THE APPROXIMATE FUNCTIONAL EQUATION OF HECKE'S DIRICHLET SERIES(1)

BY

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1. **Introduction.** In 1921 and subsequently, Hardy and Littlewood [5; 6; 7] developed the following approximate functional equation for the Riemann zeta function:

$$\zeta(s) = \sum_{n \leq x} n^{-s} + \chi(s) \sum_{n \leq y} n^{s-1} + O(x^{-\sigma}) + O(|t|^{1/2-\sigma} y^{\sigma-1}), \tag{1.1}$$

where $s=\sigma+it$, $2\pi xy=|t|$, $x>h>0$, $y>h>0$, $-k<\sigma<k$, and

$$\chi(s) = 2(2\pi)^{s-1} \sin \frac{1}{2} \pi s \Gamma(1-s).$$

They regarded formula (1.1) as a "compromise" between the series expansion

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

and the functional equation $\zeta(s)=\chi(s)\zeta(1-s)$.

The approximate functional equation (1.1) has proved to be particularly valuable in studying the behavior of $\zeta(s)$ in the critical strip $0<\sigma<1$, and especially on the critical line $\sigma=1/2$. Further refinements, in which the O-terms of (1.1) are replaced by an asymptotic series, have been carried out by Siegel [13]. (See also Titchmarsh [19, §§4.16, 4.17].) The special case of this with $x=y$ and $\sigma=1/2$ was already known to Riemann.

Analogous approximate functional equations have since been obtained for other Dirichlet series by Suetuna [14; 15], Hardy and Littlewood [7], Wilton [21], Potter [10], Titchmarsh [17], Čudakov [2], Tatazuwa [16], and Wiebelitz [20].

In this paper we derive approximate functional equations for a large class of interesting Dirichlet series, namely Hecke's series of signature $(\lambda, \kappa, \gamma)$. These are defined in §2. Although our methods are applicable to a rather larger class of series having functional equations, we will confine ourselves here to a discussion of Hecke series only, partly for the sake of simplicity, but mainly because this class of series is important enough to warrant special consideration. Many of the Dirichlet series which occur naturally in analytic number theory are Hecke series. (See [8].) Among these are the Riemann

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zeta function itself, in the form $\zeta(2s)$; the zeta-functions of imaginary quadratic fields; the series whose coefficients are the divisor functions $\sigma_{2k+1}(n)$ $(k>0)$; and the series whose coefficients are $r_k(n)$, the number of representations of n as a sum of k squares.

In §3 we derive an exact identity which is basic to all the subsequent work. This identity is of interest in itself and can be viewed as giving an exact formula for the error made in approximating a Hecke Dirichlet series by its partial sums in regions where the series fails to converge. In §4 we restate some well-known lemmas of Hardy and Littlewood in forms suitable for our purposes. These are then applied in §§5 and 6 in conjunction with the identity of §3. In §5 they yield a simple approximation theorem (Theorem 2) analogous to the following approximate formula for $\zeta(s)$:

$$\zeta(s) = \sum_{n\leq x} n^{-s} + \frac{x^{1-s}}{s-1} + O(x^{-\sigma}), \tag{1.2}$$

valid for $\sigma \geqq \sigma_0 > 0$, $x > C|t|$, $C > 1/2\pi$. (See [19, Theorem 4.11].) In §6 we obtain the approximate functional equation proper (Theorem 3).

2. **Hecke series.** The Dirichlet series considered here will be denoted by

$$\sum_{n=1}^{\infty} a(n)n^{-s}, \qquad s = \sigma + it,$$

with abscissae of convergence and absolute convergence σ_0 and σ_a, respectively. Hecke series can be characterized by the following four properties:

1. $\sigma_0 < +\infty$.
2. The function ϕ, defined for $\sigma > \sigma_0$ by the equation

$$\phi(s) = \sum_{n=1}^{\infty} a(n)n^{-s},$$

can be continued analytically as a meromorphic function in the entire s-plane.

3. There exist two positive constants λ and κ such that

$$(\lambda/2\pi)^s\Gamma(s)\phi(s) = \gamma(\lambda/2\pi)^{\kappa-s}\Gamma(\kappa - s)\phi(\kappa - s)$$

where $\gamma = \pm 1$. We will write this functional equation in the form

$$\phi(s) = \chi(s)\phi(\kappa - s), \tag{2.1}$$

where

$$\chi(s) = \gamma(\lambda/2\pi)^{\kappa-2s}\Gamma(\kappa - s)/\Gamma(s). \tag{2.2}$$

The triple $(\lambda, \kappa, \gamma)$ is called the *signature* of ϕ.

4. $(s-\kappa)\phi(s)$ is an entire function of finite order. The residue of $\phi(s)$ at $s=\kappa$ will be denoted by ρ.

If ϕ has a pole at $s=\kappa$ it follows that $\sigma_0 \geqq \kappa$ and hence $\sigma_a \geqq \kappa$. If ϕ is an entire function then the inequality $\sigma_a<\kappa$ is possible. However, we can easily prove that $\sigma_a \geqq \kappa/2$ in any case. In fact, the general theory of Dirichlet series (see, e.g., Titchmarsh [18, Chapter 9]) tells us that

$$\mu(\sigma) = 0 \qquad \text{for } \sigma \geqq \sigma_a,$$

where $\mu(\sigma)=\inf\left\{\alpha \mid \phi(\sigma+it)=O(|t|^{\alpha})\right\}$. In addition, the functional equation (2.1) in conjunction with Stirling's formula implies

$$\phi(\sigma+it) = O\left(|t|^{\kappa-2\sigma}\right) \qquad \text{for } \sigma \leqq \kappa-\sigma_a$$

and this means $\mu(\sigma)\leqq\kappa-2\sigma$ for $\sigma\leqq\kappa-\sigma_a$. Since μ is decreasing we must have $0\leqq\mu(\kappa-\sigma_a)\leqq 2\sigma_a-\kappa$ and this implies $\sigma_a\geqq\kappa/2$. At the end of this paper (§7) we prove that, in fact, we always have $\sigma_a\geqq\kappa/2+1/4$.

