CHARACTERISTIC CLASSES OF HOMOGENEOUS SPACES

BY

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Introduction. It is the object of this paper to prove the results of §§11, 12, and 13; for a homogeneous space G/K (G a compact Lie group, K a closed connected subgroup) it is shown that:

if K is abelian, the Pontrjagin classes of G/K are zero;

if G/K is symmetric, or simply-connected complex homogeneous, all characteristic classes of degree greater than (dimension G/K) $-$ (rank $G-$rank K) are zero.

Finally, it is shown that the characteristic classes of G/K depend, in a sense, only on the abelian part of K.

The canonical connection of the second kind on G/K is used to demonstrate that the characteristic classes of G/K with respect to its bundle of oriented frames can be considered characteristic classes of G/K with respect to its principal K-bundle. The characteristic classes of the K-bundle structure are then computed by means of the well-known algebraic operations in the Lie algebra of a semi-simple Lie group.

The first sections of this paper are devoted to an exposition of the needed facts concerning characterisic classes, in particular those of homogeneous spaces; much of the material is based on [3]. The numbers in brackets refer to the bibliography.

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1. **Invariants.** Let $\mathfrak{g}$ be the Lie algebra of a Lie group G of dimension N. A real-valued r-linear function T on $\mathfrak{g}$ is called a *symmetric r-tensor* if $T(\cdots, X, \cdots, Y, \cdots) = T(\cdots, Y, \cdots, X, \cdots)$ for any two elements X, Y of $\mathfrak{g}$. It is called *invariant under* G (or, simply, *invariant*) if $T(\text{ad } gX, \cdots, \text{ad } gZ) = T(X, \cdots, Z)$ for any r elements $X, \cdots, Z$ of $\mathfrak{g}$ and any element g of G. The set of all invariant symmetric r-tensors on $\mathfrak{g}$ is denoted by $I^r(\mathfrak{g})$, and the set of all invariant symmetric tensors on $\mathfrak{g}$ is denoted by $I(\mathfrak{g})$.

If T, T' are symmetric r-tensors on $\mathfrak{g}$ and S is a symmetric s-tensor on $\mathfrak{g}$, we define symmetric r- and $r+s$-tensors $T+T'$ and $T\cdot S$ on $\mathfrak{g}$ as follows:

$$(T + T')(X, \cdots, Z) = T(X, \cdots, Z) + T'(X, \cdots, Z),$$

$$X, \cdots, Zr \text{ elements of } \mathfrak{g}.$$

$$(T\cdot S)(X_1, \cdots, X_{r+s}) = ((r + s)!)^{-1} \sum T(X_{i_1}, \cdots, X_{i_r})S(X_{i_{r+1}}, \cdots, X_{i_{r+s}}),$$

$X_1, \cdots, X_{r+s}$ being any elements of $\mathfrak{g}$ and the summation extending over all permutations $i_1, \cdots, i_{r+s}$ of $1, \cdots, r+s$.

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These operations define a commutative ring structure in the set of all symmetric tensors on $\mathfrak{g}$, and in fact this ring is isomorphic to the ring of all polynomials in N variables. If T, T', and S are invariant, so are $T+T'$ and $T\cdot S$; hence the above operations define a ring structure in $I(\mathfrak{g})$.

PROPOSITION. *If G is connected, then an r-linear real-valued function f on $\mathfrak{g}$ is invariant under G if and only if the following equality holds for any choice of elements $X_1, \cdots, X_r$ and Z of $\mathfrak{g}$:*

$$f([Z, X_1], X_2, \cdots, X_r) + f(X_1, [Z, X_2], X_3, \cdots, X_r) + \cdots + f(X_1, \cdots, X_{r-1}, [Z, X_r]) = 0.$$

Proof. Let V be the vector space of all r-linear functions on $\mathfrak{g}$, $f\in V$. Then $\operatorname{ad}(\exp tZ)(f)=f$ for all $Z\in\mathfrak{g}$ if and only if $\operatorname{ad} Z(f)=0$ for all $Z\in\mathfrak{g}$. Letting $g=\exp tZ$, it follows, from $\operatorname{ad} Z=\lim_{t\to 0}(\operatorname{ad} g-I)/t$, that $(\operatorname{ad} g-I)(f)=0$ for all $Z\in\mathfrak{g}$ if and only if $(\lim(\operatorname{ad} g-I)/t)(f)=0$ for all $Z\in\mathfrak{g}$. The equality below will thus show: $(\operatorname{ad} g-I)(f(X_1, \cdots, X_r))=0$ for all $X_i\in\mathfrak{g}$ and all g of the form $\exp tZ$, $Z\in\mathfrak{g}$, if and only if

$$f([Z, X_1], X_2, \cdots, X_r) + \cdots + f(X_1, \cdots, X_{r-1}, [Z, X_r]) = 0$$

for all X_i and Z in $\mathfrak{g}$. Since G is connected (hence generated by a neighborhood of the identity element), the proposition is proved:

$$\begin{aligned}
&(\lim (\operatorname{ad} g - I)/t)(f(X_1, \cdots, X_r)) \\
&\qquad = \lim (1/t)(f(\operatorname{ad} gX_1, \cdots, \operatorname{ad} gX_r) - f(X_1, \cdots, X_r)) \\
&\qquad = \lim (1/t)((f(\operatorname{ad} gX_1, \cdots, \operatorname{ad} gX_r) - f(X_1, \operatorname{ad} gX_2, \cdots, \operatorname{ad} gX_r)) \\
&\qquad\quad + (f(X_1, \operatorname{ad} gX_2, \cdots, \operatorname{ad} gX_r) \\
&\qquad\quad - f(X_1, X_2, \operatorname{ad} gX_3, \cdots, \operatorname{ad} gX_r)) + \cdots) \\
&\qquad = f(\lim (\operatorname{ad} gX_1 - X_1)/t, \lim (\operatorname{ad} gX_2), \cdots, \lim (\operatorname{ad} gX_r)) \\
&\qquad\quad + f(X_1, \lim (\operatorname{ad} gX_2 - X_2)/t, \lim (\operatorname{ad} gX_3), \cdots) + \cdots \\
&\qquad = f([Z, X_1], X_2, \cdots, X_r) + f(X_1, [Z, X_2], \cdots, X_r) + \cdots .
\end{aligned}$$

2. Connections and curvature [1]. For a detailed discussion, see [1]. All vector fields and all forms will be assumed of class C^∞.

Let M be a C^∞-manifold, G a Lie group, $\mathfrak{g}$ the Lie algebra of G. Let (M, B, G, π, Φ) denote the principal bundle with base space M, bundle space B, group G, projection π of B onto M, and maximal family Φ of strip maps (a strip map ϕ of Φ is a 1-1mapping of some set $\theta\times G$ onto $\pi^{-1}\theta$—where θ is an open submanifold of M—with ϕ and ϕ^{-1} both of class C^∞.) If ϕ is a strip map of $\theta\times G$ onto $\pi^{-1}\theta$ and if m is a point of θ, we denote by ϕ_m the mapping of G into B which assigns to each $g\in G$ the point $\phi(m, g)$ of B. It is assumed that for any two strip maps ϕ, ϕ' for which ϕ_m and ϕ'_m are both defined, the mapping $\phi_m^{-1}\circ\phi'_m$ is a left-translation by some element of G.

A tangent vector t at a point of B is called *vertical* if $\pi(t)=0$. There is a natural mapping q of the elements X of $\mathfrak{g}$ into vertical vector fields qX on B, defined as follows: Let b be any point of B, let $m=\pi b$, let θ be an open submanifold containing m, let ϕ be a strip map of $\theta\times G$ onto $\pi^{-1}\theta$. Then $b=\phi(m,g)$ for some $g\in G$, and we define $(qX)(b)$ to be the tangent vector $\phi_m X(g)$—where X is considered here to be a left-invariant vector field on G. Since any two strip maps, when restricted to $\pi^{-1}(m)$, differ only by a left-translation in G, this definition is independent of the choice of strip map ϕ.

DEFINITION. A rule which assigns to each point of B an alternating r-linear function from the tangent space at that point into $\mathfrak{g}$, is called a *$\mathfrak{g}$-valued r-form on B.*

DEFINITION. A *connection* on (M, B, G, π, Φ) is a $\mathfrak{g}$-valued 1-form ω on B which satisfies the following conditions:

(a) if t is a vertical tangent vector at a point b of B, then $\omega(t)$ is the unique element X of $\mathfrak{g}$ with $(qX)(b)=t$;

(b) let R_g denote right-translation on the fibres $\pi^{-1}(m)$ of B induced by an element g of G, and let t be a tangent vector at a point of B; then $(R_g^*\omega)(t)=\operatorname{ad} g^{-1}(\omega(t))$.

A tangent vector t at a point of B is called *horizontal* if $\omega(t)=0$. Any tangent vector t at a point of B decomposes into the vector sum of a vertical vector (denoted by Vt) and a horizontal vector (denoted by Ht). This decomposition depends, of course, on the choice of connection ω.

DEFINITION. The *covariant derivative* $D\Delta$ of an r-form Δ on B is the $(r+1)$-form on B defined by $(D\Delta)(t, \cdots, t')=d\Delta(Ht, \cdots, Ht')$—where $t, \cdots, t'$ are $r+1$ tangent vectors at a point of B.

