SPECTRAL THEORY FOR OPERATORS ON A BANACH SPACE(1)

BY

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0. **Introduction.** The purpose of this paper is to study the spectral theory of a closed linear transformation T on a reflexive Banach space B. This will be done by means of certain vector-valued measures which are related to the transformation. (A set function m from the Borel sets of the complex plane to B will be called a vector-valued measure if the series $\sum_{i=1}^{\infty} m(S_i)$ converges to $m(\bigcup_i S_i)$ for every sequence $\{S_i\}$ of disjoint Borel sets. The relevant properties of vector-valued measures are briefly derived in §1(2). A vector-valued measure m will be called a T-measure if $Tm(S)=\int_S z dm(z)$ for all bounded Borel sets S. The properties of T-measures are studied in §2.

The results of §2 are applied in §3 to a class of transformations which have been called scalar-type transformations by Dunford [5], and which we call simply scalar transformations. A scalar operator as defined by Dunford is essentially one which admits a representation of the type $t=\int z dE(z)$, where E is a spectral measure. Unbounded scalar transformations have been studied by Taylor [16].

The main result of §3 is Theorem 3.2, in which properties of the closures of certain sets of scalar transformations are derived. This theorem is actually a rather general spectral-type theorem, which has applications to several problems in the theory of linear transformations. As a corollary we obtain a well-known theorem, which might be called the spectral theorem for symmetric transformations, as given in Stone [15]. We also derive as a corollary the spectral theorem for self-adjoint transformations. Other corollaries of Theorem 3.2, which apply to results of Bade [2] and Halmos [9] are derived.

In §4 a functional calculus is developed for a class Γ of transformations T for which both T-measures and T^*-measures exist in sufficient abundance. This is a very general functional calculus, so that correspondingly the usual theorems of functional calculus must be weakened if they are to remain true.

There is a generalization of the concept of a T-measure introduced in §5. Many theorems proved in §2 have analogues which hold after the generalization. The new type of vector-valued measures have much the same relation

Received by the editors November 11, 1955.

(1) A paper which treats this subject in some detail is Dunford, Bartle, and Schwartz [4].

(2) This paper is a part of a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the University of Chicago. The author wishes to thank Professors P. R. Halmos, J. L. Kelley and F. Wolf for their advice and encouragement.

to spectral transformations (generalizations of scalar transformations, defined by Dunford [5]) as T-measures have to scalar transformations.

1. **Vector-valued measures.** Let X be a set, $\mathfrak{S}$ a σ-ring of subsets of X, and m a function from $\mathfrak{S}$ to a Banach space B with the property that for every sequence $\{S_i\}$ of disjoint measurable sets (measurable sets are sets in $\mathfrak{S}$) we have $m(\bigcup_i S_i) = \sum_{i=1}^{\infty} m(S_i)$, where the convergence is convergence in norm. Such a set function m is called a vector-valued measure. By a theorem of Orlicz, proved by Pettis [14], unconditional convergence of series $\sum_{i=1}^{\infty} x_i$ of vectors to a vector x is equivalent to unconditional weak convergence of the series to x, i.e., convergence of $\sum_{i=1}^{\infty} \langle x_i, u\rangle$ to $\langle x, u\rangle$ for each u in B^*. Thus m is a vector-valued measure if and only if $\langle m(\bigcup_i S_i), u\rangle = \sum_{i=1}^{\infty} \langle m(S_i), u\rangle$ for each disjoint sequence $\{S_i\}$ of measurable sets and each u in B^*. Equivalently, the set function m_u defined by $m_u(S) = \langle m(S), u\rangle$ must be a complex-valued measure for each u in B^*.

If m is a vector-valued measure and S_0 any measurable set, then the equation $\tilde{m}(S) = m(S \cap S_0)$ obviously defines a vector-valued measure $\tilde{m}$, called the restriction of m to S_0, and $\tilde{m}$ is said to live on S_0.

We define the norm $\|m\|$ of a vector-valued measure m to be the sup of the quantities $\|\sum_{i=1}^{n} \lambda_i m(S_i)\|$, where $S_1, \cdots, S_n$ is any finite sequence of disjoint measurable sets and $\lambda_1, \cdots, \lambda_n$ is a corresponding sequence of complex numbers with $|\lambda_i| \leqq 1$. To show that $\|m\|$ is finite, let u be any vector in B^*. Then it is known that the quantities $|\sum_{i=1}^{n} \lambda_i m_u(S_i)|$ defined for the finite numerical measure m_u considered above are bounded. By the uniform boundedness theorem (see Banach [3]), $\|m\|$ is finite.

It is easily seen that the norm $\|m\|$ makes the set Q of vector-valued measures into a normed linear space. Indeed Q is a Banach space, although we do not give the simple proof because for the special case which will interest us this is a corollary of a later theorem. A useful fact is that

$$\|m\| = \sup_{u \in B^*} (\|m_u\| / \|u\|),$$

where the measure m_u is defined by $m_u(S) = \langle m(S), u\rangle$. This follows easily from the definition of $\|m\|$.

It is now simple to define what is meant by the integral of a bounded measurable complex-valued function f with respect to the vector-valued measure m. We first consider the case of a simple function f (a finite linear combination of characteristic functions of measurable sets), $f = \sum_{i=1}^{n} \lambda_i \psi_{S_i}$, and define $\int f(z) dm(z) = \sum_{i=1}^{n} \lambda_i m(S_i)$. The definition is easily seen to be unique. Letting $\|f\|$ denote $\sup |f(z)|$, we see immediately that $\|\int f(z) dm(z)\| \leqq \|f\| \, \|m\|$, and in fact that

$$\|m\| = \sup_f \frac{\left\| \int f(z) dm(z) \right\|}{\|f\|}$$

where the sup is taken over all simple functions. Thus $f \to \int f(z)dm(z)$ is a linear transformation of norm $\|m\|$ from the set of simple functions to B, and therefore has a unique bounded linear extension to the set of bounded measurable functions. The value of this extension at f we call $\int f(z)dm(z)$, so that the integral is a linear transformation of norm $\|m\|$ on the set of bounded measurable functions.

Again defining m_u by $m_u(S) = \langle m(S), u\rangle$, we obviously have $\int f(z)dm_u(z) = \langle \int f(z)dm(z), u\rangle$ for every simple function f. By continuity the equality holds for all bounded measurable f.

For a bounded measurable function f and a vector-valued measure m, consider the set function $\tilde{m}$ defined by $\tilde{m}(S) = \int_S f(z)dm(z)$. Then for every sequence $\{S_i\}$ of disjoint measurable sets and every u in B^* we have

$$\begin{aligned}\langle \tilde{m}(\cup_i S_i), u\rangle &= \left\langle \int_{\cup_i S_i} f(z)dm(z), u\right\rangle = \int_{\cup_i S_i} f(z)dm_u(z) \\ &= \sum_i \int_{S_i} f(z)dm_u(z) = \sum_i \left\langle \int_{S_i} f(z)dm(z), u\right\rangle \\ &= \sum_i \langle \tilde{m}(S_i), u\rangle,\end{aligned}$$

so that $\tilde{m}$ is a vector-valued measure.

If f is an arbitrary measurable function, write $\tilde{m}(S) = \int_S f(z)dm(z)$ for any measurable set S on which f is bounded, so that as we have just seen $\tilde{m}(S) = \sum_i \tilde{m}(S_i)$ if $\{S_i\}$ is a sequence of disjoint measurable sets with $\cup_i S_i = S$ and if f is bounded on S. We say that f is integrable with respect to m if $\|\tilde{m}(S)\|$ is less than some constant K for all Borel sets S on which f is bounded. This implies for each sequence $\{S_i\}$ of disjoint measurable sets on each of which f is bounded and for each u in B^* that $\sum_i \langle \tilde{m}(S_i), u\rangle$ converges, since a series of complex numbers converges if there is a uniform bound for the sums of finite subseries. Thus for each measurable set S we may represent S as the union of a sequence $\{S_i\}$ of disjoint measurable sets on each of which f is bounded, and define a linear functional $\tilde{m}(S)$ on B^* whose value on u is $\sum_{i=1}^{\infty} \langle \tilde{m}(S_i), u\rangle$. This linear functional is bounded because $|\sum_{i=1}^{\infty} \langle \tilde{m}(S_i), u\rangle| \leq \sup_n \|\tilde{m}(\cup_{i=1}^{n} S_i)\| \|u\| \leq K\|u\|$. It is seen by the usual methods that $\tilde{m}(S)$ is unique, i.e., independent of the choice of the sequence $\{S_i\}$, and that it defines a vector-valued measure $\tilde{m}$ with values in B^{**} if B is reflexive. For $\tilde{m}(X)$ we write $\int f(z)dm(z)$, the integral of f with respect to m, which we have defined under the hypothesis that f is integrable with respect to m. If B is reflexive, then $\int f(z)dm(z)$ may be considered to be in B itself.

Under the assumption that B is reflexive, that m is a vector-valued measure, that f is integrable with respect to m, and that g is integrable with respect to the vector-valued measure $\tilde{m}$ defined by $\tilde{m}(S) = \int_S f(z)dm(z)$, it follows that fg is integrable with respect to m and that $\int_S f(z)g(z)dm(z) = \int_S g(z)d\tilde{m}(z)$.

This is easily proved for the case of a simple function g and a bounded measurable function f. Then since both sides depend continuously on g in the uniform topology, it follows that the equation holds for all bounded measurable functions f and g. This implies that if f and g are any functions satisfying the hypothesis, then the equation holds for sets S on which both f and g are bounded. Then it follows easily for all measurable sets S, because both sides of the equation are countably additive set functions.

In addition to the σ-ring $\mathfrak{S}$ of subsets of X consider a σ-ring $\mathfrak{S}'$ of subsets of X' and let λ be a measurable function from X to X', that is, a function such that $\lambda^{-1}(S')$ is in $\mathfrak{S}$ for each S' in $\mathfrak{S}'$. Then it is easy to see that for any vector-valued measure m on S the set function $\hat{m}$ on $\mathfrak{S}'$ defined by $\hat{m}(S') = m(\lambda^{-1}(S'))$ is a vector-valued measure on $\mathfrak{S}'$. Moreover if f is a simple function on S', it is easy to check that $\int f(z)d\hat{m}(z) = \int f(\lambda(z))dm(z)$. From this it can be first proved for bounded measurable functions f and then for any function f integrable with respect to $\hat{m}$ that $\int f(\lambda(z))dm(z)$ exists and equals $\int f(z)d\hat{m}(z)$.

We now let X be a locally compact Hausdorff space and let $\mathfrak{S}$ be the Baire subsets of X. Let $\mathfrak{M}(X)$ denote the set of continuous complex-valued functions on X vanishing at ∞. The set $\mathfrak{M}(X)$ is a Banach space under the norm $\|f\|$ defined previously. For each m in Q we define ϕ_m to be the bounded linear map from $\mathfrak{M}(X)$ to B which takes f into $\int f(z)dm(z)$, so that $\|\phi_m\| \leq \|m\|$ and $m \rightarrow \phi_m$ is a bounded linear transformation from Q to the space of bounded linear transformations from $\mathfrak{M}(X)$ to B.

Conversely for a reflexive Banach space B, which we consider henceforth, let ϕ be a bounded linear transformation from $\mathfrak{M}(X)$ to B. We shall show that $\phi = \phi_m$ for some m in Q. For each u in B^* define $\phi_u(f) = \langle \phi(f), u\rangle$ so that ϕ_u is a bounded linear functional on $\mathfrak{M}(X)$ and $\|\phi\| = \sup_u (\|\phi_u\|/\|u\|)$. By the Riesz representation theorem for linear functionals there exists for each u in B^* a unique complex-valued measure m_u on X such that $\phi_u(f) = \int f(z)dm_u(z)$ for all f in $\mathfrak{M}(X)$, which has the property $\|m_u\| = \|\phi_u\|$. Since m_u is unique, it must depend linearly on u. Therefore for each Borel set S, $m_u(S)$ is a linear function of u. Since B is reflexive, there exists $m(S)$ in B such that $m_u(S) = \langle m(S), u\rangle$ for all u in B^*. The set function m is a vector-valued measure because m_u is a complex-valued measure for each u. Also

$$\left\langle \int f(z)dm(z),\ u \right\rangle = \int f(z)dm_u(z) = \phi_u(f) = \langle \phi(f), u\rangle$$

for all u in B^*, so that $\phi(f) = \int f(z)dm(z)$ for all f in $\mathfrak{M}(X)$. Thus $\phi = \phi_m$. Moreover, $\|m\| = \sup_u (\|m_u\|/\|u\|) = \sup_u (\|\phi_u\|/\|u\|) = \|\phi\|$. Thus $m \rightarrow \phi_m$ is a metric isomorphism of Q with the Banach space of operators from $\mathfrak{M}(X)$ to B. In particular, Q is a Banach space and $\|m\| = \sup_{f\in\mathfrak{M}(X)} (\|\int f(z)dm(z)\|/\|f\|)$. In this equation we may even take the sup over those f with compact support, since they are dense in $\mathfrak{M}(X)$.

DEFINITION 1.1. The weak operator topology of Q is that topology obtained by considering Q as the bounded linear transformations from $\mathfrak{M}(X)$ to B, and then defining the weak topology in the usual fashion.

The unit sphere of Q is compact in this topology.

