PEANO SPACES WHICH ARE EITHER STRONGLY CYCLIC OR TWO-CYCLIC(1)

BY

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1. **Introduction.** Let M be a Peano space(2) $p \in M$, and $\{p_n\}$ any infinite sequence of distinct points of M converging to p. The space M is said to be *strongly arcwise connected* at p provided there is in M a simple arc containing infinitely many of the points $\{p_n\}$, *strongly cyclic* at p provided there is in M a simple closed curve containing infinitely many of the points $\{p_n\}$, and *two-cyclic* at p provided M contains two arcs Γ_1 and Γ_2 such that each contains infinitely many of the points $\{p_n\}$ and $\Gamma_1 \cap \Gamma_2 = p$. If M is strongly arcwise connected, strongly cyclic, or two-cyclic at each of its points, then M is said to be strongly arcwise connected, strongly cyclic, or two-cyclic respectively. Strongly arcwise connected Peano spaces have been studied by Hall and Puckett [1; 2](3).

It is clear that if M is either strongly cyclic or two-cyclic at p, then M is strongly arcwise connected at p; and that if M is a cyclic Peano space which is two-cyclic at p, then M is strongly cyclic at p. Moreover, it is easily shown that if M is either strongly cyclic or two-cyclic at p, then p lies in at least one, and at most a finite number, of true cyclic elements of M. Thus only cyclic Peano spaces will be considered.

An example of a cyclic Peano space which is strongly arcwise connected at a point p, but which fails to be strongly cyclic at p follows. Let M be the point set in the Euclidean plane consisting of the closed interval from 0 to 1 on the x-axis, the closed line segment of $y=x$ from $(0, 0)$ to $(1, 1)$, and the closed line segments from $(1/n, 0)$ to $(1/n, 1/n)$ on the lines $x=1/n$, $n \in I$. Considering the sequence of points $\{p_n\}$ where $p_n=(1/n, 1/2n)$, $n \in I$, it is seen that M fails to be strongly cyclic at the point $(0, 0)$.

In this paper two characterizations of strongly cyclic Peano spaces will be obtained. The first will show that a cyclic Peano space which is strongly arcwise connected at p, but not strongly cyclic at p, is essentially like the example above. The second states that a cyclic Peano space M is strongly cyclic if and only if for any infinite collection $\{V_i\}$ of open sets in M, there exists a simple closed curve in M intersecting infinitely many of these sets.

For an example of a cyclic Peano space which is strongly cyclic at p, but

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(2) All terms and symbols not explicitly defined in this paper may be found in [4] or [5].

(3) Numbers in brackers refer to the bibliography at the end of the paper.

not two-cyclic at p, let M denote the square in the Euclidean plane with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$, either with or without its interior. Considering the sequence of points $(1/n, 0)$, $n\in I$, it is seen that M is not two-cyclic at the point $(0, 0)$. If M is considered without its interior, then M fails to be two-cyclic at $(0, 0)$ because this point lies on a free arc of M. If M is considered with its interior, define in M an arc Γ consisting of p and the union of the line segments of $x=1/n$ from $(1/n, 0)$ to $(1/n, 1/n)$, the line segments from $(1/2n, 0)$ to $(1/2n-1, 0)$ on the x-axis, and the line segments from $(1/2n+1, 1/2n+1)$ to $(1/2n, 1/2n)$ on the line $y=x$, $n\in I$. Thus given $\epsilon>0$, there exists an arc $\gamma\subset\Gamma$ such that γ has p as one endpoint, has diameter less than ϵ, and $M-\gamma$ has an infinite collection of components converging to p.

It will also be shown that a cyclic Peano space which is strongly cyclic at p, but not two-cyclic at p, is essentially like the above example.

2. **Lemmas.** In this section several lemmas are given which will be useful in obtaining the principal results.

LEMMA 2.1. *If a limit point p of an arcwise connected set R is not regularly accessible from R, then there exists a positive number η such that $R\cap S(p, \eta)$ has an infinite collection of distinct components $\{B_i\}$ such that $p\in\liminf \{B_i\}$. Moreover, $\overline{C}\cap F[S(p\eta,)]\neq\emptyset$ for every component C of $R\cap S(p, \eta)$.*

Proof. This is a consequence of a lemma of E. E. Betz [3, p. 128].