DEFINITION. The *curvature form* Ω of the connection ω is the $\mathfrak{g}$-valued 2-form $D\omega$.

REMARK. Suppose a rule is given which assigns to each point b of B a linear subspace $H(b)$ of the tangent space to B at b, and which satisfies the following conditions:

(a) $H(bg)=R_gH(b)$, for any $b\in B$ and $g\in G$;

(b) at any point b of B, $H(b)$ is a linear complement to the set of all vertical tangent vectors at b;

(c) if X is a C^∞ vector field on B, then the vector field resulting from projecting X on $H(b)$ at every point b, is also C^∞.

Then H defines a connection ω on B in the following way:

(a) if t is a vertical tangent vector at a point b of B, then $\omega(t)$ is the unique element X of $\mathfrak{g}$ with $(qX)(b)=t$.

(b) if t is a tangent vector at a point b of B, then $H(b)$ defines a decomposition of t into a vertical part Vt and a horizontal part $Ht\in H(b)$; and we define $\omega(t)$ to be $\omega(Vt)$—$\omega(Vt)$ having already been defined in (a).

We will need the following facts:

(a) If $g\in G$ and t, t' are tangent vectors at a point of B, then $(R_g^*\Omega)(t, t')=\operatorname{ad} g^{-1}(\Omega(t, t'))$.

(b) If X, X' are horizontal vector fields in a neighborhood of a point b of B, and if t, t' are their values at b, then $\Omega(t, t') = -(1/2)\omega([X, X'](b))$.

(c) $D\Omega = 0$; this is the *Bianchi identity*.

(d) If t, t' are tangent vectors at a point of B, then

$$d\omega(t, t') = \Omega(t, t') - (1/2)[\omega(t), \omega(t')];$$

this is the *equation of structure*.

3. **The characteristic ring** [3]. Let ω be a connection on the principal bundle (M, B, G, π, Φ), with covariant derivative operator D and curvature form Ω. We now define a mapping of $I(\mathfrak{g})$ into the cohomology ring $H(M)$ of M.

Let r be any positive integer and let T be an element of $I^r(\mathfrak{g})$. We define a real-valued $2r$-form $\bar{\Omega}_T$ on B as follows: If $t_1, \cdots, t_{2r}$ are tangent vectors at a point of B, then $\bar{\Omega}_T(t_1, \cdots, t_{2r})$ is to be the real number Alt $(T(\Omega(t_1, t_2), \cdots, \Omega(t_{2r-1}, t_{2r})) = ((2r)!)^{-1}\Sigma\epsilon T(\Omega(t_{i_1}, t_{i_2}), \cdots, \Omega(t_{i_{2r-1}}, t_{i_{2r}}))$, where ϵ is the sign of the permutation taking $1, \cdots, 2r$ into $i_1, \cdots, i_{2r}$, and where the summation extends over all such permutations.

The form $\bar{\Omega}_T$ has three important properties:

(1) It is horizontal—that is, it is zero if one of its arguments is vertical.

(2) $R_g^*\bar{\Omega}_T = \bar{\Omega}_T$, for any $g \in G$.

(3) $\bar{\Omega}_T$ is closed—that is, $d\bar{\Omega}_T = 0$.

Proof of (1). Ω is horizontal.

Proof of (2). Since $R_g^*\Omega = \mathrm{ad}g^{-1}\Omega$, it is clear that $R_g^*\bar{\Omega}_T = \bar{\Omega}$ $(\mathrm{ad}g^{-1})^*T$. This in turn is equal to $\bar{\Omega}_T$, since T is invariant under G.

Proof of (3). From (1) and (2) it follows that $d\bar{\Omega}_T = D\bar{\Omega}_T$. Now if we restrict our attention to horizontal forms on B, D is an antiderivation; hence $d\bar{\Omega}_T = \mathrm{Alt}\,(T(D\Omega, \Omega, \cdots, \Omega) + \cdots + T(\Omega, \cdots, \Omega, D\Omega))$, which is zero since $D\Omega = 0$.

$\bar{\Omega}_T$ gives rise in the following way to a real-valued $2r$-form Ω_T on M: If $x, \cdots, z$ are $2r$ tangent vectors at a point m of M, choose a point $b \in B$ with $\pi b = m$, choose tangent vectors $X, \cdots, Z$ at b with $\pi X = x, \cdots, \pi Z = z$, and define $\Omega_T(x, \cdots, z)$ to be the number $\bar{\Omega}_T(X, \cdots, Z)$. Since $\bar{\Omega}_T$ is horizontal and invariant under right translation by G, this definition is independent of the choice of b in $\pi^{-1}(m)$, and of the choice of tangent vectors $X, \cdots, Z$ at b projecting into $x, \cdots, z$ under π. Furthermore, $d\Omega_T = 0$ since $d\bar{\Omega}_T = 0$.

Thus any element $T \in I^r(\mathfrak{g})$ defines, in the sense of de Rham, an element of $H^{2r}(M)$ whose representative is the cocycle Ω_T.

Definition. This mapping of $I(\mathfrak{g})$ into $H(M)$ is called the *Weil mapping*. It is a ring homomorphism. The images of elements of $I^r(\mathfrak{g})$ under the Weil mapping are called $2r$th *characteristic* (*cohomology*) *classes of M with respect to the bundle* (M, B, G, π, Φ), and the image of $I(\mathfrak{g})$ is called the *characteristic ring of M with respect to the bundle* (M, B, G, π, Φ).

Note that if r is greater than half the dimension of M, the image of $I^r(\mathfrak{g})$

is zero—for, any differential form on M of degree greater than the dimension of M, is the zero-form.

4. **The Weil theorem** [3]. Let α and β be, respectively, $\mathfrak{g}$-valued p- and q-forms on the bundle space B of the principal bundle (M, B, G, π, Φ). We define a $\mathfrak{g}$-valued $p+q$ form $\alpha\wedge\beta$ on B as follows: If $t_1, \cdots, t_{p+q}$ are tangent vectors at a point of B, then $(\alpha\wedge\beta)(t_1, \cdots, t_{p+q}) = \mathrm{Alt}\,([\alpha(t_1, \cdots, t_p), \beta(t_{p+1}, \cdots, t_{p+q})])$—the alternation being over the vectors $t_1, \cdots, t_{p+q}$, and $[\alpha(\ ), \beta(\ )]$ being the bracket operation in $\mathfrak{g}$.

It follows from the Jacobi identity that if α, β are $\mathfrak{g}$-valued 1-forms on B, then $\alpha\wedge(\alpha\wedge\beta) = (1/2)(\alpha\wedge\alpha)\wedge\beta$.

It can also be checked that if α, β are, respectively, $\mathfrak{g}$-valued p- and q-forms, then $\beta\wedge\alpha = (-1)^{pq+1}\alpha\wedge\beta$.

Suppose ω_0 and ω_1 are connections on (M, B, G, π, Φ), with covariant derivative operators D_0 and D_1 and curvature forms Ω_0 and Ω_1; then the equations of structure are $\Omega_i = d\omega_i + (1/2)\omega_i\wedge\omega_i$, $i = 0, 1$. If we define, for $0 \leqq t \leqq 1$, a form $\omega_t = t\omega_1 + (1-t)\omega_0$, we get a "homotopy" of $\omega_{t=0} = \omega_0$ and $\omega_{t=1} = \omega_1$. The form ω_t is a connection on (M, B, G, π, Φ) for any such t, since both ω_0 and ω_1 coincide on vertical vectors of B with the inverse of the mapping q defined in §2; we denote the curvature form of ω_t by Ω_t, and we let $u = \omega_0 - \omega_1$.

LEMMA 1. *If ω is a connection with covariant derivative operator D, and if β is a horizontal $\mathfrak{g}$-valued form on B satisfying the rule $R_g^*\beta = \mathrm{ad}\, g^{-1}\beta$ for all $g \in G$, then $D\beta = d\beta + \omega\wedge\beta$.*

LEMMA 2. (a) *$\Omega_t = \Omega_0 - tD_0u + (t^2/2)u\wedge u$.*

(b) *$d\Omega_t/dt = -(D_0u - tu\wedge u)$.*

(c) *$D_0\Omega_t = -\Omega_t\wedge u$.*

Proof of (a). u is horizontal, hence Lemma 1 shows that $\Omega_0 - tD_0u + (t^2/2)u\wedge u = d\omega_0 + (1/2)\omega_0\wedge\omega_0 - td\omega_0 + td\omega_1 - t\omega_0\wedge\omega_0 + t\omega_0\wedge\omega_1 + (t^2/2)\omega_0\wedge\omega_0 + (t^2/2)\omega_1\wedge\omega_1 - t^2\omega_0\wedge\omega_1 = d\omega_t + (1/2)\omega_t\wedge\omega_t = \Omega_t$.

Proof of (b). This follows from (a).