The strip $\kappa-\sigma_a<\sigma<\sigma_a$ in which neither $\sum a(n)n^{-s}$ nor $\sum a(n)n^{s-\kappa}$ is absolutely convergent is called the *critical strip* of $\phi(s)$ and its central line $\sigma=\kappa/2$ is the *critical line*. The width of the critical strip is $2\sigma_a-\kappa$.

Later on we will make use of the following sums involving the coefficients of a Hecke series:

$$(2.3) \qquad S(x) = \sum_{n\leqq x} a(n), \qquad A_1(x) = \sum_{n\leqq x} |a(n)|, \qquad A_2(x) = \sum_{n\leqq x} |a(n)|^2.$$

The value of the abscissa σ_a is, of course, governed by the order of magnitude of these sums. Specifically, the general theory of Dirichlet series implies that for every Hecke series we have

$$(2.4) \qquad \sigma_a = \inf\left\{\epsilon \mid A_1(x) = O(x^{\epsilon})\right\}.$$

Moreover, the Schwarz inequality gives us $A_1^2(x)\leqq xA_2(x)$. Hence when the order of $A_2(x)$ is known, say $A_2(x)=O(x^{\sigma_2})$, we can write $A_1(x)=O(x^{(\sigma_2+1)/2})$, from which we obtain

$$(2.5) \qquad \sigma_a \leqq \frac{1}{2}(\sigma_2+1).$$

In this case the width of the critical strip does not exceed $\sigma_2+1-\kappa$.

Of particular interest are those Hecke series that satisfy one or the other of the following relations:

$$(2.6) \qquad A_1(x) = C_1 x^{\sigma_a} + O(x^{\sigma_a-\alpha}) \qquad C_1 \neq 0,\ \alpha > 0,$$

$$(2.7) \qquad A_2(x) = C_2 x^{\sigma_2} + O(x^{\sigma_2-\beta}) \qquad C_2 \neq 0,\ \beta > 0.$$

The Riemann zeta-function $\zeta(2s)$ is an example of a function satisfying (2.6). Many other instances may be found by use of a general theorem of Landau [9, Hauptsatz]. In the case of Hecke series, Landau's theorem may be stated as follows: *A Hecke series $\phi(s)$ with $\kappa>1/2$, such that $a(n)\geqq 0$ for all n, satisfies* (2.6) *with $C_1=\rho/\kappa$, $\sigma_a=\kappa$, $\alpha=2\kappa/(2\kappa+1)-\epsilon$, where $\epsilon>0$ is arbitrary, and ρ is the residue of $\phi(s)$ at $s=\kappa$.*

R. A. Rankin [11] has shown that for an extensive class of functions (2.7) is satisfied with $\sigma_2=\kappa$, $\beta\geqq 2/5$. In addition, Rankin has shown that the functions he considered satisfy the further relation

$$S(x) = O(x^{\kappa/2-\delta}) \tag{2.8}$$

with $\delta\geqq 1/10$ (see [12]). For instance, Rankin's work shows that (2.8) holds for every Hecke series with $\lambda=1$ and $\rho=0$. Conversely, if (2.8) holds then the general theory of Dirichlet series implies $\sigma_0\leqq\kappa/2-\delta<\kappa$ and hence $\rho=0$. One of the most important cases of a Hecke series satisfying (2.8) is the series whose coefficients $a(n)$ are Ramanujan's function $\tau(n)$. In this case the signature is (1, 12, 1).

3. **The basic identity.** This section is devoted to a proof of the following theorem:

THEOREM 1. *Let $\phi(s)=\sum a(n)n^{-s}$ be a Hecke series of signature $(\lambda, \kappa, \gamma)$. For real $x>0$ and integer $j\geqq 0$ define*

$$Q(x,j) = \frac{\phi(0)}{j!} + \rho\,\frac{\Gamma(\kappa)}{\Gamma(\kappa+j+1)}\,x^{\kappa} - \frac{1}{j!}\sum_{n\leqq x} a(n)\left(1-\frac{n}{x}\right)^{j}. \tag{3.1}$$

Let q be an integer greater than $c-1/2$, where $c=2\sigma_a-\kappa$ is the width of the critical strip. Then for every $x>0$ and for $\sigma>(\kappa-q-1/2)/2$ we have the identity

$$\begin{aligned}\phi(s) &= \sum_{n\leqq x} a(n)n^{-s} + \frac{\rho}{s-\kappa}\,x^{\kappa-s} + x^{-s}\sum_{j=0}^{q-1}\frac{\Gamma(s+j)}{\Gamma(s)}\,Q(x,j)\\ &\quad + \gamma\left(\frac{\lambda}{2\pi}\right)^{\kappa-2s}\frac{\Gamma(s+q)}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{a(n)}{n^{\kappa-s}}\int_{\xi}^{\infty}\left(\frac{u}{2}\right)^{\kappa-q-2s}J_{\kappa+q-1}(u)du,\end{aligned} \tag{3.2}$$

where $\xi=4\pi(nx)^{1/2}/\lambda$ and J_ν is the usual Bessel function of order ν. The series on the right of (3.2) *is absolutely convergent. The region of validity of* (3.2) *is a half-plane which includes the critical strip, since $(\kappa-q-1/2)/2<\kappa-\sigma_a$.*

Proof. We begin with the formula

$$\frac{\Gamma(s)\Gamma(q+1)}{\Gamma(s+q+1)}\,n^{-s} = \int_{n}^{\infty} v^{-q-s-1}(v-n)^{q}dv, \tag{3.3}$$

valid for real $q>-1$ and $\sigma>0$. This formula is an easy consequence of entry 6.2.19 in the table of Mellin transforms which occurs in [4]. Taking $\sigma>\sigma_a$ and $x>0$, (3.3) leads to

$$\begin{aligned}\frac{\Gamma(s)\Gamma(q+1)}{\Gamma(s+q+1)}\sum_{n>x} a(n)n^{-s} &= \sum_{n>x} a(n)\int_{n}^{\infty} v^{-q-s-1}(v-n)^{q}dv\\ &= \int_{x}^{\infty} v^{-q-s-1}\sum_{x<n\leqq v} a(n)(v-n)^{q}dv.\end{aligned} \tag{3.4}$$

The interchange of summation and integration in (3.4) can be justified as follows: For $X > x$ we have

$$\sum_{x<n\leq X} a(n)\int_n^\infty v^{-q-s-1}(v-n)^q dv = \sum_{x<n\leq X} a(n)\int_X^\infty v^{-q-s-1}(v-n)^q dv + \int_x^X v^{-q-s-1}\sum_{x<n\leq v} a(n)(v-n)^q dv. \tag{3.5}$$

Using (2.4) we find that the first term on the right of (3.5) is majorized by

$$\sum_{x<n\leq X} |a(n)| \int_X^\infty v^{-q-\sigma-1}v^q dv = \sigma^{-1}X^{-\sigma}\sum_{x<n\leq X} |a(n)| = O(X^{-\sigma+\sigma_a+\epsilon}).$$

This term is $o(1)$ as $X\to\infty$ since $\sigma>\sigma_a$ and $\epsilon>0$ is arbitrary.

