FRAGMENTS OF MANY-VALUED STATEMENT CALCULI

BY

ALAN ROSE AND J. BARKLEY ROSSER

1. Introduction. In 1930 (see [1]), Łukasiewicz and Tarski discussed the axiomatization of many-valued logics of a more general sort than originally axiomatized by Post in 1921 (see [2]). We propose to develop these ideas a bit further.

In particular, let us consider statement calculi based on a set, $\mathfrak{J}$, of truth-values. We make the following assumptions about $\mathfrak{J}$. If x is in $\mathfrak{J}$, then $0 \leqq x \leqq 1$. $\mathfrak{J}$ is nonempty. $\mathfrak{J}$ is closed under application of the functions c and n, where

(1.1) $c(x, y) = \min(1, 1 - x + y)$,

(1.2) $n(x) = 1 - x$.

Obviously $\mathfrak{J}$ must contain 0 and 1, and may perhaps contain only these. If $\mathfrak{J}$ contains M members, then $\mathfrak{J}$ must consist of the rational numbers

$$0, \frac{1}{M-1}, \frac{2}{M-1}, \cdots, \frac{M-2}{M-1}, 1,$$

as one can conclude by an analysis like that given by McNaughton (see [3]).

A similar analysis shows that if $\mathfrak{J}$ has an infinite number of members, then these must be everywhere dense in the interval $[0, 1]$. Possibilities are that $\mathfrak{J}$ might consist of all rationals in this interval, or of all reals in this interval. Many other possibilities exist, such as that of choosing an irrational θ and letting $\mathfrak{J}$ consist of all reals of the form $a + b\theta$ with $0 \leqq a + b\theta \leqq 1$; here one may set such requirements as that a and b should be integers, or that a and b should be rationals.

The members of $\mathfrak{J}$ are commonly called truth-values; we shall usually refer to them just as values.

It is common to separate $\mathfrak{J}$ into designated and undesignated values. This is done as follows. One chooses a real number $\mathcal{S}$ with $0 \leqq \mathcal{S} \leqq 1$. All members of $\mathfrak{J}$ less than $\mathcal{S}$ are undesignated and all members greater than $\mathcal{S}$ are designated. If $\mathcal{S}$ is itself a member of $\mathfrak{J}$, one must also specify whether $\mathcal{S}$ is designated or undesignated; however we set the requirement that 0 shall always be undesignated and 1 shall always be designated. If there are M values of which S are designated, then $1 \leqq S < M$, and the designated values are just

$$\frac{M-S}{M-1}, \frac{M-S+1}{M-1}, \cdots, \frac{M-2}{M-1}, 1.$$

Received by the editors August 20, 1956.

An obvious linear transformation will identify this case with that considered by Rosser and Turquette (see [4]).

In many of the later sections, we leave $\mathfrak{I}$ and $\mathfrak{S}$ quite general. In other sections, we impose special conditions, such as that $\mathfrak{I}$ have M members or that $\mathfrak{S}=1$.

In various sections, we shall make use of selected ones of the statement functions, C, N, J, T, and D. With each of these, we associate truth-value functions c, n, j, t, and d, as follows:

(1.1) $c(x, y)=\min(1, 1-x+y)$,

(1.2) $n(x)=1-x$,

$$j(x) = \begin{cases} 1 \text{ if } x = 1, \\ 0 \text{ if } x \neq 1, \end{cases} \tag{1.3}$$

$$t(x) = \frac{M-2}{M-1}, \tag{1.4}$$

$$d(x) = \begin{cases} \text{an undesignated value if } x \text{ is designated,} \\ \text{a designated value if } x \text{ is undesignated.} \end{cases} \tag{1.5}$$

In the above, x and y are restricted to lie in $\mathfrak{I}$.

Since $\mathfrak{I}$ contains both 0 and 1, it is closed under application of j. We will use T only when $\mathfrak{I}$ has M members, in which case $\mathfrak{I}$ is closed under application of t. Clearly (1.5) does not define d uniquely, but only puts certain restrictions on it. For our uses, it suffices that d be some specified one of the functions satisfying (1.5). Clearly $\mathfrak{I}$ is closed under application of d. Not uncommonly, d is made precise as follows

$$d(x) = \begin{cases} 0 \text{ if } x \text{ is designated,} \\ 1 \text{ if } x \text{ is undesignated.} \end{cases}$$

However, we do not need to be so specific.

Of these functions, we shall take C, N, and T as undefined if we use them at all. If $\mathfrak{I}$ has M members, then J and D can be defined in terms of C and N (see [3] or [4], for example) and we shall consider them as so defined. Note that the definition depends on M, and in the case of D on S also. If $\mathfrak{I}$ has an infinite number of members, then J and D are taken as undefined if they are used.

In various places, we will make use of certain of the functions A, K, B, L, I, &, and E defined as follows:

(1.6) APQ for $CCPQQ$,

(1.7) KPQ for $NANPNQ$,

(1.8) BPQ for $CNPQ$,

(1.9) LPQ for $NCPNQ$,

(1.10) IPQ for $ADPQ$,

(1.11) $\&PQ$ for $DADPDQ$,
(1.12) EPQ for $LCPQCQP$.

Then each of these has an associated truth-value function as follows:

(1.13) $a(x, y) = \max (x, y)$,
(1.14) $k(x, y) = \min (x, y)$,
(1.15) $b(x, y) = \min (1, x+y)$,
(1.16) $l(x, y) = \max (0, x+y-1)$,
(1.17) $a(x, y)$ is designated if either of x or y is designated,
(1.18) $a(x, y)$ is undesignated if both of x and y are undesignated,
(1.19) $i(x, y)$ is designated if x is undesignated or y is designated,
(1.20) $i(x, y)$ is undesignated if x is designated and y is undesignated,
(1.21) $\&(x, y)$ is designated if both of x and y are designated,
(1.22) $\&(x, y)$ is undesignated if either of x or y is undesignated,
(1.23) $e(x, y) = \min (1-x+y, 1-y+x)$.

In particular, $e(x, y) = 1$ if and only if $x = y$.

Note that A and B are analogous to the inclusive "or" function of the usual two-valued statement calculus. Each will be seen to have some, but not all, of the familiar properties of the two-valued "or." Similarly K, L, and $\&$ are analogous to the two-valued "and," C and I are analogous to the two-valued "implies," N and D are analogous to the two-valued "not," and E serves as an equivalence relation.

If α is a non-negative integer, we allow ourselves to indicate α repetitions of a block of symbols by enclosing the block in parentheses and adjoining the exponent α. Thus we may write

$(CP)^0Q$ for Q,

$(CP)^1Q$ for CPQ,

$(CP)^2Q$ for $CPCPQ$,

$(CP)^3Q$ FOR $CPCPCPQ$

etc.

Then with $(CP)^\alpha Q$, $(BP)^\alpha Q$, and $(LP)^\alpha Q$ there are associated truth-value functions as follows:

(1.24) $c_\alpha(x, y) = \min (1, y+\alpha(1-x))$ $(0 \leqq \alpha)$,
(1.25) $b_\alpha(x, y) = \min (1, y+\alpha x)$ $(0 \leqq \alpha)$,
(1.26) $l_\alpha(x, y) = \max (0, y-\alpha(1-x))$ $(0 \leqq \alpha)$.

We define

(1.27) $B_\alpha P$ for $(BP)^\alpha NCPP$ $(0 \leqq \alpha)$,
(1.28) $L_\alpha P$ for $(LP)^\alpha CPP$ $(0 \leqq \alpha)$,
(1.29) $V_\alpha P$ for $EPNB_{\alpha-1}P$ $(1 \leqq \alpha)$,
(1.30) $W_\alpha PQ$ for $EPB_\alpha Q$ $(0 \leqq \alpha)$.

These have associated truth-value functions as follows:

(1.31) $b_\alpha(x) = \min (1, \alpha x)$ $(0 \leqq \alpha)$,

(1.32) $l_\alpha(x) = \max(1, 1-\alpha(1-x))$ $(0 \leqq \alpha)$,
(1.33) $v_\alpha(x) = 1$ if and only if $\alpha x = 1$ $(1 \leqq \alpha)$,
(1.34) $w_\alpha(x, y) = 1$ if and only if $x = \min(1, \alpha y)$ $(0 \leqq \alpha)$.
As in [4], we define a chain symbol Γ by the following recursion:
(1.35) If $\beta < \alpha$, then $\Gamma_{i=\alpha}^{\beta} P_i Q$ denotes Q.
(1.36) If $\beta \geqq \alpha$, then $\Gamma_{i=\alpha}^{\beta} P_i Q$ denotes $CP_\beta \Gamma_{i=\alpha}^{\beta-1} P_i Q$.
The associated truth function is:

$$(1.37) \qquad \gamma_{i=\alpha}^{\beta}(x_i, y) = \begin{cases} y & (\alpha \geqq \beta + 1), \\ \min\left(1, \beta + 1 - \alpha + y - \sum_{i=\alpha}^{\beta} x_i\right) & (\alpha \leqq \beta). \end{cases}$$

We note that if all the P_i's are identical with P, then

$$(1.38) \qquad \Gamma_{i=\alpha}^{\beta} P_i Q \text{ is } (CP)^{\beta+1-\alpha} Q \qquad (\alpha \leqq \beta + 1).$$

We also define a generalized summation by the following recursion:
(1.39) If $\beta = \alpha$, then $\sum_{i=\alpha}^{\beta} P_i$ denotes P_α.
(1.40) If $\beta > \alpha$, then $\sum_{i=\alpha}^{\beta} P_i$ denotes $AP_\beta \sum_{i=\alpha}^{\beta-1} P_i$.
The associated truth-value function is
(1.41) $\max(x_\alpha, x_{\alpha+1}, \cdots, x_\beta)$.

A perennial problem is to make some choice of $\mathfrak{J}$, $\mathfrak{S}$, and of a set of undefined functions, and then to ask for a set of axioms and rules from which one can derive exactly those statement formulas (in the sense of [4, pp. 13–14]) whose corresponding truth-value functions take only designated values. We shall present some fragments of a general theory, and then enlarge these to give complete solutions in a number of special cases.

In [1], it is stated that Łukasiewicz conjectured that if $\mathfrak{J}$ has an infinite number of members and $\mathfrak{S} = 1$, then the following rule and set of axiom schemes give a solution when C and N are chosen as the undefined functions.

Rule C. If P and CPQ, then Q.

Axiom schemes:

$CPCQP$.

$CCPQCCQRCPR$.

$CAPQAQP$.

$ACPQCQP$.

$CCNPNQCQP$.

In §13 we shall prove this conjecture. Incidentally, we note that C. A. Meredith and C. C. Chang have recently shown how to derive the fourth of these axioms from the rest.

In [8], on p. 240, M. Wajsberg announced that he had a proof of Łukasiewicz's conjecture. However, apparently Wajsberg's proof was never published, since in [9], on p. 51, Tarski refers to Wajsberg's proof but cites only [8].

When $\mathfrak{S}=1$ and $\mathfrak{J}$ has an infinite number of members, then the set of formulas based on C and N which take designated values exclusively is independent of the further details of the composition of $\mathfrak{J}$. This enables us to assume that $\mathfrak{J}$ consists of the rationals in the interval $[0, 1]$ when dealing with axiomatization in the case when $\mathfrak{S}=1$ and $\mathfrak{J}$ has an infinite number of members and C and N are the undefined functions.

If $\mathfrak{S}<1$, the situation is not so simple. For instance, take $\mathfrak{S}=1/2$, take $\mathfrak{J}$ to consist of the rationals and take $1/2$ as not designated; then $C(CP)^2QCPQ$ can take an undesignated value, namely $1/2$. Alternatively, take $\mathfrak{S}=1/2$ again but take $\mathfrak{J}$ to consist of all reals of the form $a+b\theta$ with $0\leqq a+b\theta\leqq 1$, where θ is a fixed irrational and a and b are integers. As $1/2$ is not a member of $\mathfrak{J}$, we need not specify if it is designated or not. In any case, whatever we decide about making $1/2$ designated, we conclude that $C(CP)^2QCPQ$ must take only designated values, since its minimum possible value is $1/2$ and it cannot take that value since (with this specification of $\mathfrak{J}$) the only rational value that $C(CP)^2QCPQ$ can assume is 1.

We do not make an effort to furnish an axiomatization for any of the cases where $\mathfrak{S}<1$ and $\mathfrak{J}$ has an infinite number of members. We mention that if $\mathfrak{S}<1$, then Rule C is not acceptable. Possible alternatives are:

Rule JC. If JP and $JCPQ$, then JQ.

Rule I. If P and IPQ, then Q.

When one considers the case where $\mathfrak{J}$ has a finite number of members, the situation changes a bit. Even when $\mathfrak{S}=1$ the set of statement formulas which take designated values exclusively depends on the number of members of $\mathfrak{J}$. In particular, if $\mathfrak{S}=1$ then $C(CP)^{\alpha}Q(CP)^{\alpha-1}Q$ takes only designated values if and only if $\mathfrak{J}$ has α or fewer members.

In [4] have been given systems of axioms for each case in which $\mathfrak{J}$ has a finite number of members. These axiom systems are very general with regard to which statement functions are taken as undefined. If C and N, or C and N and T, are taken as undefined, one can get systems of axioms with fewer axioms than are used in [4]. This we do, and the results are summarized herewith.