Proof of (c). It will be shown below in (1), (2), and (3) that $D_0^2u = \Omega_0\wedge u$, $D_0(u\wedge u) = 2D_0u\wedge u$, and $(u\wedge u)\wedge u = 0$. Hence

$$\begin{aligned} D_0\Omega_t &= 0 - tD_0u + (t^2/2)D_0(u\wedge u) \\ &= -t\Omega_0u + t^2D_0u\wedge u - (t^3/2)(u\wedge u)\wedge u \\ &= -(\Omega_0 - tD_0u + (t_2/2)u\wedge u)\wedge tu = -\Omega_t\wedge tu. \end{aligned}$$

(1) Since D_0u is horizontal, Lemma 1 shows that

$$\begin{aligned} D_0(D_0u) &= d(D_0u) + \omega_0\wedge D_0u = d(du + \omega_0\wedge u) + \omega_0\wedge(du + \omega_0\wedge u) \\ &= 0 + d\omega_0\wedge u - \omega_0\wedge du + \omega_0\wedge du + \omega_0\wedge(\omega_0\wedge u) \\ &= (d\omega_0 + (1/2)\omega_0\wedge\omega_0)\wedge u = \Omega_0\wedge u. \end{aligned}$$

(2) u is horizontal; hence $D_0(u\wedge u)=D_0u\wedge u-u\wedge D_0u=D_0u\wedge u+D_0u\wedge u=2D_0u\wedge u$.

(3) $(u\wedge u)\wedge u=(1/2)u\wedge(u\wedge u)$, and $(u\wedge u)\wedge u=-u\wedge(u\wedge u)$. Thus $(u\wedge u)\wedge u=-(1/2)(u\wedge u)\wedge u$, i.e. $(u\wedge u)\wedge u=0$.

Suppose now that T is an invariant symmetric tensor of $I^r(\mathfrak{g})$, and that Ω_T^0 and Ω_T^1 are the characteristic forms on M defined by T, and respectively, Ω_0 and Ω_1. Then we have the following theorem, due to A. Weil:

THEOREM. *Ω_T^0 and Ω_T^1 are cohomologous on M—and hence, the Weil mapping is independent of the choice of connection (and curvature form) on (M, B, G, π, Φ).*

Proof. If $A_1, \cdots, A_r$ are $\mathfrak{g}$-valued forms on B, we define a real-valued form $\overline{T}(A_1, \cdots, A_r)$ on B as follows: $\overline{T}(A_1, \cdots, A_r)=\mathrm{Alt}\,(T(A_1, \cdots, A_r))$, the alternation being over the vector arguments of $A_1, \cdots, A_r$. We denote $\overline{T}(\Omega_t, \cdots, \Omega_t)$ by $\overline{\Omega}_T^t$ (this is consistent with the previous definition of $\overline{\Omega}_T^0$ and $\overline{\Omega}_T^1$.) It will be shown in Lemma 3 that $d\overline{\Omega}_T^t/dt=-r\cdot d\overline{T}(u, \Omega_t, \cdots, \Omega_t)$; hence if we define a real-valued form $R(V, X, Y, Z)$ on B by

$$R(V, X, Y, Z) = -r\int_0^1 \overline{T}(V, X-tY+(t^2/2)Z, \cdots, X-tY+(t^2/2)Z)dt,$$

then we have:

$$\overline{\Omega}_T^1-\overline{\Omega}_T^0=\int_0^1 (d\overline{\Omega}_T^t/dt)\,dt = -r\int_0^1 d\overline{T}(u, \Omega_t, \cdots, \Omega_t)dt,$$

that is, $\overline{\Omega}_T^1-\overline{\Omega}_T^0=dR(u, \Omega_0, D_0u, u\wedge u)$.

Clearly, $R(u, \Omega_0, D_0u, u\wedge u)$ is a horizontal real-valued form on B, invariant under right translation by elements of G. Thus $R(u, \Omega_0, D_0u, u\wedge u)$ defines a differential form R on M (just as $\overline{\Omega}_T$ defined Ω_T); and so $\Omega_T^1-\Omega_T^0=dR$, i.e. Ω_T^1 and Ω_T^0 are cohomologous on M.

LEMMA 3. *$d\overline{\Omega}_T/dt=-r\cdot d\overline{T}(u, \Omega_t, \cdots, \Omega_t)$.*

Proof. $\overline{T}(u, \Omega_t, \cdots, \Omega_t)$ is invariant under right translation by elements of G, and is horizontal with respect to ω_0; hence $d\overline{T}(u, \Omega_t, \cdots, \Omega_t)=D_0\overline{T}(u, \Omega_t, \cdots, \Omega_t)$, and we have:

$$\begin{aligned} d\overline{T}(u, \Omega_t, \cdots, \Omega_t) &= D_0\overline{T}(u, \Omega_t, \cdots, \Omega_t)\\ &= \overline{T}(D_0u, \Omega_t, \cdots, \Omega_t)-\overline{T}(u, D_0\Omega_t, \Omega_t, \cdots, \Omega_t)-\cdots\\ &\quad -\overline{T}(u, \Omega_t, \cdots, \Omega_t, D_0\Omega_t)\\ &= \overline{T}(D_0u, \Omega_t, \cdots, \Omega_t)+\overline{T}(u, \Omega_t\wedge tu, \Omega_t, \cdots, \Omega_t)+\cdots\\ &\quad +\overline{T}(u, \Omega_t, \cdots, \Omega_t, \Omega_t\wedge tu)\\ &=^* \overline{T}(D_0,u, \Omega_t, \cdots, \Omega_t)-\overline{T}(u\wedge tu, \Omega_t, \cdots, \Omega_t)\\ &= \overline{T}(D_0u-u\wedge tu, \Omega_t, \cdots, \Omega_t) = -\overline{T}(d\Omega_t/dt, \Omega_t, \cdots, \Omega_t). \end{aligned}$$

The equality* follows from the following extension of the proposition of §1: If A_i $(i=1, \cdots, r)$ are $\mathfrak{g}$-valued a_i-forms, and if X is a $\mathfrak{g}$-valued 1-form (on B), then the invariance of T under G implies that

$$\sum (-1)^{a_1+\cdots+a_{i-1}}\overline{T}(A_1, \cdots, A_{i-1}, A_i \wedge X, A_{i+1}, \cdots, A_r) = 0.$$

Thus we have:

$$\begin{aligned} d\overline{\Omega}_T^t/dt &= \overline{T}(d\Omega_t/dt, \Omega_t, \cdots, \Omega_t) + \cdots + \overline{T}(\Omega_t, \cdots, \Omega_t, d\Omega_t/dt) \\ &= r\cdot \overline{T}(d\Omega_t/dt, \Omega_t, \cdots, \Omega_t) = -r\cdot d\overline{T}(u, \Omega_t, \cdots, \Omega_t). \end{aligned}$$

5. **Transgressions** [2; 3].

THEOREM. *If ω is a connection on (M, B, G, π, Φ) with covariant derivative operator D and curvature form Ω; and if T is an element of $I^r(\mathfrak{g})$; then $\overline{\Omega}_T = -d\overline{R}(\omega, \Omega, d\omega, \omega\wedge\omega)$ (the notation being the same as in §4.)*

Proof. The form $\Delta_t = \Omega - td\omega + (t^2/2)\omega\wedge\omega$ is a "homotopy" of $\Delta_{t=0}=\Omega$ and $\Delta_{t=1}=0$. We have:

$$\begin{aligned} d\Delta_t &= d\Omega - tdd\omega + (t^2/2)d(\omega\wedge\omega) \\ &= d\Omega - td\Omega - (t/2)d(\omega\wedge\omega) + (t^2/2)d(\omega\wedge\omega) \\ &= (1-t)d\Omega - (t/2)(1-t)2d\omega\wedge\omega \\ &= (1-t)(D\Omega - \omega\wedge\Omega) - (t/2)(1-t)(2d\omega\wedge\omega) \\ &= (1-t)\Omega\wedge\omega - t(1-t)d\omega\wedge\omega + (t^2/2)(1-t)(\omega\wedge\omega)\wedge\omega \\ &= (\Omega - td\omega + (t^2/2)\omega\wedge\omega)\wedge(1-t)\omega = \Delta_t\wedge(1-t)\omega. \end{aligned}$$

Thus $d\Delta_t = \Delta_t\wedge(1-t)\omega$, and so an argument similar to the one used in the proof of the theorem of §4 shows that

$$\begin{aligned} dR(\omega, \Omega, d\omega, \omega\wedge\omega) &= \overline{T}(\Delta_1, \cdots, \Delta_1) - \overline{T}(\Delta_0, \cdots, \Delta_0) \\ &= 0 - \overline{T}(\Delta_0, \cdots, \Delta_0) = -\overline{\Omega}_T. \end{aligned}$$

Thus $\overline{\Omega}_T$ is a coboundary on B, although not on M itself ($R(\omega, \Omega, d\omega, \omega\wedge\omega)$ is not horizontal on B, hence does not define a form on M). If we identify G with a fibre $\pi^{-1}(m)$ of B, and denote the inclusion mapping by $i: G\to B$, then the form $i^*R(\omega, \Omega, d\omega, \omega\wedge\omega)$ is closed and hence defines an element of $H(G)$.

Let $A(\mathfrak{g})$ denote the set of all left-invariant differential forms on $\mathfrak{g}$. Then the above procedure, taking us from T to $\overline{\Omega}_T$ and then to $i^*R(\omega, \Omega, d\omega, \omega\wedge\omega)$, defines a mapping δ of $I(\mathfrak{g})$ into $A(\mathfrak{g})$.