LEMMA 2.2. *Let $\alpha=qxp$ be an arc of a Peano space M and $\{p_n\}$ an infinite sequence of distinct points of $M-\alpha$ converging to p. If, for every $\epsilon>0$, $S(p, \epsilon)$ contains an arc spanning $\alpha'=\alpha-(q\cup p)$ and containing a point of the sequence $\{p_n\}$, then M contains an arc α^* from q to p containing infinitely many of the points $\{p_n\}$.*

LEMMA 2.3. *Let M be a cyclic Peano space and $J=\alpha\cup\beta$ a simple closed curve in M where $\alpha=qxp$ and $\beta=qyp$ are arcs such that $\alpha\cap\beta=q\cup p$, $q\neq p$. Let $\{p_n\}$ be an infinite sequence of distinct points of $M-J$ converging to p, $\{x_n\}$ and $\{y_n\}$ infinite sequences of distinct points converging monotonically to p on α and β respectively. Assume that M contains a collection $\{\lambda_n\}$ of distinct arcs converging to p such that λ_n is an arc $x_np_ny_n$ spanning J, and that there is a positive number η for which $(M-J)\cap S(p, \eta)$ has an infinite collection $\{B_i\}$ of components such that $\lambda_i\subset B_i$ and $\overline{B}_i\cap F[S(p, \eta)]\neq\emptyset$ for every i. Then there is a simple closed curve in M containing infinitely many of the points $\{p_n\}$.*

Proof. There is no loss of generality in assuming that the collection $\{B_i\}$ converges to a connected set L, and that L contains a nondegenerate subarc zp of α. Let δ be a positive number such that $S(p, \delta)\cap\alpha\subset zp$. There exists a point $r\in S(p, \delta)\cap\alpha$ such that $r\neq x_i$ for all i. There exists an integer n_1, such that that in the natural order from q to p on α, $r<x_n$ for $n\geqq n_1$, and $x_{n_1-1}<r$. Let U be a neighborhood of r such that $\overline{U}\cap\beta=\emptyset$, $\overline{U}\subset S(p, \delta)$, and $\overline{U}\cap\alpha\subset x_{n_1-1}x_{n_1}$. There exists an integer $n_2\geqq n_1+2$ such that $B_{n_2}\cup s$ contains

an arc stw where $s \in \overline{U} \cap \alpha$ and $w = \lambda_{n_2} \cap stw$. Let Γ_1 be the union of the subarc qs of α, the arc stw, the subarc wy_{n_2} of λ_{n_2}, and the subarc $y_{n_2}p$ of β; Γ_2 the union of the subarc qy_{n_1+1} of β, the arc λ_{n_1+1}, and the subarc $x_{n+1}p$ of α. Then $\Gamma_1 \cup \Gamma_2$ is a simple closed curve containing the point p_{n_1+1}.

Let δ_2 be a positive number such that $\overline{S}(p, \delta_2) \cap (\Gamma_1 \cup \Gamma_2)$ contains only points of the subarcs $x_{n_2}p$ and $y_{n_2}p$ of α and β respectively. The above construction may be repeated in $S(p, \delta_2)$, thereby yielding a simple closed curve containing a second point of the sequence $\{p_n\}$. The proof of the lemma follows by continuing this process inductively.

LEMMA 2.4. *Let M be a Peano space, $p \in M$, $\alpha = qxp$ an arc of M, and $\{p_n\}$ an infinite sequence of distinct points of M converging monotonically to p on α. Assume there is a positive number η and an infinite collection $\{D_i\}$ of components of $(M-\alpha) \cap S(p, \eta)$ such that* (a) $p \in \liminf \{D_i\}$; (b) $\overline{D}_i \cap F[S(p, \eta)] \neq \emptyset$ *for every i. Then there exist arcs Γ_1 and Γ_2 in M each of which contains infinitely many of the points $\{p_n\}$ and such that $\Gamma_1 \cap \Gamma_2 = p$.*

Proof. There is no loss of generality in assuming that the collection $\{D_i\}$ converges to a connected set L, and that L contains a nondegenerate subarc zp of α. Let σ_1 be a positive number such that $\overline{S}(p, \sigma_1) \cap \alpha \subset zp$. In the natural order from q to p on α let t_1 be the last point of $\alpha \cap [\overline{S}(p, \sigma_1) - S(p, \sigma_1)]$, and p_1^* the first point of the sequence $\{p_n\}$ such that $t_1 < p_1^*$. Let σ_2 be a positive number such that $\sigma_2 < \sigma_1/2$ and $\overline{S}(p, \sigma_2) \cap \alpha$ contains only points of α which follow p_1^* in the natural order from q to p on α. In the natural order from q to p on α let t_2 be the last point of $\alpha \cap [\overline{S}(p, \sigma_2) - S(p, \sigma_2)]$, and p_2^* the first point of the sequence $\{p_n\}$ such that $t_2 < p_2^*$. Continuing this process inductively yields sequences $\{\sigma_i\}$, $\{t_i\}$, and $\{p_i^*\}$ with the following properties:

(i) $\{\sigma_i\}$ is a null sequence of positive numbers such that $\sigma_{i+1} < \sigma_i/2$, and $\overline{S}(p, \sigma_{i+1}) \cap \alpha$ contains only points of α which follow p_i^* in the natural order from q to p on α.

(ii) $\{p_i^*\}$ is a subsequence of the $\{p_i\}$ such that $p_i^* \in S(p, \sigma_i) - \overline{S}(p, \sigma_{i+1})$.

(iii) t_i is the last point of $\alpha \cap [\overline{S}(p, \sigma_i) - S(p, \sigma_i)]$ in the natural order from q to p on α.

(iv) In the natural order from q to p on α, $t_i < p_i^* < t_{i+1}$, $i \in I$.