If C, N, and T are taken as undefined and $\mathfrak{J}$ has M members, then:

(a) Rule C and six axiom schemes suffice if $\mathfrak{S}=1$ (see §5),

(b) Rule I and eight axiom schemes suffice if $\mathfrak{S}<1$ (see §8).

If C and N are taken as undefined and $\mathfrak{J}$ has M members, then:

(a) Rule C and five axiom schemes suffice if $\mathfrak{S}=1$ (see §14). However, in §6 we give an alternative treatment involving Rule C and seven axiom schemes because the alternative development seems of interest and is fairly short. Also the alternative development is far simpler than the development depending on five axiom schemes, and avoids the excessive metamathematical difficulties of the more sophisticated development.

(b) If n is the number of axiom schemes required when $\mathcal{S}=1$, then Rule I and $3+n$ axiom schemes suffice if $\mathcal{S}<1$ (see §9).

2. **A fragment of the C-calculus.** In this section, we derive a number of consequences of the following rule and axiom schemes.

Rule C. If P and CPQ, then Q.

A1. $CPCQP$.

A2. $CCPQCCQRCPR$.

A3. $CAPQAQP$.

We introduce the usual yields sign, $\vdash$, (see [4, p. 34]) and let its signification depend on the current set of axioms and rules. Thus throughout this section, we shall use $\vdash$ as depending on Rule C and axiom schemes A1, A2, A3. As we change axioms or rules, we shall make the corresponding change in the signification of $\vdash$ without comment.

We introduce the special notation

$$P_1, \cdots, P_n \vdash Q \equiv R$$

to denote that both of

$$P_1, \cdots, P_n \vdash CQR,$$
$$P_1, \cdots, P_n \vdash CRQ$$

are valid. Obviously we have $P_1, \cdots, P_n \vdash Q \equiv R$ if and only if we have $P_1, \cdots, P_n \vdash R \equiv Q$; also, from A2 and Rule C, we infer that if $P_1, \cdots, P_n \vdash Q \equiv R$ and $P_1, \cdots, P_n \vdash R \equiv S$, then $P_1, \cdots, P_n \vdash Q \equiv S$. We shall use these properties without comment.

Another principle which we shall usually use without comment is

(2.1) $CPQ, CQR \vdash CPR$,

which follows from A2 and Rule C.

By interchanging P and Q in A3, we infer:

(2.2) $\vdash APQ \equiv AQP$.

Since A2 gives $CPQ \vdash CCQRCPR$ and $CQP \vdash CCPRCQR$, we infer:

(2.3) If $S_1, \cdots, S_n \vdash P \equiv Q$, then $S_1, \cdots, S_n \vdash CPR \equiv CQR$.

By taking Q to be CQP in A1, we infer

(2.4) $\vdash CPAQP$,

whence we get

(2.5) $\vdash CPAPQ$

by A3. Consequently, we infer $\vdash CQAQR$, from which by A2 we get

$$\vdash CCAQRCPRCQCPR.$$

However, by putting CQR for Q in A2, we get

$$\vdash CCPCQRCAQRCPR.$$

By these two formulas we get

(2.6) $\vdash CCPCQRCQCPR$.

Interchanging P and Q gives

(2.7) $\vdash CPCQR \equiv CQCPR$.

By applying (2.6) to A2, we get

(2.8) $\vdash CCQRCCPQCPR$.

Using this, we may reason as in our derivation of (2.3) to infer:

(2.9) If $S_1, \cdots, S_n \vdash P \equiv Q$, then $S_1, \cdots, S_n \vdash CRP \equiv CRQ$.

In (2.6) take R to be P and use A1. This gives $\vdash CQCPP$. By taking Q to be any proved result, we get

(2.10) $\vdash CPP$,

(2.11) $\vdash P \equiv P$.

By use of (2.3), (2.9), and (2.11), we can prove the standard type of equivalence and substitution theorems to the effect that if $S_1, \cdots, S_n \vdash P \equiv Q$, then under the hypotheses $S_1, \cdots, S_n$ one can replace occurrences of P by Q at will in statement formulas built up by use of C alone. We shall make such substitutions without comment. In particular, because of (2.2), we now have full commutativity of A, and will use it freely.

THEOREM 2.1. *Let $Q_1, \cdots, Q_q$ denote an ordered set of statements among which each of $P_1, \cdots, P_p$ occurs at least once. Then*

(2.12) $\vdash C\Gamma_{i=1}^{p} P_i R \Gamma_{i=1}^{q} Q_i R$.

We can use the proof given for Lemma 3.1.4 on p. 35 of [4].

Similarly, by using the proof given for Lemma 3.1.3 on p. 35 of [4], we can infer:

(2.13) $\vdash CCPQC\Gamma_{i=1}^{\alpha} R_i P \Gamma_{i=1}^{\alpha} R_i Q$.

Suppose we have $\Gamma_{i=1}^{\beta} S_i CPQ$. Then we get $CP\Gamma_{i=1}^{\beta} S_i Q$ by Theorem 2.1, and then $C\Gamma_{i=1}^{\alpha} R_i P \Gamma_{i=1}^{\alpha} R_i \Gamma_{i=1}^{\beta} S_i Q$ by (2.13). Consequently

(2.14) $\Gamma_{i=1}^{\alpha} R_i P, \Gamma_{i=1}^{\beta} S_i CPQ \vdash \Gamma_{i=1}^{\alpha} R_i \Gamma_{i=1}^{\beta} S_i Q$.

By means of this, we can prove a generalized version of the familiar Deduction Theorem.

THEOREM 2.2. *If $R_1, \cdots, R_n, P \vdash Q$, then there is a non-negative integer α such that $R_1, \cdots, R_n \vdash (CP)^{\alpha} Q$.*

As we have $\vdash CQCPQ$ by A1 and $P \vdash CCPQQ$ by (2.5), we infer

(2.15) $P \vdash Q \equiv CPQ$.

Taking P to be CQQ and using (2.10) gives

(2.16) $\vdash Q \equiv AQQ$.

By A2,

$$\vdash CCCQRCPRCAPRAQR.$$

Combining this with A2 itself gives

(2.17) $\vdash CCPQCAPRAQR$.

Commutativity of A gives

(2.18) $\vdash CCPQCARPARQ$.

These give

$$\vdash CCPRCAPQARQ$$

$$\vdash CCQSCARQARS.$$

From these last two by (2.14)

$$\vdash CCPRCAPQCCQSARS.$$

Then by (2.7)

(2.19) $\vdash CCPRCCQSCAPQARS$,

from which by (2.16)

(2.20) $CPR,\ CQR \vdash CAPQR$.

THEOREM 2.3. *If* $P_1, \cdots, P_p, R \vdash T$ *and* $Q_1, \cdots, Q_q, S \vdash T$, *then* $P_1, \cdots, P_p, Q_1, \cdots, Q_q, ARS \vdash T$.

Proof. From $P_1, \cdots, P_p, R \vdash T$ and $Q_1, \cdots, Q_q, S \vdash T$ we get

(a) $P_1, \cdots, P_p \vdash (CR)^\alpha T$,

(b) $Q_1, \cdots, Q_q \vdash (CS)^\beta T$

by Theorem 2.2. For any W, we write $\mathcal{E}, ARS \vdash W$ as shorthand for $P_1, \cdots, P_p, Q_1, \cdots, Q_q, ARS \vdash W$. We now prove by induction on γ the following lemma:

If γ is a positive integer, and $\gamma \leqq \alpha+\beta$, and $U_1, \cdots, U_{\alpha+\beta-\gamma}$ are formulas each of which is either R or S, then

(c) $\mathcal{E}, ARS \vdash \Gamma_{i=1}^{\alpha+\beta-\gamma} U_i T$.

First let $\gamma = 1$.

Case 1. There are fewer than α R's among $U_1, \cdots, U_{\alpha+\beta-1}$. Then there are at least β S's. So by Theorem 2.1

$$\vdash C(CS)^\beta T \Gamma_{i=1}^{\alpha+\beta-1} U_i T.$$

Then (c) holds by (b).

Case 2. There are at least α R's among $U_1, \cdots, U_{\alpha+\beta-1}$. Then we can get (c) from (a) by similar reasoning.

Now assume the lemma for γ. Using this and (1.36) we get both of

$$\mathcal{E},\ ARS \vdash CR\Gamma_{i=1}^{\alpha+\beta-\gamma-1} U_i T,$$

$$\mathcal{E},\ ARS \vdash CS\Gamma_{i=1}^{\alpha+\beta-\gamma-1} U_i T.$$

From these by (2.20) we get

$$\mathcal{E},\ ARS \vdash \Gamma_{i=1}^{\alpha+\beta-\gamma-1} U_i T,$$

so that the induction is established.

Finally, we conclude our theorem by taking $\gamma = \alpha+\beta$ in the lemma.

We can prove the associative law for A, namely

(2.21) $\vdash APAQR \equiv AAPQR$,

by the methods used to prove Formel (14) and Formel (15) of §11 of Chapter 1 of [5].

We close with some miscellaneous results that will be needed later.

By (2.7)

$$\vdash CCPQCRQ \equiv CRAPQ.$$

Consequently, commutativity of A gives

(2.22) $\vdash CCPQCRQ \equiv CCQPCRP$.

By (2.5)

(2.23) $\vdash CCPQCCCPQRR$.

Taking R to be Q gives

$$\vdash CCPQCAPQQ.$$

Also, by (2.5) and A2

$$\vdash CCAPQQCPQ.$$

Thus we have shown

(2.24) $\vdash CPQ \equiv CAPQQ$.

By A2

$$\vdash CCCPQRCCRQAPQ.$$

Using commutativity of A followed by (2.7) gives

(2.25) $\vdash CCCPQRCCQPCCRQP$.

By (2.8) and (2.7), $\vdash CCRSCPCCPRS$. So by (2.14)

$$CCCPRSQ \vdash CCRSCPQ.$$

Thence we infer

(2.26) $CCQSCPR,\ CSQ \vdash CCRSCPQ$

by putting Q, S, and CPR respectively for P, Q, and R in (2.25).

3. **A fragment of the $C-N$-calculus.** In this section, we add one more axiom scheme, namely

$$\text{A4. } CCNPNQCQP,$$

to the three considered in the preceding section, and derive a number of consequences involving N.

By A1, $\vdash CNNPCNNQNNP$. By two uses of A4, we get successively $\vdash CNNPCNPNQ$ and $\vdash CNNPCQP$. Then (2.7) gives $\vdash CQCNNPP$. Taking Q to be any proved result gives

(3.1) $\vdash CNNPP$.

From this, by A2, we get $\vdash CCPNQCNNPNQ$. Using A4 gives

(3.2) $\vdash CCPNQCQNP$.

Interchanging P and Q gives

(3.3) $\vdash CPNQ \equiv CQNP$.

Putting NP for Q in (3.3) and using (2.10) we get $\vdash CPNNP$, so that by (3.1)

(3.4) $\vdash P \equiv NNP$.

Using $\vdash Q \equiv NNQ$ with (2.9) gives $\vdash CPQ \equiv CPNNQ$. Putting NQ for Q in (3.3) gives $\vdash CPNNQ \equiv CNQNP$. Thus

(3.5) $\vdash CPQ \equiv CNQNP$.

Consequently

(3.6) If $R_1, \cdots, R_n \vdash P \equiv Q$, then $R_1, \cdots, R_n \vdash NP \equiv NQ$.

This enables us to extend the equivalence and substitution theorems to formulas involving N as well as C. We can thus get many results by familiar transformations involving (3.4) and (3.5). We list a number of such, leaving the details to the reader. The first three are

(3.7) $\vdash APQ \equiv NKNPNQ$.

(3.8) $\vdash BPQ \equiv NLNPNQ$.

(3.9) $\vdash LPQ \equiv NBNPNQ$.

By applying (3.6) to (3.3), we get

(3.10) $\vdash LPQ \equiv LQP$,

whence we get

(3.11) $\vdash BPQ \equiv BQP$.

From the corresponding results for A come

(3.12) $\vdash KPQ \equiv KQP$,

(3.13) $\vdash CKQPP$,

(3.14) $\vdash CKPQP$,

(3.15) $\vdash Q \equiv KQQ$,

(3.16) $\vdash CCPQCKPRKQR$,

(3.17) $\vdash CCPQCKRPKRQ$,

(3.18) $\vdash CCPRCCQSCKPQKRS$,

(3.19) $CPQ, CPR \vdash CPKQR$,

(3.20) $\vdash KPKQR \equiv KKPQR$,

(3.21) $\vdash CQP \equiv CQKPQ$.

Since $\vdash CPCQP$ by A1, we can use (3.21) to infer

(3.22) $\vdash CPCQKPQ$.

By (2.8)

(3.23) $\vdash CCPQCBRPBRQ$,

whence we get

(3.24) $\vdash CCPQCBPRBQR$,

(3.25) $\vdash CCPRCCQSCBPQBRS$,

(3.26) $\vdash CCPQCLRPLRQ$,

(3.27) $\vdash CCPQCLPRLQR$,

(3.28) $\vdash CCPRCCQSCLPQLRS$.

Putting NP and NQ for P and Q in (2.7) gives

$$\vdash BPBQR \equiv BQBPR.$$

Interchanging Q and R gives

$$\vdash BPBRQ \equiv BRBPQ.$$

Then commutativity of B gives

(3.29) $\vdash BPBQR \equiv BBPQR$,

whence we get

(3.30) $\vdash LPLQR \equiv LLPQR$.