DEFINITION. Any element of $\delta(I(\mathfrak{g}))$—i.e. any $i^*R(\omega, \Omega, d\omega, \omega\wedge\omega)$—is called *transgressive*. Any linear mapping λ of $\delta(I(\mathfrak{g}))$ into $I(\mathfrak{g})$ satisfying $\lambda\circ\delta=$identity, is called a *transgression.*

6. **Special principal bundles.** Suppose M is a real orientable Riemannian manifold of dimension N. Then the *bundle of oriented frames of M* is the following principal bundle:

(1) G is the group $O^+(N)$ of all orthogonal real $N\times N$ matrices of determinant 1.

(2) B is the set of all $(N+1)$-tuples $(m, e_1, \cdots, e_N)$, with $m\in M$ and $e_1, \cdots, e_N$ a positively-oriented orthonormal basis of the tangent space at m.

(3) π maps the point $(m, e_1, \cdots, e_N)$ of B into the point m of M.

(4) Φ consists of all mappings ϕ defined as follows: Let m be a point of M, let $X_1, \cdots, X_N$ be vector fields in a neighborhood of m whose values at any point furnish an orthonormal properly-oriented set of tangent vectors at that point, and let g be an element of G (that is, an orthogonal $N\times N$ matrix (g_{ij}) of determinant 1). Then let $\phi(m, g)$ be the point $(m, \sum g_{1j}X_j(m), \cdots, \sum g_{Nj}X_j(m))$ of B.

If M is a complex hermitian manifold of complex dimension N, then the *unitary bundle of M* is the following principal bundle:

(1) G is the unitary group $U(N)$.

(2) B is the set of all $(N+1)$-tuples $(m, e_1, \cdots, e_N)$, with $m\in M$ and $e_1, \cdots, e_N$ a complex orthonormal set of tangent vectors at m.

(3) π and Φ are defined as in the bundle of oriented frames of a real manifold.

REMARK. Suppose Ω is a curvature form on one of these bundles. Then we can define a function $\tilde{\Omega}$ on M as follows: Let t, t' be a pair of tangent vectors at a point $m\in M$. Choose a point $b=(m, e_1, \cdots, e_N)$ in the bundle space, and choose a pair of tangent vectors T, T' at b with $\pi T=t$ and $\pi T'=t'$. Then $\tilde{\Omega}(t, t')$ is to be the linear transformation on the tangent space at m which, with respect to the basis $e_1, \cdots, e_N$, has the matrix $\Omega(T, T')$. We will let $\tilde{\Omega}_{ij}(t, t')$ denote the (i, j)th entry of this matrix.

7. **Some characteristic classes** [3]. The bundle of oriented frames of an N-dimensional real orientable Riemannian manifold, has as its group the Lie group $O^+(N)$; the Lie algebra $\mathfrak{g}$ of this group is the set of all real skew-symmetric $N\times N$ matrices (matrices (a_{ij}) satisfying $a_{ji}=-a_{ij}$). Consider the tensors T_r $(r=1, 2, \cdots)$ defined as follows:

$$T_r(A, B, \cdots, E) = \sum \epsilon a_{i_1j_1}b_{i_2j_2}\cdots e_{i_rj_r}$$

where $A=(a_{ij}), B=(b_{ij}), \cdots, E=(e_{ij})$ are r skewsymmetric $N\times N$ matrices. (Here, the summation extends over all choices of integers $i_1, \cdots, i_r$ from among $1, \cdots, N$, and over all permutations $j_1, \cdots, j_r$ of $i_1, \cdots, i_r$; ϵ is the sign of the permutation.)

T_r is invariant under $O^+(N)$, hence gives rise to a characteristic form Ω_{T_r} on M. Ω_{T_r} is called the *2rth Pontrjagin form of M*. Symbolically, it can be written as $\tilde{\Omega}_{T_r}=\sum\epsilon\tilde{\Omega}_{i_1j_1}\cdots\tilde{\Omega}_{i_rj_r}$, with Ω a curvature form on the bundle of oriented frames of M. $\tilde{\Omega}(t, t')$ is a skewsymmetric matrix (t, t' any tangent vectors at a point of B); hence Ω_{T_r} is zero unless r is even.

Note. If A is any $N\times N$ matrix, the polynomials $p_r(A)$ of the expansion $\det(\lambda I-A)=\lambda^N+p_1(A)\lambda^{N-1}+\cdots+p_N(A)$ are:

$$p_r(A) = (-1)^r \sum \epsilon a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_r j_r}.$$

Thus $T_r(A, \cdots, A) = (-1)^r \cdot p_r(A)$.

Suppose now that M has even dimension $2n$. The tensor S which assigns to any n skewsymmetric $2n \times 2n$ matrices $A, B, \cdots, H$ the value

$$S(A, B, \cdots, H) = \sum \epsilon a_{i_1 i_2} b_{i_3 i_4} \cdots h_{i_{2n-1} i_{2n}}$$

is invariant under $O^+(2n)$ (here ϵ is the sign of the permutation taking $1, \cdots, 2n$ into $i_1, \cdots, i_{2n}$, and the summation extends over all such permutations). Thus S gives rise to a characteristic form Ω_S on M, called the *Euler-Poincaré form of M*. Symbolically, it can be written as $\Omega_S = \sum \epsilon \tilde{\Omega}_{i_1 i_2} \cdots \tilde{\Omega}_{i_{2n-1} i_{2n}}$.

If M has odd dimension, we define the Euler-Poincaré form of M to be the zero-form on M.

A proof of the following theorem can be found in [3].

THE GAUSS-BONNET THEOREM. *If M is a compact orientable Riemannian manifold, then $\int_M \Omega_X$ is equal to the Euler-Poincaré characteristic of M. (Ω_X is defined to be Ω_S if M has odd dimension, and $(-1)^n/n!(4\pi)n \cdot \Omega_S$ if M has even dimension $2n$.)*

Note. If A is a real skewsymmetric $2N \times 2N$ matrix, then $(S(A, \cdots, A))^2 = 4^N \cdot \det A$.

Finally, suppose M' is a complex hermitian manifold of complex dimension N'. We consider its unitary bundle. The group of this bundle is $U(N')$, and the Lie algebra $\mathfrak{g}$ of $U(N')$ is the set of all $N' \times N'$ skew hermitian matrices (complex matrices (a_{ij}) satisfying $a_{ji} = -\bar{a}_{ij}$). Each tensor $T_r(A, B, \cdots, E) = \sum \epsilon a_{i_1 j_1} b_{i_2 j_2} \cdots e_{i_r j_r}$ is an invariant symmetric tensor on $\mathfrak{g}$ (here $A, B, \cdots, E$ are r elements of $\mathfrak{g}$), and so T_r defines a characteristic form Ω_{T_r} on M'. Ω_{T_r} is called the $2r$th *Chern form of M'*. Symbolically, it can be written as $\Omega_{T_r} = \sum \epsilon \tilde{\Omega}'_{i_1 j_1} \cdots \tilde{\Omega}'_{i_r j_r}$, with Ω' a curvature form on the unitary bundle of M'.

8. **Semi-simple Lie algebras** [4]. From now on, i will denote the square-root of -1.

The *fundamental bilinear form* of a Lie algebra $\mathfrak{g}$ is the form $(X, Y) = \operatorname{trace}(\operatorname{ad} X \circ \operatorname{ad} Y)$, $X, Y \in \mathfrak{g}$. $\mathfrak{g}$ is called *semi-simple* if the fundamental bilinear form is nondegenerate on $\mathfrak{g}$. It is known that the Lie algebra of a compact Lie group is always the algebraic direct sum of a semi-simple Lie algebra and an abelian Lie algebra.

The *rank* of a compact Lie group G is the dimension of any maximal abelian subgroup of G.

Suppose G is a compact semi-simple Lie group (that is, the Lie algebra $\mathfrak{g}$ of G is semi-simple), and suppose $H(\mathfrak{g})$ is a maximal abelian subalgebra of $\mathfrak{g}$. So the dimension of $H(\mathfrak{g})$ is the rank of G. Let $\mathfrak{g}'$ denote the complexification

$\mathfrak{g}+i\mathfrak{g}$ of $\mathfrak{g}$, and let $H(\mathfrak{g}')$ denote the complexification $H(\mathfrak{g})+iH(\mathfrak{g})$ of $H(\mathfrak{g})$. Then $H(\mathfrak{g}')$ is a maximal abelian subalgebra of $\mathfrak{g}'$, and $\mathfrak{g}'$ has a vector-space decomposition $\mathfrak{g}'=\sum\mathfrak{g}_\alpha$ satisfying:

(a) α is a complex-valued linear function (called a *root*) on $H(\mathfrak{g}')$, and $\mathfrak{g}_\alpha$ is the set of all eigenvectors of α: $[H, X]=\alpha(H)X$ for any $H\in H(\mathfrak{g}')$ and $X\in\mathfrak{g}_\alpha$.

(b) $\mathfrak{g}_0=H(\mathfrak{g}')$.

(c) Each $\mathfrak{g}_\alpha$ is one-dimensional.

(d) If α, β, and $\alpha+\beta$ are roots, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]=\mathfrak{g}_{\alpha+\beta}$; if $\alpha+\beta$ is not a root, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]=0$.

