FOURIER-STIELTJES SERIES OF WALSH FUNCTIONS

BY

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1. It is known(1) that a trigonometric series is a Riemann-Stieltjes series if and only if its $(C, 1)$ sums are bounded in the L_1 norm. The analogous problem for the Walsh system needs a slight reformulation, since the Walsh functions are not continuous. Morgenthaler(2) has proved that a Walsh series is a Riemann-Stieltjes series corresponding to a continuous determining function of bounded variation if and only if

$$\int_0^1 \left| \sigma_n(x) \right| dx = O(1) \tag{1}$$

and

$$\frac{s_n(x)}{n} \to 0 \qquad \textit{uniformly in } [0, 1], \tag{2}$$

where s_n and σ_n are the partial sums and the $(C, 1)$ sums, respectively, of the given series. This still leaves open the general case, in which a Lebesgue-Stieltjes integral is used, the determining function being merely of bounded variation. It is the purpose of this paper to settle the question by giving necessary and sufficient conditions, and to show how the determining function may be recovered from the given series. It turns out that (1), which is necessary, is not sufficient. However, the analogy with the trigonometric case can be restored completely by transferring attention to the dyadic group, of which the Walsh functions are essentially the characters. It is then an easy matter to return to the unit interval.

2. The dyadic group G consists of all sequences $\bar{x} = (x_1, x_2, \cdots)$, $x_i = 0, 1$, where addition is defined coordinatewise mod 2. The product topology is assigned to G, and with it G becomes compact and totally disconnected. For a discussion of G and its connection with the Walsh functions, we refer the reader to [2]. We define the mapping

$$\lambda(\bar{x}) = \sum_{i=1}^{\infty} x_i 2^{-i} \qquad (\bar{x} \in G) \tag{3}$$

from G to I, the reals mod 1. It is clear that λ maps G continuously onto I in an almost one-to-one fashion, the exception being that each dyadic

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(1) See [8, p. 79].

(2) See [6, Theorem 16].

rational in I has two pre-images. We make the inverse μ unique by choosing the finite expansion in case of doubt. For a dyadic rational ρ, we shall write $\bar{\rho}$ for $\mu(\rho)$ and $\bar{\rho}'$ for the other element of G which is mapped on ρ by λ. We denote the denumerable set $\{\bar{\rho}'\}$ by E.

If f is a real-valued function on I, there is a corresponding function $\bar{f}$ on G, given by

$$\begin{aligned} \bar{f}(\bar{x}) &= f(\lambda(\bar{x})) && (\bar{x} \in G - E), \\ &= \limsup_{\bar{y}\to\bar{x}} \bar{f}(\bar{y}) && (\bar{x} \in E), \end{aligned} \tag{4}$$

where the approach is over those $\bar{y}$ corresonding to dyadic irrationals. We shall indicate that (4) holds by writing $\bar{f}\sim f$. If f is continuous, so is $\bar{f}$, but not conversely. For example, the characters $\bar{\psi}_k$ are continuous but the corresponding Walsh functions ψ_k are not. These matters have been discussed in some detail by Morgenthaler, in the paper previously mentioned.

Throughout this paper, by a *measure* on G (on I) we shall mean a real, finite, signed measure defined on the Borel sets in G (in I)(3). Every measure on G decomposes uniquely into a *usual* measure, vanishing on all subsets of E, and an *unusual* measure, vanishing on all Borel subsets of $G-E$. There is a one-to-one correspondence, denoted by $\bar{m}\sim m$, between the usual measures on G and the measures on I, given by

$$\begin{aligned} \bar{m}(A) &= m\lambda(A) && (A \subset G - E), \\ &= 0 && (A \subset E), \end{aligned} \tag{5}$$

or by

$$m(B) = \bar{m}\mu(B) \qquad (B \subset I). \tag{5'}$$

A character series $\bar{S}=\sum a_k\bar{\psi}_k$ is a (Fourier-) Stieltjes series on G if there exists a measure $\bar{m}$ for which

$$a_k = \int_G \bar{\psi}_k d\bar{m}, \tag{6}$$

and we write $\bar{S}=\bar{S}(d\bar{m})$. Similarly, a Walsh series $S=\sum a_k\psi_k$ is a (Fourier-) Stieltjes series on I if there exists a measure m for which

$$a_k = \int_I \psi_k dm, \tag{7}$$

and we write $S=S(dm)$. In either case the measure is determined uniquely on Borel sets by the sequence $\{a_k\}$.