2. **Measures associated with operators.** We turn now to the study of a closed linear transformation T whose domain is a dense subset of the reflexive Banach space B and whose range is a subset of B. The class of such transformations we call $\mathfrak{T}$. It is well known (see, for instance, von Neumann [17]) that each T in $\mathfrak{T}$ has an adjoint which is a closed linear transformation T^* on B^* with dense domain and that $T^{**}=T$. This means that $\langle Tx, u\rangle = \langle x, T^*u\rangle$ for each x in $\mathfrak{D}(T)$ and each u in $\mathfrak{D}(T^*)$. Thus $\mathfrak{T}^*$, the set of T^* for T in $\mathfrak{T}$, is the set of closed linear transformations on B^* with dense domain.

A vector-valued measure m on the Borel sets $\mathfrak{S}$ of the complex plane X will be called a T-measure if for each bounded Borel set S we have $m(S)\in\mathfrak{D}(T)$ and $Tm(S)=\int_S z dm(z)$. The vector $x=m(X)$ will be said to have the T-measure m under these conditions. It is clear that the set of all T-measures is a linear subset of Q. Later we shall see that this set is closed.

As an illustration, let m be a T-measure and z_0 be a point with $m(\{z_0\})\neq 0$. Then $Tm(\{z_0\})=\int_{\{z_0\}} z dm(z)=z_0 m(\{z_0\})$, so that $m(\{z_0\})$ is a characteristic vector of T. Conversely if m is a vector-valued measure which lives on a countable set S_0, then it is easy to see that if $Tm(\{z\})=zm(\{z\})$ for each z in S_0 then m is a T-measure. Thus the notion of a T-measure is a generalization of the notion of a characteristic vector.

As another example, let B be a Hilbert space and let $T=\int z dE(z)$ be a normal transformation on B. Select any x in B and consider the set function m on $\mathfrak{S}$ defined by $m(S)=E(S)x$. From the properties of the spectral measure E it follows that m is a vector-valued measure. Moreover

$$Tm(S) = TE(S)x = \int zdE(z)E(S)x = \int_S zdE(z)x = \int_S zdm(z),$$

so that m is a T-measure. We shall show later that x has no T-measures other than m.

In the following pages we collect a few simple properties of T-measures which will be useful later and then prove the fundamental theorems about T-measures.

Given a T-measure m and a Borel set S_0, it is obvious that the restriction of m to S_0, i.e., the vector-valued measure $\tilde{m}$ defined by $\tilde{m}(S)=m(S\cap S_0)$, is also a T-measure.

If m is a T-measure and f a bounded measurable function, and if the Borel set S is bounded, then $\int_S f(z)dm(z) \in \mathfrak{D}(T)$ and $T\int_S f(z)dm(z) = \int_S zf(z)dm(z)$. For a simple function $f=\sum_{i=1}^n \lambda_i\psi_{S_i}$, where ψ_{S_i} is the characteristic function of the set S_i, this amounts to a quick computation:

$$T\int_S f(z)dm(z)$$
$$= \sum_{i=1}^{n} \lambda_i T(m(S \cap S_i))$$
$$= \sum_{i=1}^{n} \lambda_i \int_{S\cap S_i} z dm(z)$$
$$= \int_S zf(z)dm(z).$$

For an arbitrary bounded measurable function f it can be proved by approximating f in the uniform topology by a simple function g and noting that $\int_S g(z)dm(z)$ approximates $\int_S f(z)dm(z)$ and that $\int_S zg(z)dm(z)$ approximates $\int_S zf(z)dm(z)$, so that by the closure of T the assertion follows. If it is assumed merely that f is integrable with respect to m, then the above proof holds only on bounded sets S on which f is bounded. To prove the equality for all bounded Borel sets S, it is necessary to make use again of the fact that T is closed.

Conversely if the vector-valued measure m has the property that for every function f in $\mathfrak{M}(X)$ with compact support it is true that $\int f(z)dm(z) \in \mathfrak{D}(T)$ and $T\int f(z)dm(z) = \int zf(z)dm(z)$, then m is a T-measure. To prove this define a numerical measure m_u for each u in B^* by $m_u(S) = \langle m(S), u\rangle$. Then for each u in $\mathfrak{D}(T^*)$ we have

$$\int f(z)dm_{T^*u}(z) = \left\langle \int f(z)dm(z),\ T^*u \right\rangle = \left\langle T\int f(z)dm(z),\ u \right\rangle$$
$$= \left\langle \int zf(z)dm(z),\ u \right\rangle = \int zf(z)dm_u(z).$$

Since the functions with compact support are dense in $\mathfrak{M}(X)$ and since the representation of a linear functional on $\mathfrak{M}(X)$ as a measure is unique, we must have $m_{T^*u}(S) = \int_S z dm_u(z)$ for all Borel sets S. For S bounded this becomes $\langle m(S), T^*u\rangle = \langle \int_S z dm(z), u\rangle$. Since u is any vector in $\mathfrak{D}(T^*)$ this implies that $m(S) \in \mathfrak{D}(T)$ and $Tm(S) = \int_S z dm(z)$, as was to be proved.

We have seen that $\tilde{m}(S) = \int_S f(z)dm(z)$ is a vector-valued measure if f is integrable with respect to m. On the other hand if m is a T-measure it was shown above that $\tilde{m}(S) \in \mathfrak{D}(T)$ and $T\tilde{m}(S) = \int_S zf(z)dm(z) = \int_S z d\tilde{m}(z)$ for all bounded Borel sets S, so that $\tilde{m}$ is also a T-measure. In particular, we see again by taking f to be a characteristic function that the restriction of m to a Borel set S_0 is a T-measure.

To note that the concept has suitable invariance properties, let m be a T-measure which lives on the spectrum σ of the operator T. (We shall see later that every T-measure has this property.) Let f be analytic on an open set about σ, and let C be a finite collection of simple closed rectifiable curves lying in the domain of f and surrounding σ. Then the operator $f(T)$ is defined by $f(T)=(1/2\pi i)\int_C f(\lambda)d\lambda/(\lambda-T)$ (see Hille [10]). We omit the easy proof that the vector-valued measure $\hat{m}$ defined by $\hat{m}(S)=m(f^{-1}(S))$ is an $f(T)$-measure.

For a Hilbert space B it will at times be necessary to state theorems about T-measures in terms of the Hilbert space adjoint T^0 and the inner product in B rather than in terms of T^* and the bilinear product $\langle x, u\rangle$. There exists a unique 1-1 norm-preserving anti-linear map τ from B onto B^* such that $(x, y)=\langle x, \tau y\rangle$ for all x and y in B. If T^0 denotes the Hilbert space adjoint of T, then $T^0=\tau^{-1}T^*\tau$.

If μ is a vector-valued measure with values in B^* let μ^0 be the vector-valued measure with values in B defined by $\mu^0(S)=\tau^{-1}\mu(S^*)$. Then if μ is a T^*-measure, μ^0 is a T^0-measure. Conversely, if μ^0 is a T^0-measure then μ is a T^*-measure.

We now prove some of the more fundamental properties of T-measures.

LEMMA 2.1. *Let S_1 and S_2 be compact disjoint subsets of the complex plane. Let T be in $\mathfrak{T}$, let x_λ be an analytic function from S_1' to B such that $x_\lambda\in\mathfrak{D}(T)$ for all λ in S_1' and $(T-\lambda)x_\lambda=x$, a constant. Similarly let u_λ be an analytic function from S_2' to B^* such that $u_\lambda\in\mathfrak{D}(T^*)$ for λ in S_2' and $(T^*-\lambda)u_\lambda=u$, a constant. Then if $x_\lambda\to 0$ as $|\lambda|\to\infty$ it follows that $\langle x_\lambda, u\rangle=0$.*

Proof. Define the function $f(\lambda)$ to be $\langle x_\lambda, u\rangle$ for λ in S_1' and $\langle x, u_\lambda\rangle$ for λ in S_2'. This definition is consistent since for λ in $S_1'\cap S_2'$ we have

$$\langle x, u_\lambda\rangle = \langle (T-\lambda)x_\lambda, u_\lambda\rangle = \langle x_\lambda, (T^*-\lambda)u_\lambda\rangle = \langle x_\lambda, u\rangle.$$

Since x_λ is analytic in S_1' and u_λ in S_2' it follows that $f(\lambda)$ is everywhere analytic and therefore a constant. But $f(\lambda)\to 0$ as $\lambda\to\infty$ since $x_\lambda\to 0$ as $\lambda\to\infty$, so that $f(\lambda)=0$ identically.

COROLLARY. *If in addition to the hypotheses of the theorem T is bounded, then $\langle x, u\rangle=0$.*

Proof. If T is bounded, then $x_\lambda=(T-\lambda)^{-1}x$ for $|\lambda|>\|T\|$ and $\lim_{|\lambda|\to\infty} -\lambda x_\lambda =x$. Thus $\langle x, u\rangle=\lim_{|\lambda|\to\infty} -\lambda\langle x_\lambda, u\rangle=0$.

THEOREM 2.1. *Let T be in $\mathfrak{T}$ and let m and μ be T and T^*-measures respectively which live on Borel sets S_1 and S_2 respectively with $S_1\cap S_2=0$. Then $m\perp\mu$, i.e., $\langle m(U), \mu(V)\rangle=0$ for all Borel sets U and V.*

Proof. Replacing U by $U\cap S_1$ and V by $V\cap S_2$ if necessary, we may assume that $U\subset S_1$ and $V\subset S_2$. As a function of the Borel set S, $\langle m(U), \mu(S\cap V)\rangle$ is a numerical measure, so to show that $\langle m(U), \mu(V)\rangle=0$ it is enough to prove that $\langle m(U), \mu(\tilde{V})\rangle=0$ for all compact subsets $\tilde{V}$ of V. Thus we may assume that V is compact. Similarly we may assume that U is compact. Define

$x = m(U)$, $u = \mu(V)$, $x_\lambda = \int_U (1/(z-\lambda))dm(z)$ for λ in U', and $u_\lambda = \int_V (1/(z-\lambda))d\mu(z)$ for λ in V'. Then x_λ is analytic on U' and $(T-\lambda)x_\lambda = x$. Also u_λ is analytic on V' and $(T^*-\lambda)u_\lambda = u$. Since x_λ vanishes at infinity, Lemma 2.1 states that $\langle x_\lambda, u\rangle = 0$. Since $\lim_{|\lambda|\to\infty} -\lambda x_\lambda = m(U) = x$, this gives $\langle x, u\rangle = 0$ or $\langle m(U), \mu(V)\rangle = 0$, as was to be proved.

To see the significance of this theorem, let m be a T-measure concentrated at the point z_1, so that $m(\{z_1\})$ is a characteristic vector and $Tm(\{z_1\}) = z_1 m(\{z_1\})$. Also let μ be a T^*-measure concentrated at the point $z_2 \neq z_1$, so that $T^*\mu(\{z_2\}) = z_2\mu(\{z_2\})$. Then the theorem says that $\langle m(\{z_1\}), \mu(\{z_2\})\rangle = 0$, i.e., that characteristic vectors of T are orthogonal to characteristic vectors of T^* which belong to different characteristic values. This is a well-known and trivial result.

Corollary 1. *If m is a T-measure and μ a T^*-measure, then $\langle m(S_1), \mu(S_2)\rangle = 0$ for disjoint Borel sets S_1 and S_2.*

Proof. Let $\tilde{m}$ be the T-measure defined by $\tilde{m}(S) = m(S \cap S_1)$, so that $\tilde{m}$ lives on S_1. Let $\tilde{\mu}$ similarly be defined by $\tilde{\mu}(S) = \mu(S \cap S_2)$, so that $\tilde{\mu}$ lives on S_2. By Theorem 2.1, $\tilde{m} \perp \tilde{\mu}$, so that $\langle m(S_1), \mu(S_2)\rangle = \langle \tilde{m}(S_1), \tilde{\mu}(S_2)\rangle = 0$, as was to be proved.

Corollary 2. *If m is any T-measure and μ is a T^*-measure for which $\mu(X) = 0$ (X is the entire complex plane), then $m \perp \mu$, i.e., $\langle m(U), \mu(V)\rangle = 0$ for all Borel sets U and V.*

Proof. We have

$$\begin{aligned}\langle m(U), \mu(V)\rangle &= \langle m(U \cap V), \mu(V)\rangle + \langle m(U \cap V'), \mu(V)\rangle \\ &= -\langle m(U \cap V), \mu(V')\rangle + \langle m(U \cap V'), \mu(V)\rangle \\ &= 0\end{aligned}$$

by Corollary 1.

Corollary 3. *If the values of T^*-measures are dense in B^*, then each x in B has at most one T-measure.*

Proof. If x had the two T-measures m_1 and m_2, then $m = m_1 - m_2$ would be a nontrivial T-measure with $m(X) = 0$. By the previous corollary (with the roles of m and μ interchanged), for every Borel set S the vector $m(S)$ is orthogonal to the values of all T^*-measures. Thus $m(S) = 0$, which contradicts the fact that m is nontrivial.