Let $\{V_i\}$ be a null sequence of regions such that $t_i \in V_i$ and $\overline{V}_i \cap \alpha$ contains only points interior to the subarc $p_{i-1}^* p_i^*$ of α. Then there exists a collection of distinct arcs $\{e_i f_{i+3}\}$ spanning α and such that $e_i f_{i+3}$ lies in $V_i \cup [\overline{S}(p, \sigma_i) - S(p, \sigma_{i+3})] \cup V_{i+3}$. The proof of the lemma is completed by defining the arcs Γ_1 and Γ_2 as follows:

$$\Gamma_1 = p \cup \left[\bigcup_{n \in I} (e_{4n-3} f_{4n}) \right] \cup \left[\bigcup_{n \in I} (f_{4n} p_{4n}^* e_{4n+1}) \right],$$

$$\Gamma_2 = p \cup \left[\bigcup_{n \in I} (e_{4n-1} f_{4n+2}) \right] \cup \left[\bigcup_{n \in I} (f_{4n+2} p_{4n+2}^* e_{4n+3}) \right]$$

where the collections $\{f_{4n}p^*_{4n}e_{4n+1}\}$ and $\{f_{4n+2}p^*_{4n+2}e_{4n+3}\}$ are subarcs of α.

3. **Separation theorem.** In this section the following theorem is proved.

THEOREM 3.1. *Let M be a cyclic Peano space which is strongly arcwise connected at p but not strongly cyclic at p, and let J be any simple closed curve in M containing p. Let $J=\alpha\cup\beta$ where $\alpha=qxp$ and $\beta=qyp$ are arcs such that $\alpha\cap\beta=q\cup p$, $q\neq p$. Then there exist in M an infinite sequence of distinct points $\{p_n\}$, and a sequence of simple arcs $\{\lambda_n\}$ having the following properties:*

(i) *$p_n\in M-J$ for every n.*

(ii) *$\{p_n\}$ converges to p.*

(iii) *No simple closed curve in M contains infinitely many points of the sequence $\{p_n\}$.*

(iv) *Every simple arc in M containing infinitely many points of the sequence $\{p_n\}$ intersects each of the arcs α and β in a set of points having p as a limit point.*

(v) *Each λ_n is an arc x_ny_n spanning J and containing the point p_n.*

(vi) *$\lambda_m\cap\lambda_n=\emptyset$ for $m\neq n$.*

(vii) *$\{\lambda_n\}$ converges to p.*

(viii) *$\bigcup_{n\in I}x_n\subset\alpha'=\alpha-(q\cup p)$, $\bigcup_{n\in I}y_n\subset\beta'=\beta-(q\cup p)$.*

(ix) *The sequences $\{x_n\}$ and $\{y_n\}$ converge monotonically to p on the arcs α and β respectively.*

(x) *For $m\neq n$, the sets $\lambda'_m=\lambda_m-(x_m\cup y_m)$ and $\lambda'_n=\lambda_n-(x_n\cup y_n)$ lie in different components of $M-J$. Furthermore, these components converge to p. Thus for any simple closed curve J of M containing p, every arc of J having p as an interior point separates M into infinitely many distinct components.*

(xi) *If $\{R_i\}$ is the collection of components of $M-J$ such that $\lambda'_i\subset R_i$, then for no value of i is there an arc of $\overline{R}_i-\beta$ spanning α and containing the point p_i, or an arc of $\overline{R}_i-\alpha$ spanning β and containing the point p_i.*

Proof. Since M is not strongly cyclic at p, there exists in M an infinite sequence of distinct points $\{p_n\}$ satisfying (ii) and (iii). From (iii) it follows that J contains at most a finite number of the points $\{p_n\}$. Hence no generality is lost in assuming that (i) holds.

Since M is strongly arcwise connected at p, there exists in M an arc containing infinitely many of the points $\{p_n\}$. Let λ be an arbitrary arc satisfying this condition. No generality is lost in assuming that λ contains p_n for every n, that λ has p as one endpoint, and that the sequence $\{p_n\}$ converges monotonically to p on λ. Let $\lambda=bdp$ where b is the endpoint of λ distinct from p.

If one of the arcs α and β, say α, contains a subarc $\alpha_1=stp$ such that $\alpha_1\cap\lambda=p$, then M is strongly cyclic at p. This contradiction proves (iv).

There exists a positive number ϵ such that in $S(p,\epsilon)$ no subarc of λ containing a point of the sequence $\{p_n\}$ spans one of the arcs α and β without intersecting the other. If this is not true, then with the aid of Lemma 2.2 it is seen that M becomes strongly cyclic at p. Thus there exists a sequence

$\{\lambda_n\}$ of distinct arcs converging to p such that $\lambda_n = x_n y_n$ is a subarc of λ spanning J and containing a point of the sequence $\{p_n\}$, $x_n \in \alpha'$, $y_n \in \beta'$, and the sequences $\{x_n\}$ and $\{y_n\}$ converge monotonically to p on α and β respectively. Since there is no loss of generality in assuming that $p_n \in \lambda_n$ for every n, this proves (v), (vi), (vii), (viii), and (ix).