If a character series $\bar{S}$ and a Walsh series S have the same coefficients, we write $\bar{S}\sim S$, or $S\sim\bar{S}$.

(3) For the measure-theoretic concepts used here, see [4].

3. The following theorem is easy to prove, and is stated merely for convenient reference.

THEOREM 1. (i) *If* $\bar{f}\sim f$ *and* $\bar{m}\sim m$, *then* $\int_G \bar{f}d\bar{m} = \int_I f dm$.

(ii) *If* $\bar{f}\sim f$, *then* $\int_G \bar{f}d\bar{x} = \int_I f dx$, *where* $d\bar{x}$ *denotes the normalized Haar measure on* G *and* dx *denotes Lebesgue measure on* I(4).

(iii) $S(dm_1) = S(dm_2)$ *implies* $m_1 = m_2$, *and* $\bar{S}(d\bar{m}_1) = \bar{S}(d\bar{m}_2)$ *implies* $\bar{m}_1 = \bar{m}_2$ (*on Borel sets*).

(iv) $S = S(dm)$ *and* $\bar{m}\sim m$ *imply* $S(dm)\sim \bar{S}(d\bar{m})$.

(v) $S(dm)\sim\bar{S}(d\bar{m})$ *implies* $\bar{m}\sim m$.

The next theorem has a precise analogue in the trigonometric case(5).

THEOREM 2. *A necessary and sufficient condition that a character series* $\bar{S}$ *be a Stieltjes series on* G *is that its* $(C, 1)$ *sums*

$$\bar{\sigma}_n = \sum_{k=0}^{n-1}\left(1 - \frac{k}{n}\right) a_k \bar{\psi}_k \tag{8}$$

satisfy

$$\int_G |\bar{\sigma}_n| \, d\bar{x} = O(1). \tag{9}$$

Proof. If $\bar{S} = \bar{S}(d\bar{m})$, then

$$\bar{\sigma}_n(\bar{x}) = \int_G \bar{K}_n(\bar{x} \dotplus \bar{t}) d\bar{m}(\bar{t}),$$

where $\bar{K}_n$ is the $(C, 1)$ kernel. By Fubini's theorem,

$$\int_G |\bar{\sigma}_n(\bar{x})| \, d\bar{x} \leqq \int_G \left[\int_G |\bar{K}_n(\bar{x} \dotplus \bar{t})| \, d\bar{x}\right] |d\bar{m}(\bar{t})|.$$

But

$$\int_G |\bar{K}_n(\bar{x} \dotplus \bar{t})| \, d\bar{x} = \int_G |\bar{K}_n(\bar{x})| \, d\bar{x} \leqq 2,$$

by the invariance of Haar measure and a result of Yano's(6). Therefore

$$\int_G |\bar{\sigma}_n(\bar{x})| \, d\bar{x} \leqq 2V,$$

where V is the total variation of $\bar{m}$ over G.

(4) For a proof of this part, see [6, Theorem 1].

(5) See [8, p. 79].

(6) See [7, Lemma 9].