Writing $\sum_{x<n\leq v}=\sum_{n\leq v}-\sum_{n\leq x}$ in (3.4) we now obtain

$$\begin{aligned}\sum_{n>x} a(n)n^{-s} &= \frac{\Gamma(s+q+1)}{\Gamma(s)\Gamma(q+1)}\int_x^\infty v^{-q-s-1}\sum_{n\leq v} a(n)(v-n)^q dv \\ &\quad - \frac{\Gamma(s+q+1)}{\Gamma(s)\Gamma(q+1)}\sum_{n\leq x} a(n)\int_x^\infty v^{-q-s-1}(v-n)^q dv \\ &= I_1 - I_2,\end{aligned} \tag{3.6}$$

say. When q is an integer we can simplify I_2 by means of the formula

$$\frac{\Gamma(s+q+1)}{\Gamma(q+1)}\int_x^\infty v^{-q-s-1}(v-n)^q dv = x^{-s}\sum_{j=0}^{q}\frac{\Gamma(s+j)}{\Gamma(j+1)}\left(1-\frac{n}{x}\right)^j$$

(which is easily verified by using integration by parts repeatedly). We then obtain

$$I_2 = x^{-s}\sum_{j=0}^{q}\frac{\Gamma(s+j)}{\Gamma(s)\Gamma(j+1)}\sum_{n\leq x} a(n)\left(1-\frac{n}{x}\right)^j. \tag{3.7}$$

To deal with I_1 we use the identity (see formula (1.1) of [1])

$$\begin{aligned}\sum_{n\leq v} a(n)(v-n)^q &= \phi(0)v^q + \rho\frac{\Gamma(\kappa)\Gamma(q+1)}{\Gamma(\kappa+q+1)}v^{\kappa+q} \\ &\quad + \gamma\left(\frac{\lambda}{2\pi}\right)^q\Gamma(q+1)\sum_{n=1}^\infty a(n)\left(\frac{v}{n}\right)^{(\kappa+q)/2} J_{\kappa+q}\left(\frac{4\pi}{\lambda}(nv)^{1/2}\right).\end{aligned} \tag{3.8}$$

Absolute convergence of the series in (3.8) is insured by the restriction(2) $q>c-1/2$. When (3.8) is substituted in the integral defining I_1 we find

(2) Formula (1.1) in [1] is stated with the restriction $q>\kappa-1/2$. However it is also valid for $q>c-1/2$.

$$
(3.9)\quad \begin{aligned}
I_1 = {} & \frac{\Gamma(s+q+1)}{\Gamma(s)\Gamma(q+1)}\phi(0)x^{-s} + \rho\frac{\Gamma(s+q+1)\Gamma(\kappa)}{\Gamma(s)\Gamma(\kappa+q+1)}\int_x^\infty v^{\kappa-s-1}dv \\
& + \gamma\left(\frac{\lambda}{2\pi}\right)^q \frac{\Gamma(s+q+1)}{\Gamma(s)} \\
& \cdot \int_x^\infty v^{-q-s-1}\sum_{n=1}^\infty a(n)\left(\frac{v}{n}\right)^{(\kappa+q)/2} J_{\kappa+q}\left(\frac{4\pi}{\lambda}(nv)^{1/2}\right)dv.
\end{aligned}
$$

We note that the second term on the right of (3.9) is not present unless $\rho\neq 0$. But for $\rho\neq 0$ we have $\sigma_a\geqq\kappa$ and, since we are assuming $\sigma>\sigma_a$, the integral in this term converges to $(s-\kappa)^{-1}x^{\kappa-s}$. Therefore the second term can always be written as

$$
\rho\frac{\Gamma(s+q+1)\Gamma(\kappa)}{(s-\kappa)\Gamma(s)\Gamma(\kappa+q+1)}x^{\kappa-s}.
$$

In the third term of (3.9) absolute convergence enables us to interchange summation and integration. Therefore, when (3.7) and (3.9) are used in conjunction with (3.6) we obtain the identity

$$
(3.10)\quad \begin{aligned}
\phi(s) = {} & \sum_{n\leqq x} a(n)n^{-s} + \frac{\Gamma(s+q+1)\phi(0)}{\Gamma(s+1)\Gamma(q+1)}x^{-s} + \rho\frac{\Gamma(s+q+1)\Gamma(\kappa)}{(s-\kappa)\Gamma(s)\Gamma(\kappa+q+1)}x^{\kappa-s} \\
& - x^{-s}\sum_{j=0}^q \frac{\Gamma(s+j)}{\Gamma(s)\Gamma(j+1)}\sum_{n\leqq x} a(n)\left(1-\frac{n}{x}\right)^j \\
& + \gamma\left(\frac{\lambda}{2\pi}\right)^q\frac{\Gamma(s+q+1)}{\Gamma(s)}\sum_{n=1}^\infty a(n)n^{-(\kappa+q)/2} \\
& \times\int_x^\infty v^{(\kappa-q)/2-s-1}J_{\kappa+q}\left(\frac{4\pi}{\lambda}(nv)^{1/2}\right)dv,
\end{aligned}
$$

valid for $\sigma>\sigma_a$. The sum of the second, third and fourth terms in the right of (3.10) can be written in the form

$$
\rho\frac{x^{\kappa-s}}{s-\kappa} + x^{-s}\sum_{j=0}^q\frac{\Gamma(s+j)}{\Gamma(s)}Q(x,j),
$$

by using (3.1) and the identities

$$
(3.11)\quad \frac{\Gamma(s+q+1)}{(s-\kappa)\Gamma(\kappa+q+1)} = \sum_{j=0}^q\frac{\Gamma(s+j)}{\Gamma(\kappa+j+1)} + \frac{\Gamma(s)}{(s-\kappa)\Gamma(\kappa)}
$$

and

$$
(3.12)\quad \frac{\Gamma(s+q+1)}{s\Gamma(q+1)} = \sum_{j=0}^q\frac{\Gamma(s+j)}{\Gamma(j+1)}.
$$

