{pmatrix}$$

with at least one nonzero coefficient. Hence det $(C_{\alpha_\nu r}) > 0$ and $y(\alpha)$ cannot vanish for each of the values $\alpha = \alpha_1, \cdots, \alpha_{k+1}$. On the other hand $y(\alpha)$ is linear in each of the intervals $i \leqq \alpha \leqq i+1$. Consequently the sequence $y = Px$ has at most k changes of sign. This proves the following.

THEOREM 22. *If $P = (P_{ij}(t))$ is a strictly totally positive solution then the transformations*

$$y_i = \sum_j P_{ij}(t)x_j, \qquad u_j = \sum_i P_{ij}(t)v_i$$

of l_∞ to l_∞ and l_1 to l_1 respectively, are sign variation diminishing.

For the first of these transformations a sharper result can be established when $\sum_j P_{ij}(t) \equiv 1$; namely, if $x_j = \lambda$ for at most k values of j then $y_i = \lambda$ for at most k values of i. These properties strongly spell out the diffusive nature of the process. The results of this section should be compared with the corresponding variation diminishing properties possessed by convolution transformations defined by means of Pólya frequency functions ([15], see also [16 Chap. 4]).

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