Now suppose that (9) holds. Let $C(G)$ denote the Banach space of real-valued continuous functions on G, with the usual norm. We define the linear functionals T_n on $C(G)$ by

$$T_n(\bar{f}) = \int_G \bar{\sigma}_n(\bar{x})\bar{f}(\bar{x})d\bar{x}. \tag{10}$$

Then $\|T_n\| = \int_G |\bar{\sigma}_n| d\bar{x} = O(1)$. By a theorem of Banach[7], there is a subsequence T_{n_i} which converges weakly to a bounded linear functional T, that is,

$$T_{n_i}(\bar{f}) \to T(\bar{f}) \qquad (\bar{f} \in C(G)).$$

Since every bounded linear functional T on $C(G)$ has the representation[8]

$$T(\bar{f}) = \int_G \bar{f}(\bar{x})d\bar{m},$$

where $\bar{m}$ is a measure on G, we have

$$\int_G \bar{\sigma}_{n_i}(\bar{x})\bar{f}(\bar{x})d\bar{x} \to \int_G \bar{f}(\bar{x})d\bar{m}.$$

Putting $\bar{f} = \bar{\psi}_k$, we get

$$a_k = \lim_{i\to\infty}\left(1 - \frac{k}{n_i}\right)a_k = \int_G \bar{\psi}_k d\bar{m}. \tag{11}$$

Hence $\bar{S} = \bar{S}(d\bar{m})$ and the proof is complete.

We observe that (9) may be replaced by

$$\liminf_{n\to\infty} \int_G |\bar{\sigma}_n| d\bar{x} < \infty, \tag{12}$$

since we can apply the reasoning above to any bounded subsequence of $\{T_n\}$ to obtain (11). By the necessity part of the theorem, we see that (12) implies (9). Also, we can show that the entire sequence $\{T_n\}$ converges weakly to T. For if not, there exist an $\bar{f} \in C(G)$ and two subsequences $\{n_i\}$ and $\{n_i'\}$ such that $T_{n_i}(\bar{f}) \to a$, $T'_{n_i}(\bar{f}) \to b$, and $a \neq b$. By taking sub-subsequences, we find $a = \int_G \bar{f} d\bar{m}_1$, $b = \int_G \bar{f} d\bar{m}_2$. But, as in (11), $\bar{S}(d\bar{m}_1) = \bar{S}(d\bar{m}_2)$, so $\bar{m}_1 = \bar{m}_2$ by Theorem 1 (iii). This implies the contradiction $a = b$.

It is of course not necessary that T_n converge strongly to T. In fact, this is precisely the condition that $\bar{m}$ be absolutely continuous with respect to Haar measure, or that $\bar{S}$ be a Fourier series[9]. It is not true, either, that $\|T_n\| \to \|T\|$, as in the trigonometric case, the difference being that the Walsh

(7) See [1, p. 123].
(8) See [5, Theorem 10] or [4, pp. 247–249].
(9) See [6, Theorem 17].

$(C, 1)$ kernels are merely quasi-positive. However, we can show that $\|T\| \leqq \liminf \|T_n\| \leqq \limsup \|T_n\| \leqq 2\|T\|$, and it seems likely that the constant 2 is best possible. This would be true if

$$\limsup \int_G | \bar{K}_n | \, d\bar{x} = 2.$$

We shall now show how to isolate the discrete component of $\bar{m}$ (and of m).

THEOREM 3. *If $\bar{S} = \bar{S}(d\bar{m})$, then the partial sums satisfy*

$$\bar{s}_n(\bar{x})/n \to \bar{m}(\{\bar{x}\}). \tag{13}$$

If $S = S(dm)$, then

$$s_n(x)/n \to m(\{x\}). \tag{14}$$

Proof. We have

$$\frac{\bar{s}_n(\bar{x})}{n} = \int_G \frac{\bar{D}_n(\bar{x} \dotplus \bar{t})}{n} \, d\bar{m}(\bar{t}),$$

where $\bar{D}_n$ is the Dirichlet kernel. The integrand is bounded by 1 and converges to 1 at $\bar{t} = \bar{x}$, to 0 elsewhere. The result (13) follows from Lebesgue's convergence theorem. To prove (14), let $\bar{m} \sim m$ and apply Theorem 1 (iv). Then, by (13),

$$\frac{s_n(x)}{n} = \frac{\bar{s}_n(\mu(x))}{n} \to \bar{m}(\{\mu(x)\}) = m(\{x\}).$$