Corollary 4. *Let f be a Borel function which is integrable with respect to the T-measure m and the T^*-measure μ. Then for each Borel set S,*

$$\left\langle \int f(z)dm(z), \mu(S) \right\rangle = \left\langle m(S), \int f(z)d\mu(z) \right\rangle.$$

Proof. Let the measure $\bar{m}$ be defined by $\bar{m}(C)=\langle m(C),\, \mu(S)\rangle$ and let the measure $\bar{\mu}$ be defined by $\bar{\mu}(C)=\langle m(S),\, \mu(C)\rangle$. Then for each Borel set C we have

$$\begin{aligned} \bar{m}(C) &= \langle m(C),\, \mu(S)\rangle = \langle m(C\cap S) + m(C\cap S'),\, \mu(S)\rangle \\ &= \langle m(C\cap S),\, \mu(S)\rangle \qquad \text{by Corollary 1} \\ &= \langle m(C\cap S),\, \mu(C\cap S)\rangle = \langle m(S),\, \mu(C\cap S)\rangle \\ &= \langle m(S),\, \mu(C)\rangle = \bar{\mu}(C), \end{aligned}$$

so that $\bar{m}=\bar{\mu}$. Thus

$$\left\langle \int f(z)dm(z),\, \mu(S)\right\rangle = \int f(z)d\bar{m}(z) = \int f(z)d\bar{\mu}(z)$$
$$= \left\langle m(S),\ \int f(z)d\mu(z)\right\rangle.$$

COROLLARY 5. *Let T be the operator on the Hilbert space B which acts on the complete orthonormal set $\{x_i\}_{i=1}^{\infty}$ by $Tx_i=x_{i+1}$. Then there are no nontrivial T-measures.*

Proof. In the dual space B^* there exists a complete orthonormal set $\{u_i\}_{i=1}^{\infty}$ with $\langle x_i,\, u_j\rangle=\delta_{ij}$. Also $T^*u_i=u_{i-1}$ for $i>1$ and $T^*u_1=0$. Thus for each α with $|\alpha|<1$ the vector $v_\alpha=\sum_{i=1}^{\infty}\alpha^i u_i$ is a characteristic vector of T^* corresponding to the characteristic value α. This means that the measure consisting of a point mass v_α concentrated at the point α is a T^*-measure. Thus if $S\subset\{\lambda: |\lambda|\geqq 1/2\}$ and if $|\alpha|<1/2$, Corollary 1 implies that $\langle m(S), v_\alpha\rangle=0$ for each T-measure m. Now it is easy to see that the vectors v_α with $|\alpha|<1/2$ span B^*. Thus $m(S)=0$. Similarly it can be shown that $m(S)=0$ if $S\subset\{\lambda: |\lambda|<1/2\}$. But every Borel set S can be written in the form $S=S_1\cup S_2$ with $S_1\subset\{\lambda: |\lambda|\geqq 1/2\}$ and $S_2\subset\{\lambda: |\lambda|<1/2\}$. Thus $m(S)=m(S_1)+m(S_2)=0$. Hence $m(S)=0$. Therefore m is trivial, as was to be proved.

It is easy to see that in case B is a Hilbert space Theorem 2.1 can be stated as follows: Let T be in $\mathfrak{T}$ and let m and μ be T- and T^0-measures respectively, which live on Borel sets S_1 and S_2 respectively with $S_1\cap S_2^*=0$. Then $m\perp\mu$, i.e., $(m(U),\, \mu(V))=0$ for all Borel sets U and V.

THEOREM 2.2. *If S_1 and S_2 are compact disjoint sets such that each component of $S_1'\cap S_2'$ contains a point of the resolvent set of T, then for any two T-measures m_1 and m_2 it is true that $m_1(S_1)\neq m_2(S_2)$ unless $m_1(S_1)=m_2(S_2)=0$.*

Proof. Suppose $m_1(S_1)=m_2(S_2)=x$. Define

$$x_\lambda = \int_{S_1}\frac{1}{z-\lambda}\,dm_1(z) \qquad \text{for } \lambda \text{ in } S_1',$$

$$x_\lambda = \int_{S_2} \frac{1}{z-\lambda}\, dm_2(z) \qquad \text{for } \lambda \text{ in } S_2'.$$

To see that this definition is consistent for λ in $S_1' \cap S_2'$, note that it is consistent for λ in the resolvent set since in that case there is only one vector x_λ with $(T-\lambda)x_\lambda = x$. Then note that the definition must be consistent on each component of $S_1' \cap S_2'$ because each component intersects the resolvent set in an open region. Thus x_λ is everywhere analytic and $x_\lambda \to 0$ as $\lambda \to \infty$, so that x_λ is identically zero. This gives $x = (T-\lambda)x_\lambda = 0$, as desired.

THEOREM 2.3. *Every T-measure m lives on $\sigma(T)$.*

Proof. Let R_λ be the resolvent of T and K the resolvent set. Let S be any compact subset of K, and write $x = m(S)$. Then $x_\lambda = \int_S (1/(z-\lambda))dm(z)$ is analytic in S' and $(T-\lambda)x_\lambda = x$. On the other hand $R_\lambda x$ is analytic in K and $(T-\lambda)R_\lambda x = x$. Moreover, $R_\lambda x = x_\lambda$ for λ in $K \cap S'$. Thus x_λ can be extended to be analytic in the whole complex plane. Thus $x_\lambda = 0$ since $x_\lambda \to 0$ as $\lambda \to \infty$. Therefore $x = (T-\lambda)x_\lambda = 0$. Thus the measure m is zero on all compact subsets of K. It follows that the measure m_u defined by $m_u(S) = \langle m(S), u\rangle$ is zero on compact subsets of K, for all u in B^*. By regularity m_u is zero on all subsets of K. Hence m is zero on all subsets of K. Thus m lives on $\sigma(T)$, as was to be proved.

It is natural to define the basic sets $O(T_0, M, \epsilon)$ of a topology on $\mathfrak{T}$ as follows: let T_0 be any member of $\mathfrak{T}$, M any finite-dimensional subspace of $\mathfrak{D}(T_0)$, and ϵ any positive number; write $O(T_0, M, \epsilon) = \{T: M \subset \mathfrak{D}(T), \|(T-T_0)x\| < \epsilon\|x\| \text{ for all } x \text{ in } M\}$. It is easy to see that the conditions that these sets be the basic open sets of some topology are satisfied. The resulting topology is called the strong topology because it agrees with the strong topology on the set of operators. Convergence of a directed set $\{T_\alpha\}$ to T means convergence of $\{T_\alpha x\}$ to Tx for each x in $\mathfrak{D}(T)$. The strong topology is not well behaved because for T_1 and T_2 in $\mathfrak{T}$ with $T_1 \subset T_2$ every neighborhood of T_1 will also be a neighborhood of T_2.

The next theorem might be called the continuity property of T-measures, that is, continuity as a function of T. It says that under certain conditions a T-measure can be obtained as a limit of T_α-measures, if $\{T_\alpha\}$ is a directed set such that $T_\alpha^* \to T^*$ strongly. It will be the principal tool in the investigation of §3.

THEOREM 2.4. *Let T be in $\mathfrak{T}$ and let $\{T_\alpha\}$ be a directed set of transformations in $\mathfrak{T}$ such that $T_\alpha^* \to T^*$ in the strong topology of $\mathfrak{T}^*$. Let m_α be a T_α-measure with $\|m_\alpha\| \leqq c$ for all α. Let m in Q be a cluster point of the directed set $\{m_\alpha\}$ in the weak operator topology of Q (such cluster points exist because those measures m in Q with $\|m\| \leqq c$ are compact in the weak operator topology). Then m is a T-measure.*

Proof. It will be enough to show that for each f in $\mathfrak{M}(X)$ with compact support we have $\int f(z)dm(z) \in \mathfrak{D}(T)$ and $T\int f(z)dm(z) = \int zf(z)dm(z)$, or equivalently that

$$\left\langle \int f(z)dm(z),\ T^*u \right\rangle = \left\langle \int zf(z)dm(z), u \right\rangle$$

for all u in $\mathfrak{D}(T^*)$. Since $T_\alpha^* \to T^*$, there exists for each $\epsilon > 0$ an index α_0 such that $u \in \mathfrak{D}(T_\alpha^*)$ and $\|T^*u - T_\alpha^* u\| < \epsilon$ for all $\alpha > \alpha_0$. Then $\langle \int f(z)dm_\alpha(z),\ T_\alpha^* u\rangle = \langle \int zf(z)dm_\alpha(z),\ u\rangle$ for $\alpha > \alpha_0$ because m_α is a T_α-measure. Since m is a cluster point of $\{m_\alpha\}$ in the weak operator topology, we can choose a particular $\alpha > \alpha_0$ such that $|\langle \int f(z)dm_\alpha(z),\ T^*u\rangle - \langle \int f(z)dm(z),\ T^*u\rangle| < \epsilon$ and $|\langle \int zf(z)dm_\alpha(z),\ u\rangle - \langle \int zf(z)dm(z),\ u\rangle| < \epsilon$. It follows that

$$\begin{aligned}
&\left|\left\langle \int f(z)dm(z),\ T^*u \right\rangle - \left\langle \int zf(z)dm(z),\ u \right\rangle\right| \\
&\qquad \leqq \left|\left\langle \int f(z)dm(z),\ T^*u \right\rangle - \left\langle \int f(z)dm_\alpha(z),\ T^*u \right\rangle\right| \\
&\qquad\quad + \left|\left\langle \int f(z)dm_\alpha(z),\ T^*u \right\rangle - \left\langle \int f(z)dm_\alpha(z),\ T_\alpha^* u \right\rangle\right| \\
&\qquad\quad + \left|\left\langle \int zf(z)dm_\alpha(z),\ u \right\rangle - \left\langle \int zf(z)dm(z),\ u \right\rangle\right| \\
&\qquad \leqq \epsilon + \epsilon\left\| \int f(z)dm_\alpha(z) \right\| + \epsilon \\
&\qquad \leqq 2\epsilon + \epsilon\|m_\alpha\|\,\|f\| \\
&\qquad = 2\epsilon + \epsilon c\|f\|.
\end{aligned}$$

Since ϵ is arbitrary, this gives $\langle \int f(z)dm(z),\ T^*u\rangle = \langle \int zf(z)dm(z),\ u\rangle$, as was to be proved.

COROLLARY. *The set of T-measures m with $\|m\| \leqq 1$ is closed in the weak operator topology of Q.*

Proof. Call the set in question C. If m is in $\overline{C}$ then there exists a directed set $\{m_\alpha\}$ of elements in C converging to m. Define a directed set $\{T_\alpha\}$ of elements of $\mathfrak{T}$ by $T_\alpha = T$, so that $\{T_\alpha^*\}$ converges to T^* and m_α is a T_α-measure. By the theorem, m is a T-measure, as was to be proved.

To make Theorem 2.4 useful there must be some way of showing that the T-measure m is nontrivial. An example of how this can be done may be seen in Theorem 3.2.

3. **Application to scalar transformations.** Consider a function $E(\ )$ on the Borel sets of the complex plane whose values are operators on the reflexive Banach space B, such that for every disjoint sequence $\{S_i\}$ of Borel sets we

have $E(\bigcup_{i=1}^{\infty} S_i) = \sum_{i=1}^{\infty} E(S_i)$ in the strong topology, and such that $E(X) = I$, and such that $E(S_1 \cap S_2) = E(S_1)E(S_2)$ for all S_1 and S_2. This function is called a spectral measure. It is not hard to show (see Dunford [5]) that the norms $\|E(S)\|$ are uniformly bounded. The reason we use $E(\ )$ instead of the more usual symbol E for a spectral measure will become apparent.

It follows directly from the definition that for each x in B the set function $E(\ )x$ defined by $(E(\ )x)(S) = E(S)x$ is a vector-valued measure. We define a transformation $T = \int z dE(z)$, which has for its domain the set of all x in B for which z is integrable with respect to $E(\ )x$, by setting $Tx = \int z dE(z)x$. It is obvious that T is a linear transformation. Dunford calls T a scalar-type transformation, which we abbreviate to scalar transformation. The domain of T is dense because $E(S)x \in \mathfrak{D}(T)$ for each bounded Borel set S and each x in B. Since the set function $E^*(\ )$, defined by $E^*(S) = (E(S))^*$ for each Borel set S, is also a spectral measure, we may analogously define a transformation $U = \int z dE^*(z)$. It is also linear and has dense domain. For x in $\mathfrak{D}(T)$ and u in $\mathfrak{D}(U)$ we have

$$\begin{aligned} \langle Tx, u \rangle &= \left\langle \int z dE(z)x, u \right\rangle = \int z d\langle E(z)x, u \rangle \\ &= \int z d\langle x, E^*(z)u \rangle \\ &= \left\langle x, \int z dE^*(z)u \right\rangle = \langle x, Uu \rangle, \end{aligned}$$

so that T has an adjoint and $U \subset T^*$. For x in $\mathfrak{D}(T)$ and u in $\mathfrak{D}(T^*)$ it is true that $\langle \int z dE(z)x, u \rangle = \langle x, T^*u \rangle$. In particular, for every bounded S, every u in $\mathfrak{D}(T^*)$, and every x in B we have $\langle x, \int_S z dE^*(z)u \rangle = \langle \int_S z dE(z)x, u \rangle = \langle \int z dE(z)E(S)x, u \rangle = \langle E(S)x, T^*u \rangle = \langle x, E^*(S)T^*u \rangle$, so that $\int_S z dE^*(z)u = E^*(S)T^*u$. This shows that z is integrable with respect to $E^*(\ )u$ and that $\int z dE^*(z)u = T^*u$. Thus $u \in \mathfrak{D}(U)$ and $Uu = T^*u$. It follows that $U = T^*$. Similarly, $T = U^*$. Hence T and U are closed. Thus T and U are in $\mathfrak{T}$ and $\mathfrak{T}^*$ respectively and are the adjoints of each other.

In case B is a Hilbert space, it is a theorem of Mackey and Lorch that every scalar transformation T is similar to a normal transformation, i.e., there exists an invertible operator A such that $A^{-1}TA$ is normal. A good place to read about scalar transformations is in Dunford [5].