Putting NQ for Q in A1 gives

(3.31) $\vdash CPBQP$,

whence we get

(3.32) $\vdash CPBPQ$,

(3.33) $\vdash CLQPP$,

(3.34) $\vdash CLPQP$.

By (3.5)

$$\vdash CPCQR \equiv CPCNRNQ.$$

Then by (2.7)

$$\vdash CPCQR \equiv CNRCPNQ.$$

Finally by (3.11)

(3.35) $\vdash CPCQR \equiv CLPQR$.

Applying this to $\vdash CLPQLPQ$, we get

(3.36) $\vdash CPCQLPQ$.

By this and (3.33)

(3.37) $P \vdash Q \equiv LPQ$.

THEOREM 3.1. *$R_1, \cdots, R_n \vdash P \equiv Q$ if and only if $R_1, \cdots, R_n \vdash EPQ$.*

Proof. By (3.36)

$$CPQ, CQP \vdash EPQ,$$

and by (3.34) and (3.33)

$$EPQ \vdash P \equiv Q.$$

By use of this theorem, we can get

(3.38) $\vdash EPP$,

(3.39) $EPQ, EQR \vdash EPR$

directly, and

(3.40) $EPQ \vdash ENPNQ$,

(3.41) $EPQ, ERS \vdash ECPRCQS$

by appealing respectively to (3.6), and to both of (2.3) and (2.9). By (3.10), we have

(3.42) $\vdash CEPQEQP$.

With both A and B serving as disjunctions and both K and L serving as conjunctions, one can write a number of possible distributive laws. Some are not valid, and of the valid ones we have been able to prove only two from axiom schemes A1–A4. We now give the proofs.

By (2.5) and the commutativity of L

$$\vdash CLQPALPQLPR.$$

Then by (3.35)

$$\vdash CQCPALPQLPR.$$

Similarly

$$\vdash CRCPALPQLPR.$$

Then by (2.20)

$$\vdash CAQRCPALPQLPR.$$

Finally by (3.35) and the commutativity of L

(a) $\vdash CLPAQRALPQLPR.$

By (2.5) and (3.26)

$$\vdash CLPQLPAQR.$$

By (2.4) and (3.26)

$$\vdash CLPRLPAQR.$$

Then by (2.20)

$$\vdash CALPQLPRLPAQR.$$

From this and (a) we get

(3.43) $\vdash LPAQR \equiv ALPQLPR.$

By replacing P, Q, and R by NP, NQ, and NR, we get

(3.44) $\vdash BPKQR \equiv KBPQBPR.$

By (2.15)

(3.45) $NP \vdash Q \equiv BPQ.$

By (2.10) and A1, $\vdash CPCRR$. So by (3.36)

$$\vdash CCCRRPEPCRR.$$

But by A1, $\vdash CPCCRRP$, so that

(3.46) $\vdash CPEPCRR.$

Then by Theorem 3.1,

(3.47) $P \vdash P \equiv CRR,$

whence, by the transitivity of $\equiv$,

(3.48) $P, Q \vdash P \equiv Q.$

By (3.47), (3.6), and (3.4)

(3.49) $NP \vdash P \equiv NCRR,$

whence

(3.50) $NP, NQ \vdash P \equiv Q$.

We close with some miscellaneous results that will be needed later.

Negating all variables of (2.22) and applying (3.5) gives

(3.51) $\vdash CCQPCQR \equiv CCPQCPR$.

We raise the question if this can be proved from A1, A2, and A3 alone.

By (3.32) and A2

$$\vdash CCBPQQCPQ.$$

This is

(3.52) $\vdash CANPQCPQ$.

By A1, $\vdash CNSCNRNS$, so that by (3.5)

(3.53) $\vdash CNSCSR$.

A simple application of (3.4) gives

(3.54) $\vdash BLPQR \equiv CCPNQR$.

If we put NQ and NS for Q and S in (2.26) and use (3.5), we get

$$CQS, CCSQCPR \vdash CCRNSCPNQ.$$

Another use of (3.5) gives

(3.55) $CQS, CCSQCPR \vdash CLPQLRS$.

By (3.32), $P \vdash BPR$, so that by (3.36) $P, Q \vdash LBPRQ$. However, by (3.52)

$$ANPQ, P \vdash Q.$$

So

(3.56) $ANPQ, P \vdash LBPRQ$.

THEOREM 3.2.

(3.57) $ANPQ \vdash LBPRQ \equiv BPLQR$.

Proof. By (3.37)

$$Q \vdash BPR \equiv BPLQR,$$

$$Q \vdash LQBPR \equiv BPR.$$

So by the commutativity of L

(a) $Q \vdash LBPRQ \equiv BPLQR$.

By (3.45)

$$NP \vdash LQR \equiv BPLQR,$$

$$NP \vdash LBPRQ \equiv LRQ.$$

So by the commutativity of L

(b) $NP \vdash LBPRQ \equiv BPLQR$.

Our theorem follows from (a) and (b) by Theorem 2.3.

4. **Special results for use in the finite-valued case.** We continue with the same four axiom schemes as in the preceding section.

By (2.10) and (3.4)

(4.1) $\vdash NB_0P$.

Then by (3.50)

(4.2) $\vdash B_0P \equiv B_0Q$,

and by (3.45) and the commutativity of B

(4.3) $\vdash P \equiv BPB_0Q$.

Taking Q to be P in this gives

(4.4) $\vdash P \equiv B_1P$.

THEOREM 4.1. *If α and β are non-negative integers, then*

(4.5) $\vdash B_{\alpha+1}P \equiv BPB_\alpha P$,

(4.6) $\vdash B_{\alpha+1}P \equiv CNB_\alpha PP$,

(4.7) $\vdash B_{\alpha+\beta}P \equiv BB_\alpha PB_\beta P$.

Proof. We infer (4.5) by (1.27), and then deduce (4.6) by the commutativity of B. To prove (4.7), we use induction on α. When $\alpha=0$, use (4.3) and the commutativity of B. For the induction step, use (4.5) and the associativity of B.

THEOREM 4.2. *If α and β are non-negative integers, then*

(4.8) $\vdash B_{\alpha\beta}P \equiv B_\alpha B_\beta P$.

Proof by induction on α. When $\alpha=0$, use (4.2). For the induction step, use (4.7) and (4.5).

THEOREM 4.3. *If α and β are non-negative integers, then*

(4.9) $\vdash CB_\alpha PB_{\alpha+\beta}P$.

Proof. By (3.32)

$$\vdash CB_\alpha PBB_\alpha PB_\beta P.$$

Now use (4.7).

THEOREM 4.4. *If α and β are positive integers, then*

(4.10) $CB_\alpha PNB_\beta P \vdash CB_{\alpha-1}PNB_{\beta+1}P$.

Proof. Assume

(a) $CB_\alpha PNB_\beta P$.

By A2

$$CCNB_\beta PPCB_\alpha PP.$$

So by (4.6)

(b) $CB_{\beta+1}PCB_\alpha PP$.

By (a) and (3.3)

(c) $CB_\beta PNB_\alpha P$.

By (4.4) and (4.9)

(d) $CPB_\beta P$.

By (4.9) and (3.5)
(e) $CNB_{\alpha}PNB_{\alpha-1}P$.
By (c), (d), and (e)

$$CPNB_{\alpha-1}P.$$

By (2.24), this gives

$$CAPNB_{\alpha-1}PNB_{\alpha-1}P.$$

By the commutativity of A

$$CCCNB_{\alpha-1}PPPNB_{\alpha-1}P.$$

Then by (4.6)

$$CCB_{\alpha}PPNB_{\alpha-1}P,$$

whence by (b)

$$CB_{\beta+1}PNB_{\alpha-1}P.$$

By (3.3), we conclude our theorem.

THEOREM 4.5. *If α and β are positive integers, and γ is a non-negative integer, and $\gamma \leqq \alpha$, then*
(4.11) $CB_{\alpha}PNB_{\beta}P \vdash CB_{\alpha-\gamma}PNB_{\beta+\gamma}P$.

Proof by induction on γ, using Theorem 4.4 for the induction step.

THEOREM 4.6. *If α and β are positive integers, then*
(4.12) $CPNB_{\alpha+\beta-1}P \vdash CB_{\alpha}PNB_{\beta}P$,
(4.13) $CB_{\alpha}PNB_{\beta}P \vdash CPNB_{\alpha+\beta-1}P$.

Proof. First assume $CPNB_{\alpha+\beta-1}P$. By (3.3) and (4.4), $CB_{\alpha+\beta-1}PNB_{1}P$, so that by (4.11) $CB_{\alpha+\beta-1-\gamma}PNB_{1+\gamma}P$. Then we get (4.12) by taking $\gamma=\beta-1$. To get (4.13), we take $\gamma=\alpha-1$ in (4.11) and use (4.4).

THEOREM 4.7. *If α is a positive integer, then*
(4.14) $V_{\alpha}P \vdash B_{\alpha}P$.

Proof. This follows by (3.33) and (4.6).

THEOREM 4.8. *If α and β are positive integers, then*
(4.15) $V_{\alpha+\beta}P \vdash EB_{\alpha}PNB_{\beta}P$,
(4.16) $EB_{\alpha}PNB_{\beta}P \vdash V_{\alpha+\beta}P$.

Proof. First assume $V_{\alpha+\beta}P$. Then by (3.34) and (3.33), we get $CPNB_{\alpha+\beta-1}P$ and $CNB_{\alpha+\beta-1}PP$. From the first, we get $CB_{\alpha}PNB_{\beta}P$ by (4.12), and from the second we get $BB_{\beta}PB_{\alpha}P$ by (4.6) and (4.7). Then we get $EB_{\alpha}PNB_{\beta}P$ by (3.36). To get (4.16), we merely reverse the steps just given.

THEOREM 4.9. *If* α *is a positive integer, and* γ *is a non-negative integer, and* $\gamma \leqq \alpha$, *then*

(4.17) $V_\alpha P \vdash EB_\gamma PNB_{\alpha-\gamma}P$,

(4.18) $V_\alpha P \vdash EB_{\alpha-\gamma}PNB_\gamma P$.

Proof. We note that if (4.17) can be proved for all γ with $0 \leqq \gamma \leqq \alpha$, then (4.18) follows by replacing γ by $\alpha - \gamma$. If $0 < \gamma < \alpha$, then both (4.17) and (4.18) follow from (4.15). To handle the remaining cases, we note first that by (4.14) and (3.46)

(a) $V_\alpha P \vdash EB_\alpha PCPP$.

By (3.4), this gives

$$V_\alpha P \vdash EB_\alpha PNB_0 P,$$

which gives (4.17) for the case $\gamma = \alpha$. By applying (3.40) and (3.42) to (a), we get

$$V_\alpha P \vdash EB_0 PNB_\alpha P,$$

which gives (4.17) for the case $\gamma = 0$.

THEOREM 4.10. *If* α *is a positive integer, and* γ *is a non-negative integer, and* $\gamma \leqq \alpha$, *then*

(4.19) $V_\alpha R,\ W_\gamma PR \vdash W_{\alpha-\gamma} NPR$.

Proof. By (3.40)

$$W_\gamma PR \vdash ENPNB_\gamma R$$

and by (4.18)

$$V_\alpha R \vdash EB_{\alpha-\gamma} RNB_\gamma R.$$

Combining these by (3.39) and (3.42) gives (4.19).

THEOREM 4.11. *If* α *is a positive integer, and* β *and* γ *are non-negative integers, and* $\beta \leqq \alpha$, *then*

(4.20) $V_\alpha P \vdash W_{\alpha-\beta+\gamma} CB_\beta PB_\gamma PP$.

Proof. By (4.17) and Theorem 3.1

$$V_\alpha P \vdash B_\beta P \equiv NB_{\alpha-\beta} P.$$

Then by (2.3) and Theorem 3.1

$$V_\alpha P \vdash ECB_\beta PB_\gamma PBB_{\alpha-\beta} PB_\gamma P.$$

Finally we use (4.7) and (1.30).

THEOREM 4.12. *If* α *is a positive integer, and* β *and* γ *are non-negative integers, and* $\beta \leqq \alpha$, *and* $\eta = \min(\alpha, \alpha - \beta + \gamma)$, *then*

(4.21) $V_\alpha R,\ W_\beta PR,\ W_\gamma QR \vdash W_\eta CPQR$.

Proof. Assume $V_\alpha R$, $W_\beta PR$, and $W_\gamma QR$. As we have

$$W_\beta PR \vdash P \equiv B_\beta R,$$
$$W_\gamma QR \vdash Q \equiv B_\gamma R,$$

by Theorem 3.1, we infer

(a) $W_{\alpha-\beta+\gamma}CPQR$

by (4.20), and we infer

(b) $B_\alpha R$

by (4.14). If $\alpha-\beta+\gamma\leqq\alpha$, then (a) gives the desired result. So assume $\alpha<\alpha-\beta+\gamma$. Then by (b) and (4.9) we get $B_{\alpha-\beta+\gamma}R$, while by (a) and (3.33) we get $CB_{\alpha-\beta+\gamma}RCPQ$, so that we can conclude

(c) CPQ.

Then we conclude

(d) $W_\alpha CPQR$

by (b), (c), (3.48), and Theorem 3.1. In this case, (d) gives the desired result.