THEOREM 4. *Necessary and sufficient conditions that a Walsh series S on I be a Stieltjes series $S(dm)$ are*

$$\int_I | \sigma_n | \, dx = O(1), \tag{15}$$

$$\frac{s_n(\rho - 0)}{n} \to 0 \qquad (\rho = \textit{dyadic rational}). \tag{16}$$

Proof. Let $\bar{S} \sim S$. By Theorem 1 (ii),

$$\int_G | \bar{\sigma}_n | \, d\bar{x} = \int_I | \sigma_n | \, dx,$$

so by Theorem 2, (15) is necessary and sufficient that $\bar{S} = \bar{S}(d\bar{m})$. Again by Theorem 1 (iv and v), $S = S(dm)$ is equivalent to $\bar{m}$ being a usual measure and $\bar{m} \sim m$. By Theorem 3, $\bar{m}$ is usual if and only if

$$\bar{s}_n(\bar{\rho}')/n \to 0 \tag{17}$$

for every dyadic rational ρ. But $\bar{s}_n(\bar{\rho}') = s_n(\rho - 0)$, so (17) is equivalent to (16), and the proof is complete.

A simple example to show that (16) may fail even if (15) holds is the following. Let $a_k=\psi_k(1-0)$, $S=\sum a_k\psi_k$. Then $S\sim\bar{S}(d\bar{m})$, where $\bar{m}$ is the unit measure concentrated at $\bar{0}'=(1, 1, 1, \cdots)$. It is easy to verify directly that there is no measure m on I for which $S=S(dm)$, that $\sigma_n(x)=K_n(x\dotplus(1-0))$, so that (15) holds, and that $s_n(1-0)/n=1$.

This situation is typical. Given a Walsh series S satisfying (15), the limits

$$c(\rho) = \lim_{n\to\infty}\frac{s_n(\rho-0)}{n} \qquad (\rho = \textit{dyadic rational}) \tag{18}$$

always exist, the series $\sum_\rho c(\rho)$ converges absolutely, and the Walsh series with coefficients

$$a_k - \sum_\rho c(\rho)\psi_k(\rho-0) \tag{19}$$

is a Stieltjes series $S(dm)$. In fact, $S\sim\bar{S}(d\bar{m})$, $\bar{m}=\bar{m}_1+\bar{m}_2$, where $\bar{m}_1$ is usual, $\bar{m}_2$ unusual, and $\bar{m}_1\sim m$. The sum in (19) is the kth coefficient of $\bar{S}(d\bar{m}_2)$, and $c(\rho)=\bar{m}_2(\{\bar{\rho}'\})$.

4. We have mentioned earlier a result due to Morgenthaler, which in our notation is equivalent to a characterization of those $S(dm)$ with m nonatomic. Theorem 5 corresponds to his result, although the method differs somewhat.

THEOREM 5. (i) *Necessary and sufficient conditions that $\bar{S}=\bar{S}(d\bar{m})$ with $\bar{m}$ nonatomic are*

$$\int_G |\bar{\sigma}_n|\,d\bar{x} = O(1), \tag{20}$$

$$\bar{s}_n(\bar{x})/n \to 0 \textit{ uniformly in } G. \tag{21}$$

(ii) *Necessary and sufficient conditions that $S=S(dm)$ with m nonatomic are*

$$\int_I |\sigma_n|\,dx = O(1), \tag{20$'$}$$

$$s_n(x)/n \to 0 \textit{ uniformly in } I. \tag{21$'$}$$

Proof. (i) The sufficiency of (20) and (21) follows immediately from Theorems 2 and 3. The necessity of (20) also follows from Theorem 2. Suppose, then, that $\bar{S}=\bar{S}(d\bar{m})$ with $\bar{m}$ nonatomic. For each $\bar{x}\in G$ and $\epsilon>0$, there is a neighborhood $N_r(\bar{x})$, consisting of all $\bar{y}\in G$ which coincide with $\bar{x}$ in the first r places, such that $|m|(N_r(\bar{x}))<\epsilon/2$. Since G is compact, we may take r to depend only on ϵ. Then