THEOREM 3.1. *A transformation T in $\mathfrak{T}$ is a scalar transformation if and only if each x in B has a T-measure and each u in B^* has a T^*-measure. If T is a scalar transformation, then each x in B has a unique T-measure. If each x in B has a unique T-measure, then T is an extension of a scalar transformation, and if T is bounded it is a scalar operator.*

Proof. If $T=\int zdE(z)$ is a scalar transformation and if x is an arbitrary vector in B, then for each bounded Borel set S the function z is integrable with respect to $E(\ )(E(S)x)$ and the integral is $\int_S zdE(z)x$. Thus $E(S)x\in\mathfrak{D}(T)$ and $TE(S)x=\int_S zdE(z)x$. It follows that x has the T-measure $E(\ )x$. Similarly every u in B^* has the T^*-measure $E^*(\ )u$.

If every x in B has a T-measure and every u in B^* has a T^*-measure, then by Corollary 3 to Theorem 2.1 every x in B has a unique T-measure.

If every x in B has a unique T-measure m_x, then define the transformation W from B to Q by $Wx=m_x$. Then since $\lambda m_x+\mu m_y$ is a T-measure for $\lambda x+\mu y$ it follows that $m_{\lambda x+\mu y}=\lambda m_x+\mu m_y$. Hence W is a linear transformation. If $x_n\to x$ in B and $Wx_n=m_{x_n}\to m$ in Q, then m is a T-measure by the corollary to Theorem 2.4. Since $m_{x_n}(X)\to m(X)$ we have $m(X)=x$. Hence $Wx=m$. Thus W is closed. By the closed graph theorem W is bounded. For all x in B we have $\|m_x\|=\|Wx\|\leqq\|W\|\,\|x\|$.

For each x in B and each Borel set S define $E(S)x=m_x(S)=(Wx)(S)$. $E(S)$ is an operator because W is an operator. It follows immediately that $E(\bigcup_{i=1}^\infty S_i)=\sum_{i=1}^\infty E(S_i)$ in the strong topology for every disjoint sequence $\{S_i\}$ of Borel sets. Finally note that the vector $E(S)x$ has a T-measure which is the restriction of m_x to S. Hence $E(S)E(S)x=m_{E(S)x}(S)=m_{E(S)x}(X)$ $=E(S)x$. Thus $E(S)$ is an idempotent operator. The set function $E(\ )$ is therefore a spectral measure. If $x\in B$ and S is a bounded Borel set, then $E(S)x\in\mathfrak{D}(T)$ and $TE(S)x=Tm_x(S)=\int_S zdm_x(z)=\int_S zdE(z)x$. It follows from the closure of T that Tx exists and $Tx=\int zdE(z)x$ whenever $x\in\mathfrak{D}(\int zdE(z))$, i.e., whenever $\int zdE(z)x$ exists. Thus $\int zdE(z)\subset T$. Therefore T is an extension of a scalar transformation.

Now if each x in B has a T-measure and each u in B^* a T^*-measure, then by what we have just proved there is some scalar transformation $\int zdE(z)$ of which T is an extension. For the same reasons there is some scalar transformation $\int zdF(z)$ of which T^* is an extension. Therefore $\int zdF(z)\subset T^*$ $\subset\int zdE^*(z)$. Thus for each u in B^* the function $F(\ )u$ is a $\int zdE^*(z)$-measure. Since u has the unique $\int zdE^*(z)$-measure $E^*(\ )u$, it follows that $F(\ )u$ $=E^*(\ )u$. Hence $F(\ )=E^*(\ )$. Thus $\int zdF(z)=T^*=\int zdE^*(z)$, so that T $=\int zdE(z)$ is a scalar transformation. This proves the first statement of the theorem. The second statement has already been proved.

It has been shown that T is the extension of a scalar transformation if every x in B has a unique T-measure. If T is bounded, the scalar transformation is also bounded, and so must be equal to T. This completes the proof of the theorem.

If $T=\int zdE(z)$ is a scalar transformation, define

$$|||T||| = \sup_{x\in B}(\|E(\ )x\|/\|x\|).$$

Then

$$|||T||| = \sup\left|\left\langle \sum_i \lambda_i E(S_i)x, u\right\rangle\right| = \sup\left|\left\langle x, \sum_i \lambda_i E(S_i)^* u\right\rangle\right|,$$

where the sup is taken over all x and u with $\|x\| = \|u\| = 1$ and over all finite disjoint sequences $\{S_i\}$ of Borel sets and corresponding finite sequences $\{\lambda_i\}$ with $|\lambda_i| \leqq 1$. By the uniform boundedness theorem (see Banach [3]), $|||T|||$ is finite if the above sup is finite for each fixed x. This is the case because for fixed x the sup is equal to $\|E(\ )x\|$. Let G_c be the set of all scalar transformations T with $|||T||| \leqq c$. It is easy to see that then $|||T^*||| \leqq c$, so that G_c^*, the adjoint of the set G_c, consists of all scalar transformations U on B^* with $|||U||| \leqq c$. It will be seen later that the following theorem about G_c is generalization of the spectral theorem.

THEOREM 3.2. *Let T be in the closure $\overline{G}_c$ of G_c in the strong topology. Then there exists a function $E(\ )$ from the Borel sets of X to operators on B such that*

(1) *for each u in B^*, $E^*(\ )u$ is a T^*-measure for u and $\|E^*(\ )u\| \leqq c\|u\|$;*

(2) *if $x \in \mathfrak{D}(T)$, then $Tx = \int z dE(z)x$;*

(3) *if T is in the closure of the subset of G_c consisting of those transformations whose spectrum is in a given closed set C, then $E(S) = 0$ if S is disjoint from C;*

(4) *if B is a Hilbert space and $c = 1$, then $E(S)$ is a positive Hermitian operator and $\|Tx\| = \int |z|^2 d(E(z)x, x)$ for all x in $\mathfrak{D}(T)$.*

Proof. Since $T \in \overline{G}_c$, there exists a directed set $\{T_\alpha\}$ from G_c converging to T, and under the assumption of (3) we may choose the transformations T_α to have their spectra in C. Let $T_\alpha = \int z dE_\alpha(z)$ so that $T_\alpha^* = \int z dE_\alpha^*(z)$. For each index α and each u in B^* consider the T_α^*-measure $E_\alpha^*(\ )u$, which depends linearly on u. The inequality $|||T_\alpha^*||| \leqq c$ implies $\|E_\alpha^*(\ )u\| \leqq c\|u\|$. Let D_u be the set of all m in Q with $\|m\| \leqq c\|u\|$. Then D_u is compact in the weak operator topology. Hence the Cartesian product space $D = \prod_{u \in B^*} D_u$ is also compact. For each index α, $E_\alpha^*(\ )$ can be identified with the point of D whose coordinate in D_u is $E_\alpha^*(\ )u$.

Since D is compact the directed set $\{E_\alpha^*(\ )\}$ has a cluster point in D which we call $E^*(\ )$. The coordinate of $E^*(\ )$ in D_u will be written $E^*(\ )u$. The value of $E^*(\ )u$ on the Borel set S will be written $E^*(S)u$. The transformation on B which takes u into $E^*(S)u$ will be written $E^*(S)$.

By Theorem 2.4, $E^*(\ )u$ is a T^*-measure for each u in B^*. To prove the equality $E^*(\ )(\lambda u + \mu v) = \lambda E^*(\ )u + \mu E^*(\ )v$, note that we can simultaneously approximate $E^*(\ )(\lambda u + \mu v)$ by $E_\alpha^*(\ )(\lambda u + \mu v)$, $E^*(\ )u$ by $E_\alpha^*(\ )u$, and $E^*(\ )v$ by $E_\alpha^*(\ )v$ in the weak operator topology because $E^*(\ )$ is a cluster point of the directed set $E_\alpha^*(\ )$. Hence the equality in question is a consequence of the equalities $E_\alpha^*(\ )(\lambda u + \mu v) = \lambda E_\alpha^*(\ )u + \mu E_\alpha^*(\ )v$.

Now $E^*(\ )$ we may consider to be the function on the Borel sets of the complex plane whose value at S is $E^*(S)$. For each Borel set S, $E^*(S)$ is a linear operator on B^* because $E^*(\ )u$ is a linear function of u and $\|E^*(\ )u\| \leqq c\|u\|$. Under the assumption of (3) we have $\langle x, \int f(z)dE_\alpha^*(z)u\rangle = 0$ for each index α, each x in B, each u in B^*, and each f in $\mathfrak{M}(X)$ with compact support disjoint from C. Passing to the limit we obtain $\langle x, \int f(z)dE^*(z)u\rangle = 0$. Since this holds for all x and f the measure $E^*(\ )u$ lives on C. Thus $E^*(S) = 0$ if S is disjoint from C. Hence $E(S) = (E^*(S))^* = 0$ if S is disjoint from C.

To show that $E^*(\ )u$ is a T^*-measure for u, i.e., that $E^*(X)u = u$, assume otherwise. Choose x in $\mathfrak{D}(T)$ such that $\langle x, u - E^*(X)u\rangle \neq 0$. Define $S_r = \{z: |z| \leqq r\}$. Then

$$
\begin{aligned}
\|E_\alpha(\ )E_\alpha(S_r')x\| &= \sup_{\|f\|=1} \left\| \int f(z)dE_\alpha(z)E_\alpha(S_r')x \right\| \\
&= \sup_{\|f\|=1} \left\| \int_{S_r'} \frac{zf(z)}{z} dE_\alpha(z)x \right\| \\
&= \sup_{\|f\|=1} \left\| \int_{S_r'} \frac{f(z)}{z} dE_\alpha(z)T_\alpha x \right\| \\
&\leqq \sup_{\|f\|=1/r} \left\| \int f(z)dE_\alpha(z)T_\alpha x \right\| \\
&= \frac{1}{r} \|E_\alpha(\ )T_\alpha x\| \\
&\leqq \frac{c}{r} \|T_\alpha x\| \\
&\leqq \frac{c}{r} (\|Tx\| + 1)
\end{aligned}
$$

if α is sufficiently large, say $\alpha > \alpha_0$.

For each positive number r choose a function f_r in $\mathfrak{M}(X)$ with compact support which is 1 on S_r and for which $\|f_r\| = 1$. Since $E^*(\ )u$ is a vector-valued measure, the vector $\int f_r(z)dE^*(z)u - u$ converges to $E^*(X)u - u$ as $r \to \infty$. Therefore for each $\epsilon > 0$ a number $r_\epsilon > 1/\epsilon$ exists for which

$$\left| \left\langle x, \int f_{r_\epsilon}(z)dE^*(z)u - u \right\rangle - \langle x, E^*(X)u - u\rangle \right| < \epsilon.$$

Since $E_\alpha^*(\ )u$ has $E^*(\ )u$ as a cluster point in the weak operator topology, an index $\alpha_\epsilon > \alpha_0$ exists for each $\epsilon > 0$ for which

$$\left| \left\langle x, \int f_{r_\epsilon}(z)dE^*(z)u - u \right\rangle - \left\langle x, \int f_{r_\epsilon}(z)dE_{\alpha_\epsilon}^*(z)u - u \right\rangle \right| < \epsilon.$$

We have

$$
\begin{aligned}
|\langle x, E^*(X)u - u\rangle| &< \epsilon + \left|\left\langle x, \int f_{r_\epsilon}(z)dE^*(z)u - u\right\rangle\right| \\
&< 2\epsilon + \left|\left\langle x, \int f_{r_\epsilon}(z)dE^*_{\alpha_\epsilon}(z)u - u\right\rangle\right| \\
&= 2\epsilon + \left|\left\langle \int (f_{r_\epsilon}(z) - 1)dE_{\alpha_\epsilon}(z)x, u\right\rangle\right| \\
&= 2\epsilon + \left|\left\langle \int (f_{r_\epsilon}(z) - 1)dE_{\alpha_\epsilon}(z)E_{\alpha_\epsilon}(S'_{r_\epsilon})x, u\right\rangle\right| \\
&\qquad (\text{since } f_{r_\epsilon}(z) - 1 = 0 \text{ for } z \text{ in } S'_{r_\epsilon}) \\
&\leqq 2\epsilon + \|u\| \| f_{r_\epsilon}(z) - 1\| \|E_{\alpha_\epsilon}(\ )E_{\alpha_\epsilon}(S'_{r_\epsilon})x\| \\
&\leqq 2\epsilon + \frac{2c}{r_\epsilon}[\|Tx\| + 1]\|u\| \\
&\leqq 2\epsilon + \frac{2c}{1/\epsilon}[\|Tx\| + 1]\|u\| \\
&\leqq \epsilon[2 + 2c(\|Tx\| + 1)\|u\|].
\end{aligned}
$$

Since ϵ is arbitrary this gives the contradiction $\langle x, E^*(u)-u\rangle=0$. Thus $E^*(\ )u$ is a T^*-measure for u.

For x in $\mathfrak{D}(T)$, u in B^*, and each bounded Borel set S we have

$$
\begin{aligned}
\langle Tx, E^*(S)u\rangle = \langle x, T^*E^*(S)u\rangle &= \left\langle x, \int_S zdE^*(z)u\right\rangle \\
&= \int_S zd\langle x, E^*(z)u\rangle = \int_S zd\langle E(z)x, u\rangle.
\end{aligned}
$$

Since this holds for all bounded Borel sets S, $\int zd\langle E(z)x, u\rangle$ exists and equals $\langle Tx, u\rangle$. Since this holds for all u in B^* it follows that $\int zdE(z)x$ exists and equals Tx. We have thus proved (1), (2), and (3).