To prove (x), assume that R is a component of $M-J$ containing infinitely many of the sets $\{\lambda_i'\}$, and hence p is a limit point of R. If p is accessible from R, it is easily shown that M becomes strongly cyclic at p. Thus p is not regularly accessible from R. Thus by the lemma of E. E. Betz referred to for the proof of Lemma 2.1 and by Lemma 2.1 there exists a positive number $\epsilon > 0$ such that for every positive number η, $\eta < \epsilon$, $R \cap S(p, \eta)$ has an infinite number of components $\{B_i\}$ with $p \in \liminf B_i$, $\overline{B}_i \cap F[S(p, \eta)] \neq \emptyset$ for every i, and for every positive number δ such that $\delta < \eta$, $B_i \cap S(p, \delta) \neq \emptyset$ for all but a finite number of values of i. There exists a value η_0 of η such that no component of $R \cap S(p, \eta_0)$ contains more than one element of the collection $\{\lambda_i'\}$. If this is not the case then with the aid of Lemma 2.2 it is easily shown that M is strongly cyclic at p. Thus it may be assumed that $R \cap S(p, \eta_0)$ has an infinite collection of components $\{B_i\}$ such that $\lambda_i' \subset B_i$ for every i. But this makes M strongly cyclic at p since the hypothesis of Lemma 2.3 is satisfied. This contradiction shows that each component of $M-J$ can contain at most a finite number of the sets $\{\lambda_i'\}$. Hence it may be assumed that $M-J$ has an infinite collection of components $\{R_i\}$ such that $\lambda_i' \subset R_i$ for every i. The proof of (x) is complete if it is observed that if the sequence $\{R_i\}$ does not converge to p, then again the hypothesis of Lemma 2.3 is satisfied.

The proof of the theorem is complete if it is noted that if (xi) is not true, then M becomes strongly cyclic at p by an application of Lemma 2.2.

4. **The properties** $P(r, \alpha)$ **and** $P(r, \beta)$. In this section let M be a cyclic Peano space, $J = \alpha \cup \beta$ a simple closed curve in M where $\alpha = qxp$ and $\beta = qyp$ are arcs such that $\alpha \cap \beta = q \cup p$, $q \neq p$, and $r \in M-J$. Let $\alpha' = \alpha - (q \cup p)$, $\beta' = \beta - (q \cup p)$. The next two theorems are conveniently stated in terms of a set $R(J, r)$ and properties $P(r, \alpha)$ or $P(r, \beta)$ which are defined as follows.

DEFINITION 4.1. By the set $R(J, r)$ is meant the component of $M-J$ containing r.

DEFINITION 4.2. The arc α is said to possess the properties $P(r, \alpha)$ with respect to the set $R(J, r)$ provided it possesses both of the following properties:

$P_1(r, \alpha)$: No point of $R(J, r)$ separates r from $F[R(J, r)] \cap \alpha'$ in $\overline{R}(J, r) - \beta$.

$P_2(r, \alpha)$: $F[R(J, r)] \cap \alpha'$ is nondegenerate. The properties $P(r, \beta)$ are defined by an interchange of α and β.

THEOREM 4.3 (SPANNING THEOREM). *In order that there exists an arc in M which spans α, contains r, and is disjoint from β, it is necessary and sufficient that α possesses the properties $P(r, \alpha)$ with respect to $R(J, r)$.*

Proof. The necessity is immediate. To prove the sufficiency observe that the arc β and the points of $M-\beta$ give an upper semicontinuous decomposition of M. Let M' be the hyperspace of this decomposition and $f(M)=M'$ the associated monotone map. Let $f(\beta)=b$, and note that $f(M-\beta)=M'-b$ is one-to-one.

By property $P_2(r,\alpha)$ there exist distinct points z_1 and z_2 belonging to $\alpha'\cap F[R(J,r)]$. Thus z_i does not separate r from z_j $(i\neq j)$ in $M-\beta$, hence no such separation occurs in $M'-b$. By property $P_1(r,\alpha)$ no point separates r and $z_1\cup z_2$ in $M-\beta$, hence no such separation occurs in $M'-b$. Thus by a theorem of Hall and Puckett [**2**, Theorem 2.2, p. 555], $M'-b$ contains an arc from z_1 to z_2 having r as an interior point. Clearly the inverse under f of this arc is an arc which contains a subarc a_1ra_2 spanning α' and such that $a_1ra_2-(a_1\cup a_2)\subset R(J,r)$.

THEOREM 4.4. *Let each of the arcs α and β possess at most one of the properties $P_i(r,\alpha)$ and $P_i(r,\beta)$ respectively, $i=1,2$. If there exists an arc $\gamma=arb$ such that $\gamma\cap\alpha'=a$, $\gamma\cap\beta'=b$, and $\gamma-(a\cup b)\subset R(J,r)$, then there exists a region V in $R(J,r)$ and two distinct points c and d of the arc γ such that the following conditions hold:*

(i) $r\in V$.