THEOREM 4.13. *Let α be a positive integer, let n be a non-negative integer, and let β_r $(0\leqq r\leqq n)$ be non-negative integers such that $\beta_r\leqq\alpha$ $(0\leqq r\leqq n)$. Let $\phi(P_0, \cdots, P_n)$ be a statement formula built up from $P_0, \cdots, P_n$ by means of C and N. Let μ be that non-negative integer with $\mu\leqq\alpha$ such that if P_r is assigned the value β_r/α $(0\leqq r\leqq n)$, then $\phi(P_0, \cdots, P_n)$ takes the value μ/α. Then*

(4.22) $V_\alpha R, W_{\beta_0}P_0R, \cdots, W_{\beta_n}P_nR \vdash W_\mu\phi(P_0, \cdots, P_n)R.$

Proof by induction on the structure of ϕ, using Theorem 4.10 and Theorem 4.12.

THEOREM 4.14. *If α and β are positive integers and $\beta\leqq\alpha$, and γ and δ are non-negative integers, then*

(4.23) $LV_\beta SW_\delta PS, W_\gamma RP \vdash \sum_{r=0}^{\alpha} (W_rRS).$

Proof. Assume $LV_\beta SW_\delta PS$ and $W_\gamma RP$. By (3.33), (3.34), Theorem 3.1, (4.8) and (4.14)

(a) $W_{\gamma\delta}RS$,

(b) $B_\beta S$.

If $\gamma\delta\leqq\alpha$, then we have

$$W_{\gamma\delta}RS \vdash \sum_{r=0}^{\alpha} (W_rRS)$$

by (2.4) and (2.5), so that our theorem follows by (a). So let $\alpha<\gamma\delta$. Then $\beta<\gamma\delta$, so that by (b) and (4.9) we get $B_{\gamma\delta}S$, while by (a) and (3.33) we get $CB_{\gamma\delta}SR$; thence we get

(c) R.

Then we conclude

(d) $W_\beta RS$

by (b), (c), (3.48), and Theorem 3.1. As $\beta\leqq\alpha$, we have

$$W_\beta RS \vdash \sum_{r=0}^{\alpha} (W_r RS)$$

by (2.4) and (2.5), so that our theorem follows from (d) in this case.

THEOREM 4.15. *If α is a positive integer, then*

(4.24) $V_\alpha B_0 P \vdash Q$.

Proof. By (4.1) and (3.53)
(4.25) $\vdash CB_0PQ$.
By (4.8), $\vdash B_0P \equiv B_\alpha B_0 P$, and by (4.14), $V_\alpha B_0 P \vdash B_\alpha B_0 P$, so that our theorem follows.

THEOREM 4.16. *If α is a positive integer and β is a non-negative integer, then*
(4.26) $V_\alpha P,\ V_\alpha Q,\ W_\beta PQ \vdash EPQ$.

Proof. *Case* 1. $\beta = 0$. Then by Theorem 3.1

$$V_\alpha P,\ W_\beta PQ \vdash V_\alpha B_0 Q.$$

Then by Theorem 4.15,

$$V_\alpha P,\ W_\beta PQ \vdash EPQ.$$

Case 2. $\beta = 1$. Then by (4.4),

$$W_\beta PQ \vdash EPQ.$$

Case 3. $\alpha = 1$. Then by (1.29) and Theorem 3.1

$$V_\alpha P \vdash P \equiv NB_0P,$$

$$V_\alpha Q \vdash Q \equiv NB_0Q.$$

Then by (4.2), (3.6), and Theorem 3.1, $V_\alpha P,\ V_\alpha Q \vdash EPQ$.
Case 4. $\alpha \geqq 2$ and $\beta \geqq 2$. Then $(\alpha-1)\beta \geqq \alpha$, so that by (4.9) and (4.14)

$$V_\alpha Q \vdash B_{(\alpha-1)\beta} Q.$$

However, by (4.8)

$$W_\beta PQ \vdash B_{\alpha-1} P \equiv B_{(\alpha-1)\beta} Q.$$

The last two results give
(a) $V_\alpha Q,\ W_\beta PQ \vdash B_{\alpha-1} P$.
By (4.18) and (4.4)
(b) $V_\alpha P \vdash EB_{\alpha-1} PNP$.
Then by (a), (b), and (3.34),

$$V_\alpha P,\ V_\alpha Q,\ W_\beta PQ \vdash NP.$$

From this by (3.49)

$$V_\alpha P,\ V_\alpha Q,\ W_\beta PQ \vdash P \equiv B_0 R.$$

Thus

$$V_\alpha P,\ V_\alpha Q,\ W_\beta PQ \vdash V_\alpha B_0 R,$$

so that by Theorem 4.15

$$V_\alpha P,\ V_\alpha Q,\ W_\beta PQ \vdash EPQ.$$

THEOREM 4.17. *If α is a positive integer and β is a non-negative integer, then* (4.27) $V_\alpha P,\ V_\alpha Q,\ \sum_{r=0}^{\beta} (W_r PQ) \vdash EPQ.$

Proof. Use Theorem 4.16 and Theorem 2.3.

5. **The case when $\mathfrak{I}$ has M members, $\mathcal{S}=1$, and C, N, and T are taken as undefined.** We make the assumptions just listed, and use Rule C, axiom schemes A1–A4 and also the two following axiom schemes:

AT1. $V_{M-1}NTP$.

AT2. $\sum_{r=0}^{M-1} (W_r PNTQ)$.

Since $M \geqq 2$, we get by (4.27), Theorem 3.1, AT1, and AT2

(5.1) $V_{M-1}P \vdash P \equiv NTQ$,

whence by AT1, (3.6), and (3.4)

(5.2) $\vdash TP \equiv TQ$.

By AT1, (3.40), and (3.4), we get $\vdash W_{M-2}TPNTP$, whence we get

(5.3) $\vdash W_{M-2}TPNTQ$

by (5.2). By (5.2) we can extend the equivalence and substitution theorems to the case where T is used as well as C and N.

THEOREM 5.1. *Let n be a non-negative integer, and let β_r $(0 \leqq r \leqq n)$ be non-negative integers such that $\beta_r \leqq M-1$ $(0 \leqq r \leqq n)$. Let $\phi(P_0, \cdots, P_n)$ be a statement formula built up from $P_0, \cdots, P_n$ by means of C, N, and T. Let μ be that non-negative integer with $\mu \leqq M-1$ such that if P_r is assigned the value $\beta_r/(M-1)$ $(0 \leqq r \leqq n)$, then $\phi(P_0, \cdots, P_n)$ takes the value $\mu/(M-1)$. Then*

(5.4) $W_{\beta_0}P_0NTQ, \cdots, W_{\beta_n}P_nNTQ \vdash W_\mu \phi(P_0, \cdots, P_n)NTQ.$

Proof by induction on the structure of ϕ, using AT1, Theorem 4.10, Theorem 4.12, and (5.3).

As a temporary definition, we introduce a generalized product by the following recursion:

(5.5) If $\beta = \alpha$, then $\prod_{i=\alpha}^{\beta} P_i$ denotes P_α.

(5.6) If $\beta > \alpha$, then $\prod_{i=\alpha}^{\beta} P_i$ denotes $LP_\beta \prod_{i=\alpha}^{\beta-1} P_i$.

By (3.33) and (3.34), we can rewrite (5.4) as

(5.7) $\prod_{r=0}^{n} (W_{\beta_r}P_rNTQ) \vdash W_\mu \phi(P_0, \cdots, P_n)NTQ.$

By (4.14) and AT1,

(5.8) $\vdash B_{M-1}NTQ$.

Therefore, by (3.33)

(5.9) $W_{M-1}PNTQ \vdash P$.

THEOREM 5.2. *Let $\phi(P_0, \cdots, P_n)$ be a statement formula built up from*

$P_0, \cdots, P_n$ by means of C, N, and T. Then $\vdash\phi$ if and only if the corresponding truth-value function takes only designated values.

Proof. For the "only if" part, we use the standard type of proof. So assume that the truth-value function corresponding to ϕ takes only designated truth-values. As 1 is the only designated truth-value, we infer from Theorem 5.1 that for each set of non-negative integers β_r with $\beta_r \leqq M-1$ $(0 \leqq r \leqq n)$

$$\prod_{r=0}^{n} (W_{\beta_r}P_rNTQ) \vdash W_{M-1}\phi NTQ.$$

Then by (5.9)

$$\prod_{r=0}^{n} (W_{\beta_r}P_rNTQ) \vdash \phi.$$

From this, by Theorem 2.3, AT2, and the distributive law for A and L, we can infer $\vdash\phi$.

6. **The case when $\mathfrak{J}$ has M members, $\mathfrak{S}=1$, and C and N are taken as undefined.** With these assumptions, it follows from a theorem of McNaughton (see [3]) that one can define a function F whose corresponding truth-value function $f(x, y)$ has the following property:

Let b and d be divisors of $M-1$. Let $x=a/b$ and $y=c/d$, where $(a, b) = (c, d) = 1$ (we regard 0 as being 0/1 for this purpose). Then $f(x, y) = 1/\{b, d\}$, where $\{b, d\}$ denotes the least common multiple of b and d.

Clearly the definition of F depends on the value of M.

We use Rule C, axiom schemes A1–A4, and also the three following axiom schemes:

AF1. $CFPQFQP$,

AF2. $\sum_{r=0}^{M-1} (W_rPFPQ)$,

AF3. $\sum_{j=1}^{d} (V_{\alpha(j)}FPQ)$,

where d denotes the number of positive divisors of $M-1$ and $\alpha(j)$ denotes the jth positive divisor of $M-1$, starting with the least and counting up.

Interchanging P and Q in AF1 gives

(6.1) $\vdash FPQ \equiv FQP$,

so that by AF2

(6.2) $\vdash \sum_{r=0}^{M-1} (W_rPFQP)$.

THEOREM 6.1. *If γ is a non-negative integer, then*

(6.3) $W_\gamma RP \vdash \sum_{r=0}^{M-1} (W_rRFQP)$.

Proof. If $1 \leqq j \leqq d$, then by (4.23)

$$LV_{\alpha(j)}FQPW_\delta PFQP,\ W_\gamma RP \vdash \sum_{r=0}^{M-1} (W_rRFQP).$$

From this by Theorem 2.3, AF3, (6.2) and the distributive law for A and L, we can infer (6.3).

Let us define Φ by the following recursion:

(6.4) If $\alpha=\beta$, then $\Phi_{i=\alpha}^{\beta} P_i$ denotes P_α.
(6.5) If $\alpha<\beta$, then $\Phi_{i=\alpha}^{\beta} P_i$ denotes $FP_\beta\Phi_{i=\alpha}^{\beta-1} P_i$.

THEOREM 6.2. *Let β and n be non-negative integers with $\beta\leqq n$. Then*
(6.6) $\vdash \sum_{r=0}^{M-1} (W_rP_\beta\Phi_{i=0}^{n} P_i)$.

Proof by induction on n. By (3.38) and (4.4),
(6.7) $\vdash W_1PP$.
Taking P to be P_0, and using (6.4), (2.4), and (2.5), we conclude that (6.6) holds when $n=0$. Assume (6.6) for n.

Case 1. $\beta=n+1$. Then (6.6) holds for $n+1$ by AF2 and (6.5).

Case 2. $\beta\leqq n$. By (6.5) and (6.3).

$$W_\gamma P_\beta\Phi_{i=0}^{n}P_i \vdash \sum_{r=0}^{M-1} (W_rP_\beta\Phi_{i=0}^{n+1}P_i).$$

Then by Theorem 2.3

$$\sum_{r=0}^{M-1} (W_rP_\beta\Phi_{i=0}^{n}P_i) \vdash \sum_{r=0}^{M-1} (W_rP_\beta\Phi_{i=0}^{n+1}P_i).$$

Thus, since we are assuming (6.6) for n, we get (6.6) for $n+1$.

THEOREM 6.3. *Let α be a positive integer and let β and n be non-negative integers with $\beta\leqq n$. Then*
(6.8) $V_\alpha\Phi_{i=0}^{n} P_i \vdash \sum_{r=0}^{\alpha} (W_rP_\beta\Phi_{i=0}^{n} P_i)$.

Proof. In Theorem 4.14, take $\beta=\alpha$, $\delta=1$, P and S to be Φ, and R to be P_β. Then by (6.7), we have

$$V_\alpha\Phi,\ W_\gamma P_\beta\Phi \vdash \sum_{r=0}^{\alpha} (W_rP_\beta\Phi).$$

Then by Theorem 2.3 and (6.6), we infer (6.8).

THEOREM 6.4. *Let α be a positive integer, let γ be a non-negative integer with $\gamma\leqq\alpha$, and let n be a non-negative integer. Let $\phi(P_0, \cdots, P_n)$ be a statement formula built up from $P_0, \cdots, P_n$ by means of C and N. Suppose that whenever β_r $(0\leqq r\leqq n)$ are non-negative integers with $\beta_r\leqq\alpha$, and P_r is given the value β_r/α $(0\leqq r\leqq n)$, the corresponding value of $\phi(P_0, \cdots, P_n)$ is greater than or equal to γ/α. Then*
(6.9) $V_\alpha\Phi_{i=0}^{n} P_i,\ B_\gamma\Phi_{i=0}^{n} P_i \vdash \phi(P_0, \cdots, P_n)$.