If B is a Hilbert space and $c=1$, then $E_\alpha(S)$ and $E^*_\alpha(S)$ are idempotents of norm 1 and therefore projections. Thus for each non-negative function f in $\mathfrak{M}(X)$ with compact support and each x in B we have

$$
\begin{aligned}
0 \leqq \left(\int f(z)dE_\alpha(z)x, x\right) &= \left\langle \int f(z)dE_\alpha(z)x, \tau x\right\rangle \\
&= \left\langle x, \int f(z)dE^*_\alpha(z)\tau x\right\rangle.
\end{aligned}
$$

Passing to the limit this gives

$$\left(\int f(z)dE(z)x, x\right) = \left\langle x, \int f(z)dE^*(z)\tau x\right\rangle \geqq 0.$$

It follows that $(E(S)x, x) \geqq 0$ for all Borel sets S. Hence $E(S)$ is a positive Hermitian operator. Since $E^*(\ )\tau x$ is a T^*-measure, the measure m_x defined by

$$m_x(S) = \tau^{-1}E^*(S^*)\tau x = \tau^{-1}(\tau E^0(S^*)\tau^{-1})\tau x = E^0(S^*)x = E(S^*)x$$

is a T^0-measure. Therefore

$$\begin{aligned} T^0 \int_S f(z)dE(z)x &= T^0 \int_{S^*} f(z^*)dE(z^*)x \\ &= \int_{S^*} zf(z^*)dE(z^*)x \\ &= \int_S z^*f(z)dE(z)x \end{aligned}$$

for each bounded Borel set S and each bounded measurable function f. Thus for each x in $\mathfrak{D}(T)$ we have

$$\begin{aligned} (Tx, Tx) &= \left(\int zdE(z)x, Tx\right) = \lim_{r\to\infty}\left(\int_{|z|<r} zdE(z)x, Tx\right) \\ &= \lim_{r\to\infty}\left(T^0\int_{|z|<r} zdE(z)x, x\right) \\ &= \lim_{r\to\infty}\left(\int_{|z|<r} |z|^2dE(z)x, x\right) \\ &= \int |z|^2d(E(z)x, x). \end{aligned}$$

This completes the proof of the theorem.

The double strong topology on $\mathfrak{T}$ is defined after von Neumann by taking the open sets in the strong topology of $\mathfrak{T}$ and the adjoints of the open sets in the strong topology of $\mathfrak{T}^*$ and letting these form a sub-basis.

COROLLARY 1. *The set G_c is closed in the double strong topology.*

Proof. If T is in the closure of G_c in the double strong topology, then by part (1) of the theorem each u in B^* has a T^*-measure, since T is in the closure of G_c in the strong topology. Similarly each x in B has a T-measure. By Theorem 3.1 it follows that T is a scalar transformation. By part (1) of Theorem 3.2 it follows that $|||T^*||| \leqq c$, or $|||T||| \leqq c$. Thus $T \in G_c$ as was to be proved.

COROLLARY 2. *If T is a symmetric transformation on the Hilbert space B (that is, $T \in \mathfrak{T}$ and $T \subset T^0$), then there exists a set function $E(\ )$ (called a generalized spectral measure) from the Borel sets of the real line to the positive Hermitian operators on B such that*

(1) *for each x in B, $E(\ )x$ is a T^0-measure for x,*

(2) *if $x \in \mathfrak{D}(T)$, then $Tx = \int z dE(z)x$ and $\|Tx\|^2 = \int |z|^2 d(E(z)x, x)$.*

Proof. For any finite-dimensional subspace $M \subset \mathfrak{D}(T)$ let P be the projection whose range is M. Then PTP is a Hermitian operator whose range is finite-dimensional. By the theory of finite-dimensional matrices this implies that $PTP \in G_1$ (see [15]). It is clear that the operators PTP form a directed set if we let P_1TP_1 precede P_2TP_2 whenever $P_1P_2 = P_1$, and that this directed set converges to T strongly. Thus $T \in \overline{G}_1$. Hence there exists a set function $E(\ )$ with the properties described in Theorem 3.2. Since for each M the spectrum of PTP is included in the real line, $E(\ )$ lives on the real line by (3) of Theorem 3.2. Since B is a Hilbert space and $c = 1$, part (4) of Theorem 3.2 is true. It remains only to prove that $E(\ )x$ is a T^0-measure. This follows from the fact that $E^*(\ )\tau x$ is a T^*-measure, since the measure μ^0 defined by $\mu^0(S) = \tau^{-1}E^*(S^*)\tau x$ is therefore a T^0-measure and

$$\mu^0(S) = \tau^{-1}E^*(S^*)\tau x = \tau^{-1}(\tau E^0(S^*)\tau^{-1})\tau x = E^0(S^*)x = E(S^*)x = E(S)x,$$

since $E(S) = E(S^*)$ (because $E(\ )$ lives on the real line).

COROLLARY 3. *If H is a self-adjoint transformation on the Hilbert space B, then H is a scalar transformation, $H = \int z dE(z)$, where the operators $E(S)$ are projections and $E(\ )$ lives on the real line.*

Proof. By Corollary 2 there exists a generalized spectral measure $E(\ )$ which lives on the real line such that $Hx = \int z dE(z)x$ for x in $\mathfrak{D}(H)$ and such that $E(\ )x$ is a $H^0 = H$-measure for each x in B. Thus H is a scalar transformation since every x in B has an H-measure as well as an H^0-measure (which implies that every u in B^* has an H^*-measure). By the unicity of H-measures, $E(\ )$ must be the spectral measure for H, so that $H = \int z dE(z)$. Therefore $E(S)$ is an idempotent which is positive Hermitian, or a projection. This completes the proof.

Professor F. Wolf has called the attention of the author to an interesting result of Bade [2]. Bade considers a directed set $\{T_\alpha\}$ of operators in G_c such that $T_\alpha^* \to T^*$ strongly for some operator T on B, much as in Theorem 3.2. In addition he assumes that there is a closed set C of complex numbers such that the spectrum of each T_α lies in C and such that linear combinations of functions of the type $1/(z - z_0)$ with z_0 in C' can be used to approximate any continuous function on C. The conclusion is that $T \in G_c$. This result can be derived from Theorem 3.2, with the hypothesis that the spectrum of each T_α lies in C replaced by the weaker hypothesis that the spectrum of T lies in C. To see this, let T be such an operator in $\overline{G}_c$, so that (1) and (2) of Theorem 3.2 hold for some function $E(\ )$. The T^*-measure $E^*(\ )u$ lives on C because it lives on the spectrum of T^*. If μ is any other T^*-measure for u, then μ also lives on C, so that $E^*(\ )u-\mu$ is a T^*-measure for 0 which lives on C. For each λ in C' define $x_\lambda=\int_C d(E^*(\ )u-\mu)(z)/(z-\lambda)$, so that $(T-\lambda)x_\lambda=0$ for all λ in C'. Since λ is in the resolvent set of C this implies that $x_\lambda=0$ for all λ in C'. By the assumptions about C this means that $\int_C f(z)d(E^*(\ )u-\mu)(z)=0$ for every continuous function $f(z)$ on C. Therefore the measure $E^*(\ )u-\mu$ is zero. Thus u has the unique T^*-measure $E^*(\ )u$. It follows from Theorem 3.1 that T^* is a scalar operator.

The author knows of no way to show directly that $\overline{G}_1$ includes the set of normal transformations if B is a Hilbert space. Therefore Theorem 3.2 cannot be used to prove the spectral theorem for normal transformations directly, but instead the usual proofs based on Corollary 3 may be given. For the rest of the paper we assume that the spectral theorem for normal transformations is known.

Halmos [9] has defined an operator T on a Hilbert space B to be subnormal if there exists an extension $\tilde{B}$ of B and a normal operator $\tilde{T}$ on $\tilde{B}$ which is an extension of T. We define similarly a transformation T in $\mathfrak{T}$ to be subnormal if there exists an extension $\tilde{B}$ of B and a normal transformation $\tilde{T}$ on $\tilde{B}$ which is an extension of T. The following theorem gives a new characterization of the subnormal transformations.

THEOREM 3.3. *The transformation T is subnormal if and only if $T\in\overline{G}_1$. Since G_1 is the set of normal transformations, this means that the subnormal transformations are the strong closure of the set of normal transformations. The transformation T is subnormal if and only if there exists a generalized spectral measure $E(\ )$ such that*

(1) *for each u in B, $E^*(\ )u$ is a T^*-measure,*
(2) *$Tx=\int zdE(z)x$ for all x in $\mathfrak{D}(T)$,*
(3) *$\|Tx\|^2=\int|z|^2d(E(z)x,x)$ for all x in $\mathfrak{D}(T)$.*

Proof. We may assume that B is infinite-dimensional since otherwise the various transformations mentioned in the theorem can easily be shown to be normal. Let T be subnormal, so that $\tilde{B}$ and $\tilde{T}$ exist. We may assume that $\tilde{B}$ has the same dimension as B, since the least subspace of $\tilde{B}$ including B and invariant under T and T^* will have this property; and on this subspace T is normal. For any finite-dimensional subspace M of $\mathfrak{D}(T)$, we may therefore find a unitary map U of B onto $\tilde{B}$ which takes each vector of $M\cup T(M)$ onto itself. Thus $U^{-1}\tilde{T}U$ is a normal transformation on B, which agrees with T on M. The existence of such a transformation for arbitrary M means that $T\in\overline{G}_1$.

If $T\in\overline{G}_1$, conditions (1), (2), and (3) follow from Theorem 3.2.

If there is a generalized spectral measure $E(\ )$ satisfying (1), (2), and (3), then by a theorem of Neumark [13] there exists an extension $\tilde{B}$ of B and a spectral measure $\tilde{E}(\ )$ on $\tilde{B}$ such that $E(\ )=P\tilde{E}(\ )P$, where P is the projection whose range is B. Let $\tilde{T} = \int zd\tilde{E}(z)$. Then for x in $\mathfrak{D}(T)$, $\|Tx\|^2 =\int|z|^2d(E(z)x, x)=\int|z|^2 d(PE(z)Px, x)=\int|z|^2 d(\tilde{E}(z)x, x)$. Since the latter integral converges, $x\in\mathfrak{D}(\tilde{T})$, and so $\|Tx\|=\|\tilde{T}x\|$. Moreover

$$P\tilde{T}(x) = P\int zd\tilde{E}(z)x = \int zdP\tilde{E}(z)Px = Tx.$$

Since $\|Tx\|=\|\tilde{T}x\|$, we must have $\tilde{T}x=Tx$. Thus $\tilde{T}$ is a normal extension of T, so that T is a subnormal transformation. This completes the proof of the theorem. See [1] for this type of argument.

COROLLARY 1. *The subnormal operators are the closure of the normal operators in the strong operator topology.*

Proof. It follows from the theorem that the closure of the set of normal operators is included in the set of subnormal operators. To prove the converse, it is only necessary to repeat the first part of the proof of Theorem 3.3 for operators instead of transformations.

4. **Functional calculus.** A functional calculus for certain transformations T in $\mathfrak{T}$ will be developed. It is clear by induction that if m is a T-measure and S is a bounded Borel set then $m(S)\in\mathfrak{D}(T^n)$ for each positive integer n and $T^nm(S)=\int_S z^n dm(z)$. More generally if $p(T)$ is a polynomial in T, then under the same assumptions $m(S)\in\mathfrak{D}(p(T))$ and $p(T)m(S)=\int_S p(z)dm(z)$. This gives one hopes of establishing a functional calculus by defining, for a given Borel function f, $f(T)x=y$ to mean that $x=m(S)$ for some Borel set S over which f is integrable with respect to some T-measure m and that $y=\int_S f(z)dm(z)$.

There are two requirements which such a definition of $f(T)$ might not fulfill. First, there might not be a sufficient number of T-measures, in which case the domain of $f(T)$ would not be dense. Second, it might turn out that $f(T)$ so defined is multiple-valued.

Neither of these things can happen, as we shall see, for the following class Γ of transformations.

DEFINITION 4.1. The class Γ consists of those T in $\mathfrak{T}$ for which the set F of values of T-measures is dense in B and the set F^* of values of T^*-measures is dense in B^*.

LEMMA 4.1. *If $T\in\Gamma$, then each x in F has a unique T-measure m_x and each u in F^* has a unique T^*-measure μ_u. The sets F and F^* are linear, and m_x and μ_u are linear functions of x and u respectively.*

Proof. If $x\in F$ there exists a T-measure m and a Borel set S_0 such that $x=m(S_0)$. Defining m_1 by $m_1(S)=m(S\cap S_0)$, we see that x has the T-measure m_1. Since the values of T^*-measures are dense, there is no other T-measure for x by Corollary 3 of Theorem 2.1. If x has the T-measure m_x and y the T-measure m_y, then $\lambda_1x+\lambda_2y$ has the T-measure $\lambda_1m_x+\lambda_2m_y$; hence F is linear and m_x is a linear function of x. The statements about F^* are proved similarly.

We now consider any Borel measurable function f on the complex plane and any T in Γ. Let F_f be all those x in F for which f is integrable with respect to m_x. Define F_f^* similarly.

LEMMA 4.2. *The set F_f is linear and dense in B. The set F_f^* is linear and dense in B^*.*

Proof. If x and y are in F_f, then f is integrable with respect to m_x as well as m_y, and so is integrable with respect to $m_{\lambda_1 x+\lambda_2 y}=\lambda_1 m_x+\lambda_2 m_y$. Thus $\lambda_1 x+\lambda_2 y\in F_f$, so that F_f is linear. Write $U_n=\{z: |f(z)|<n\}$. Then for each x in F we have $x=m_x(X)=\lim_n m_x(U_n)$. Now $m_x(U_n)$ has the T-measure m_n defined by $m_n(S)=m_x(U_n\cap S)$, which lives on the set U_n on which $f(z)$ is bounded. Thus $m_x(U_n)\in F_f$, so that F is included in the closure of F_f. Since F is dense, this implies F_f is dense. The statements about F_f^* are proved similarly.