(ii) $F(V)\supset c\cup d$ *where c and d are distinct from p and q.*

(iii) $F(V)\subset c\cup d\cup q\cup p$.

(iv) $\overline{V}$ *is locally connected, hence a Peano space.*

Proof. If there exists a region V of $R(J,r)$ satisfying (i), (ii), and (iii), then $\overline{V}$ is locally connected since $\overline{V}-V$ is a finite set.

Since α possesses at most one of the properties $P_i(r,\alpha)$, $i=1,2$, the subarc ar of γ contains a point c which either separates r from $F[R(J,r)]\cap\alpha'$ in $\overline{R}(J,r)-\beta$, or is the single point of $F[R(J,r)]\cap\alpha'$. Similarly the subarc br of γ contains a point d which either separates r from $F[R(J,r)]\cap\beta'$ in $\overline{R}(J,r)-\alpha$, or is the single point of $F[R(J,r)]\cap\beta'$. Thus the component of $R(J,r)-(c\cup d)$ which contains r is the region V.

5. **Principal theorems.** In this section the principal theorems on strongly cyclic Peano spaces are obtained.

THEOREM I. *In order that a cyclic Peano space M which is strongly arcwise connected at a point p fail to be strongly cyclic at p, it is necessary and sufficient that there exists in M a closed set D containing p and a separation $M-D=(\bigcup_{i\in I}V_i)\cup N$, where $\{V_i\}$ is a collection of distinct components of $M-D$ having the following properties:*

(i) $V_i\rightarrow p$.

(ii) *For each i, $F(V_i)=c_i\cup d_i$ where $c_i\cup d_i\subset D-p$, $c_i\neq d_i$.*

(iii) *No simple closed curve of M intersects infinitely many of the regions V_i.*

Proof. The sufficiency is immediate. To prove the necessity let $J=\alpha\cup\beta$

be any simple closed curve in M containing p, where $\alpha = qxp$ and $\beta = qyp$ are arcs such that $\alpha \cap \beta = q \cup p$, $q \neq p$. Let $\{p_n\}$ and $\{\lambda_n\}$ respectively be the sequences of points and arcs of Theorem 3.1. By Theorem 3.1 there exists an infinite collection of distinct components $\{R_i\}$ of $M - J$ converging to p with $\lambda_i' \subset R_i$. Thus R_i is the set $R(J, p_i)$ of Definition 4.1. By Theorems 3.1 (xi) and 4.4 each component R_i contains a region V_i such that $p_i \in V_i$, $F(V_i) \subset c_i \cup d_i \cup q \cup p$, and $F(V_i) \supset c_i \cup d_i$ where $c_i \cup d_i \subset \lambda_i$, $c_i \neq d_i$. Since the sequence $\{R_i\}$ converges to p, it may be assumed $q \notin F(V_i)$ for any i. If $p \in F(V_k)$ for some integer k, then p is accessible from V_k by Theorem 4.4 (iv). But this makes M strongly cyclic at p contrary to assumption. Thus it may be assumed that $F(V_i) = c_i \cup d_i$ for every i. The necessity of (i) and (ii) is established by letting $D = p \cup [\bigcup_{i \in I} (c_i \cup d_i)]$.

To prove the necessity of (iii), let J_1 be a simple closed curve in M containing p. If $J_1 \cap V_k \neq \emptyset$ for some integer k, then J_1 contains the points c_k and d_k. If the subarc $c_k d_k$ of J_1 lying in $\overline{V}_k$ is replaced by the subarc $c_k p_k d_k$ of λ_k, then a simple closed curve containing the points p_k and p is obtained. If J_1 intersects infinitely many of the sets V_i, then it is seen that M is strongly cyclic at p. This contradiction completes the proof of the theorem.

THEOREM II. *In order that a cyclic Peano space M be strongly cyclic it is necessary and sufficient that for every infinite collection of open sets $\{V_i\}$ in M there exists in M a simple closed curve intersecting infinitely many of these sets.*

Proof. The necessity is immediate. To prove the sufficiency, observe that M is strongly arcwise connected. If M is not strongly cyclic, then by Theorem I there is an infinite collection of open sets in M such that no simple closed curve of M intersects infinitely many of these sets. Since this would contradict the hypothesis, it follows that M is strongly cyclic.