(e) If α is a root, then $k\alpha$ is a root if and only if $k=0$, 1, or -1.

$H(\mathfrak{g}')$ is called a *Cartan subalgebra of* $\mathfrak{g}'$; an element of $\mathfrak{g}_\alpha$ is called a *root vector* of $\mathfrak{g}'$ with respect to $H(\mathfrak{g}')$.

The following relations hold: $(H, \mathfrak{g}_\alpha)=0$, $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta)=0$, and $(H, H)\neq 0$, for any $H\in H(\mathfrak{g}')$ and any roots α, β with $\beta\neq-\alpha$. It is possible to choose one element e_α from each $\mathfrak{g}_\alpha$ in such a way that $(e_\alpha, e_{-\alpha})=-1$, and that the numbers $N_{\alpha\beta}$ defined by $[e_\alpha, e_\beta]=N_{\alpha\beta}e_{\alpha+\beta}$ satisfy the relations: (1) $N_{-\alpha,-\beta}=N_{\alpha\beta}$ (2) $N_{\beta\alpha}=-N_{\alpha\beta}$ (3) $N^2_{\alpha\beta}$ is a non-negative rational number.

From now on, it will be assumed that the e_α's have been chosen in this manner. We then define, for each root α, elements X_α, Y_α, H^α, $\overline{H}^\alpha$ of $\mathfrak{g}'$:

$$X_\alpha = e_\alpha + e_{-\alpha}, \qquad H^\alpha = -[e_\alpha, e_{-\alpha}],$$
$$Y_\alpha = i(e_\alpha - e_{-\alpha}), \qquad \overline{H}^\alpha = iH^\alpha.$$

The vectors X_α, Y_α are called *quasi-root vectors* of $\mathfrak{g}'$. The following facts are known.

(a) The elements X_α, Y_α, $\overline{H}^\alpha$ span $\mathfrak{g}$ and lie in $\mathfrak{g}$. So a vector $\sum a_\alpha e_\alpha + \sum b_\alpha H^\alpha$ of $\mathfrak{g}'$(a_α, b_α complex numbers) is in $\mathfrak{g}$ if and only if each b_α is pure imaginary and $a_{-\alpha}=\bar{a}_\alpha$ (complex conjugate).

(b) If α is a root, $H\in H(\mathfrak{g}')$, and $h\in H(\mathfrak{g})$, then $(H^\alpha, H)=\alpha(H)$, $\alpha(H^\alpha)$ is a positive real number, and $\alpha(h)$ is pure imaginary.

It can easily be seen that for any $H\in H(\mathfrak{g}')$ and any root α, $[H, X_\alpha] = -i\alpha(H)Y_\alpha$ and $[H, Y_\alpha]=i\alpha(H)X_\alpha$.

From now on, we will be dealing only with compact semi-simple Lie groups; this serves to simplify the notation. All results hold, however, for compact Lie groups, since every compact Lie group has a decomposition of its Lie algebra into the algebraic direct sum of an abelian and a semi-simple Lie algebra. Only slight modifications of the proofs are needed.

9. **Homogeneous spaces** [6]. Hereafter, G will denote a compact semi-simple Lie group, and K a closed connected subgroup of G. The Lie algebras of G and K will be denoted by $\mathfrak{g}$ and $\mathfrak{k}$ respectively, and $\mathfrak{m}$ will denote the set of all $X\in\mathfrak{g}$ for which $(X, k)=0$, all $k\in\mathfrak{k}$. Then ad K is a set of linear transformations of $\mathfrak{m}$ (and so $[\mathfrak{m}, \mathfrak{k}]$ lies in $\mathfrak{m}$), and $\mathfrak{g}$ is the vector-space direct sum of $\mathfrak{m}$ and $\mathfrak{k}$.

We can associate $\mathfrak{g}$ with G_e (the tangent space to G at the identity e). The fundamental bilinear form thus can be considered to be on G_e, and is invariant under ad G. If we define a bilinear form on the tangent spaces at other points of G by right-translating the fundamental bilinear form at e, the resulting bilinear form on G will be both left and right invariant under G. It induces a metric on G/K in the natural way (see [6]).

We can also associate $\mathfrak{m}$ with the tangent space $(G/K)_{eK}$ to G/K at the point eK. Then K induces a group of orientation-preserving isometries of $(G/K)_{eK}$, and in fact the effect of any ad k on $\mathfrak{m}$ ($k \in K$) is the same as the effect of left-translation by k on $(G/K)_{eK}$.

If t is a tangent vector at a point of G, we define an element $t_{\mathfrak{k}}$ of $\mathfrak{g}$ as follows: Extend t to a left-invariant vector field on G, thus defining an element of $\mathfrak{g}$; $t_{\mathfrak{k}}$ is to be the projection of this element on $\mathfrak{k}$.

10. **The canonical connection of the second kind.** There are now two principal bundles to be considered: The bundle $(G/K, B, O^+(N), \pi, \Phi)$ of oriented frames of G/K (here N denotes the dimension of G/K), and the coset bundle $(G/K, G, K, p, \Phi')$—where p is the natural projection of G onto G/K, and where the strip maps Φ' are the natural ones.

Let ω be the connection on the coset bundle of G/K defined as follows: If t is a tangent vector at a point of G, then $\omega(t) = t_{\mathfrak{k}}$. This choice of connection makes horizontal the elements of G_e corresponding to $\mathfrak{m}$; and so if Ω is the curvature form of ω and if X, Y are tangent vectors on G which, when extended left-invariantly, generate elements of $\mathfrak{m}$, then

$$\Omega(X, Y) = -(1/2)[X, Y]_{\mathfrak{k}}.$$

The connection ω induces a connection on the bundle of oriented frames of G/K in the following way:

A. Let $X_1, \cdots, X_N$ be horizontal left-invariant vector fields on G which, at any point of G, define a set of orthonormal tangent vectors whose orientation is consistent with that of G/K. Then there is a mapping Δ of G into B: $\Delta(g) = (gK, pX_1(g), \cdots, pX_N(g))$. $\Delta(g)$ can be interpreted as the left-action of g on $(G/K)_{eK}$; Δ maps K into $O^+(N)$ by taking any element k of K into the matrix of ad k acting on the vectors $X_1(e), \cdots, X_N(e)$—i.e. acting on $\mathfrak{m}$. Clearly, $\pi \circ \Delta = p$.

B. Δ can be used to define a connection in B. $H(\Delta(g))$ is to be the image under Δ of the space of horizontal tangent vectors (with respect to ω) to G at g, and H at other points of $\pi^{-1}(gK)$ is to be defined by the relations $H(bo) = R_0 H(b)$, $b \in B$ and $o \in O^+(N)$. It is clear that this H is a connection on the bundle of oriented frames of G/K; let ad ω denote the corresponding 1-form on B with values in the Lie algebra of $O^+(N)$, and let ad Ω denote the curvature form of ad ω. ad ω is called the *canonical connection of the second kind on* G/K.

C. We have the following simple results:

(a) If t is a vertical tangent vector at a point $g \in G$, then $\Delta(t)$ is a vertical tangent vector at $\Delta(g)$, with $(\mathrm{ad}\,\omega)(\Delta(t)) = \Delta(\omega(t))$. Also, $\Delta(\omega(t))$ is the matrix of ad $t_{\mathfrak{k}}$ with respect to the basis $X_1, \cdots, X_N$ of $\mathfrak{m}$.

(b) If x, y are horizontal tangent vectors at a point $\Delta(g)$, then there exist horizontal vector fields X, Y at g with $\Delta(X)$, $\Delta(Y)$ horizontal vector fields whose values at $\Delta(g)$ are x, y.

Proposition 1. *If x, y, X, Y are as above, then*

$$(\mathrm{ad}\,\Omega)(x, y) = \mathrm{ad}\,(\Omega(X(g),\ Y(g)))[\mathfrak{m}]$$

(*the symbol* ad $Z[\mathfrak{m}]$, $Z \in \mathfrak{k}$, *denotes the matrix of* ad Z *with respect to the basis* $X_1, \cdots, X_N$ of $\mathfrak{m}$).

Proof. $(\mathrm{ad}\,\Omega)(x, y) = -\mathrm{ad}\,\omega([\Delta(X), \Delta(Y)](\Delta(g))) = -\Delta(\omega[X, Y](g)) = \Delta(\Omega(X(g), Y(g)))$.

Now ad Ω induces the form $\widetilde{\mathrm{ad}}\,\Omega$ on G/K. Let g be a point of G. We will denote pX_j by Y_j. Then $L_g Y_j(e) = Y_j(g)$—here L_g denotes left-translation by g—and $(\widetilde{\mathrm{ad}}\,\Omega)(L_g Y_j(e), L_g Y_k(e))$ is the linear transformation whose matrix with respect to the basis $L_g Y_r(e)$ is $(\mathrm{ad}\,\Omega)(\Delta(Y_j(g)), \Delta(Y_k(g))) = -\Delta(\omega[Y_j, Y_k](g))$. Similarly, $(\widetilde{\mathrm{ad}}\,\Omega)(Y_j(e), Y_k(e))$ is the linear transformation whose matrix with respect to the basis $Y_r(e)$ is $-\Delta(\omega[Y_j, Y_k](e))$. Thus, since $\widetilde{\mathrm{ad}}\,\Omega$ is bilinear, we have:

Proposition 2. *If t, t' are tangent vectors at eK and if $g \in G$, then* $(\widetilde{\mathrm{ad}}\,\Omega)(L_g t, L_g t')$ *w.r.t.* $\{L_g Y_r(e)\} = (\widetilde{\mathrm{ad}}\,\Omega)(t, t')$ *w.r.t.* $\{Y_r(e)\}$.