Now consider the transformation $f_0(T)$ from F_f to B defined by $f_0(T)x=\int f(z)dm_x(z)$, where the integral exists because $x\in F_f$. This transformation $f_0(T)$ is linear because m_x is a linear function of x. Similarly define $f_0(T^*)$.

LEMMA 4.3. *We have $f_0(T^*)\subset(f_0(T))^*$.*

Proof. For x in F_f and u in F_f^* we must prove $\langle f_0(T)x, u\rangle=\langle x, f_0(T^*)u\rangle$, or $\langle\int f(z)dm_x(z), u\rangle=\langle x, \int f(z)d\mu_u(z)\rangle$. The latter equation however is just Corollary 4 to Theorem 2.1, with $S=X$.

The lemma tells us that $f_0(T)$ has an adjoint with dense domain, since the domain F_f^* of $f_0(T^*)$ is dense. As is proved in [15], the transformation $f_0(T)$ therefore has a closure $(f_0(T))^{**}$.

DEFINITION 4.2. The transformation $f(T)$ is the closure of $f_0(T)$, i.e., is $(f_0(T))^{**}$.

We have now defined a map $f(z)\to f(T)$ from the set of Borel functions on X to the set $\mathfrak{T}$. Some of the usual properties of the functional calculus for self-adjoint transformations, such as are expounded in [15], [17], or [1], can be demonstrated.

THEOREM 4.1. *Let f and g be Borel functions and let T be in Γ. Then*

(1) *if $f(z)=z$, then $f(T)\subset T$;*

(2) $f(\lambda T)=\lambda f(T)$;

(3) $f(T^*)\subset(f(T))^*$;

(4) *if $x\in F_f$ and if the measure $\tilde{m}_x$ is defined by $\tilde{m}_x(S)=m_x(f^{-1}(S))$, then $\tilde{m}_x$ is a $f(T)$-measure for x;*

(5) $f(T)\in\Gamma$;

(6) *if $\{S_n\}_{n=1}^{\infty}$ is an increasing sequence of sets with $\bigcup_n S_n=X$, if C is the set of all x in F for which m_x lives on one of the sets S_n, and if $C\subset F_f$, then $f(T)$ is the closure of $f(T)|C$;*

(7) *if* $f(z)$ *is bounded, then* $F=F_f$;

(8) *if* $(f\circ g)(z)=f(g(z))$, *then* $f(g(T))=(f\circ g)(T)$;

(9) *there exists a dense linear set* $\mathfrak{L}\subset\mathfrak{D}(f(T))\cap\mathfrak{D}(g(T))\cap\mathfrak{D}((fg)(T))\cap\mathfrak{D}((f+g)(T))$ *for which the closures of the transformations* $f(T)|\mathfrak{L}$, $g(T)|\mathfrak{L}$, $(f+g)(T)|\mathfrak{L}$, and $(fg)(T)|\mathfrak{L}$ *are respectively* $f(T)$, $g(T)$, $(f+g)(T)$, *and* $(fg)(T)$, *and for which* $(f+g)(T)|\mathfrak{L}=f(T)|\mathfrak{L}+g(T)|\mathfrak{L}$ *and* $(fg)(T)|\mathfrak{L}=f(T)g(T)|\mathfrak{L}$;

(10) *if* $f_n(z)\rightarrow f(z)$ *as* $n\rightarrow\infty$ *for all* z, *then there exists a dense linear subset* $\mathfrak{N}$ *of* B *such that the closure of* $f(T)|\mathfrak{N}$ *is* $f(T)$ *and such that* $f_n(T)x\rightarrow f(T)x$ *as* $n\rightarrow\infty$ *for all* x *in* $\mathfrak{N}$.

Proof. If $f(z)=z$ and $x\in F_f=\mathfrak{D}(f_0(T))$, then $f_0(T)x=\int z dm_x(z)=Tx$, as was shown in §2. Thus $f_0(T)\subset T$, and since T is closed, $f(T)\subset T$, proving (1).

It is obvious that $f(\lambda T)=\lambda f(T)$.

Since $(f(T))^*=(f_0(T))^{***}=(f_0(T))^*$ and since $(f_0(T))^*\supset f_0(T^*)$, we have $(f(T))^*\supset f_0(T^*)$. Since $(f(T))^*$ is closed, this gives $(f(T))^*\supset f(T^*)$.

If $x\in F_f$ and if S is a bounded Borel set, then we saw in §1 that $\int_S z d\tilde{m}_x(z)=\int_{f^{-1}(S)}f(z)dm_x(z)$. Since $f(z)$ is bounded on $f^{-1}(S)$ it follows that $\tilde{m}_x(S)=m_x(f^{-1}(S))\in F_f$ and $f(T)\tilde{m}_x(S)=f(T)m_x(f^{-1}(S))=f_0(T)m_x(f^{-1}(S))=\int_{f^{-1}(S)}f(z)dm_x(z)=\int_S z d\tilde{m}_x(z)$. Thus $\tilde{m}_x$ is a $f(T)$-measure.

We have just seen that each x in F_f has a $f(T)$-measure. Similarly each u in F_f^* has a $f(T^*)$-measure, which is at the same time a $(f(T))^*$-measure since $f(T^*)\subset(f(T))^*$. Since F_f is dense in B and F_f^* is dense in B^*, this implies that $f(T)\in\Gamma$.

To prove (6) it is necessary only to show that $f_0(T)$ is included in the closure of $f(T)|C$, since the closure of $f_0(T)$ is $f(T)$. If $x\in F_f$ let $x_n=m_x(S_n)$. By hypothesis $x_n\in C$. We have $x_n\rightarrow x$ and $f_0(T)x_n=\int_{S_n}f(z)dm_x(z)\rightarrow\int f(z)dm_x(z)=f_0(T)x$ as $n\rightarrow\infty$ since $x\in F_f$. Thus $f_0(T)$ is included in the closure of $f(T)|C$, as was to be proved.

If $f(z)$ is bounded, then for each x in F the integral $\int f(z)dm_x(z)$ exists so that $F=F_f$.

To prove that $f(g(T))=(f\circ g)(T)$, we consider the sequence of sets $\{S_n\}$ defined by $S_n=g^{-1}\{z: |f(z)|\leqq n\}\cap\{z: |g(z)|\leqq n\}$, which satisfies the hypothesis of (6). Take any x in C, so that there exists an n for which m_x lives on S_n, and define $\tilde{m}_x$ by $\tilde{m}_x(S)=m_x(g^{-1}(S))$. Then $\tilde{m}_x$ is a $g(T)$-measure for x by (4). Moreover for any subset S of $\{z: |f(z)|>n\}$ we have $\tilde{m}_x(S)=m_x(g^{-1}(S))=0$, since m_x lives on S_n and $S_n\cap g^{-1}(S)$ is void. Thus $\tilde{m}_x$ lives on $\{z: |f(z)|\leqq n\}$. Hence $x\in\mathfrak{D}(f(g(T)))$ and $f(g(T))x=\int f(z)d\tilde{m}_x(z)=\int f(g(z))dm_x(z)=\int(f\circ g)(z)dm_x(z)=(f\circ g)(T)x$. Therefore, $f(g(T))|C=(f\circ g)(T)|C=A$. It follows from (6) that $f(g(T))$ as well as $(f\circ g)(T)$ is the closure of A, so that $f(g(T))=(f\circ g)(T)$.

To prove (9), let $S_n=\{z: |f(z)|\leqq n \text{ and } |g(z)|\leqq n\}$, so that $\{S_n\}$ satisfies the hypothesis of (6). Write $\mathfrak{L}=C$. Then $\mathfrak{L}\subset F_f$, $\mathfrak{L}\subset F_g$, $\mathfrak{L}\subset F_{f+g}$, and $\mathfrak{L}\subset F_{fg}$. It follows from (6) that $f(T)$, $g(T)$, $(f+g)(T)$, and $(fg)(T)$ are the closures respectively of $f(T)|\mathfrak{L}$, $g(T)|\mathfrak{L}$, $(f+g)(T)|\mathfrak{L}$, $(fg)(T)|\mathfrak{L}$. Moreover, for x in $\mathfrak{L}$

we have $f(T)x + g(T)x = \int f(z)dm_x(z) + \int g(z)dm_x(z) = \int (f+g)(z)dm_x(z) = (f+g)(T)x$. Choose n for which m_x lives on S_n. Define $\hat{m}_x$ by $\hat{m}_x(S) = \int_S g(z)dm_x(z)$. As was shown in §2, $\hat{m}_x$ is a T-measure. By definition $g(T)x = \hat{m}_x(S_n) = \hat{m}_x(X)$, and hence $\hat{m}_x$ is a T-measure for $g(T)x$. Since $f(T)$ is bounded on S_n we have $g(T)x \in \mathfrak{D}(f(T))$ and $f(T)g(T)x = f(T)\hat{m}_x(S_n) = \int_{S_n} f(z)d\hat{m}_x(z) = \int_{S_n} f(z)g(z)dm_x(z) = (fg)(T)x$. Thus $(fg)(T)x = f(T)g(T)x$ for x in $\mathfrak{L}$. This completes the proof (9).

To prove (10), define the sequence S_n of (6) by $S_n = \{z: |f(z)| \leqq n$ and $|f_i(z) - f(z)| \leqq 1$ for all $i \geqq n\}$ and write $\mathfrak{R}$ for C. Clearly S_n is a monotone increasing sequence. Also $\bigcup_n S_n = X$ because $\lim_n f_n(z) = f(z)$ pointwise. Any element x of $\mathfrak{R}$ is in F_f because $f(z)$ is bounded on each S_n. Thus by (6), $f(T)$ is the closure of $f(T)|\mathfrak{R}$. For any x in $\mathfrak{R}$, m_x lives on S_n for some n, so that $x \in F_{f_i}$ for $i \geqq n$. Moreover $f_i(T)x = \int_{S_n} f_i(z)dm_x(z)$ converges to $\int_{S_n} f(z)dm_x(z) = f(T)x$ as $i \to \infty$ by Lebesgue's bounded convergence theorem (which is easily shown to hold for vector-valued measures). This finishes the proof of Theorem 4.1.

At several points the calculus just developed falls short of the functional calculus for self-adjoint transformations. One would like to have equality instead of inclusion in (1) and (3). It would also be desirable to strengthen (9), and prove for instance $f(T) + g(T) \subset (f+g)(T)$, or at least that $f(T) + g(T)$ and $(f+g)(T)$ agree on their common domain. The author was not able to decide these questions.

In analogy with the case of a self-adjoint transformation, it might be expected that $f(T)$ is bounded in case $f(z)$ and T are bounded. That this is not true can be seen from very simple examples.

We merely state, without giving the somewhat lengthy proof, that an operator T in Γ is a scalar operator if and only if $f(T)$ is bounded for every bounded Borel function f.

5. **Weak T-measures.** For certain operators T there are no nontrivial T- or T^*-measures. For instance, let T be quasi-nilpotent. Then any T-measure m must be concentrated at the origin. Its value at the origin must be in the null space of T. Thus there are no nontrivial T- or T^*-measures in case the null spaces of T and T^* are trivial. On the other hand it would be desirable to generalize the notion of T-measure in order to have every measure concentrated at the origin be a T-measure when T is quasi-nilpotent.

Before giving the generalization in question, we must prove some preliminary lemmas.

LEMMA 5.1. *If T is a closed linear transformation on the reflexive Banach space B, and if m is a vector-valued measure on the set of positive integers with values in $\mathfrak{D}(T)$, then Tm is a vector-valued measure.*

Proof. We first note that if $\{x_i\}$ is a sequence from $\mathfrak{D}(T)$ converging to x in $\mathfrak{D}(T)$ for which $\|Tx_i\| < K$ for all i, then $Tx_i \to Tx$ in the weak topology as $i \to \infty$. To see this, note that for all u in $\mathfrak{D}(T^*)$ we have $\langle Tx, u\rangle = \langle x, T^*u\rangle = \lim \langle x_i, T^*u\rangle = \lim \langle Tx_i, u\rangle$. This, together with $\|Tx_i\| < K$, implies the result.

Let C be the range of m, so that $C \subset B$. To show that C is closed, let $\{m(S_i)\}$ be a convergent sequence from C. By passing to a subsequence if necessary, we may assume that the sequence $\{S_i\}$ of sets is convergent to a set S. Since m is a vector-valued measure, $\{m(S_i)\}$ will then converge to $m(S)$. Thus C is closed. By the Baire category theorem, there exists a K such that $\{m(S): \|Tm(S)\| < K\}$ is dense in some C-neighborhood U of a point $m(S_0)$ of C. We may assume that S_0 is finite, say $n < n_0$ for all n in S_0. If $x \in U$, we may therefore find a sequence $\{x_n\}$ converging to x with $\|Tx_n\| < K$. Then $\{Tx_n\}$, by the above remark, converges weakly to Tx. Therefore $\|Tx\| \leqq K$ for all x in U.

Since m is a vector-valued measure, an integer $n_1 > n_0$ can be chosen so that $m(S_1 \cup S_0) \in U$ whenever $S_1 \subset \{n: n > n_1\}$. From this and the above we see that $\|Tm(S)\| \leqq K_0$ for some constant K_0 independent of S. It follows that $\sum_{i=1}^{\infty} Tm(S_i)$ converges weakly to $Tm(\bigcup_i S_i)$ for every disjoint sequence $\{S_i\}$ of Borel sets. In other words, $(Tm)_u$ is a vector-valued measure for each u. This proves the lemma.