[Identities (3.11) and (3.12) can be proved, e.g., by induction on q.] Formula (3.10) now becomes

$$
\begin{aligned}
\phi(s) &= \sum_{n\leq x} a(n)n^{-s} + \rho\frac{x^{\kappa-s}}{s-\kappa} + x^{-s}\sum_{j=0}^{q}\frac{\Gamma(s+j)}{\Gamma(s)}Q(x,j) \\
&\quad + \gamma\left(\frac{\lambda}{2\pi}\right)^{q}\frac{\Gamma(s+q+1)}{\Gamma(s)}\sum_{n=1}^{\infty} a(n)n^{-(\kappa+q)/2} \qquad (3.13)\\
&\quad \times \int_x^\infty v^{(\kappa-q)/2-s-1}J_{\kappa+q}\left(\frac{4\pi}{\lambda}(nv)^{1/2}\right)dv.
\end{aligned}
$$

Although (3.13) was proved under the restriction $\sigma>\sigma_a$, its validity can be extended by analytic continuation to that region of the s-plane for which the integral on the right is absolutely convergent. Since $J_\nu(z)=O(z^{-1/2})$ it is easily seen that this region is the half-plane $\sigma>(\kappa-q-1/2)/2$.

It is now easy to complete the proof. An integration by parts yields

$$
\begin{aligned}
&\sum_{n=1}^{\infty} a(n)n^{-(\kappa+q)/2}\int_x^\infty v^{(\kappa-q)/2-s-1}J_{\kappa+q}\left(\frac{4\pi}{\lambda}(nv)^{1/2}\right)dv \\
&= \frac{x^{(\kappa-q)/2-s}}{s+q}\sum_{n=1}^{\infty} a(n)n^{-(\kappa+q)/2}J_{\kappa+q}\left(\frac{4\pi}{\lambda}(nx)^{1/2}\right) \qquad (3.14)\\
&\quad + \frac{2\pi/\lambda}{s+q}\sum_{n=1}^{\infty} a(n)n^{-(\kappa+q-1)/2}\int_x^\infty v^{(\kappa-q-1)/2-s}J_{\kappa+q-1}\left(\frac{4\pi}{\lambda}(nv)^{1/2}\right)dv.
\end{aligned}
$$

This is valid for $\sigma>(\kappa-q-1/2)/2$ because the first series on the right is absolutely convergent for $q>2\sigma_a-\kappa-1/2$. This and the absolute convergence of the series on the left imply the absolute convergence of the second series on the right.

When the first term on the right of (3.14) is multiplied by $\gamma(\lambda/2\pi)^q \cdot\Gamma(s+q+1)/\Gamma(s)$ it becomes $-x^{-s}\Gamma(s+q)Q(x,q)/\Gamma(s)$, because of (3.8) and (3.1). Hence, when (3.14) is substituted into (3.13) the identity (3.2) follows after making the change of variable $u=4\pi(nv)^{1/2}/\lambda$ in the integral. This completes the proof of Theorem 1.

For the Riemann zeta-function we can take $q=1$ in (3.2) and replace x by x^2 to get the well-known identity [5, Equation (2.111)]

$$
\begin{aligned}
\zeta(s) &= \sum_{n\leq x} n^{-s} + \frac{x^{1-s}}{s-1} - \left(x-[x]-\frac{1}{2}\right)x^{-s} \\
&\quad + 2(2\pi)^{s-1}s\sum_{n=1}^{\infty} n^{s-1}\int_{2\pi nx}^{\infty} u^{-s-1}\sin u\,du.
\end{aligned}
$$

For functions satisfying (2.7) with $\sigma_2=\kappa$ and (2.8) with $\delta>0$ we have

$\rho=\phi(0)=0$. By (2.5) we have $c\leqq 1$ and we can again take $q=1$ in (3.2) to obtain the identity

$$\phi(s) = \sum_{n\leq x} a(n)n^{-s} - x^{-s}\sum_{n\leq x} a(n)$$

$$+ \gamma\left(\frac{\lambda}{2\pi}\right)^{\kappa-2s} s\sum_{n=1}^{\infty} a(n)n^{s-\kappa}\int_{\xi}^{\infty}\left(\frac{u}{2}\right)^{\kappa-2s-1} J_{\kappa}(u)du. \tag{3.15}$$

4. **The Hardy-Littlewood lemmas.** Throughout this section A and B will denote real constants (not necessarily the same at each occurrence) satisfying the inequalities $0<A<1<B$.

We begin by re-stating parts of Lemmas 12 and 13 of [5, pp. 298–301] and some related formulas.

LEMMA 1. *If $a<0$, $b>0$, τ real, $T=|\tau|>A$, then we have*

$$\int_{\xi}^{\infty} u^{a+i\tau} \begin{matrix}\sin\\ \cos\end{matrix} u\, du = O(\xi^{a}), \qquad (T < BT < \xi) \tag{4.1}$$

$$= O\left(\frac{\xi^{a+1}}{\xi - T}\right), \qquad (T < \xi < BT) \tag{4.2}$$

$$= O(\xi^{a}T^{1/2}), \qquad (T \leqq \xi), \tag{4.3}$$

$$\int_{0}^{\xi} u^{b+i\tau} \begin{matrix}\sin\\ \cos\end{matrix} u\, du = O(\xi^{b+1}T^{-1}), \qquad (\xi < AT < T) \tag{4.4}$$

$$= O\left(\frac{\xi^{b+1}}{T - \xi}\right), \qquad (AT < \xi < T) \tag{4.5}$$

$$= O(\xi^{b}T^{1/2}), \qquad (A < \xi \leqq T). \tag{4.6}$$

Equation (4.1) is the same as Hardy-Littlewood's (3.15) and (4.2) is the same as their (3.14). As remarked in their paper, these hold for $a<0$. Equation (4.3) is a consequence of (4.2) and Hardy-Littlewood's (3.18) and is valid for $a<0$. Similarly, (4.4) and (4.5) follow from their (3.17) whereas (4.6) is a consequence of (4.5) and their (3.18). These are valid for $b>0$.