Proof. Using the product notation of (5.5) and (5.6), we get by (4.22)

$$V_\alpha\Phi, \prod_{r=0}^{n} (W_{\beta_r}P_r\Phi) \vdash W_\mu\phi\Phi.$$

Thus by (3.33)

(a) $V_\alpha\Phi, \prod_{r=0}^{n}(W_{\beta_r}P_r\Phi)\vdash CB_\mu\Phi\phi$.
By the hypothesis of the theorem, $\gamma \leqq \mu$. So by (4.9)

$$B_\gamma\Phi \vdash B_\mu\Phi.$$

So by (a)
(b) $V_\alpha\Phi, B_\gamma\Phi, \prod_{r=0}^{n}(W_{\beta_r}P_r\Phi)\vdash\phi$.
Since this holds for each choice of β_r with $0\leqq\beta_r\leqq\alpha$ $(0\leqq r\leqq n)$, we can use Theorem 2.3, Theorem 6.3, and the distributive law for A and L to infer (6.9).

THEOREM 6.5. *Let $\phi(P_1, \cdots, P_n)$ be a statement formula built up from $P_1, \cdots, P_n$ by means of C and N. Then $\vdash\phi$ if and only if the corresponding truth-value function takes only designated values.*

Proof. Assume that the truth-value function corresponding to ϕ takes only designated truth-values. Write $\theta(P_0, \cdots, P_n)$ for $CCP_0P_0\phi(P_1, \cdots, P_n)$. Then θ takes only designated truth-values for any assignment of values to $P_0, P_1, \cdots, P_n$. Let $\alpha(j)$ be a divisor of $M-1$. Then we may take both α and γ equal to $\alpha(j)$ in Theorem 6.4, so that by (4.14)

$$V_{\alpha(j)}\Phi_{i=0}^{n}P_i \vdash \theta(P_0, \cdots, P_n).$$

Since $n\geqq 1$, we may use (6.5), AF3, and Theorem 2.3 to infer

$$\vdash\theta(P_0, \cdots, P_n).$$

Finally, by (2.10) and the definition of θ, we conclude $\vdash\phi$.

7. **A fragment of the C-N-J-D calculus.** In this section we take C and N as undefined, and we assume that J and D are either undefined or are definable in terms of C and N. We use Rule I and the axiom schemes:

AJ1. $JCPCQP$.
AJ2. $JCCPQCCQRCPR$.
AJ3. $JCAPQAQP$.
AJ4. $JCCNPNQCQP$.
AJ5. $IJCPQIJPJQ$.
AJ6. $IJCPQIPQ$.
AJ7. $IIQRIAPQAPR$.

By Rule I and AJ5, we infer the following rule:
Rule JC. If JP and $JCPQ$, then JQ.
Using this and AJ1–AJ4, we can easily prove the following theorem.

THEOREM 7.1. *If $P_1, \cdots, P_n\vdash Q$ can be derived on the basis of Rule* C *and axiom schemes* A1–A4, *then $JP_1, \cdots, JP_n\vdash JQ$.*

From this by (2.16) and (2.5) we get

$$\vdash JCAPPP,$$

$$\vdash JCPAPQ.$$

From these and AJ3 we get

(7.1) $\vdash IAPPP$,

(7.2) $\vdash IPAPQ$,

(7.3) $\vdash IAPQAQP$,

by means of AJ6.

THEOREM 7.2. *If we count I, D, A, and & as the two-valued implication, negation, disjunction, and conjunction, we have the full two-valued statement calculus.*

Proof. Rule I is the standard rule, and (7.1), (7.2), (7.3), and AJ7 are the standard axiom schemes for the two-valued calculus (for example, see [5]).

In particular, we can get such results as the two-valued commutativity and associativity of &, and we can get the two-valued distributive laws for & and A. Moreover, we can get such standard results as the following.

THEOREM 7.3. *If $P_1, \cdots, P_p, R \vdash T$ and $Q_1, \cdots, Q_q, S \vdash T$, then $P_1, \cdots, P_p, Q_1, \cdots, Q_q, ARS \vdash T$.*

By Theorem 7.1, (2.4), and (2.5), we have for $\alpha \leqq \gamma \leqq \beta$

$$JP_\gamma \vdash J \sum_{i=\alpha}^{\beta} P_i.$$

Then by Theorem 7.3, we can infer the following theorem.

THEOREM 7.4. *If α and β are integers with $\alpha \leqq \beta$, then*

(7.4) $\sum_{i=\alpha}^{\beta} (JP_i) \vdash J \sum_{i=\alpha}^{\beta} P_i$.

By Rule JC and AJ1

$$JP \vdash JCJPP.$$

Then by Rule I and AJ6

$$JP \vdash IJPP.$$

So by Rule I

(7.5) $JP \vdash P$.

8. **The case when $\mathfrak{I}$ has M members, $\mathfrak{S} < 1$, and C, N, and T are taken as undefined.** Let J and D be defined in terms of C and N (see [3] or [4]). Let H be the least integer such that $H/(M-1)$ is designated. We use Rule I, axiom schemes AJ1–AJ5 and also the three following axiom schemes:

ATJ1. $JV_{M-1}NTP$.

ATJ2. $J\sum_{r=0}^{M-1} (W_rPNTQ)$.

ATJ3. $IJCB_HNTQPP$.

Inasmuch as only Rule I and axiom schemes AJ1–AJ5 were used in proving Theorem 7.1, we see that we can prove a theorem analogous to Theorem 7.1 except that it refers to results derivable on the basis of Rule C and axiom schemes A1–A4 and axiom schemes AT1–AT2.

We now prove a theorem whose statement is identical with that of Theorem 5.2. We assume that ϕ is a formula whose truth-value is always designated. Then Theorem 5.2 tells us that we can derive $CB_HNTQ\phi$ from axiom schemes A1–A4 and AT1–AT2 by Rule C. So by our generalized Theorem 7.1, we get

$$\vdash JCB_HNTQ\phi.$$

Then $\vdash\phi$ by axiom scheme ATJ3.

9. **The case when $\mathfrak{J}$ has M members, $\mathcal{S}<1$, and C and N are taken as undefined.** As in §§6 and 8, we let F, J, and D be defined in terms of C and N. We also take d and $\alpha(j)$ as in §6, and if $1\leqq j\leqq d$, we take $\gamma(j)$ to be the least integer such that $\gamma(j)/\alpha(j)$ is designated. We take $J_n(P)$ as defined in [4] and use $G(P)$ to designate

$$\sum_{j=1}^{d} KJ_{M-\alpha(j)}(P)B_{\gamma(j)}P.$$

We use Rule I, axiom schemes AJ5–AJ6, the following axiom scheme

AG. $G(FPQ)$,

and a set of auxiliary axiom schemes built up as follows:
Choose a set of axiom schemes such that from them by means of Rule C one can derive exactly those statement formulas built up by means of C and N whose corresponding truth-value functions take only the truth-value 1. Then prefix a J to each of these axiom schemes. The resulting set of axiom schemes is the set of auxiliary axiom schemes.

In view of Theorem 6.5, the auxiliary axiom schemes could be got by prefixing a J to each of A1–A4 and AF1–AF3. Alternatively, the auxiliary axiom schemes could be got by prefixing a J to each of the five axiom schemes appearing in §14.

By Rule I and AJ5, we infer Rule JC. By Rule JC and the auxiliary axiom schemes, we can prove:

THEOREM 9.1. *Let $\phi(P_0, \cdots, P_n)$ be a statement formula built up from $P_0, \cdots, P_n$ by means of C and N such that the corresponding truth-value function takes only the truth-value* 1. *Then*

$$\vdash J\phi(P_0, \cdots, P_n).$$

We now prove a theorem whose statement is identical with that of Theorem 6.5. We assume that $\phi(P_0, \cdots, P_n)$ is a formula whose truth-value is always designated. Then

$$CG(FP_0\Phi_{i=0}^{n}P_i)\phi(P_0, \cdots, P_n)$$

always takes the value unity. So by Theorem 9.1

$$\vdash JCG(FP_0\Phi_{i=0}^{n}P_i)\phi(P_0, \cdots, P_n).$$

Now by axiom scheme AJ6,

$$\vdash IG(FP_0\Phi_{i=0}^{n}P_i)\phi(P_0, \cdots, P_n)$$

so that we get $\vdash\phi$ by axiom scheme AG.

10. **Special results for use in the infinite-valued case.** We adjoin an additional axiom scheme A5 to the four used in §§3 and 4. Actually, C. A. Meredith and later independently C. C. Chang discovered that axiom scheme A5 is a consequence of Rule C and axiom schemes A1–A4, so that it would suffice to assume the latter. The proofs of Meredith and Chang appear in notes after the end of the present paper, but for the present it is convenient merely to refer to the result in question as the fifth one of our axiom schemes. For the reader's convenience, we state in full the axiom schemes we will be using.

In this section, we use Rule C and the following axiom schemes:

A1. $CPCQP$.
A2. $CCPQCCQRCPR$.
A3. $CAPQAQP$.
A4. $CCNPNQCQP$.
A5. $ACPQCQP$.

Theorem 10.1.
(10.1) $\vdash LCCPQRCQP \equiv LCCRQPCQR$.

Proof. Temporarily let us write
(a) V for $LCCPQRCQP$
and
(b) W for $LCCRQPCQR$.
By (2.25), (3.35), and (a)

$$\vdash CVCCRQP.$$

So by (3.37), the commutativity of L, and (b)
(c) $CQR \vdash CVW$.
Interchanging P and R in (3.51) gives

$$\vdash CCCQRCQPCCRQCRP.$$

Then by (2.7)
(d) $CRQ \vdash CCCQRCQPCRP$.
By A2, we have $CPR \vdash CCRQCPQ$, whence, by A2 again, we get

$$CPR \vdash CCCPQRCCRQR.$$

Using this and $\vdash CCRPCRP$ in (2.14) gives

$$CPR \vdash CCCPQRCCRQCCRPP,$$

whence by two uses of (2.7) we get

$$CPR \vdash CCRPCCCPQRCCRQP.$$

Using this and (d) gives

(e) $CPR, CRQ \vdash CCCQRCQPCCCPQRCCRQP.$

By (2.8)

(f) $CPR \vdash CCQPCQR.$

By (f), (e), (3.55), (a), and (b)

$$CPR, CRQ \vdash CVW.$$

By this, (c), A5, and Theorem 2.3

(g) $CPR \vdash CVW.$

By (3.51) and (2.7)

(h) $\vdash CCPCCPQRCCQPCQR.$

By A1, we have $\vdash CPCCRQP$, whence by A2

$$\vdash CCCCRQPCCPQRCPCCPQR.$$

By this and (h)

(i) $\vdash CCCCRQPCCPQRCCQPCQR.$

By (2.25) and (2.7)

(j) $CQP \vdash CCCPQRCCRQP.$

By A1, $CRP \vdash CCRQCRP$, so that by (2.7)

$$CRP \vdash CRCCRQP.$$

Also by (2.5)

$$CPQ \vdash CCCPQRR.$$

By the last two results

$$CPQ, CRP \vdash CCCPQRCCRQP.$$

By this, (j), A5, and Theorem 2.3

$$CRP \vdash CCCPQRCCRQP.$$

By this, (i), and (3.55)

$$CRP \vdash CLCQPCCPQRLCQRCCRQP.$$

By the commutative law for L and (a) and (b)

$$CRP \vdash CVW.$$

By this, (g), A5, and Theorem 2.3

$$\vdash CVW.$$

Interchanging P and R in this gives (10.1).

THEOREM 10.2.
(10.2) $\vdash LBLPQRBQP \equiv LBLRQPBQR.$

Proof. Replace Q by NQ in (10.1) and use (3.54).

In the succeeding theorems of this section, the letter T will not denote the Słupecki operator characterized by (1.4), but will take the place of an unspecified statement, in the same role as P, Q, R, $\cdots$.

THEOREM 10.3. *If*
(a) $\vdash ANVW,$
(b) $\vdash R \equiv LBVZW,$
(c) $\vdash T \equiv LBVYW,$
then
(d) $\vdash LBRYBWZ \equiv LBTZBWY.$

Proof. By (3.45)

$$NV \vdash Z \equiv BVZ,$$

so that by (b)
(e) $NV \vdash LBRYBWZ \equiv LBLZWYBWZ.$
Interchanging Y and Z in the above reasoning gives
(f) $NV \vdash LBTZBWY \equiv LBLYWZBWY.$
From (e) and (f) by (10.2), we get
(g) $NV \vdash LBRYBWZ \equiv LBTZBWY.$
By (3.37), (b), and the commutativity of L

$$W \vdash R \equiv BVZ.$$

Thus

$$W \vdash BRY \equiv BBVZY,$$

so that by the associativity of B
(h) $W \vdash BRY \equiv BVBZY.$
By (3.32), $W \vdash BWZ$, so that by (3.37) and the commutativity of L

$$W \vdash LBRYBWZ \equiv BRY.$$

Thus by (h)
(i) $W \vdash LBRYBWZ \equiv BVBZY.$
If we interchange Y and Z in the proof of (i), we get
(j) $W \vdash LBTZBWY \equiv BVBYZ.$
By (i), (j), and the commutativity of B,
(k) $W \vdash LBRYBWZ \equiv LBTZBWY.$
By (g), (k), (a), and Theorem 2.3, we conclude (d).

THEOREM 10.4. *If*
(a) $\vdash ANVW$,
(b) $\vdash R \equiv LBVZW$,
(c) $\vdash S \equiv LBWZX$,
(d) $\vdash T \equiv LBVYW$,
(e) $\vdash U \equiv LBWYX$,
then
(f) $\vdash LBRYS \equiv LBTZU$.

Proof. By Theorem 10.3, we have

$$\vdash LBRYBWZ \equiv LBTZBWY.$$

So

$$\vdash LLBRYBWZX \equiv LLBTZBWYX.$$

Now use the associativity of L, and (c) and (e).