Proposition 2 shows that one need consider characteristic forms on G/K (with respect to the bundle of oriented frames) only on vectors at eK: For, if one uses the connection ad ω, then ad Ω at any point gK can be expressed in terms of ad Ω at eK.

Proposition 3. *Let $c(K)$ and $c(O^+)$ denote respectively the characteristic rings of G/K with respect to the coset bundle and the bundle of oriented frames. Then $c(O^+)$ is contained in $c(K)$.*

Proof. We use the connections ω and ad ω. Let Ω_T be a form of $c(O^+)$, and define a tensor $\overline{T}$ on $\mathfrak{k}$ by: $\overline{T}(X, \cdots, Z) = T(\mathrm{ad}\,X[\mathfrak{m}], \cdots, \mathrm{ad}\,Z[\mathfrak{m}])$, $X, \cdots, Z \in \mathfrak{k}$. It follows from Proposition 1 that $\Omega_{\overline{T}}$ (using the curvature form Ω) is identical with Ω_T (using the curvature form ad Ω). Since $\Omega_{\overline{T}}$ is an element of $c(K)$ if $\overline{T}$ is an invariant symmetric tensor on $\mathfrak{k}$, it remains to show that $\overline{T}$ is invariant (clearly, it is symmetric): So suppose $X, Y, \cdots, Z$ and A are elements of $\mathfrak{k}$; then:

$$\begin{aligned}\overline{T}([A, X], Y, \cdots, Z) + \cdots + \overline{T}(X, Y, \cdots, [A, Z])& \\ &= T(\mathrm{ad}\,[A, X][\mathfrak{m}], \mathrm{ad}\,Y[\mathfrak{m}], \cdots, \mathrm{ad}\,Z[\mathfrak{m}]) + \cdots \\ &= T([\mathrm{ad}\,A[\mathfrak{m}], \mathrm{ad}\,X[\mathfrak{m}]], \mathrm{ad}\,Y[\mathfrak{m}], \cdots, \mathrm{ad}\,Z[\mathfrak{m}]) + \cdots\end{aligned}$$

and this final sum is zero since T is invariant under $O^+(N)$ and since ad $K[\mathfrak{m}]$ is a subset of $O^+(N)$. Since K is connected, it follows that $\overline{T}$ is invariant under K.

PROPOSITION 4 (See [2, p. 70]). *Let J be the set of all elements of $I(\mathfrak{k})$ which can be written in the form $\sum_i T_iS_i$, where T_i and S_i are elements of $I(\mathfrak{k})$ of degree greater than zero, and where each S_i can be extended to an element of $I(\mathfrak{g})$. Then a characteristic form Ω_T of $c(K)$ is cohomologous to zero on G/K if and only if T is an element of J.*

Proof. We will prove here only that every $T\in J$ gives rise to a form Ω_T which is cohomologous to zero on G/K; the other half of the proposition will not be needed here, and its proof is considerably more difficult.

So suppose $T=\sum T_iS_i$, with each S_i extendable to an element of $I(\mathfrak{g})$. It will be shown below that each Ω_{S_i} is cohomologous to zero—that is, $\Omega_{S_i}=d\Delta_i$, for some form Δ_i on G/K. Then, since the Weil mapping is a ring homomorphism, and since $d\Omega_{T_i}=0$, it follows that $\Omega_T=d(\sum\Omega_{T_i}\wedge\Delta_i)$.

Suppose then that S is an element of $I(\mathfrak{k})$, extendable to an element of $I(\mathfrak{g})$. The principal bundle with fibre G associated with the principal bundle $(G/K, G, K, p)$, is the bundle $(G/K, G\times_K G, G, p')$, where $G\times_K G$ is the set of equivalence classes of $G\times G$ under the equivalence relation $gk\times g'=g\times k^{-1}g'$. Let ω be the connection on $(G/K, G, K, p)$ defined previously, and let Ω be its curvature form. An argument similar to the one used in the proof of Proposition 1, above, shows that ω and Ω, and the natural mapping of G into $G\times_K G$, give rise to a connection ω' on $(G/K, G\times_K G, G, p')$, with curvature form Ω', satisfying: $S(\Omega, \cdots, \Omega)=S(\Omega', \cdots, \Omega')$ (both of these forms are considered to be on the base space G/K.)

Now $S\in I(\mathfrak{g})$, hence $S(\Omega', \cdots, \Omega')$ is a characteristic form of G/K with respect to the bundle $(G/K, G\times_K G, G, p')$. Since the Weil mapping is independent of the choice of connection, and since this bundle is trivial ($f(gK)=g\times g^{-1}$ is a cross-section), it follows that $S(\Omega', \cdots, \Omega')$ is cohomologous to zero on G/K, hence that $\Omega_S=S(\Omega, \cdots, \Omega)$ is cohomologous to zero on G/K.

PROPOSITION 5. *Let $H(\mathfrak{k})$ be a maximal abelian subalgebra of $\mathfrak{k}$, let $\mathfrak{g}'$, $\mathfrak{k}'$, $\mathfrak{m}'$, and $H(\mathfrak{k}')$ denote respectively the complexifications of $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{m}$, and $H(\mathfrak{k})$; so $H(\mathfrak{k}')$ is a maximal abelian subalgebra of $\mathfrak{k}'$. Extend $H(\mathfrak{k}')$ to a maximal abelian subalgebra $H(\mathfrak{g}')$ of $\mathfrak{g}'$. Let $H(\mathfrak{m}')=H(\mathfrak{g}')\cap\mathfrak{m}'$. Then $H(\mathfrak{g}')=H(\mathfrak{k}')+H(\mathfrak{m}')$.*

Proof. Suppose $H\in H(\mathfrak{g}')$, with $H=k+m$ $(k\in\mathfrak{k}', m\in\mathfrak{m}')$. Then $0=[H, H(\mathfrak{k}')]=[k, H(\mathfrak{k}')]+[m, H(\mathfrak{k}')]$. Since $[k, H(\mathfrak{k}')]\in\mathfrak{k}'$ and $[m, H(\mathfrak{k}')]\in\mathfrak{m}'$, it follows that $[k, H(\mathfrak{k}')]=0$. But $H(\mathfrak{k}')$ is a maximal abelian subalgebra of $\mathfrak{k}'$; hence $k\in H(\mathfrak{k}')$, and $m=H-k\in\mathfrak{m}'\cap H(\mathfrak{g}')=H(\mathfrak{m}')$. Thus $H\in H(\mathfrak{k}')+H(\mathfrak{m}')$.

11. The abelian case.

THEOREM. *If K is abelian, all Pontrjagin forms of G/K are cohomologous to zero.*

Proof. Let N, N' denote the dimensions of G/K, G. Since K is abelian, $\mathfrak{k}$ lies in $H(\mathfrak{g})$. If we choose an orthonormal basis of $\mathfrak{g}$ whose first N elements are an orthonormal properly-oriented basis of $\mathfrak{m}$, then with respect to this basis we have

$$\operatorname{ad} H[\mathfrak{g}] = \left(\begin{array}{c|c} \operatorname{ad} H[\mathfrak{m}] & 0 \\ \hline 0 & 0 \end{array}\right)$$

for any element H of $\mathfrak{k}$.

Define T, T' as follows:

(a) If $A=(a_{ij})$, $B=(b_{ij})$, $\cdots$, $E=(e_{ij})$ are r skewsymmetric real $N\times N$ matrices, then $T(A, B, \cdots, E)=\sum \epsilon a_{i_1j_1}b_{i_2j_2}\cdots e_{i_rj_r}$ ($i_1, \cdots, i_r$ chosen from among the integers $1, \cdots, N$).

(b) If $A'=(a'_{ij})$, $B'=(b'_{ij})$, $\cdots$, $E'=(e'_{ij})$ are r skewsymmetric real $N'\times N'$ matrices, then $T'(A', B', \cdots, E')=\sum \epsilon a'_{i_1j_1}b'_{i_2j_2}\cdots e'_{i_rj_r}$ ($i_1, \cdots, i_r$ chosen from among the integers $1, \cdots, N'$).