For our purposes we require similar estimates for integrals involving Bessel functions. These are given in the following lemma.

LEMMA 2. *If $a<1/2$, τ real, $T=|\tau|>A$, then we have*

$$\int_{\xi}^{\infty} u^{a+i\tau}J_{\nu}(u)du = O(\xi^{a-1/2}), \qquad (T < BT < \xi) \tag{4.7}$$

$$= O\left(\frac{\xi^{a+1/2}}{\xi - T}\right), \qquad (T < \xi < BT) \tag{4.8}$$

$$= O(\xi^{a-1/2}T^{1/2}), \qquad (T \leqq \xi). \tag{4.9}$$

If, in addition, $\nu+a+1>0$, *then we also have*

$$\int_\xi^\infty u^{a+i\tau}J_\nu(u)du = G_\nu(a,\tau) + O(\xi^{a+1/2}T^{-1}), \quad (\xi < AT < T) \tag{4.10}$$

$$= G_\nu(a,\tau) + O\left(\frac{\xi^{a+1/2}}{T-\xi}\right), \quad (AT < \xi < T) \tag{4.11}$$

$$= G_\nu(a,\tau) + O(\xi^{a-1/2}T^{1/2}), \quad (A < \xi \leqq T) \tag{4.12}$$

where

$$G_\nu(a,\tau) = 2^a \frac{\Gamma\left(\dfrac{\nu+a+1+i\tau}{2}\right)}{\Gamma\left(\dfrac{\nu-a+1-i\tau}{2}\right)}.$$

Proof. Using the following formula for J_ν (see [3, 7.13.1]),

$$\begin{aligned} J_\nu(u) &= \left(\frac{2}{\pi u}\right)^{1/2}\cos\frac{(2\nu+1)\pi}{4}\cos u \\ &+ \left(\frac{2}{\pi u}\right)^{1/2}\sin\frac{(2\nu+1)\pi}{4}\sin u + O(u^{-3/2}), \end{aligned} \tag{4.13}$$

the relations (4.7), (4.8) and (4.9) follow at once from (4.1), (4.2) and (4.3), respectively. In the remaining cases we have

$$\int_\xi^\infty u^{a+i\tau}J_\nu(u)du = \int_0^\infty - \int_0^\xi = G_\nu(a,\tau) - \int_0^\xi u^{a+i\tau}J_\nu(u)du,$$

by formula (6.8.1) in [4] which is valid if $\nu+a+1>0$. If we write

$$I(\nu, a) = \int_0^\xi u^{a+i\tau}J_\nu(u)du,$$

integration by parts yields

$$\begin{aligned} I(\nu, a) &= \frac{\xi^{a+1+i\tau}J_\nu(\xi)}{a+1-\nu+i\tau} - \frac{I(\nu-1, a+1)}{a+1-\nu+i\tau} \\ &= O(T^{-1}\xi^{a+1/2}) + I(\nu-1, a+1)O(T^{-1}). \end{aligned}$$

Similarly, m integrations by parts lead to the relation

$$\begin{aligned} I(\nu, a) &= \sum_{n=1}^{m} O(T^{-n}\xi^{n+a-1/2}) + I(\nu-m, a+m)O(T^{-m}) \\ &= O(T^{-1}\xi^{a+1/2}) + I(\nu-m, a+m)O(T^{-m}), \end{aligned}$$

provided $\xi \leqq T$. Taking m large enough so that $m+a-1/2>0$ we may substitute (4.13) in the integral defining $I(\nu-m, a+m)$ and use (4.4), (4.5) and (4.6) to obtain (4.10), (4.11) and (4.12), respectively.

Later on we will have $\xi=T(n/y)^{1/2}$ where y is fixed and positive. It is therefore convenient to rephrase Lemma 2 in terms of n.

LEMMA 3. *If $a<1/2$, τ real, $T=|\tau|>A$, $n\geqq 1$, $y>0$, $\xi=T(n/y)^{1/2}$, then we have*

$$(4.14)\quad \int_\xi^\infty u^{a+i\tau}J_\nu(u)du = O(T^{a-1/2}y^{1/4-a/2}n^{a/2-1/4}), \qquad (y < By < n)$$

$$(4.15)\quad = O\left(T^{a-1/2}y^{3/4-a/2}\frac{n^{a/2-1/4}}{n^{1/2}-y^{1/2}}\right), \quad (y^{1/2} < n^{1/2} < By^{1/2})$$

$$(4.16)\quad = O(T^a y^{1/4-a/2}n^{a/2-1/4}), \qquad (y \leqq n).$$

If, in addition, $\nu+a+1>0$, then we also have

$$(4.17)\quad \int_\xi^\infty u^{a+i\tau}J_\nu(u)du = G_\nu(a,\tau) + O(T^{a-1/2}y^{-1/4-a/2}n^{a/2+1/4}), \quad (n < Ay < y)$$

$$(4.18)\quad = G_\nu(a,\tau) + O\left(T^{a-1/2}y^{3/4-a/2}\frac{n^{a/2-1/4}}{y^{1/2}-n^{1/2}}\right),$$

$$(Ay^{1/2} < n^{1/2} < y^{1/2})$$

$$(4.19)\quad = G_\nu(a,\tau) + O(T^a y^{1/4-a/2}n^{a/2-1/4}), \qquad (n \leqq y).$$

5. **The simplest approximation theorem for Hecke series.** This section is devoted to a proof of the following theorem:

THEOREM 2. *Let $\phi(s)=\sum a(n)n^{-s}$ be a Hecke series of signature $(\lambda, \kappa, \gamma)$. Under the hypotheses of Theorem 1, we have*

$$(5.1)\quad \phi(s) = \sum_{n\leq x}\frac{a(n)}{n^s} + \frac{\rho}{s-\kappa}x^{\kappa-s} + x^{-s}\sum_{j=0}^{q-1}\frac{\Gamma(s+j)}{\Gamma(s)}Q(x,j) + O(x^{\kappa/2-\sigma-1/4})$$

uniformly for $\sigma\geqq\sigma_1>(\kappa-q-1/2)/2$, provided $x>B(\lambda/4\pi)^2t^2$ for some $B>1$.