THEOREM 10.5. *If*
(a) $\vdash ANRS$,
(b) $\vdash ANST$,
(c) $\vdash P \equiv LBRXS$,
(d) $\vdash Q \equiv LBSXT$,
then
(e) $\vdash ANPQ$.

Proof. By (3.56), (b), and (d), $S \vdash Q$. So by (2.4)
(f) $S \vdash ANPQ$.
By (c) and (3.33), $\vdash CPS$, so that by (3.5), $NS \vdash NP$. Then by (2.5)
(g) $NS \vdash ANPQ$.
By (3.45) and (c)

$$NR \vdash P \equiv LXS,$$

while by (3.37) and the commutativity of L

$$T \vdash Q \equiv BSX.$$

Then (using (3.4)),

$$NR, T \vdash ANPQ \equiv ACXNSCNSX.$$

So by A5
(h) $NR, T \vdash ANPQ$.
Now by (f), (h), (a), and Theorem 2.3, we get
(i) $T \vdash ANPQ$.
By (g), (i), (b), and Theorem 2.3, we get (e).

THEOREM 10.6. *If*
(a) $\vdash ANSM$,

(b) $\vdash ANUV$,
(c) $\vdash ANVW$,
(d) $\vdash ANYZ$,
(e) $\vdash Q \equiv LBUXV$,
(f) $\vdash T \equiv LBVXW$,
(g) $\vdash R \equiv LBYXZ$,
(h) $S \vdash CVCBPUY$,
(i) $S \vdash CWCBPVZ$,
(j) $M \vdash CVCBSUZ$,

then

(k) $S \vdash CTCBPQR$.

Proof. By (a) and (3.52), $\vdash CSM$. So by (j) and (3.32), $S,\ V \vdash Z$. Then by (3.37), (g), and the commutativity of L

(l) $S,\ V \vdash R \equiv BYX$.

By (e) and (3.34), $\vdash CQBUX$. Then by (3.23), $\vdash CBPQBPBUX$. So by the associativity of B,

(m) $\vdash CBPQBBPUX$.

By (h) and (3.24)

$$S,\ V \vdash CBBPUXBYX.$$

So by (l) and (m),

$$S,\ V \vdash CBPQR,$$

whence by A1

(n) $S,\ V \vdash CTCBPQR$.

By (e), (3.33), and (3.5), $NV \vdash NQ$. So by (3.45) and the commutativity of B,

(o) $NV \vdash P \equiv BPQ$.

By (3.45) and (f)

(p) $NV \vdash T \equiv LXW$.

By (3.31), (3.27), and (g)

(q) $\vdash CLXZR$.

By (i) and (2.7), $S \vdash CBPVCWZ$, so that by (3.32), $S \vdash CPCWZ$. Then by (3.26) and (2.7)

$$S \vdash CLXWCPLXZ.$$

From this by (o) and (p)

$$S,\ NV \vdash CTCBPQLXZ,$$

so that by (q) and (2.14)

(r) $S,\ NV \vdash CTCBPQR$.

By (3.45) and (e)

(s) $NU \vdash Q \equiv LXV$.

By (3.37), (g), and the commutativity of L

(t) $Z \vdash R \equiv BYX$.

By (3.45), (h), and the commutativity of B

(u) $S, NU \vdash CVCPY$.

Then by (3.35) and the commutativity of L

$$S, NU \vdash CLPVY.$$

Then by (3.24) and (t)

$$S, Z, NU \vdash CBLPVXR.$$

Then by (3.34)

$$S, Z, NU \vdash CLBLPVXBVPR.$$

Then by (10.2)

$$S, Z, NU \vdash CLBLXVPBVXR.$$

So by (s) and the commutativity of B

$$S, Z, NU \vdash CLBPQBVXR.$$

Then by the commutativity of L and (3.35)

$$S, Z, NU \vdash CBVXCBPQR.$$

Finally by (3.34) and (f)

(v) $S, Z, NU \vdash CTCBPQR$.

By (i) and the commutativity of B

(w) $S, W \vdash CBVPZ$.

By (u), (3.5), and (2.7)

$$S, NU, NY \vdash CVNP.$$

Then by (2.5)

$$S, NU, NY \vdash CCCVNPXX.$$

Then by (3.4) and the commutativity of L

$$S, NU, NY \vdash CBLPVXX.$$

By applying (3.28) to this and (w), we get

$$S, W, NU, NY \vdash CLBLPVXBVPLXZ.$$

Then by (10.2)

$$S, W, NU, NY \vdash CLBLXVPBVXLXZ.$$

Then by (s) and the commutativity of B

$$S, W, NU, NY \vdash CLBPQBVXLXZ.$$

Then by the commutativity of L and (3.35)

$$S, W, NU, NY \vdash CBVXCBPQLXZ.$$

Then by (q) and (2.14)

$$S, W, NU, NY \vdash CBVXCBPQR.$$

Finally by (3.34) and (f)

(x) $S, W, NU, NY \vdash CTCBPQR$.

We now make a succession of uses of Theorem 2.3. In particular, if we write ϕ for $CTCBPQR$, then by (v), (x), and (d), $S, W, NU \vdash \phi$. Then by (r) and (c), $S, NU \vdash \phi$. Then by (n) and (b), $S \vdash \phi$, which is the result we wish.

THEOREM 10.7. *If*

(a) $\vdash ANUV$,

(b) $\vdash ANYZ$,

(c) $\vdash APW$,

(d) $\vdash Q \equiv LBUXV$,

(e) $\vdash R \equiv LBYXZ$,

(f) $\vdash CYBPU$,

(g) $\vdash CZBPV$,

(h) $W \vdash CZCCPYV$,

then

(i) $\vdash CRBPQ$.

Proof. By (f) and (2.7), $\vdash CNPCVU$. Then by (3.24), $\vdash CNPCBYXBUX$. Finally by (2.7)

(j) $\vdash CBYXBPBUX$.

Then by (e) and (3.34)

$$\vdash CRBPBUX.$$

However, by (d), (3.37) and the commutativity of L, we have $V \vdash BUX \equiv Q$, so that

(k) $V \vdash CRBPQ$.

From (g), by reasoning like that used to derive (j), we get

$$\vdash CLXZBPLXV.$$

However, by (3.45) and (e), $NY \vdash R \equiv LXZ$, so that

(l) $NY \vdash CRBPLXV$.

By (3.31), $\vdash CXBUX$, so that by (3.27) and (d)

$$\vdash CLXVQ.$$

By applying (2.14) to this and (l), keeping (1.8) in mind, we get

(m) $NY \vdash CRBPQ$.

From (f) by the commutativity of B and (2.7), we get

(n) $NU \vdash CYP$.

By (3.45) and (d)

(o) $NU \vdash Q \equiv LXV$.

By (h), (n), and (2.25)

(p) $W, Z, NU \vdash CCVYP$.

By (2.8)

$$\vdash CBXYCCVNXCVY.$$

By this, (p), and (2.14)

$$W, Z, NU \vdash CBXYCCVNXP.$$

Then by (3.4)

$$W, Z, NU \vdash CBXYBLVXP.$$

Then by (o) and the commutativity of L and B

$$W, Z, NU \vdash CBXYBPQ.$$

Finally by (e), (3.34), and the commutativity of B

(q) $W, Z, NU \vdash CRBPQ$.

By (3.32) and A1

(r) $P \vdash CRBPQ$.

We now use Theorem 2.3 with (a), (b), (c), (k), (m), (r), and (q) in order to infer (i).

THEOREM 10.8.

(10.3) $\vdash LBLXQNXBQX \equiv Q$.

Proof. Temporarily let us write

(a) V for $LBLXQNXBQX$.

By (a), (3.37), and the commutativity of L

$$BQX \vdash V \equiv BLXQNX.$$

Then commutativity of L gives

$$BQX \vdash V \equiv BLQXNX.$$

By (3.54)

$$BQX \vdash V \equiv AQNX.$$

Then commutativity of A gives

$$BQX \vdash V \equiv CBXQQ.$$

Finally by commutativity of B and (2.15)

(b) $BQX \vdash V \equiv Q$.

By (3.4)

$$CXNQ \vdash NLXQ.$$

Thus by (a) and (3.45)

$$CXNQ \vdash V \equiv LNXBQX.$$

By commutativity of L,

$$CXNQ \vdash V \equiv LBQXNX,$$

whence (3.4) gives

$$CXNQ \vdash V \equiv NCCNQXX.$$

Then commutativity of A gives

$$CXNQ \vdash V \equiv NAXNQ,$$

which is the same as

$$CXNQ \vdash V \equiv LCXNQQ.$$

Finally by (3.37)

(c) $CXNQ \vdash V \equiv Q$.

Now we use Theorem 2.3 with (b), (c) and A5 to infer $\vdash V \equiv Q$, which by (a) gives (10.3).

THEOREM 10.9

(10.4) $$ANPQ,\ ANQR \vdash LBLBPXQNXLBQXR \equiv Q.$$

Proof. Let us temporarily write

(a) W for $LBLBPXQNXLBQXR$.

By (3.45) and (a)

(b) $NP \vdash W \equiv LBLXQNXLBQXR$.

By (3.37) and the commutativity of L, $R \vdash BQX \equiv LBQXR$, so that by (b)

$$NP,\ R \vdash W \equiv LBLXQNXBQX.$$

So by (10.3)

(c) $NP,\ R \vdash W \equiv Q$.

By (3.5) and (3.33)

$$NQ \vdash NLXQ.$$

Then by (b) and (3.45)

$$NP,\ NQ \vdash W \equiv LNXLXR.$$

By (3.4), this reduces to

(d) $NP,\ NQ \vdash W \equiv NCNXCXNR$.

By (3.53) and (3.4)

$$\vdash NNCNXCXNR.$$

Then by (3.50)

$$NQ \vdash NCNXCXNR \equiv Q.$$

So by (d)
(e) $NP, NQ \vdash W \equiv Q$.
By Theorem 2.3, (c), and (e)
(f) $NP, ANQR \vdash W \equiv Q$.
By (3.52),

$$Q, ANQR \vdash R.$$

Then by (3.32) and (3.36)

$$Q, ANQR \vdash LBQXR.$$

Consequently, by (a), (3.37), and the commutativity of L

$$Q, ANQR \vdash W \equiv BBPXNX.$$

Commutativity and associativity of B gives

$$Q, ANQR \vdash W \equiv BNXBXP,$$

which is the same as
(g) $Q, ANQR \vdash W \equiv CNNXCNXP$.
By (3.53) and (3.48)

$$Q \vdash CNNXCNXP \equiv Q.$$

So by (g)
(h) $Q, ANQR \vdash W \equiv Q$.
By Theorem 2.3, (f), and (h)

$$ANPQ, ANQR \vdash W \equiv Q,$$

which gives (10.4) by use of (a).

11. Some properties of inequalities for nonhomogeneous polynomials over the field of rationals. The results of this section were derived for us by Theodor Motzkin. They are based on a special case of the transposition theorem (see [6, §13]); we now state this special case.

THEOREM 11.1. *Let $\mathbf{A}$ and $\mathbf{B}$ be matrices of m rows, with rational components. Let $\mathbf{x}$ be a row vector of m components, each of which is a variable over the rationals. Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be column vectors, each component of which is a variable over the rationals; let $\mathbf{y}_1$ have as many rows as $\mathbf{A}$ has columns, and $\mathbf{y}_2$ have as many rows as $\mathbf{B}$ has columns. Define two sets of conditions, as follows:*

(I) *Every component of $\mathbf{xA}$ is positive, and every component of $\mathbf{xB}$ is non-negative.*

(II) *$\mathbf{Ay}_1+\mathbf{By}_2=0$, every component of $\mathbf{y}_1$ or $\mathbf{y}_2$ is non-negative, and at least one component of $\mathbf{y}_1$ is positive.*

Then we have the result that there is an $\mathbf{x}$ *satisfying* (I) *if and only if there is no* y_1 *and* y_2 *satisfying* (II).

To prove this, one merely follows the development of [6], noting that this development holds over any ordered field, and hence over the rationals.