Clearly T and T' are symmetric and are invariant under $O^+(N)$ and $O^+(N')$ respectively. They define tensors $\overline{T}$, $\overline{T}'$ on $\mathfrak{k}$ in the following way: If $H_1, \cdots, H_r$ are r elements of $\mathfrak{k}$, then $\overline{T}(H_1, \cdots, H_r)=T(\operatorname{ad} H_1[\mathfrak{m}], \cdots, \operatorname{ad} H_r[\mathfrak{m}])$ and $\overline{T}'(H_1, \cdots, H_r)=T'(\operatorname{ad} H_1[\mathfrak{g}], \cdots, \operatorname{ad} H_r[\mathfrak{g}])$. Since K is connected, $\overline{T}$ and $\overline{T}'$ are elements of $I(\mathfrak{k})$; and from the expression given above for any matrix $\operatorname{ad} H[\mathfrak{g}]$ $(H\in\mathfrak{k})$ it follows that the summands of $\overline{T}'$ give zero contributions except in the cases when all the indices $i_1, \cdots, i_r$ lie between 1 and N, hence that $\overline{T}$ and $\overline{T}'$ coincide on $\mathfrak{k}$. The method of proof of Proposition 3, §10, shows that the $2r$th Pontrjagin characteristic form of G/K is $\Omega_{\overline{T}}$; and so the $2r$th Pontrjagin form of G/K is $\Omega_{\overline{T}'}$.

If the tensor $\overline{T}''$ on $\mathfrak{g}$ is defined by $\overline{T}''(X, \cdots, Z)=T'(\operatorname{ad} X[\mathfrak{g}], \cdots, \operatorname{ad} Z[\mathfrak{g}])$, $X, \cdots, Z$ r elements of $\mathfrak{g}$, then $\overline{T}''$ is an element of $I(\mathfrak{g})$ and also an extension of $\overline{T}'$ to all of $\mathfrak{g}$. Thus $\overline{T}'\in J$ (see Proposition 4, §10), and hence $\Omega_{\overline{T}'}$ is cohomologous to zero.

12. **Complex and symmetric homogeneous spaces** [6; 8]. A homogeneous space G/K (G a compact connected Lie group, K a closed connected subgroup of G) is called *symmetric* if there is an involutive automorphism Δ of G (that is, $\Delta^2=$ identity) for which K lies between the set K_Δ of all fixed points of Δ and the identity component of K_Δ. Δ induces an involutive automorphism of $\mathfrak{g}$ (again denoted by Δ); if $\overline{\mathfrak{m}}$ denotes the eigenspace for the eigenvalue -1 of Δ, then $\mathfrak{g}=\mathfrak{k}+\overline{\mathfrak{m}}$, and $\Delta([\overline{\mathfrak{m}}, \overline{\mathfrak{m}}])=[\Delta\overline{\mathfrak{m}}, \Delta\overline{\mathfrak{m}}]=[-\overline{\mathfrak{m}}, -\overline{\mathfrak{m}}]=[\overline{\mathfrak{m}}, \overline{\mathfrak{m}}]$, i.e. $[\overline{\mathfrak{m}}, \overline{\mathfrak{m}}]$ lies in $\mathfrak{k}$. It is easily seen that $\overline{\mathfrak{m}}$ is the orthogonal complement of $\mathfrak{k}$ with respect to the fundamental bilinear form of $\mathfrak{g}$. So we conclude that $[\mathfrak{m}, \mathfrak{m}]\subseteq\mathfrak{k}$.

By a *C-space* we will mean an even-dimensional homogeneous space G/K, with G a compact semi-simple Lie group and K a closed connected subgroup of G whose semi-simple part coincides with the semi-simple part of the centralizer of a toral subgroup of G. Wang has shown in [8] that the C-spaces are exactly the simply-connected compact complex homogeneous manifolds.

THEOREM. *If G/K is a symmetric space or a C-space, then all characteristic $c(K)$- and $c(O^+)$-forms of degree greater than (dimension G/K) $-$ (rank G $-$ rank K), are zero. If G/K is a symmetric C-space, then rank $G=$ rank K.*

Proof. If G/K is symmetric, it has been proved by H. Cartan [2] that the cohomology ring $H(G/K)$ of G/K is ring-isomorphic with a tensor product $c(K)\otimes Y$, where Y is the Grassman algebra over a vector space of dimension $N-R$ ($N=\dim G/K$, $R=\text{rank } G-\text{rank } K$) and where the degree of an element $c\otimes y$ ($c\in c(K)$, $y\in Y$) is understood to be the sum of the degrees of c and y if these are both homogeneous elements. So if Ω_T were a nonzero $c(K)$-form of degree greater than $N-R$, and if $a_1, \cdots, a_{N-R}$ were a basis of the vector space generating Y, the element $\Omega_T\otimes(a_1\cdots a_{N-R})$ would be a nonzero element of $H(G/K)$ of degree greater than N. This cannot be, since N is the dimension of G/K. It follows that all $c(K)$-forms of degree greater than $N-R$ are zero; hence all $c(O^+)$-forms of degree greater than $N-R$ are zero, since $c(O^+)$ is contained in $c(K)$.

Next, suppose G/K is a C-space. It will now be shown that every pair of root vectors e_α, $e_{-\alpha}$ lies either in $\mathfrak{k}'$ or in $\mathfrak{m}'$, hence that every pair of quasi-root vectors X_α, Y_α lies either in $\mathfrak{k}$ or in $\mathfrak{m}$. Since $[H, X_\alpha]=-i\alpha(H)Y_\alpha$ and $[H, Y_\alpha]=i\alpha(H)X_\alpha$ for any $H\in H(\mathfrak{g}')$, it follows from this that $[H(\mathfrak{m}), \mathfrak{m}]\mathfrak{k}=0$ (where $H(\mathfrak{m})=\mathfrak{m}\cap H(\mathfrak{m}')$). Thus $\Omega(H, X)=0$ for any $H\in H(\mathfrak{m})$ and $X\in\mathfrak{m}$. But then suppose Ω_T is a $c(K)$-form of degree greater than $N-R$. The dimension of $H(\mathfrak{m})$ is R, and so we can choose a basis of $\mathfrak{m}$ whose first R elements are in $H(\mathfrak{m})$. Consider any term $\Omega_T(X, \cdots, Z)$, where $X, \cdots, Z$ are elements of $\mathfrak{m}$. If this term is expanded in terms of the above basis of $\mathfrak{m}$ (that is, $X, \cdots, Z$ are written in terms of this basis), then every term $\Omega_T(\cdots)$ of this expansion will have at least one entry from $H(\mathfrak{m})$. Since $\Omega(H(\mathfrak{m}), \mathfrak{m})$ is zero, it follows that each $\Omega_T(\cdots)$ is zero, hence that $\Omega_T(X, \cdots, Z)$ is zero.

We now show that every pair e_α, $e_{-\alpha}$ lies in $\mathfrak{k}'$ or $\mathfrak{m}'$. First, $[H(\mathfrak{m}), \mathfrak{k}]=0$: For if this were not zero, there would be an element $H\in H(\mathfrak{m})$ and an element $\sum a_\alpha e_\alpha$ (a_α complex numbers) in $\mathfrak{k}$ with $[H, \sum a_\alpha e_\alpha]\neq 0$; then, since $[\mathfrak{m}, \mathfrak{k}]\subseteq\mathfrak{m}$ and since the semi-simple part of $\mathfrak{k}$ is the semi-simple part of the centralizer of a torus T of $\mathfrak{g}$ (it can easily be seen that our choice of $H(\mathfrak{g}')$ can be subjected to the condition $T\subseteq H(\mathfrak{g}')\cap\mathfrak{g}$), it follows that there is an $H'\in T$ with $[H', [H, \sum a_\alpha e_\alpha]]\neq 0$—i.e. $\sum a_\alpha\alpha(H')\alpha(H)e_\alpha\neq 0$, which cannot be since it implies that $[H', \sum a_\alpha e_\alpha]\neq 0$. But the condition $[H(\mathfrak{m}), \mathfrak{k}]=0$ implies that $[H(\mathfrak{m}'), \mathfrak{k}']=0$, hence that the root vectors of the semi-simple part of $\mathfrak{k}'$ with respect to $H(\mathfrak{k}')$ are in fact root vectors of $\mathfrak{g}'$ with respect to $H(\mathfrak{g}')$. Hence it is clear that $\mathfrak{k}'$ is spanned by the pairs of root vectors e_α, $e_{-\alpha}$ of $\mathfrak{g}'$ which lie in $\mathfrak{k}'$; and it follows from this that the root vectors of $\mathfrak{g}'$ which do not lie in $\mathfrak{k}'$, must lie in $\mathfrak{m}'$.

Finally, suppose that G/K is a symmetric C-space and that rank $G\neq$ rank K.

Then there exists a nonzero element H in $H(\mathfrak{m}')$. Let α be a root such that $\alpha(H)\neq 0$. Since G/K is a C-space, e_α lies in $\mathfrak{k}'$ or in $\mathfrak{m}'$. But the fact that $\alpha(H)\neq 0$ then leads to a contradiction: For example, if $e_\alpha\in\mathfrak{m}'$, then $[H, e_\alpha]=\alpha(H)e_\alpha$ is a nonzero element of $\mathfrak{k}'$ (since $[\mathfrak{m}', \mathfrak{m}']\subseteq\mathfrak{k}'$), which cannot be since $\mathfrak{k}'\cap\mathfrak{m}'=0$.

13. **Maximal abelian subalgebras.** Let K be a compact connected Lie group with Lie algebra $\mathfrak{k}$. Let $H(\mathfrak{k})$ be a maximal abelian subalgebra of $\mathfrak{k}$, and let $\mathfrak{k}'$, $H(\mathfrak{k}')$ be the complexifications of $\mathfrak{k}$, $H(\mathfrak{k})$. Let X_α, Y_α, X_β, $Y_\beta, \cdots$ denote the quasi-root vectors of the semi-simple part of $\mathfrak{k}$ with respect to $H(\mathfrak{k}')$. Let T be an invariant symmetric tensor on $\mathfrak{k}$.