COROLLARY 5.1. *If a Peano space M is not separated by any two of its points, then M is strongly cyclic.*

COROLLARY 5.2. *If M is a cyclic Peano space which fails to be strongly cyclic at the point p, then p is an im kleinen cycle point of M.*

THEOREM 5.3. *If M is a cyclic Peano space which is strongly arcwise connected at p, but not strongly cyclic at p, then given $\epsilon > 0$, there exists in M an arc $\gamma = txp$ having diameter less than ϵ and an infinite collection of distinct components $\{K_i\}$ of $M - \gamma$ converging to p.*

Proof. Let $\{\alpha_i\}$ and $\{\beta_i\}$ be the collection of subarcs $\{x_i x_{i+1}\}$ and $\{y_i y_{i+1}\}$ of α and β respectively. Let $\alpha_i' = \alpha_i - (x_i \cup x_{i+1})$. Consider $M - \Gamma$ where $\Gamma = p \cup (\bigcup_{i \in I} \lambda_i) \cup (\bigcup_{i \in I} \alpha_{2i-1}) \cup (\bigcup_{i \in I} \beta_{2i})$. If a component K of $M - \Gamma$ should contain infinitely many of the sets α_{2i}', then p is a limit point of K. Since M is not strongly cyclic at p, p is not regularly accessible from K. But this makes M strongly cyclic at p by applications of Lemmas 2.1 and 2.4. Thus $M-\Gamma$ has an infinite collection $\{K_i\}$ of components each of which contains a set α'_{2i}. Now $K_i\rightarrow p$, for otherwise an application of Lemma 2.4 would make M strongly cyclic at p. Thus every subarc of Γ having p as one endpoint gives the desired separation of M, thereby proving the theorem.

6. **Two-cyclic Peano spaces.** It is known that every Peano space has a basis each element of which is an open connected set having property S [4, p. 219]. Thus the closure of an arbitrary element of such a basis is a Peano space. Throughout this section we denote by Σ such a basis for the Peano space M under consideration.

THEOREM III. *Let M be a cyclic Peano space and $p\in M$. Suppose that M is strongly cyclic at p, but not two-cyclic at p. Then either p belongs to a free arc of M or given $\epsilon>0$, there exists in M an arc $\gamma=txp$ having diameter less than ϵ and an infinite collection of distinct components $\{K_i\}$ of $M-\gamma$ converging to p.*

The proof of Theorem III requires that some background material be developed.

THEOREM 6.1. *Let M be the Peano space of Theorem* III *and U an element of Σ such that $p\in U$. Then the following conditions hold*:

(i) *The components of $\overline{U}-p$ may be classified as follows*:

(a) $\{X_i\}$: *those components of $\overline{U}-p$ such that p is an endpoint of $\overline{X}_i$; or if p is not an endpoint of $\overline{X}_i$, then $\overline{X}_i$ is not strongly cyclic at p.*

(b) $\{Y_j\}$: *those components of $\overline{U}-p$ such that $\overline{Y}_j$ is strongly cyclic at p.*

(ii) *For each i and j, $\overline{X}_i\cap(\overline{U}-U)\neq\emptyset$, $\overline{Y}_j\cap(\overline{U}-U)\neq\emptyset$.*

(iii) *Each of the collections $\{X_i\}$ and $\{Y_j\}$ is finite.*

(iv) *If the collection $\{X_i\}$ is nonvacuous, then Theorem* III *is true.*

Proof. Statement (i) is just a classification of the components of $\overline{U}-p$. Statement (ii) follows from the fact that M is cyclic, while (iii) is a consequence of the local connectivity of M.

To prove (iv), let $X_1\in\{X_i\}$. If p is an endpoint of $\overline{X}_1$, Theorem III is easily seen to hold. If p is not an endpoint of $\overline{X}_1$ but $\overline{X}_1$ is not strongly cyclic at p, then Theorem III follows from Theorem 5.3.

In view of Theorem 6.1(iv), it may be assumed that for the Peano space M of Theorem III the collection $\{X_i\}$ is empty for every element U of Σ such that $p\in U$. The space M then possesses the properties which follow:

PROPERTY 1. *There exists in M an infinite sequence of distinct points $\{p_n\}$ converging to p such that if Γ_1 and Γ_2 are two arcs of M having p as one endpoint and each containing infinitely many of the points $\{p_n\}$, then $\Gamma_1\cap\Gamma_2$ consists of a set of points having p as a limit point. Moreover, there exists in M an arc $\alpha=qxp$ such that the points $\{p_n\}$ converge monotonically to p on α.*

PROPERTY 2. *If $U\in\Sigma$ and $p\in U$, then all but a finite number of the points*

$\{p_n\}$ of property 1 *lie in a true cyclic element E of the set $\overline{Y}_1$ where $\overline{U}-p = \bigcup_{j=1}^{h} Y_j$.*

Proof. Let $U \in \Sigma$ such that $p \in U$. Then $\overline{U}-p=\bigcup_{j=1}^{h} Y_j$. In view of Property 1 it may be assumed that all but a finite number of the points $\{p_n\}$ lie in Y_1. Let E be the true cyclic element of $\overline{Y}_1$ containing p. Since M is cyclic, every component of $\overline{Y}_1-E$ must intersect $\overline{U}-U$, hence the components of $\overline{Y}_1-E$ cannot have limit points in E arbitrarily close to p. Thus all but a finite number of the points $\{p_n\}$ lie in E.