Proof. Taking $\xi=4\pi(nx)^{1/2}/\lambda$ in (4.7) we have

$$\int_\xi^\infty u^{\kappa-q-2s}J_{\kappa+q-1}(u)du = O((nx)^{(\kappa-q-2\sigma-1/2)/2}).$$

Accordingly, we have

$$\frac{\Gamma(s+q)}{\Gamma(s)} \sum_{n=1}^{\infty} a(n)n^{s-\kappa} \int_{\xi}^{\infty} u^{\kappa-q-2s} J_{\kappa+q-1}(u)du$$
$$= O\left(\left| t \right|^{q} x^{(\kappa-q-2\sigma-1/2)/2} \sum_{n=1}^{\infty} \left| a(n) \right| n^{-(\kappa+q+1/2)/2}\right)$$
$$= O(\left| t \right|^{q} x^{(\kappa-q-2\sigma-1/2)/2}) = O(x^{\kappa/2-\sigma-1/4})$$

since $(\kappa+q+1/2)/2>\sigma_a$ and $|t| = O(x^{1/2})$. Using this estimate in identity (3.2) the theorem follows.

For the Riemann zeta-function Theorem 2 reduces to (1.2). For functions satisfying (2.7) and (2.8) we can use (3.15) to obtain

$$\phi(s) = \sum_{n\leq x} a(n)n^{-s} + O(x^{(\kappa/2-\sigma-\theta)})$$

where $\theta = \min(1/4, \beta)$.

6. **The approximate functional equation.** In deriving (5.1) we made essential use of the fact that the quotient t^2/x was bounded. In this section we write $y = (\lambda/(2\pi))^2 t^2/x$ and we re-estimate the last term of (3.2) in terms of y. This yields the approximate functional equation for a Hecke series having signature $(\lambda, \kappa, \gamma)$. The result is stated in Theorem 3 below. Corollaries 1 and 2 deal with special cases in which more is assumed about the coefficients $a(n)$.

Theorem 3 is stated for a general exponent $\sigma_1 \geqq \sigma_a$ because in a particular case the exact value of σ_a may not be known.

THEOREM 3. *Let $\phi(s) = \sum a(n)n^{-s}$, $s = \sigma + it$, be a Hecke series of signature $(\lambda, \kappa, \gamma)$. Let $A_1(x) = O(x^{\sigma_1})$ and let q denote the integer satisfying $2\sigma_1 - \kappa - 1/2 < q \leqq 2\sigma_1 - \kappa + 1/2$. Then for $x > 1$, $y > 1$, $2\pi(xy)^{1/2} = \lambda|t| > \lambda$, $(\kappa - q - 1/2)/2 < \sigma < \kappa$, we have*:

$$\phi(s) = \sum_{n\leq x} a(n)n^{-s} + \chi(s) \sum_{n\leq y} a(n)n^{s-\kappa} + \frac{\rho}{s-\kappa} x^{\kappa-s} + x^{-s} \sum_{j=0}^{q-1} \frac{\Gamma(s+j)}{\Gamma(s)} Q(x, j) + R \tag{6.1}$$

where $\chi(s)$ is given by (2.2), *$Q(x, j)$ is defined in* (3.1) *and*

$$R = O(x^{(\kappa-2\sigma-1/2)/2} y^{(2\sigma_1-\kappa-1/2)/2} m) + O\left(x^{(\kappa-2\sigma-1/2)/2} y^{-(\kappa+1/2)/2} \sum_{y/2<n<2y} | a(n) | f(n)\right). \tag{6.2}$$

Here m is given by

$$m = \begin{cases} 1 \text{ if } q < 2\sigma_1 - \kappa + 1/2, \\ \log y \text{ if } q = 2\sigma_1 - \kappa + 1/2 \end{cases} \tag{6.3}$$

and $f(n)$ by

$$f(u) = f(u; y, t) = \begin{cases} y/(y-u) & \text{if } u \leqq y_1, \\ 1/2\,|t|^{1/2}\{3 - |t|(1-u/y)^2\} & \text{if } y_1 \leqq u \leqq y_2, \\ y/(u-y) & \text{if } u \geqq y_2, \end{cases} \tag{6.4}$$

where

$$y_1 = y(1 - |t|^{-1/2}) \quad \text{and} \quad y_2 = y(1 + |t|^{-1/2}). \tag{6.5}$$

COROLLARY 1. *If the conditions of Theorem 3 are satisfied, and in addition $A_1(x)$ is given by* (2.6), *then* (6.1) *holds with the error term R replaced by*

$$R = O(x^{(\kappa-2\sigma-1/2)/2}y^{(c-1/2)/2}m_1) + O(x^{\kappa/2-\sigma}y^{c/2-\alpha}) \tag{6.6}$$

where α is the α of (2.6), $c = 2\sigma_a - \kappa$ (*the width of the critical strip*), *and*

$$m_1 = \max(\log |t|, \log y). \tag{6.7}$$

COROLLARY 2. *If the conditions of Theorem 3 are satisfied and in addition $A_2(x)$ is given by* (2.7), *then* (6.2) *holds with R given by*

$$R = O(x^{(\kappa-2\sigma-1/2)/2}y^{(\sigma_2-\kappa+m_2)/2}(x+y)^{1/4}m_1) \tag{6.8}$$

where m_1 is defined by (6.7) *and m_2 by*

$$m_2 = \max\left(0, \frac{1}{2} - \beta\right). \tag{6.9}$$

Proof of Theorem 3. Because of the identity (3.2) we can write

$$\begin{aligned} R &+ \chi(s) \sum_{n \leqq y} a(n) n^{s-\kappa} \\ &= \gamma\left(\frac{\lambda}{2\pi}\right)^{\kappa-2s} \frac{\Gamma(s+q)}{\Gamma(s)} \sum_{n=1}^{\infty} a(n) n^{s-\kappa} \int_{\xi}^{\infty} \left(\frac{u}{2}\right)^{\kappa-q-2s} J_{\kappa+q-1}(u)\,du \end{aligned} \tag{6.10}$$