THEOREM 11.2. *Let*

$$f_i = a_i + \sum_{j=1}^{n} b_{ij}x_j \qquad (1 \leqq i \leqq m), \tag{11.1}$$

$$g = c + \sum_{j=1}^{n} d_j x_j, \tag{11.2}$$

where the a's, b's, c, and d's are rationals. Suppose that there are sets of rational values of the x's for which

$$f_i \geqq 0 \qquad (1 \leqq i \leqq m), \tag{11.3}$$

and that $g>0$ *for all such sets of values of the x's. Then there is a positive rational constant* μ *such that whenever the x's are rationals for which* (11.3) *holds, then*

$$g \geqq \mu. \tag{11.4}$$

Proof. Assume the hypothesis of the theorem. Then (11.3) is inconsistent with $-g \geqq 0$. Define

$$\bar{f}_i = a_i x_0 + \sum_{j=1}^{n} b_{ij}x_j,$$

$$\bar{g} = c x_0 + \sum_{j=1}^{n} d_j x_j.$$

Then in the field of rationals, the set of inequalities

$$\bar{f}_i \geqq 0,$$
$$-\bar{g} \geqq 0,$$
$$x_0 > 0$$

has no solution. Let us take $\mathbf{x}$ to be the row vector with components $(x_0, x_1, \cdots, x_n)$, $\boldsymbol{A}$ to be the matrix of one column and $n+1$ rows with a 1 in the first row and 0's elsewhere, and $\boldsymbol{B}$ to be the matrix of $m+1$ columns and $n+1$ rows, whose last column consists of $-c$ and the $-d_j$'s, and whose ith column $(1 \leqq i \leqq m)$ consists of a_i and the b_{ij}'s. Then condition (I) of Theorem 11.1 cannot be fulfilled, so that condition (II) must be fulfilled. That is, there is a positive y_1 and non-negative $y_2, \cdots, y_{m+2}$ such that

$$y_1 + \sum_{i=1}^{m} y_{i+1}a_i - y_{m+2}c = 0, \tag{11.5}$$

$$\sum_{i=1}^{m} y_{i+1}b_{ij} - y_{m+2}d_j = 0 \qquad (1 \leqq j \leqq n). \tag{11.6}$$

If we multiply (11.6) by x_j, sum, and add (11.5), we conclude

$$y_{m+2}g = y_1 + \sum_{i=1}^{m} y_{i+1}f_i \tag{11.7}$$

as an identity in the x's. As $y_1 > 0$, and $y_{i+1} \geqq 0$, and there is a set of x's for which (11.3) holds, we may substitute this set of x's into (11.7) and conclude $y_{m+2} > 0$. So, writing

$$\mu = y_1/y_{m+2},$$
$$\lambda_i = y_{i+1}/y_{m+2}$$

we have

$$g = \mu + \sum_{i=1}^{m} \lambda_i f_i, \tag{11.8}$$

$$\mu > 0. \tag{11.9}$$

From these two results, our theorem follows.

THEOREM 11.3. *Let f_i and g be as in* (11.1) *and* (11.2), *with rational coefficients. Suppose that there are sets of rational values of the x's for which* (11.3) *holds, and that $g \geqq 0$ for all such sets of values. Then there are non-negative rational constants $\mu, \lambda_1, \cdots, \lambda_m$ such that*

$$g = \mu + \sum_{i=1}^{m} \lambda_i f_i \tag{11.10}$$

is an identity in the x's.

Proof. We modify slightly the proof of Theorem 11.2. We first note that the set of inequalities

$$\bar{f}_i \geqq 0,$$
$$-\bar{g} > 0,$$
$$x_0 > 0$$

has no solution. Then we use corresponding reasoning to conclude that (11.7) holds, except that now we have that all y's are non-negative and at least one of y_1 or y_{m+2} must be positive. As before, we conclude that $y_{m+2} \neq 0$, and conclude (11.8), which is just the same as (11.10). We also have the required result that the μ and λ_i's are all non-negative.

12. **Polynomial formulas.** We shall make much use of linear polynomials such as

$$f = a + \sum_{j=1}^{n} b_j x_j. \tag{12.1}$$

Here a and the b_j's are constant real numbers, and the x_j's are variables. Since we permit some or all of the b_j's to be zero, we cannot say unambiguously how many variables really occur in f. Indeed, for our purposes, it is useful to consider the number of variables as indeterminate, but always finite. Thus if $b_j = 0$ for $n+1 \leqq j \leqq N$, then we consider the polynomial

$$g = a + \sum_{j=1}^{N} b_j x_j$$

to be identical with the f given by (12.1). Perhaps a better way to look at the situation is to say that we are considering forms such as

$$a + \sum_{j=1}^{\infty} b_j x_j,$$

where there is always to be a non-negative K such that $b_j = 0$ for $j > K$. Then we allow ourselves the convenience of using the form (12.1) as a shorthand provided that $b_j = 0$ for $j > n$. We assume that x_i is distinct from x_j if $i \neq j$.

We now make some definitions.

Whenever we use the word "polynomial" throughout the remainder of the text, we shall mean a polynomial of the form (12.1) for which the constant term a and the coefficients b_j are integers.

We shall write $\sigma(f)$ for the sum of the absolute values of the coefficients of the variables in f. That is, with f as in (12.1),

$$\sigma(f) = \sum_{j=1}^{n} |b_j|. \tag{12.2}$$

If x is a real number, then we define

$$\tau(x) = \begin{cases} 1 \text{ if } 1 < x, \\ x \text{ if } 0 \leqq x \leqq 1, \\ 0 \text{ if } x < 0. \end{cases} \tag{12.3}$$

Let f be a polynomial. With f we wish to associate a class of statement formulas PF(f), called the polynomial formulas of f. If f involves variables $x_1, \cdots, x_n$, and P is in PF(f), then P is to depend on distinct statements $X_1, \cdots, X_n$, correlated with the x_j's. Just as f may not really depend on x_j (for instance, one may have $b_j = 0$), so P may not really depend on X_j; indeed there need not even be occurrences of X_j in P in some cases. The definition of PF(f) is by induction on $\sigma(f)$.

First let $\sigma(f)=0$.

Case 1. $a\geqq 1$. Then P is in PF(f) if and only if P is CX_jX_j, where x_j is one of the variables "occurring" in f.

Case 2. $a\leqq 0$. Then P is in PF(f) if and only if P is NCX_jX_j, where x_j is one of the variables "occurring" in f.

Since a is an integer, these cases cover the situation when $\sigma(f)=0$.

Now let α be a positive integer and assume that PF(f) has been defined for each f for which $\sigma(f)<\alpha$. Let f be a polynomial for which $\sigma(f)=\alpha$. There are two ways in which a statement formula P can be in PF(f).

Case 1. For some j, $b_j>0$. Choose a Q in PF$(f-x_j)$ and an R in PF$(f+1-x_j)$, and take

(12.4) $P=LBQX_jR$.

Case 2. For some j, $b_j<0$. Choose a Q in PF$(f+x_j-1)$ and an R in PF$(f+x_j)$, and take

(12.5) $P=LBQNX_jR$.

These two cases are intended to exhaust all P's in PF(f). Note that in Case 1, we allow ourselves to take any j for which $b_j>0$, any Q in PF$(f-x_j)$, and any R in PF$(f+1-x_j)$. Clearly, in this case $\sigma(f-x_j)=\sigma(f)-1$ and $\sigma(f+1-x_j)=\sigma(f)-1$, so that the classes from which we are to select Q and R have already been defined. Similar remarks hold relative to Case 2.

We say that P is a polynomial formula if there is a polynomial f such that P is in PF(f). More precisely, we take PF to be the logical sum of all the PF(f)'s.

Clearly each P in PF is a statement formula of $X_1, X_2, \cdots$. If we assign the values x_i to X_i, then there will be a value assigned to P, which we shall denote by $v(P)$.

THEOREM 12.1. *If P is in PF(f), then*

(12.6) $v(P)=\tau(f)$

whenever $0\leqq x_j\leqq 1$ $(1\leqq j\leqq n)$.

Proof by induction on $\sigma(f)$. Clearly the theorem holds if $\sigma(f)=0$. Let α be a positive integer, and assume that the theorem holds for each f for which $\sigma(f)<\alpha$. Let f be a polynomial for which $\sigma(f)=\alpha$. Let P be in PF(f).

Case 1. $b_j>0$, Q is in PF$(f-x_j)$, R is in PF$(f+1-x_j)$, and $P=LBQX_jR$.

Subcase 1. $1<f-x_j$. Then $\tau(f-x_j)=\tau(f+1-x_j)=\tau(f)=1$. So by the hypothesis of the induction, $v(Q)=1=v(R)$. Then by (12.4), $v(P)=1=\tau(f)$.

Subcase 2. $0\leqq f-x_j\leqq 1$. Then $v(Q)=\tau(f-x_j)=f-x_j$, and $v(R)=\tau(f+1-x_j)$ $=1$. Since the value x_j is assigned to X_j, we see by (12.4) that $v(P)$ $=\max(\min(f,1),0)=\min(f,1)=\tau(f)$.

Subcase 3. $-1\leqq f-x_j<0$. Then $v(Q)=\tau(f-x_j)=0$, and $v(R)=\tau(f+1-x_j)$ $=f+1-x_j$. Then $v(BQX_j)=x_j$, so that $v(P)=\max(0,f)=\tau(f)$.

Subcase 4. $f-x_j<-1$. Then $\tau(f-x_j)=\tau(f+1-x_j)=\tau(f)=0$. So $v(Q)=0$ $=v(R)$, whence $v(P)=0=\tau(f)$.

Case 2. $b_j<0$, Q is in $\mathrm{PF}(f+x_j-1)$, R is in $\mathrm{PF}(f+x_j)$, and $P=LBQNX_jR$. This case proceeds similarly to Case 1, by considering the subcases $2<f+x_j$, $1\leqq f+x_j\leqq 2$, $0\leqq f+x_j<1$, $f+x_j<0$.

It will be noted that Theorem 1 of [3] follows immediately from Theorem 12.1, so that we have incidentally furnished an alternative proof for Theorem 1 of [3]. This is probably just as well, inasmuch as the proof given in [3] for Theorem 1 is much more complicated than our proof of Theorem 12.1.

13. **The case when $\mathfrak{J}$ has an infinite number of members, $\mathfrak{S}=1$, and C and N are taken as undefined.** As in §10, we use Rule C and axiom schemes A1–A5. We remind the reader that Meredith and Chang have shown that axiom scheme A5 can be derived from the others.

THEOREM 13.1. (a) *If P and Q are both in $PF(f)$, then* $\vdash P\equiv Q$. (b) *If P is in $PF(f)$ and Q is in $PF(f+1)$, then* $\vdash ANPQ$.

Proof by induction on $\sigma(f)$. First let $\sigma(f)=0$. If $f\geqq 1$, then we infer part (a) by (2.10) and (3.48), while we infer part (b) by (2.10) and (2.4). If $f\leqq 0$, then we infer part (a) by (2.10), (3.4), and (3.50), while we infer part (b) by (2.10), (3.4), and (2.5).

Let α be a positive integer and assume that the theorem holds if $\sigma(f)<\alpha$.

LEMMA. *Part* (a) *holds for every f with $\sigma(f)=\alpha$.*

Let $\sigma(f)=\alpha$, and let both P and Q be in $\mathrm{PF}(f)$.

Case 1. $b_j>0$, R and T are both in $\mathrm{PF}(f-x_j)$, S and U are both in $\mathrm{PF}(f+1-x_j)$, $P=LBRX_jS$, and $Q=LBTX_jU$. Then by the hypothesis of the induction, $\vdash R\equiv T$ and $\vdash S\equiv U$, so that we easily get $\vdash P\equiv Q$.

Case 2. $b_j<0$, R and T are both in $\mathrm{PF}(f+x_j-1)$, S and U are both in $\mathrm{PF}(f+x_j)$, $P=LBRNX_jS$, and $Q=LBTNX_jU$. Similar to Case 1.

Case 3. $b_j>0$ and $b_k>0$, R is in $\mathrm{PF}(f-x_j)$, S is in $\mathrm{PF}(f+1-x_j)$, T is in $\mathrm{PF}(f-x_k)$, U is in $\mathrm{PF}(f+1-x_k)$, P is $LBRX_jS$, and Q is $LBTX_kU$. Let V, W, and X be in $\mathrm{PF}(f-x_j-x_k)$, $\mathrm{PF}(f+1-x_j-x_k)$, and $\mathrm{PF}(f+2-x_j-x_k)$ respectively. By part (b) of our theorem for $\alpha-2$

$$\vdash ANVW.$$

Also $LBVX_kW$ is in $\mathrm{PF}(f-x_j)$ so that by part (a) of our theorem for $\alpha-1$

$$\vdash R\equiv LBVX_kW.$$

Similarly

$$\vdash S\equiv LBWX_kX,$$
$$\vdash T\equiv LBVX_jW,$$
$$\vdash U\equiv LBWX_jX.$$

Then $\vdash P\equiv Q$ by Theorem 10.4.

Case 4. $b_j>0$ and $b_k<0$, R is in $\mathrm{PF}(f-x_j)$, S is in $\mathrm{PF}(f+1-x_j)$, T is in $\mathrm{PF}(f+x_k-1)$, U is in $\mathrm{PF}(f+x_k)$, $P=LBRX_jS$, and $Q=LBTNX_kU$. Let V, W, and X be in $\mathrm{PF}(f-1-x_j+x_k)$, $\mathrm{PF}(f-x_j+x_k)$, and $\mathrm{PF}(f+1-x_j+x_k)$ respectively. By part (b) of our theorem for $\alpha-2$

$$\vdash ANVW.$$

By part (a) for $\alpha-1$

$$\vdash R\equiv LBVNX_kW, \qquad \vdash T\equiv LBVX_jW,$$
$$\vdash S\equiv LBWNX_kX, \qquad \vdash U\equiv LBWX_jX.$$

Then $\vdash P\equiv Q$ by Theorem 10.4.

The two remaining cases, namely $b_j<0$ and $b_k>0$, or $b_j<0$ and $b_k<0$, are handled similarly.

This still leaves part (b) to be handled. So let $\sigma(f)=\alpha$, and let P be in $\mathrm{PF}(f)$ and Q be in $\mathrm{PF}(f+1)$.

Case 1. There is a $b_j>0$. Choose R, S, and T in $\mathrm{PF}(f-x_j)$, $\mathrm{PF}(f+1-x_j)$, and $\mathrm{PF}(f+2-x_j)$ respectively. Then by part (b) for $\alpha-1$

$$\vdash ANRS,$$
$$\vdash ANST.$$

Also $LBRX_jS$ is in $\mathrm{PF}(f)$, so that by our lemma

$$\vdash P\equiv LBRX_jS.$$

Similarly

$$\vdash Q\equiv LBSX_jT.$$

So $\vdash ANPQ$ by Theorem 10.5.