PROPERTY 3. *$E-\alpha=H\cup N$, where H is the set of all points $x\in E$ for which there exists in E a nondegenerate arc xp with $xp\cap\alpha=p$, N is the set of all points $x\in E$ for which no such arc exists.*

PROPERTY 4. *If, for every component W of $E-\alpha$, $\overline{W}\cap F(M-E)\neq\emptyset$, then p is not a limit point of N and the collection $\{C_i\}$ of components of H is finite. Thus there exists a subarc vp of α which does not separate M. Moreover, in this case there exists at least one component of H which has a set of limit points in $\alpha-p$ having p as a limit point.*

Proof. By the definition of N, p is not accessible from N. Thus if p is a limit point of N, the hypothesis of Lemma 2.4 is easily seen to be satisfied. Since this makes M two-cyclic at p, p is not a limit point of N. If H has an infinite collection of components, the hypothesis of Lemma 2.4 is again satisfied. Thus H has a finite number of components in this case. In the natural order from q to p on α, let d be the last point of $\overline{N}\cap\alpha$. Thus for any point v interior to the subarc dp, the subarc vp of α does not separate M. The proof of Property 4 is complete if it is observed that if H does not have a component which has a set of limit points in $\alpha-p$ having p as a limit point, then there is an element $V\in\Sigma$ containing p for which the collection $\{X_i\}$ is nonempty.

PROPERTY 5. *If, for arbitrarily small elements U of Σ such that $p\in U$ there exists a component W of $E-\alpha$ such that $\overline{W}\cap F(M-E)=\emptyset$, then Theorem* III *is true.*

The following restrictions may be imposed on all further discussion.

RESTRICTIONS 6.2. The space M under consideration is a cyclic Peano space which is strongly cyclic at the point p, but not two-cyclic at p, and for which every element U of the basis Σ such that $p\in U$ satisfies the following conditions:

(a) The collection $\{X_i\}$ is empty.

(b) For every component W of $E-\alpha$, $\overline{W}\cap F(M-E)\neq\emptyset$, where E and W are the sets of $\overline{U}$ as given in the above properties.

THEOREM 6.3. *Let M be a Peano space satisfying the Restrictions* 6.2. *Then there exists a sequence $\{U_i\}$ of elements of the basis Σ such that $p\in U_i$ for every i, and corresponding sequences $\{E_i\}$, $\{N_i\}$, and $\{H_i\}$ as given by Properties* 2 *and* 3, *such that the following conditions hold*:

(i) $\{U_i\}$ *is a null sequence,* $U_i \supset \overline{U}_{i+1}$, *and* $\overline{U}_{i+1} \cap N_i = \emptyset$.

(ii) $E_i \supset E_{i+1}$ *and* $H_i \supset \overline{H}_{i+1}$.

(iii) *If* R_i *is a component of* H_i *having a set of limit points in* $\alpha - p$ *which has* p *as a limit point, then there exists a sequence* $\{R_j\}$, $j \geqq i$, *such that* R_j *is a component of* H_j *having a set of limit points in* $\alpha - p$ *which has* p *as a limit point, and* $R_j \supset \overline{R}_{j+1}$.

Proof. The sequences $\{U_i\}$, $\{E_i\}$, $\{N_i\}$, and $\{H_i\}$ are easily seen to exist. To prove (iii), note that by Property 4 each set H_i contains at least one component having a set of limit points in $\alpha - p$ having p as a limit point. Let R_1 be such a component of H_1. In the natural order from q to p on α let d be the last point of $\overline{N}_1 \cap \alpha$. There exists a sequence $\{r_n\}$ of points interior to the subarc dp of α such that $r_n \to p$, r_n is a limit point of R_1, and $r_n \in U_2$, $n \in I$. Since H_2 has only a finite number of components, it is clear that there is a component R_2 of H_2 which has infinitely many of the points $\{r_n\}$ as limit points and $\overline{R}_2 \subset R_1$. Continuing this process inductively completes the proof of the theorem.

THEOREM 6.4. *For each sequence* $\{R_j\}$, $j \geqq i$, *of Theorem* 6.3(iii) *there exists an arc* λ *from a point* $t_i \in F(R_i) - \alpha$ *to* p, *and a sequence* $\{\mu_j\}$ *of arcs such that*

(a) $\lambda - p \subset \overline{R}_i - \alpha$; *i.e.,* $\lambda \cap \alpha = p$.

(b) $\lambda \cap R_j \neq \emptyset$ *for every* $j \geqq i$.

(c) $\mu_j = x_j y_j$ *is an arc of* $E_j - E_{j+1}$ *such that* $\mu_j \cap \alpha = x_j$, $\mu_j \cap \lambda = y_j$.

(d) *The sequences* $\{x_j\}$ *and* $\{y_j\}$ *converge monotonically to* p *on* α *and* λ *respectively.*

Proof. An arc λ satisfying (a) and (b) is readily constructed. In each set E_i, let $t_i \in \alpha$, $z_i \in \lambda$. There exists an arc pz_it_i in E_i, and this arc contains a subarc $\mu_i = x_i y_i$ such that $\mu_i \cap \alpha = x_i$, $\mu_i \cap \lambda = y_i$. It is clear that no generality is lost in assuming that (c) and (d) hold.