where $\xi = 4\pi(nx)^{1/2}/\lambda = 2|t|(n/y)^{1/2}$. In the infinite series we write

$$\gamma\left(\frac{\lambda}{2\pi}\right)^{\kappa-2s} \frac{\Gamma(s+q)}{\Gamma(s)} \sum_{n=1}^{\infty} = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5 + \Sigma_6 \tag{6.11}$$

where the sums Σ_i are defined by the respective ranges $1 \leqq n \leqq y/2$, $y/2 < n \leqq y_1$, $y_1 < n \leqq y$, $y < n \leqq y_2$, $y_2 < n \leqq 2y$, $2y < n$, y_1 and y_2 being defined by (6.5). In Σ_1 we use (4.17) to obtain

$$\Sigma_1 = \chi(s) \sum_{n \leqq y/2} a(n) n^{s-\kappa} + O\Big(|t|^{\kappa-2\sigma-1/2} y^{\sigma-\kappa/2+q/2-1/4} \sum_{n \leqq y/2} |a(n)|\, n^{-(\kappa+q-1/2)/2}\Big).$$

Using partial integration, the sum appearing in the O-term is

$$\sum_{n\le y/2} |a(n)|\, n^{-(\kappa+q-1/2)/2} = A_1(y/2)(y/2)^{-(\kappa+q-1/2)/2}$$
$$+\frac{1}{2}\left(\kappa+q-\frac{1}{2}\right)\int_1^{y/2} A_1(u)u^{-(\kappa+q-1/2)/2-1}du$$
$$= O(y^{(2\sigma_1-\kappa-q+1/2)/2}) + O\left(\left|\int_1^{y/2} u^{(2\sigma_1-\kappa-q+1/2)/2-1}du\right|\right).$$

The integral in the last O-term is $O(y^{(2\sigma_1-\kappa-q+1/2)/2})$ if $q<2\sigma_1-\kappa+1/2$. If $q=2\sigma_1-\kappa+1/2$, then the integral is

$$\log (y/2) = O(\log y) = O(y^{(2\sigma_1-\kappa-q+1/2)/2}\log y).$$

Consequently we have

$$\Sigma_1 = \chi(s)\sum_{n\le y/2} a(n)n^{s-\kappa} + O(|t|^{\kappa-2\sigma-1/2}y^{\sigma_1+\sigma-\kappa}m)$$
$$(6.12)\qquad = \chi(s)\sum_{n\le y/2} a(n)n^{s-\kappa} + O(x^{(\kappa-2\sigma-1/2)/2}y^{(2\sigma_1-\kappa-1/2)/2}m)$$

since $|t| = O(x^{1/2}y^{1/2})$. A similar argument, using (4.14), shows that

$$(6.13)\qquad \Sigma_6 = O(x^{(\kappa-2\sigma-1/2)/2}y^{(2\sigma_1-\kappa-1/2)/2}).$$

In Σ_2 we use (4.18) to obtain

$$\Sigma_2 = \chi(s)\sum_{y/2<n\le y_1} a(n)n^{s-\kappa}$$
$$+ O\left(|t|^{\kappa-2\sigma-1/2}y^{\sigma+(q+1/2-\kappa)/2}\sum_{y/2<n\le y_1}|a(n)|\,\frac{n^{-(\kappa+q-1/2)/2}}{y^{1/2}-n^{1/2}}\right).$$

Now we have

$$\sum_{y/2<n\le y_1}|a(n)|\,\frac{n^{-(\kappa+q-1/2)/2}}{y^{1/2}-n^{1/2}} = \sum_{y/2<n\le y_1}|a(n)|\,\frac{n^{-(\kappa+q-1/2)/2}(y^{1/2}+n^{1/2})}{y-n}$$
$$= O\left(y^{-(\kappa+q+1/2)/2}\sum_{y/2<n\le y_1}|a(n)|\,\frac{y}{y-n}\right)$$
$$= O\left(y^{-(\kappa+q+1/2)/2}\sum_{y/2<n\le y_1}|a(n)|\,f(n)\right).$$

Accordingly,

$$\Sigma_2 = \chi(s)\sum_{y/2<n\le y_1} a(n)n^{s-\kappa}$$
$$(6.14)\qquad + O\left(x^{(\kappa-2\sigma-1/2)/2}y^{-(\kappa+1/2)/2}\sum_{y/2<n\le y}|a(n)|\,f(n)\right).$$

Similarly, using (4.19), (4.16) and (4.15) we obtain

$$\Sigma_3 = \chi(s) \sum_{y_1<n\leq y} a(n)n^{s-\kappa} \tag{6.15}$$
$$+ O\left(x^{(\kappa-2\sigma-1/2)/2}y^{-(\kappa+1/2)/2} \sum_{y_1<n\leq y} |a(n)| f(n)\right),$$

$$\Sigma_4 = O\left(x^{(\kappa-2\sigma-1/2)/2}y^{-(\kappa+1/2)/2} \sum_{y<n\leq y_2} |a(n)| f(n)\right), \tag{6.16}$$

$$\Sigma_5 = O\left(x^{(\kappa-2\sigma-1/2)/2}y^{-(\kappa+1/2)/2} \sum_{y_2<n\leq 2y} |a(n)| f(n)\right). \tag{6.17}$$

Combining (6.10) through (6.17) yields (6.2) and completes the proof of Theorem 3.