Lemma 5.2. *If T is a closed linear transformation on the reflexive Banach space B and if m is a vector-valued measure with values in $\mathfrak{D}(T)$, then Tm is a vector-valued measure.*

Proof. If $\{S_i\}$ is a disjoint sequence of measurable sets, define the vector-valued measure μ on the positive integers C by $\mu(R) = m(\bigcup_{i \in R} S_i)$. Then $T\mu$ is a vector-valued measure by Lemma 5.1, so that $\sum_{i=1}^{\infty} Tm(S_i) = \sum_{i=1}^{\infty} T\mu(\{i\}) = T\mu(C) = Tm(\bigcup_i S_i)$, as was to be proved.

If f is a simple function and Tm exists (i.e., $Tm(S)$ exists for all Borel sets S) we obviously have $\int f(z)dm(z) \in \mathfrak{D}(T)$ and $T\int f(z)dm(z) = \int f(z)d(Tm)(z)$. For an arbitrary bounded Borel function f take a sequence $\{f_n\}$ of simple functions converging uniformly to f. Then $\int f_n(z)dm(z) \to \int f(z)dm(z)$ and $\int f_n(z)d(Tm)(z) \to \int f(z)d(Tm)(z)$ as $n \to \infty$. Since T is closed, this gives $\int f(z)dm(z) \in \mathfrak{D}(T)$ and $T\int f(z)dm(z) = \int f(z)d(Tm)(z)$. Thus if we define the measure m^0 by $m^0(S) = \int_S f(z)dm(z)$, it follows that Tm^0 exists and that $Tm^0 = (Tm)^0$.

It is clear that if T is an operator, then $\|Tm\| \leqq \|T\| \|m\|$.

Definition 5.1. The set Q_0 is the set of those ν in Q which have compact support, i.e., those ν which live on bounded sets. The transformation Ω from Q_0 into Q_0 is defined by $(\Omega\nu)(S) = \int_S z d\nu(z)$. A measure ν in Q_0 is said to be a slice of a measure m in Q if there exists a bounded Borel set S_0 with $\nu(S) = m(S \cap S_0)$ for each S; ν is called the slice of m lying on S_0, i.e., the restriction of m to S_0.

It is clear that Ω is a linear transformation from Q_0 to Q_0. If ν in Q_0 lives on the set $S=\{z: |z| \leqq c\}$, then for each Borel function f with $\|f\| \leqq 1$ we have

$$\left\| \int f(z) d(\Omega\nu)(z) \right\| = \left\| \int z f(z) d\nu(z) \right\| \leqq \|zf\| \|\nu\| \leqq c\|\nu\|.$$

Therefore by definition $\|\Omega\nu\| \leqq c\|\nu\|$.

Let ν be in Q_0 and let f be integrable with respect to ν. Let $\bar{\nu}$ be the measure defined by $\bar{\nu}(S)=\int_S f(z)d\nu(z)$. Since $\bar{\nu}$ lives on the same sets as ν, $\bar{\nu}\in Q_0$. We have

$$(\Omega\bar{\nu})(S) = \int_S z d\bar{\nu}(z) = \int_S z f(z) d\nu(z) = \int_S f(z) d(\Omega\nu) z = (\Omega\,\nu)^{-}(S).$$

Thus $\Omega\bar{\nu}=(\Omega\,\nu)^{-}$.

If $m\in Q_0$ and Tm exists then taking $f(z)=z$ and taking m^0 as above we have seen that $Tm^0=(Tm)^0$. This means that $T\Omega M=\Omega Tm$, assuming that Tm exists.

Definition 5.2. A measure m in Q is a weak T-measure if for each slice ν of m and each positive integer n, $T^n\nu$ exists and $\|(T-\Omega)^n\nu\|^{1/n}\rightarrow 0$ as $n\rightarrow\infty$.

The notion of a weak T-measure includes that of a T-measure. It is clear that any slice of a weak T-measure is a weak T-measure, and that if $m\in Q_0$ is a weak T-measure, then Ωm is a weak T-measure. More generally let f be a bounded Borel function and define m^0 as above. For any slice ν of m, ν^0 is the corresponding slice of m^0. Thus $\|(T-\Omega)^n\nu^0\|^{1/n}=\|((T-\Omega)^n\nu)^0\|^{1/n} \leqq \|f\|^{1/n}\|(T-\Omega)^n\nu\|^{1/n}\rightarrow 0$ as $n\rightarrow\infty$, so that m^0 is also a weak T-measure. Furthermore, if Tm exists, then each slice of Tm is of the form $T\nu$ for some slice ν of m and

$$\|(T-\Omega)^n T\nu\|^{1/n} \leqq \|(T-\Omega)^{n+1}\nu\|^{1/n} + \|\Omega(T-\Omega)^n\nu\|^{1/n} \rightarrow 0$$

as $n\rightarrow\infty$ so that Tm is a weak T-measure.

As an example consider a weak T-measure m and a point $\{z_0\}$ which has measure x. Let ν be the measure of mass x concentrated at z_0, i.e., the slice of m on $\{z_0\}$. It is easy to see that

$$\|(T-\Omega)^n\nu\| = \|(T-z_0)^n x\|,$$

so that $\|(T-z_0)^n x\|^{1/n}\rightarrow 0$ as $n\rightarrow\infty$ because ν is a slice of m. Conversely it follows easily that a measure of mass x concentrated at a point $\{z_0\}$ is a weak T-measure if $\|(T-z_0)^n x\|^{1/n}\rightarrow 0$ as $n\rightarrow\infty$. Applied to a quasi-nilpotent operator T this means that every measure concentrated at the origin is a weak T-measure.

Theorem 5.1. *If m is a weak T-measure, then for every bounded Borel set S there exists an analytic function x_λ on $\bar{S}'$ such that $(T-\lambda)x_\lambda=m(S)$ for each λ in $\bar{S}'$, defined by $x_\lambda=-\sum_{n=0}^{\infty}\int_S (1/(\lambda-z)^{n+1})d[(T-\Omega)^n m](z)$. It has the property $x_\lambda\rightarrow 0$ as $|\lambda|\rightarrow\infty$ and $-\lambda x_\lambda\rightarrow m(S)$ as $|\lambda|\rightarrow\infty$.*

Proof. Since $1/(\lambda-z)$ considered as a function of z is bounded on S for each λ in $\bar{S}'$, the integral $\int_S(1/(\lambda-z)^{n+1})d[(T-\Omega)^n m](z)$ defines an analytic function x_n on $\bar{S}'$ and

$$\|x_n(\lambda)\| \leqq \max_{z\in S}\left|\frac{1}{(\lambda-z)^{n+1}}\right|\|(T-\Omega)^n\nu\| = \frac{1}{r^{n+1}}\|(T-\Omega)^n\nu\|,$$

where r is the distance from λ to S and ν is the slice of m lying on S. Since $\|(T-\Omega)^n\nu\|^{1/n}\to 0$ as $n\to\infty$, the series $x_\lambda=-\sum_{n=0}^{\infty}x_n(\lambda)$ converges uniformly on compact subsets of $\bar{S}'$ and therefore defines an analytic function on $\bar{S}'$. Obviously $x_\lambda\to 0$ and $-\lambda x_\lambda\to m(S)$ as $\lambda\to\infty$. We shall prove that $(T-\lambda)\cdot(-\sum_{n=0}^{N}x_n(\lambda))\to m(S)$ as $N\to\infty$. Since T is closed, this will show that $(T-\lambda)x_\lambda=m(S)$. We have

$$\begin{aligned}
(T-\lambda)\left(-\sum_{n=0}^{N}x_n(\lambda)\right) &= -\sum_{n=0}^{N}T\int_S\frac{1}{(\lambda-z)^{n+1}}d[(T-\Omega)^n m](z)\\
&\quad+\sum_{n=0}^{N}\int_S\frac{(\lambda-z)+z}{(\lambda-z)^{n+1}}d[(T-\Omega)^n m](z)
\end{aligned}$$

$$\begin{aligned}
&= -\sum_{n=0}^{N}\int_S\frac{1}{(\lambda-z)^{n+1}}d[T(T-\Omega)^n m](z)+\sum_{n=0}^{N}\int_S\frac{1}{(\lambda-z)^{n}}d[(T-\Omega)^n m](z)\\
&\quad+\sum_{n=0}^{N}\int_S\frac{1}{(\lambda-z)^{n+1}}d[\Omega(T-\Omega)^n m](z)\\
&= -\sum_{n=0}^{N}\int_S\frac{1}{(\lambda-z)^{n+1}}d[(T-\Omega)^{n+1} m](z)+\sum_{n=0}^{N}\int_S\frac{1}{(\lambda-z)^{n}}d[(T-\Omega)^n m](z)\\
&= \int_S dm(z)-\int_S\frac{1}{(\lambda-z)^{N+1}}d[(T-\Omega)^{N+1}m](z)\\
&= m(S)-\int_S\frac{1}{(\lambda-z)^{N+1}}d[(T-\Omega)^{N+1}m](z).
\end{aligned}$$

Since

$$\left\|\int_S\frac{1}{(\lambda-z)^{N+1}}d[(T-\Omega)^{N+1}m](z)\right\| \leqq \frac{1}{r^{N+1}}\|(T-\Omega)^{N+1}\nu\|\to 0$$

as $N\to\infty$, this completes the proof.

This theorem has as some consequences results analogous to results for T-measures which were demonstrated in §2. The most important of these is the analog of Theorem 2.1.

THEOREM 5.2. *If m is a weak T-measure, μ a weak T^*-measure, and S_1 and S_2 are disjoint Borel sets, then* $\langle m(S_1),\mu(S_2)\rangle=0$.

Proof. Since $\langle m(S_1), \mu(C)\rangle$ is a numerical measure as a function of C, $\langle m(S_1), \mu(S_2)\rangle$ will be zero if $\langle m(S_1), \mu(C)\rangle$ is zero for all closed bounded subsets C of S_2. Thus we may assume that S_2, and similarly S_1, is closed and bounded. By Theorem 5.1 there exists a function x_λ analytic on S_1' and 0 at infinity such that $(T-\lambda)x_\lambda = m(S_1)$. Similarly there exists u_λ analytic on S_2' such that $(T^*-\lambda)u_\lambda = \mu(S_2)$. By Lemma 2.1, $\langle m(S_1), \mu(S_2)\rangle = 0$, since $-\lambda x_\lambda \to m(S_1)$ as $\lambda \to \infty$.

Corollary 1. *If m is a weak T-measure and μ is a weak T^*-measure with $\mu(X)=0$, then $\langle m(S_1), \mu(S_2)\rangle = 0$ for all S_1 and S_2. If those x having weak T-measures are dense in B then each u has at most one weak T^*-measure. If f is integrable with respect to the weak T-measure m and the weak T^*-measure μ, then $\langle \int f(z)dm(z), \mu(X)\rangle = \langle m(X), \int f(z)d\mu(z)\rangle$.*

Proof. If $\mu(X)=0$, then $\mu(S_2') = -\mu(S_2)$. Therefore

$$\begin{aligned}\langle m(S_1), \mu(S_2)\rangle &= \langle m(S_1\cap S_2'), \mu(S_2)\rangle + \langle m(S_1\cap S_2), \mu(S_2)\rangle \\ &= \langle m(S_1\cap S_2'), \mu(S_2)\rangle - \langle m(S_1\cap S_2), \mu(S_2')\rangle \\ &= 0\end{aligned}$$

by Theorem 5.2.

If u in B^* has weak T-measures μ_1 and μ_2, then $\langle m(X), (\mu_1-\mu_2)(S)\rangle = 0$ for each weak T-measure m and each Borel set S, as has just been demonstrated. Thus under the assumption that a dense set in B has weak T-measures, this shows that $\mu_1=\mu_2$.

To verify that last assertion of the corollary, put $x=m(X)$ and $u=\mu(X)$. Define the measures m_u and μ_x by $m_u(S)=\langle m(S), u\rangle$ and $\mu_x(S)=\langle x, \mu(S)\rangle$. Thus

$$\begin{aligned}m_u(S) &= \langle m(S), u\rangle = \langle m(S), \mu(S)+\mu(S')\rangle = \langle m(S), \mu(S)\rangle \\ &= \langle m(S)+m(S'), \mu(S)\rangle = \langle x, \mu(S)\rangle = \mu_x(S),\end{aligned}$$

and so $m_u=\mu_x$. This gives

$$\begin{aligned}\left\langle \int f(z)dm(z), \mu(X)\right\rangle &= \int f(z)dm_u(z) = \int f(z)d\mu_x(z) \\ &= \left\langle x, \int f(z)d\mu(z)\right\rangle = \left\langle m(X), \int f(z)d\mu(z)\right\rangle.\end{aligned}$$

The next two theorems are not proved because the proofs follow exactly the proofs of Theorems 2.3 and 2.2.

Theorem 5.3. *If m is a weak T-measure it lives on the spectrum of T.*

Theorem 5.4. *Let m_1 and m_2 be weak T-measures and S_1 and S_2 be closed disjoint sets such that each component of $S_1'\cap S_2'$ contains a point of the resolvent set of T. Then $m_1(S_1)=m_2(S_2)$ implies that $m_1(S_1)=m_2(S_2)=0$.*

The exact analog of Theorem 2.4 does not hold. Instead we have the following result.