THEOREM 6.5. *Each set* H_i, $i \in I$, *of Theorem* 6.3 *contains exactly one component* R_i *having a set of limit points in* $\alpha - p$ *which has* p *as a limit point. Thus the sequence* $\{R_i\}$, $i \in I$, *of Theorem* 6.3(iii) *is unique, as is the sequence* $\{G_i\}$ *where* G_i *is the set of all points* $x \in R_i$ *which can be joined to* $\alpha - p$ *by an arc of* E_i *not meeting* λ.

Proof. Assume that there is a set H_i which has two components having a set of limit points in $\alpha - p$ each having p as a limit point. It may be assumed that H_1 is such a set. By Theorem 6.3(iii) there exist two distinct sequences $\{R_i\}$ and $\{S_i\}$ as there described. As described in Theorem 6.4, there exist arcs λ and $\{\mu_i\}$ for the sequence $\{R_i\}$, and arcs λ^* and $\{\mu_i^*\}$ for the sequence $\{S_i\}$. It is easily seen that this makes M two-cyclic at p. This contradiction proves the theorem.

A review of Properties 1–5 and Theorems 6.3–6.5 reveals that there is no loss of generality in making the following assumptions.

ASSUMPTIONS 6.6. For the Peano space M of Properties 1–5 and Theorems 6.3–6.5 the following conditions hold:

(a) In the natural order from q to p on α, $x_i < p_i < x_{i+1}$, $i \in I$.

(b) $p_i \in E_i - E_{i+1}$, $i \in I$.

(c) $\mu_i \subset E_i - E_{i+1}$.

(d) λ is an arc from a point $t \in F(R_1) - \alpha$ to the point p such that λ satisfies Theorem 6.4 for the sequence $\{R_i\}$.

THEOREM 6.7. *Let M be a Peano space satisfying the restrictions* 6.2 *and subject to the Assumptions* 6.6. *Let $\{\alpha_i\}$ be the collection of subarcs of α defined by $\alpha_i = x_i p_i x_{i+1}$, $\{\lambda_i\}$ the collection of subarcs of λ defined by $\lambda_i = y_i y_{i+1}$. Let $\alpha_i' = \alpha_i - (x_i \cup x_{i+1})$, $\lambda_i' = \lambda_i - (y_i \cup y_{i+1})$. For each set G_j of the sequence $\{G_i\}$ of sets defined in Theorem* 6.5, *consider the set $G_j - \bigcup_{i \in I} \mu_i$, $j \in I$. Then there exists a positive integer k such that for $j \geqq k$ the following hold*:

(i) *No component of $G_j - \bigcup_{i \in I} \mu_i$ has limit points in two distinct sets α_m' and α_n' where $|m-n| \geqq 2$.*

(ii) *If a component of $G_j - \bigcup_{i \in I} \mu_i$ has limit points in a set α_m' and a set λ_n', then $m = n$.*

(iii) *If a component of $G_j - \bigcup_{i \in I} \mu_i$ has limit points in a set α_m', then any limit points of this component in $\bigcup_{i \in I} \mu_i'$, where $\mu_i' = \mu_i - (x_i \cup y_i)$, must lie in $\mu_m' \cup \mu_{m+1}'$.*

Proof. If no such integer k exists, it is easy to construct two arcs Γ_1 and Γ_2 such that $\Gamma_1 \cap \Gamma_2 = p$ and each contains infinitely many of the points $\{p_n\}$.

Proof of Theorem III. Theorem 6.1 (iv) and property 5 give special cases in which Theorem III is true. It may thus be assumed that M satisfies the Restrictions 6.2 and is subject to the Assumptions 6.6. Moreover, it may be assumed in Theorem 6.7 that the integer $k=1$. In the natural order from q to p on α, let d be the last point of $\overline{N}_1 \cap \alpha$. Then there exists an integer r such that for $i \geqq r$, x_i is a point interior to the subarc dp of α. No generality is lost in assuming that $r=1$. Define an arc Γ as follows:

$$\Gamma = p \cup \left(\bigcup_{i \in I} \alpha_{2i-1} \right) \cup \left(\bigcup_{i \in I} \mu_i \right) \cup \left(\bigcup_{i \in I} \lambda_{2i} \right).$$

Note that the arc Γ lies in the set E_1 and is disjoint from $\overline{N}_1$.

Next define a collection of sets $\{K_i\}$ as follows:

$K_i = \alpha_{2i}' \cup$(all components of $G_1 - \bigcup_{i \in I} \mu_i$ having limit points in α_{2i}').

Observe that each set K_i is a component of $E_1 - \Gamma$ by Theorem 6.7. Thus E_1 is separated into infinitely many components $\{K_i\}$ by the arc Γ. Moreover, the $\{K_i\}$ converge to p, for otherwise the hypothesis of Lemma 2.4 is satisfied thereby making E_1, and hence M, two-cyclic at p. Thus Γ, and every subarc of Γ having p as an endpoint, separates M in to infinitely many components converging to p, the collection of components in such cases being a subcollection of the collection $\{K_i\}$. This completes the proof of Theorem III.

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