Proof of Corollary 1. Combining (6.2) with (2.6) yields

$$R = O(x^{(\kappa-2\sigma-1/2)/2}y^{(c-1/2)/2}m) \tag{6.18}$$
$$+ O\left(x^{(\kappa-2\sigma-1/2)/2}y^{-(\kappa+1/2)/2} \sum_{y/2<n\leq 2y} |a(n)| f(n)\right).$$

We have, by partial integration,

$$\sum_{y/2<n\leq 2y} |a(n)| f(n) = A_1(2y)f(2y) - A_1(y/2)f(y/2) - \int_{y/2}^{2y} A_1(u)f'(u)du$$
$$= O(y^{\sigma_a}) - C_1\int_{y/2}^{2y} u^{\sigma_a}f'(u)du + O\left(y^{\sigma_a-\alpha}\int_{y/2}^{2y} |f'(u)|\,du\right).$$

This is valid because, by definition (6.4), $f(u)$ has a continuous derivative. Integrating by parts, we have

$$\int_{y/2}^{2y} u^{\sigma_a}f'(u)du = (2y)^{\sigma_a}f(2y) - (y/2)^{\sigma_a}f(y/2) - \sigma_a\int_{y/2}^{2y} u^{\sigma_a-1}f(u)du$$
$$= O(y^{\sigma_a}) + O\left(y^{\sigma_a-1}\int_{y/2}^{y} f(u)du\right).$$

Direct use of (6.4) gives

$$\int_{y/2}^{2y} f(u)du = O(ym_1) \quad \text{where} \quad m_1 \text{ is given in (6.7),}$$

and

$$\int_{y/2}^{2y} |f'(u)|\,du = O(|t|^{1/2}) = O(x^{1/4}y^{1/4}).$$

Consequently, we obtain

$$\sum_{y/2<n\leq 2y} |a(n)| f(n) = O(y^{\sigma_a} m_1) + O(x^{1/4} y^{\sigma_a+1/4-\alpha}).$$

When this is put into (6.18), the result is (6.6).

Proof of Corollary 2. Given (2.7), we may, by (2.5), take $\sigma_1 = (\sigma_2+1)/2$ in (6.2), and write (6.2) in the form

$$\begin{aligned} R &= O(x^{(\kappa-2\sigma-1/2)/2} y^{(\sigma_2-\kappa+1/2)/2}) \\ &\quad + O\Big(x^{(\kappa-2\sigma-1/2)/2} y^{-(\kappa+1/2)/2} \sum_{y/2<n\leq 2y} |a(n)| f(n)\Big). \end{aligned} \tag{6.19}$$

By Schwarz's inequality

$$\sum_{y/2<n\leq 2y} |a(n)| f(n) \leqq \Big\{\sum_{y/2<n\leq 2y} |a(n)|^2 f(n)\Big\}^{1/2} \Big\{\sum_{y/2<n\leq 2y} f(n)\Big\}^{1/2}. \tag{6.20}$$

The second sum on the right becomes

$$\begin{aligned} \sum_{y/2<n\leqq 2y} f(n) &= \int_{y/2}^{2y} f(u)du + O\Big(\max_{y/2\leq u\leq 2y} f(u)\Big) \\ &= O(ym_1) + O(|t|^{1/2}) \\ &= O(ym_1) + O(x^{1/4}y^{1/4}) \\ &= O(ym_1 \max(1, x^{1/4}y^{-3/4})). \end{aligned}$$

But we have

$$\max(1, x^{1/4}y^{-3/4}) = O(1 + x^{1/4}y^{-3/4}) = O\left(1 + \left(\frac{x}{y}\right)^{1/4}\right),$$

and

$$1 + \left(\frac{x}{y}\right)^{1/4} \leqq 2^{1/2}\left(1 + \left(\frac{x}{y}\right)^{1/2}\right)^{1/2} \leqq 2\left(1 + \frac{x}{y}\right)^{1/4}.$$

Accordingly,

$$\sum_{y/2<n\leq 2y} f(n) = O(ym_1(1 + x/y)^{1/4}). \tag{6.21}$$

We can estimate the first sum on the right of (6.20) in the same manner as we estimated $\sum |a(n)| f(n)$ in the proof of Corollary 1, using (2.7) instead of (2.6). The result is

$$\sum_{y/2<n\leq 2y} |a(n)|^2 f(n) = O(y^{\sigma_2} m_1) + O(x^{1/4} y^{\sigma_2+1/4-\beta}).$$

These O-terms may be re-written as follows:

$$\begin{aligned} O(y^{\sigma_2}m_1) + O(x^{1/4}y^{\sigma_2+1/4-\beta}) &= O(y^{\sigma_2}m_1) + O(y^{\sigma_2+1/2-\beta}(x/y)^{1/4}m_1) \\ &= O(y^{\sigma_2}M_1m_1) + O(y^{\sigma_2+1/2-\beta}M_1m_1) \\ &= O(y^{\sigma_2+m_2}M_1m_1) \end{aligned}$$

where $M_1 = \max\left\{1, (x/y)^{1/4}\right\}$ and m_2 is defined by (6.9). But

$$\max(1, (x/y)^{1/4}) = O(1 + x/y)^{1/4}$$

and hence we can write

$$\sum_{y/2<n\leqq 2y} |a(n)|^2 f(n) = O(y^{\sigma_2+m_2}(1 + x/y)^{1/4}m_1). \tag{6.22}$$

Combining (6.19) through (6.22) yields (6.8).

When $\phi(s) = \zeta(2s)$, Corollary 1 reduces to an "imperfect" form of the approximate functional equation for the Riemann zeta-function, the imperfection consisting of an extra factor m in one O-term. Special cases of Corollary 2 are provided by the functions studied by Rankin. For these functions we have $\sigma_2 = \kappa$, $m_2 = 1/10$ and so (6.8) reduces to

$$R = O(x^{(\kappa-2\sigma-1/2)/2}y^{1/20}(x + y)^{1/4}m_1).$$

For $x = y = (\lambda/(2\pi))|t|$, this gives

$$R = O(|t|^{\kappa/2-\sigma+1/20}\log|t|).$$

7. **A lower bound for the width of the critical strip.** In this section we prove that for every Hecke series of signature $(\lambda, \kappa, \gamma)$ with abscissa of absolute convergence σ_a we have

$$\sigma_a \geqq \frac{\kappa}{2} + \frac{1}{4}\cdot \tag{7.1}$$

This tells us that the width of the critical strip, $2\sigma_a - \kappa$, is at least $1/2$.

To prove (7.1) we make use of identity (3.8) which is valid for $q > 2\sigma_a - \kappa - 1/2$. The series on the right of (3.8), considered as a function of v, converges uniformly on every finite interval $0 < a \leqq v \leqq b$ and hence the right member of (3.8) is a continuous function of v on every such interval. However, the left member of (3.8) cannot be continuous in v when $q = 0$. It follows that the identity in (3.8) cannot be valid for $q = 0$. Hence $2\sigma_a - \kappa - 1/2 \geqq 0$ and this is the same as (7.1).

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