LEMMA. *Consider any term $T(H_1, \cdots, H_j, X_{\alpha_1}, \cdots, X_{\alpha_n}, Y_{\beta_1}, \cdots, Y_{\beta_m})$, with the H_i's elements of $H(\mathfrak{k})$ and the X_{α_r}, Y_{β_r}'s quasi-root vectors of the semi-simple part of $\mathfrak{k}$ with respect to $H(\mathfrak{k}')$. Suppose $\alpha_1(H_1)\neq 0$. Then this term can be written as a linear combinations of the following types:*

(1) *terms $T(\cdots)$ with $j+1$ entries from $H(\mathfrak{k})$;*

(2) *terms $T(\cdots)$ with j entries from $H(\mathfrak{k})$, with H_1 in at least two entries.*

Proof. The invariance of $T(H_1, \cdots, H_j, H_1, X_{\alpha_2}, \cdots, Y_{\beta_1}, \cdots)$ under Y_{α_1} shows

$$\begin{aligned} 0 = {} & i\alpha_1(H_1)\cdot T(X_{\alpha_1}, H_2, \cdots, H_j, H_1, H_{\alpha_2}, \cdots, Y_{\beta_1}, \cdots) && \text{(a)}\\ & + i\alpha_1(H_2)\cdot T(H_1, X_{\alpha_1}, H_3, \cdots, H_j, H_1, X_{\alpha_2}, \cdots, Y_{\beta_1}, \cdots)\\ & \quad\vdots\\ & + i\alpha_1(H_j)\cdot T(H_1, \cdots, H_{j-1}, X_{\alpha_1}, H_1, X_{\alpha_2}, \cdots, Y_{\beta_1}, \cdots)\\ & + i\alpha_1(H_1)\cdot T(H_1, \cdots, H_j, X_{\alpha_1}, X_{\alpha_2}, \cdots, Y_{\beta_1}, \cdots) && \text{(b)}\\ & + \text{terms of the form } T(H_1, \cdots, H_j, H_1, \cdots). \end{aligned}$$

The terms (a) and (b) both equal $\alpha_1(H_1)\cdot T(H_1, \cdots, H_j, X_{\alpha_1}, \cdots, Y_{\beta_1}, \cdots)$. Since $\alpha_1(H_1)\neq 0$, the lemma is proved by bringing (a) and (b) to the left side of the equality sign.

THEOREM. *If $X, \cdots, Z$ are any elements of $\mathfrak{k}$, then $T(X, \cdots, Z)$ can be written as a linear combination (with real coefficients) of terms $T(\cdots)$ all of whose entries are elements of $H(\mathfrak{k})$. Thus the invariant symmetric tensors on $\mathfrak{k}$ depend on the maximal abelian part of $\mathfrak{k}$.*

Proof. Consider any term $T(H_1, \cdots, H_j, X_{\alpha_1}, \cdots, Y_{\beta_1}, \cdots)$. It will be shown that this term has property P: It is expressible as a linear combination of terms $T(\cdots)$ each having at least $j+1$ entries from $H(\mathfrak{k})$. Thus the proposition will be proved by induction on j. We will denote by (*) any linear combination of terms $T(\cdots)$ each having at least $j+1$ entries from $H(\mathfrak{k})$.

Suppose $\alpha_s(H_t)\neq 0$ or $\beta_s(H_t)\neq 0$ for some s and t (if this is not the case, see Case 2, below); and for simplicity of notation, suppose it is $\alpha_1(H_1)$ which is not zero. The preceding lemma shows that $T(H_1,\cdots,H_j,X_{\alpha_1},\cdots,Y_{\beta_1},\cdots)$ is expressible (modulo (*)) as a linear combination of terms with H_1 in at least two entries. So we must show that each of the terms in this linear combination has property P. We continue the process described in the lemma until we arrive at a point where the terms are of the form $T(H_1,\cdots,H_1,H_2',\cdots,H_k',X_{\lambda_1},\cdots,Y_{\delta_1},\cdots)$, with $\lambda_1(H_1)=\lambda_2(H_1)=\cdots=\delta_1(H_1)=\delta_2(H_1)=\cdots=0$. We then apply the process of the lemma to H_2', and so on, finally arriving at the situation $T(H_1,\cdots,H_j,X_{\alpha_1},\cdots,Y_{\beta_1},\cdots)=(*)+(**)$, with (**) a linear combination of terms of the form

$$t=T(\overbrace{H_1,\cdots,H_1,h_2,\cdots,h_2,\cdots,h_d,\cdots,h_d,\overbrace{h,\cdots,h}^{p\text{ terms}}}^{j\text{ terms}},X_{\pi_1},\cdots,Y_{\Delta_1},\cdots),$$

$h_2,\cdots,h_d$ elements of $H(\mathfrak{k})$, $h\in H(\mathfrak{k})$, and $\pi_r(h_2)=\cdots=\pi_r(h_d)=\Delta_r(h_1)=\cdots=\Delta_r(h_d)=0$ for all subscripts r. So the theorem will be proved if it can be shown that any term t of (**) has property P.

CASE 1. Suppose that for some r, $\pi_r(h)\neq 0$ or $\Delta_r(h)\neq 0$; and, for simplicity, suppose it is $\pi_1(h)$ which is not zero. Then the invariance of

$$T(\underbrace{H_1,\cdots,H_1,h_2,\cdots,h_2,\cdots,h_d,\cdots,h_d,\overbrace{h,\cdots,h}^{p+1\text{ terms}}}_{j+1\text{ terms}},X_{\pi_2},\cdots,Y_{\Delta_1},\cdots)$$

under Y_{π_1} shows that t has property P.

CASE 2. Suppose $\pi_r(h)=\Delta_r(h)=0$ for all r. Choose an H with $\pi_1(H)\neq 0$, $H\in H(\mathfrak{k})$. Then the invariance of the following term under Y_{π_1} shows that t has property P:

$$T(\underbrace{H_1,\cdots,H_1,h_2,\cdots,h_2,\cdots,h_d,\cdots,h_d,\overbrace{h,\cdots,h}^{p\text{ terms}},H}_{j+1\text{ terms}},X_{\pi_2},\cdots,Y_{\Delta_1},\cdots).$$

COROLLARY 1. *If rank $G\neq$ rank K, the Euler-Poincaré characteristic of G/K is zero.* (This is part of the theorem proved in [5].)

Proof. The Gauss-Bonnet theorem shows that it suffices to prove Ω_S is identically zero. This is true by definition if G/K has odd dimension. If the dimension of G/K is $2N$, choose $H(\mathfrak{k})$ to be a maximal abelian subalgebra of $\mathfrak{k}$ and let $H_1,\cdots,H_N$ be any elements of $H(\mathfrak{k})$; then $S(H_1,\cdots,H_N)=\sum\epsilon(\text{ad } H_1[\mathfrak{m}])_{i_1i_2}\cdots(\text{ad } H_N[\mathfrak{m}])_{i_{2N-1}i_{2N}}$. It will be shown that

$S(H_1, \cdots, H_N)$ is zero. It then follows from the preceding theorem that S is identically zero on $\mathfrak{k}$, hence that Ω_S is identically zero. Let $H(\mathfrak{k}') = H(\mathfrak{k}) + iH(\mathfrak{k})$. Extend $H(\mathfrak{k}')$ to a maximal abelian subalgebra $H(\mathfrak{g}')$ of $\mathfrak{g}'$, and let $H(\mathfrak{m}) = H(\mathfrak{g}')\cap\mathfrak{m}$ and $H(\mathfrak{m}') = H(\mathfrak{g}')\cap\mathfrak{m}'$. Then $H(\mathfrak{g}') = H(\mathfrak{k}') + H(\mathfrak{m}')$, and $H(\mathfrak{m}') \neq 0$ since rank $G \neq$ rank K. If we choose an orthonormal properly-oriented basis $H^1, \cdots, H^s, X, \cdots, Z$ of $\mathfrak{m}$ (with $H^1, \cdots, H^s$ in $H(\mathfrak{m})$), then, with respect to this basis,

$$\operatorname{ad} H_j[\mathfrak{m}] = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A_j \end{array}\right) \qquad j = 1, \cdots, N$$

with A_j a skewsymmetric $(2N-s)\times(2N-s)$ matrix.

In the definition of $S(H_1, \cdots, H_N)$, one of $i_1, \cdots, i_{2N}$ must in each summand be equal to the number 1. From the above matrix it can be seen that the corresponding term $(\operatorname{ad} H_j[\mathfrak{m}])_{1,i_k}$ is zero. Thus $S(H_1, \cdots, H_N) = 0$.

COROLLARY 2. *If rank G = rank K, the natural mapping of $I(\mathfrak{g})$ into $I(\mathfrak{k})$ is 1-1.*

Proof. Choose any maximal abelian subalgebra of $\mathfrak{g}$ which lies entirely in $\mathfrak{k}$. The preceding theorem shows that any element of $I(\mathfrak{g})$ is then determined by its values on $\mathfrak{k}$.

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