$$\frac{|\bar{s}_n(\bar{x})|}{n} \leqq \int_G \frac{|\bar{D}_n(\bar{x}\dotplus\bar{t})|}{n}\,|d\bar{m}(\bar{t})| = \int_{N_r(\bar{x})} + \int_{G-N_r(\bar{x})} = J_1 + J_2,$$

say. We have $J_1 \leqq |m|(N_r(\bar{x})) < \epsilon/2$. But $|\bar{D}_n(\bar{x} \dot{+} \bar{t})| \leqq 2^{r+1}$ for $\bar{t} \in G - N_r(\bar{x})$, so

$$J_2 \leqq 2^{r+1} |\bar{m}|(G)/n < \epsilon/2$$

for $n > n_0(r, \epsilon) = n_0'(\epsilon)$. This completes the proof of (i).

(ii) Given (20′) and (21′), we observe that $s_n(\rho - 0) = s_n(\rho - 2^{-n})$, so (15) and (16) hold, and $S = S(dm)$, by Theorem 4. That m is nonatomic then follows from Theorem 3. Conversely, if $S = S(dm)$ with m nonatomic, then $S(dm) \sim \bar{S}(d\bar{m})$ with $\bar{m} \sim m$, and $\bar{m}$ is nonatomic. By (i), $\bar{s}_n(\bar{x})/n \to 0$ uniformly in G, so $s_n(x)/n = \bar{s}_n(\mu(x))/n \to 0$ uniformly in I, and (21′) holds. Of course (20′) holds, by Theorem 4, and the proof of the theorem is complete.

Our next result replaces (21) and (21′) in Theorem 5 by a condition which depends more directly on the coefficients.

THEOREM 6. *If $\bar{S} \sim S = \sum a_k \psi_k$, then S and $\bar{S}$ are Stieltjes series corresponding to nonatomic measures if and only if*

$$\int_I |\sigma_n| \, dx = O(1), \tag{22}$$

$$\frac{1}{n} \sum_{k=0}^{n-1} a_k^2 \to 0. \tag{23}$$

Proof. (22) is equivalent to $\bar{S} = \bar{S}(d\bar{m})$. By Theorem 3, and Lebesgue's convergence theorem([10])

$$\int_G \frac{\bar{s}_n(\bar{x})}{n} d\bar{m}(\bar{x}) \to \int_G \bar{m}(\{\bar{x}\}) d\bar{m}(\bar{x}) = \sum_{\bar{x} \in G} \bar{m}^2(\{\bar{x}\}),$$

the last sum containing only countably many nonzero terms. But

$$\begin{aligned} \int_G \frac{\bar{s}_n(\bar{x})}{n} d\bar{m}(\bar{x}) &= \lim_{N \to \infty} \frac{1}{n} \int_G \bar{s}_n(\bar{x}) \bar{\sigma}_N(\bar{x}) d\bar{x} \\ &= \lim_{N \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k^2 \left(1 - \frac{k}{N}\right) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} a_k^2. \end{aligned}$$

Therefore

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k^2 = \sum_{\bar{x} \in G} \bar{m}^2(\{\bar{x}\}), \tag{24}$$

and the theorem is proved for $\bar{S}$. It follows for S by Theorem 1 and the fact

([10]) Clearly $|a_k| \leqq |\bar{m}|(G)$, so $|\bar{s}_n(\bar{x})/n| \leqq |\bar{m}|(G)$.

that a nonatomic $\bar{m}$ is necessarily usual.

Similar results for trigonometric series are known([11]). For example, $S=S(dm)$ implies

$$\frac{s_n(x)}{n} \to \frac{m(\{x\})}{\pi}, \tag{25}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{-n}^{n} |c_k|^2 = \frac{1}{2\pi^2} \sum_x m^2(\{x\}), \tag{26}$$

where $c_k=(1/2\pi)\int_0^{2\pi} e^{-ikt}dm(t)$.