THEOREM 5.5. *Let* $\{T_\alpha\}$ *be a sequence of operators on* B *and* T *an operator for which* $T_\alpha^* \to T^*$ *strongly. For each* α *let* m_α *be a weak* T_α*-measure with* $\|m_\alpha\| \leqq c$. *Let* m *be a cluster point of the sequence* $\{m_\alpha\}$ *in the weak operator topology. Let* $\{a_n\}$ *be a sequence of positive constants converging to* 0 *such that* $\|(T_\alpha - \Omega)^n m_\alpha\|^{1/n} \leqq a_n$ *for all* n *and* α. *Then* m *is a weak* T*-measure and* $\|(T-\Omega)^n m\|^{1/n} \leqq a_n$ *for all* n.

Proof. For each f in $\mathfrak{M}(X)$ with compact support and each u in B^* we have $\|\int f(z)d[(T_\alpha - \Omega)^n m_\alpha](z)\| \leqq \|f\| \, \|(T_\alpha - \Omega)^n m_\alpha\| \leqq \|f\| a_n^n$ and $\|\int z^i f(z) dm_\alpha(z)\| \leqq c \sup_z |z^i f(z)|$. Also

$$\left\langle \int f(z)d[(T_\alpha - \Omega)^n m_\alpha](z), u \right\rangle = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \left\langle \int z^i f(z) dm_\alpha(z), (T_\alpha^*)^{n-i} u \right\rangle$$

and

$$\left\langle \int f(z)d[(T - \Omega)^n m](z), u \right\rangle = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \left\langle \int z^i f(z) dm(z), (T^*)^{n-i} u \right\rangle.$$

Now $(T_\alpha^*)^{n-i}u \to (T^*)^{n-i}u$ as $n \to \infty$ because $T_\alpha^* \to T^*$ strongly. Thus we may choose α_0 such that $(T_\alpha^*)^{n-i}u$ is arbitrarily near to $(T^*)^{n-i}u$ for all $\alpha > \alpha_0$ and then choose $\alpha > \alpha_0$ such that $\langle \int z^i f(z) dm_\alpha(z), (T^*)^{n-i}u \rangle$ is arbitrarily near to $\langle \int z^i f(z) dm(z), (T^*)^{n-i}u \rangle$ for $1 \leqq i \leqq n$, because m_α has m as a cluster point in the weak operator topology. Thus the quantities

$$\begin{aligned} &\left| \left\langle \int z^i f(z) dm_\alpha(z), (T_\alpha^*)^{n-i} u \right\rangle - \left\langle \int z^i f(z) dm(z), (T^*)^{n-i} u \right\rangle \right| \\ &\qquad \leqq \left\| \int z^i f(z) dm_\alpha(z) \right\| \, \|(T_\alpha^*)^{n-i} u - (T^*)^{n-i} u\| \\ &\qquad\quad + \left| \left\langle \int z^i f(z) dm_\alpha(z), (T^*)^{n-i} u \right\rangle - \left\langle \int z^i f(z) dm(z), (T^*)^{n-i} u \right\rangle \right| \end{aligned}$$

can simultaneously be made arbitrarily small for all i with $1 \leqq i \leqq n$. Hence $\langle \int f(z) d[(T_\alpha - \Omega)^n m_\alpha](z), u \rangle$ can be made arbitrarily close to

$$\left\langle \int f(z)d[(T - \Omega)^n m](z), u \right\rangle$$

by an appropriate choice of α. It follows that

$$\left| \left\langle \int f(z)d[(T - \Omega)^n m](z), u \right\rangle \right| \leqq \|f\| a_n^n \|u\|.$$

Since this holds for all u in B^* we have $\|\int f(z)d[(T-\Omega)^n m](z)\| \leqq a_n^n \|f\|$. As was shown in §1 this implies that $\|(T-\Omega)^n m\| \leqq a_n^n$. Thus m is a weak T-measure, since the same inequality must also hold for all slices of m. This completes the proof.

SUPPLEMENT. *If for each positive integer α we have $\|T_\alpha m_\alpha\| \leqq K$ and $m_\alpha(X) = x$, then $m(X) = x$.*

Proof. Let f be any function with compact support in $\mathfrak{M}(X)$ for which $f(z) = 1$ for $|z| \leqq M$ and $0 \leqq f(z) \leqq 1$ for all z, so that $|(f(z)-1)/z| \leqq 1/M$ for all z. For each u in B^*, $\langle \int f(z)dm(z), u\rangle$ is a cluster point of the sequence $\{\langle \int f(z)dm_\alpha(z), u\rangle\}$, so that

$$\left|\left\langle \int f(z)dm(z),\, u\right\rangle - \langle x, u\rangle\right|$$

$$\leqq \sup_\alpha \left|\left\langle \int f(z)dm_\alpha(z),\, u\right\rangle - \langle x, u\rangle\right| = \sup_\alpha \left|\left\langle \int (f(z) - 1)dm_\alpha(z),\, u\right\rangle\right|$$

$$= \sup_\alpha \left|\left\langle \int \frac{f(z) - 1}{z}\, d(\Omega m_\alpha)(z),\, u\right\rangle\right| \leqq \sup_z \left|\frac{f(z) - 1}{z}\right| \|u\| \sup_\alpha \|\Omega m_\alpha\|$$

$$\leqq \frac{\|u\|}{M} \sup_\alpha (\|T_\alpha m_\alpha\| + \|(T_\alpha - \Omega)m_\alpha\|) \leqq \frac{\|u\|}{M} (K + a_1).$$

Letting M become infinite we obtain $|\langle m(X), u\rangle - \langle x, u\rangle| = 0$, or $\langle m(X), u\rangle = \langle x, u\rangle$. Since u is arbitrary this gives $m(X) = x$, which finishes the proof.

As defined by Dunford [6], a spectral operator T on a reflexive Banach space B is one for which there exists a spectral measure $E(\ )$ such that for all Borel sets S

(1) $TE(S) = E(S)T$,

(2) the spectrum of $T|\mathfrak{R}E(S)$ is a subset of $\bar{S}$.

The spectral measure $E(\ )$ is uniquely determined. The adjoint T^* of T is also a spectral operator and the corresponding spectral measure, denoted by $E^*(\ )$, has the value $(E(S))^*$ on the Borel set S.

If x is any vector in B there exists a function x_z analytic for z in $\bar{S}'$ with values in $\mathfrak{R}(E(S))$ such that $(T-z)x_z = E(S)x$. This follows from the fact that the spectrum of $T|\mathfrak{R}(E(S))$ is a subset of $\bar{S}$. This implies (we omit the proof) that $E(\ )x$ is a weak T-measure. This remark enables us to characterize the class of spectral operators.

THEOREM 5.6. *An operator T on a reflexive Banach space B is a spectral operator if and only if every x in B has a weak T-measure and every u in B^* has a weak T^*-measure.*

Proof. We omit the proof that if T is a spectral operator then every x in B has a weak T-measure. Every u in B^* has a weak T^*-measure because T^* is also a spectral operator.

Conversely assume that every x in B has a weak T-measure m_x and that every u in B^* has a weak T^*-measure μ_u. The measures m_x and μ_u are unique by Corollary 1 to Theorem 5.2. Therefore m_x and μ_u are linear functions of x and u respectively. The transformations $x \to m_x$ and $u \to \mu_u$ are closed. To see this let a sequence $\{x_n\}$ from B converge to x and let $\{m_{x_n}\}$ converge to a measure m in Q. Then $m(X) = \lim m_{x_n}(X) = \lim x_n = x$. By Theorem 5.5, m is a weak T-measure. Therefore $m = m_x$. This proves that the transformation $x \to m_x$ is closed. By the closed graph theorem, the transformation $x \to m_x$ is bounded. Similarly the transformation $u \to \mu_u$ is bounded.

Thus for each Borel set S the transformation $x \to m_x(S)$ is an operator $E(S)$ on B. Since $\langle m_x(S), u\rangle = \langle x, \mu_u(S)\rangle$ for all u in B^*, the adjoint $E^*(S)$ of $E(S)$ is the transformation $u \to \mu_u(S)$. If $y = E(S)x$, then m_y is defined by $m_y(U) = m_x(U \cap S)$, so that m_y lives on S. Therefore $E(S)y = y$, so that $E(S)$ is an idempotent. The set function $E(\ )$ is therefore a spectral measure since $E(X) = I$ and

$$E\left(\bigcup_i S_i\right)x = m_x\left(\bigcup_i S_i\right) = \sum_i m_x(S_i) = \sum_i E(S_i)x$$

for every disjoint sequence $\{S_i\}$ of Borel sets and all x in B. For any x, Tm_x is a weak T-measure, so that by uniqueness $Tm_x = m_{Tx}$. Thus $TE(S)x = Tm_x(S) = m_{Tx}(S) = E(S)Tx$.

It remains to prove that the spectrum of $T|\,\mathfrak{R}(E(S))$ is included in $\bar{S}$. We first calculate the adjoint of $T|\,\mathfrak{R}(E(S))$. Each u in B^* is a bounded linear functional on B and therefore on $\mathfrak{R}(E(S))$. Two such, u_1 and u_2, are equal on $\mathfrak{R}(E(S))$ if and only if $(E(S)x, u_1 - u_2) = 0$ for all x in B, or $(x, E^*(S)u_1) = (x, E^*(S)u_2)$, which implies $E^*(S)u_1 = E^*(S)u_2$. Since any bounded linear functional on $\mathfrak{R}(E(S))$ comes from such a u in B^*, by the Hahn-Banach theorem, it follows that there is a 1-1 map $u \to \tilde{u}$ from $\mathfrak{R}(E^*(S))$ onto the dual space of $\mathfrak{R}(E(S))$. It is obvious that $\|\tilde{u}\| \leqq \|u\|$. From the closed graph theorem it therefore follows that the map $u \to \tilde{u}$ is a homeomorphism and that $\mathfrak{R}(E^*(S))$ may be identified with the dual space of $\mathfrak{R}(E(S))$. For x in $\mathfrak{R}(E(S))$ and u in $\mathfrak{R}(E^*(S))$ we have $\langle Tx, u\rangle = \langle x, T^*u\rangle$. Therefore the adjoint of $T|\,\mathfrak{R}(E(S))$ is $T^*|\,\mathfrak{R}(E^*(S))$.

Let λ be any number in $\bar{S}'$. By Theorem 5.1, for every $x = m_x(S)$ in $\mathfrak{R}(E(S))$ there exists y in B with $(T-\lambda)y = x$. This gives

$$x = E(S)x = E(S)(T - \lambda)y = (T - \lambda)E(S)y,$$

so that we may suppose $y \in \mathfrak{R}(E(S))$. Thus $\mathfrak{R}((T-\lambda)|\,\mathfrak{R}(E(S))) = \mathfrak{R}(E(S))$. Similarly, $\mathfrak{R}((T^*-\lambda)|\,\mathfrak{R}(E^*(S))) = \mathfrak{R}(E^*(S))$. Since $(T^*-\lambda)|\,\mathfrak{R}(E^*(S))$ is the adjoint of $(T-\lambda)|\mathfrak{N}(E(S))$ it follows that λ is in the resolvent set of $T|\ \mathfrak{N}(E(S))$, as was to be proved.

Consider a spectral operator T and the corresponding spectral measure $E(\ )$. It is known [6] that $T-\int\lambda dE(\lambda)$ is a quasi-nilpotent operator, so that there exists a sequence $\{a_n\}$ of positive numbers converging to 0 such that $\|(T-\int\lambda dE(\lambda))^n\|^{1/n}\leqq a_n$ for all positive integers n. Since the projections $E(S)$ are uniformly bounded there also exists a constant c such that $\|E(\ )x\|\leqq c\|x\|$ for all x in B. If we define $\mathfrak{N}$ to be the set of T satisfying these conditions for fixed c and $\{a_n\}$, then the following analog to Corollary 1 of Theorem 3.2 is valid.

THEOREM 5.7. *Each operator T which is a limit in the double strong topology of a sequence $\{T_\alpha\}$ of operators in $\mathfrak{N}$ is a spectral operator.*

Proof. Let $E_\alpha(\ )$ be the spectral measure associated with T_α, so that for each x in B the set function $E_\alpha(\ )x$ is a weak T_α-measure for x. For each Borel set S we have

$$[\Omega E_\alpha(\ )x](S)=\int_S\lambda dE_\alpha(\lambda)x=\int\lambda dE_\alpha(\lambda)E_\alpha(S)x.$$

Hence $\Omega E_\alpha(\ )x=\int\lambda dE_\alpha(\lambda)E_\alpha(\ )x$. Therefore $(T-\Omega)^nE_\alpha(\ )x=(T-\int\lambda dE_\alpha(\lambda))^n\cdot E_\alpha(\ )x$. It follows that

$$\begin{aligned}\|(T-\Omega)^nE_\alpha(\ )x\|^{1/n}&\leqq\left\|\left(T-\int\lambda dE(\lambda)\right)^n\ \right\|^{1/n}\|E_\alpha(\ )x\|^{1/n}\\&\leqq a_n(c\|x\|)^{1/n}.\end{aligned}$$

By Theorem 5.5 this implies that any cluster point m_x in the weak operator topology of the sequence $\{E_\alpha(\ )x\}$ is a weak T-measure. Since $\lim_{\alpha\to\infty}\|T_\alpha x\|=\|Tx\|$, the norms $\|T_\alpha x\|$ are bounded. Thus the norms $\|T_\alpha E_\alpha(\ )x\|$ are bounded, since by the commutativity of T_α and $E_\alpha(\ )$ we see that $\|T_\alpha E_\alpha(\ )x\|=\|E_\alpha(\ )T_\alpha x\|\leqq c\|T_\alpha x\|$. By the supplement to Theorem 5.5 it follows that $m_x(X)=x$.

Thus every x in B has a weak T-measure. Similarly, every u in B^* has a weak T^*-measure. Therefore T is a spectral operator, as was to be proved.

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