5. By Theorem 3, we can isolate the atomic portion of a measure m by means of the partial sums of $S(dm)$. Our next step is to recover the distribution function $F(x) = m([0, x))$.

THEOREM 7. *If x is a dyadic rational or a point of continuity of F, then*

$$F(x) = \lim_{n\to\infty} \int_0^x s_n(t)dt. \tag{27}$$

Proof. Let $g(u)$ denote the characteristic function of $[0, x)$, and let $g(u; r)$ be the Abel sum([12]) of its Walsh-Fourier series:

$$g(u; r) = \sum_{k=0}^{\infty} J_k(x)\psi_k(u)r^k \qquad (0 \leqq r < 1), \tag{28}$$

where $J_k(x) = \int_0^x \psi_k(t)dt$. Under either hypothesis for x, $g(u; r) \to g(u)$ a.e. (m) as $r \to 1-0$, and $|g(u, r)| \leqq 1$. Hence

$$\int_I g(u; r)dm(u) \to \int_I g(u)dm(u) = F(x).$$

For fixed $r<1$, the series (28) converges uniformly in u, so

$$\int_I g(u; r)dm(u) = \sum_{k=0}^{\infty} a_k J_k(x)r^k. \tag{29}$$

Thus the series in (29) is Abel summable to $F(x)$. Now $a_k=O(1)$, $J_k(x) = O(1/k)$. By Littlewood's Tauberian theorem, the series converges at $r=1$:

$$F(x) = \sum_{k=0}^{\infty} a_k J_k(x). \tag{30}$$

Equation (30) is equivalent to (27).

The analogous theorem for trigonometric series is that for all a and b,

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([11]) See [8].

([12]) See [2, §7 and Theorem 21].

$$\frac{F(b+0)+F(b-0)}{2}-\frac{F(a+0)+F(a-0)}{2}=\lim_{n\to\infty}\int_a^b s_n(t)dt.$$

The proof can be made as in Theorem 7, or even more easily by observing that $s_n(t;\, dF)=s_n'(t;\, F)$ and using Dirichlet's theorem. The restriction on x in Theorem 7 cannot be removed, as may be seen by taking m to be the unit measure concentrated at a dyadic irrational x and referring to §8 of [2].

The problem of the recovery of F is theoretically solved by Theorem 7, but the following is also of interest, since it shows how the absolutely continuous component can be isolated.

THEOREM 8. *If* $S=S(dm)=S(dF)$, *then*

$$\text{(i)}\quad s_{2^n}(x)\to F'(x) \text{ a.e.},$$

$$\text{(ii)}\quad \sigma_n(x)\to F'(x) \text{ a.e.},$$

Proof. Define $\alpha_n(x)$, $\beta_n(x)$ by

$$p\cdot 2^{-n}=\alpha_n(x)\leqq x<\beta_n(x)=(p+1)2^{-n}.$$

Since the characteristic function of $[\alpha_n,\beta_n)$ is $2^{-n}\sum_{k=0}^{2^n-1}\psi_k(x)\psi_k(t)$,

$$\frac{F(\beta_n)-F(\alpha_n)}{\beta_n-\alpha_n}=2^n\int_{[\alpha_n,\beta_n)}dm=\int_I\sum_{k=0}^{2^n-1}\psi_k(x)\psi_k(t)dm(t)$$
$$=s_{2^n}(x),$$

from which (i) follows immediately. For F absolutely continuous, (ii) has been proved in [3]. The proof for an arbitrary F is quite similar and requires only technical modifications. We shall therefore omit it. For the trigonometric case, see [8, p. 59].

From Theorems 7 and 8 we get the following representations for the singular component of a continuous F:

$$F^*(x)=\lim_{n\to\infty}\int_0^x s_{2^n}(t)dt-\int_0^x\lim_{n\to\infty}s_{2^n}(t)dt, \tag{31}$$

$$F^*(x)=\lim_{n\to\infty}\int_0^x \sigma_n(t)dt-\int_0^x\lim_{n\to\infty}\sigma_n(t)dt. \tag{32}$$

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