Case 2. There is a $b_j<0$. Proceed as in Case 1.

THEOREM 13.2. *If P is in $PF(f)$ and Q is in $PF(1-f)$, then* $\vdash P\equiv NQ$.

Proof by induction on $\sigma(f)$. First let $\sigma(f)=0$. If $1\leqq f$, then $\vdash P$ by (2.10) and $\vdash NQ$ by (2.10) and (3.4). So $\vdash P\equiv NQ$ by (3.48). If $f\leqq 0$, then $\vdash NP$ and $\vdash NNQ$ by (2.10) and (3.4). So $\vdash P\equiv NQ$ by (3.50).

Let α be a positive integer and assume the theorem holds if $\sigma(f)<\alpha$. Let $\sigma(f)=\alpha$ and let P be in $\mathrm{PF}(f)$ and Q be in $\mathrm{PF}(1-f)$.

Case 1. There is a $b_j>0$. Choose R, S, T, and U in $\mathrm{PF}(f-x_j)$, $\mathrm{PF}(f+1-x_j)$, $\mathrm{PF}(x_j-f)$, and $\mathrm{PF}(x_j-f+1)$ respectively. By our induction hypothesis

(a) $$\vdash R\equiv NU,$$

(b) $$\vdash S\equiv NT.$$

By Theorem 13.1(a),

(c) $$\vdash P \equiv LBRX_jS,$$

(d) $$\vdash Q \equiv LBTNX_jU.$$

By Theorem 13.1(b),

(e) $$\vdash ANRS.$$

By (3.37)

$$S \vdash BRX_j \equiv BRLSX_j,$$

and by (3.37), the commutativity of L, and (c)

$$S \vdash P \equiv BRX_j.$$

So

(f) $$S \vdash P \equiv BRLSX_j.$$

By (3.45)

$$NR \vdash LSX_j \equiv BRLSX_j,$$

and by (3.45) and (c)

$$NR \vdash P \equiv LX_jS.$$

So by the commutativity of L

(g) $$NR \vdash P \equiv BRLSX_j.$$

Then by Theorem 2.3, (e), (f), and (g)

$$\vdash P \equiv BRLSX_j.$$

By the commutativity of B,

$$\vdash P \equiv BLSX_jR.$$

Then by (3.8)

$$\vdash P \equiv NLNLSX_jNR,$$

so that by (3.9)

$$\vdash P \equiv NLNNBNSNX_jNR.$$

Then by (a), (b), and (3.4),

$$\vdash P \equiv NLBTNX_jU.$$

Thus we conclude finally by (d)

$$\vdash P \equiv NQ.$$

Case 2. There is a $b_j<0$. Interchange P and Q and replace f by $1-f$. Then we are back to Case 1, and can conclude $\vdash Q\equiv NP$. Then by (3.6) and (3.4), $\vdash P\equiv NQ$.

THEOREM 13.3. *If P is in $PF(f)$ and Q is in $PF(2-f)$, then* $\vdash APQ$.

Proof. Take R in $\mathrm{PF}(1-f)$. Then $\vdash P\equiv NR$ by Theorem 13.2 and $\vdash ANRQ$ by Theorem 13.1(b).

THEOREM 13.4. *If α is a non-negative integer, P is in $PF(f)$ and Q is in $PF(\alpha+f)$, then* $\vdash CPQ$.

Proof by induction on α. If $\alpha=0$, use Theorem 13.1(a). So assume the theorem for α. Let P be in $\mathrm{PF}(f)$ and Q be in $\mathrm{PF}(\alpha+1+f)$. Choose R in $\mathrm{PF}(\alpha+f)$. Then $\vdash CPR$ by the hypothesis of the induction, and $\vdash ANRQ$ by Theorem 13.1(b). Then $\vdash CRQ$ by (3.52), so that we can infer $\vdash CPQ$.

THEOREM 13.5. *If α is a non-negative integer, P is in $PF(f)$ and Q is in $PF(1-\alpha-f)$, then for each formula R,* $\vdash CPCQR$.

Proof. Take S in $\mathrm{PF}(1-f)$. Then $\vdash CQS$ by Theorem 13.4 and $\vdash P\equiv NS$ by Theorem 13.2. Then $\vdash CCSRCQR$ by A2 and $\vdash CPCSR$ by (3.53). Combining these gives the theorem.

Let f be a polynomial in which the coefficient of x_k is zero, and let P be in $\mathrm{PF}(f)$. It is possible for P to contain occurrences of X_k. The simplest instance of this would be if $f\equiv 1$ and P is CX_kX_k. However, in any such case, the values of P will not depend on X_k. This is proved in the next theorem.

THEOREM 13.6. *Let f be a polynomial in which the coefficient of x_k is zero. Let $\Phi(X_k)$ be in $PF(f)$. Then* $\vdash\Phi(X_k)\equiv\Phi(R)$.

Proof by induction on $\sigma(f)$. First let $\sigma(f)=0$. If X_k does not occur in $\Phi(X_k)$, then the theorem follows trivially by (2.11). If X_k does occur in $\Phi(X_k)$ it must be because $\Phi(X_k)$ is either CX_kX_k or NCX_kX_k. In this case our theorem follows either by (3.48) or (3.50).

Assume the theorem for $\sigma(f)<\alpha$, and let $\sigma(f)=\alpha$.

Case 1. Some $b_j>0$. Then $j\neq k$. Choose a $\Phi_1(X_k)$ in $\mathrm{PF}(f-x_j)$ and a $\Phi_2(X_k)$ in $\mathrm{PF}(f+1-x_j)$. Then by Theorem 13.1(a), $\vdash\Phi(X_k)\equiv LB\Phi_1(X_k)X_j\Phi_2(X_k)$. So $\vdash\Phi(R)\equiv LB\Phi_1(R)X_j\Phi_2(R)$. However, by the hypothesis of the induction $\vdash\Phi_1(X_k)\equiv\Phi_1(R)$ and $\vdash\Phi_2(X_k)\equiv\Phi_2(R)$. So $\vdash\Phi(X_k)\equiv\Phi(R)$.

Case 2. Some $b_j<0$. Proceed similarly.

THEOREM 13.7. *Let f be a polynomial in which the coefficients of x_j and x_k are both zero. Let be be a non-negative integer. Let $\Phi(X_k)$ be in $PF(f+bx_k)$ and Q be in $PF(f+b-bx_j)$. Then* $\vdash\Phi(NX_j)\equiv Q$.

Proof by induction on b. First let $b=0$. Then $\vdash\Phi(X_k)\equiv\Phi(NX_j)$ by Theorem 13.6, while $\vdash\Phi(X_k)\equiv Q$ by Theorem 13.1(a).

Assume the theorem for b, and let $\Phi(X_k)$ be in $\mathrm{PF}(f+(b+1)x_k)$ and Q be in $\mathrm{PF}(f+b+1-(b+1)x_j)$. Choose a $\Phi_1(X_k)$ in $\mathrm{PF}(f+bx_k)$ and a $\Phi_2(X_k)$ in $\mathrm{PF}(f+1+bx_k)$. Then by Theorem 13.1(a),

$$\vdash \Phi(X_k) \equiv LB\Phi_1(X_k)X_k\Phi_2(X_k).$$

So

(a) $$\vdash \Phi(NX_j) \equiv LB\Phi_1(NX_j)NX_j\Phi_2(NX_j).$$

Similarly we choose an R in $\mathrm{PF}(f+b-bx_j)$ and an S in $\mathrm{PF}(f+1+b-bx_j)$ and have

(b) $$\vdash Q \equiv LBRNX_jS.$$

By the hypothesis of the induction

(c) $$\vdash \Phi_1(NX_j) \equiv R,$$

(d) $$\vdash \Phi_2(NX_j) \equiv S.$$

Then by (a), (b), (c), and (d), we get $\vdash \Phi(NX_j) \equiv Q$.

THEOREM 13.8. *Let f be a polynomial in which the coefficients of x_j and x_k are both zero. Let b and c be non-negative integers. Let $\Phi(X_j, X_k)$ be in*

$$PF(f + cx_j + (b + c)x_k)$$

and Q be in $PF(f+b+c-bx_j)$. Then $\vdash \Phi(X_j, NX_j) \equiv Q$.

Proof by induction on c. When $c=0$, our theorem reduces to Theorem 13.7.

Assume the theorem for c. Let $\Phi(X_j, X_k)$ be in $\mathrm{PF}(f+(c+1)x_j+(b+c+1)x_k)$ and Q be in $\mathrm{PF}(f+b+c+1-bx_j)$. Choose $\Phi_1(X_j, X_k)$, $\Phi_2(X_j, X_k)$, and $\Phi_3(X_j, X_k)$ in $\mathrm{PF}(f + cx_j + (b + c)x_k)$, $\mathrm{PF}(f + 1 + cx_j + (b + c)x_k)$, and $\mathrm{PF}(f+2+cx_j+(b+c)x_k)$ respectively. Also choose P and R in $\mathrm{PF}(f+b+c-bx_j)$ and $\mathrm{PF}(f+b+c+2-bx_j)$ respectively. By the hypothesis of the induction

(a) $$\vdash \Phi_1(X_j, NX_j) \equiv P,$$

(b) $$\vdash \Phi_2(X_j, NX_j) \equiv Q,$$

(c) $$\vdash \Phi_3(X_j, NX_j) \equiv R.$$

Now $LB\Phi_1X_j\Phi_2$ is in $\mathrm{PF}(f + (c + 1)x_j + (b + c)x_k)$, and $LB\Phi_2X_j\Phi_3$ is in $\mathrm{PF}(f + 1 + (c + 1)x_j + (b + c)x_k)$. So $LBLB\Phi_1X_j\Phi_2X_kLB\Phi_2X_j\Phi_3$ is in $\mathrm{PF}(f+(c+1)x_j+(b+c+1)x_k)$. So by Theorem 13.1(a)

$$\vdash \Phi(X_j, X_k) \equiv LBLB\Phi_1(X_j, X_k)X_j\Phi_2(X_j, X_k)X_kLB\Phi_2(X_j, X_k)X_j\Phi_3(X_j, X_k).$$

Then by (a), (b), and (c)

(d) $$\vdash \Phi(X_j, NX_j) \equiv LBLBPX_jQNX_jLBQX_jR.$$

Also, by Theorem 13.1(b)

(e) $$\vdash ANPQ,$$

(f) $$\vdash ANQR.$$

By (d), (e), (f), and Theorem 10.9, we conclude $\vdash \Phi(X_j, NX_j) \equiv Q$.

THEOREM 13.9. *Let P, Q, R, S, and T be in PF(f), PF(g), PF(f+g), PF(f+1), and PF(g+1) respectively. Then*

$$S \vdash CTCBPQR. \tag{13.1}$$

Proof by induction on $\sigma(g)$. First let $\sigma(g)=0$. If $g \geqq 1$, then $S \vdash R$ by Theorem 13.4. So by two uses of A1, $S \vdash CTCBPQR$. If $g=0$, then $\vdash NQ$ by (2.10) and (3.4), and $\vdash CPR$ by Theorem 13.1(a). So $\vdash CBPQR$ by (3.45) and the commutativity of B. Then $S \vdash CTCBPQR$ by A1. If $g \leqq -1$, then $\vdash NT$. So $\vdash CTCBPQR$ by (3.53).

Assume the theorem for $\sigma(g) < \alpha$. Let $\sigma(g) = \alpha$. Let

$$f = a + \sum_{j=1}^{n} b_j x_j, \tag{a}$$

$$g = c + \sum_{j=1}^{n} d_j x_j. \tag{b}$$

Case 1. There is a j for which $d_j > 0$ and $b_j + d_j > 0$. Let U, V, W, Y, Z, and M be in PF$(g - x_j)$, PF$(g+1-x_j)$, PF$(g+2-x_j)$, PF$(f+g-x_j)$, PF$(f+g+1-x_j)$, and PF$(f+2)$ respectively. By Theorem 13.1(a)

$$\vdash Q \equiv LBUX_jV,$$
$$\vdash T \equiv LBVX_jW,$$
$$\vdash R \equiv LBYX_jZ.$$

By Theorem 13.1(b),

$$\vdash ANSM,$$
$$\vdash ANUV,$$
$$\vdash ANVW,$$
$$\vdash ANYZ.$$

By the hypothesis of the induction,

$$S \vdash CVCBPUY,$$
$$S \vdash CWCBPVZ,$$
$$M \vdash CVCBSUZ.$$

Then $S \vdash CTCBPQR$ by Theorem 10.6.

Case 2. There is a j for which $d_j > 0$, but no j for which both $d_j > 0$ and $b_j + d_j > 0$. Take a j for which $d_j > 0$. Then $-b_j \geqq d_j > 0$. Take a k for which $b_k = d_k = 0$. If necessary, take $k > n$. Take $\Phi_1(X_j, X_k)$, $\Phi_2(X_j, X_k)$, and $\Phi_3(X_j, X_k)$ in PF$(f + b_j - b_j x_j - b_j x_k)$, PF$(f + g + b_j - b_j x_j - b_j x_k)$, and PF$(f + 1 + b_j - b_j x_j - b_j x_k)$ respectively. By Case 1,

$$\Phi_3(X_j, X_k) \vdash CTCB\Phi_1(X_j, X_k)Q\Phi_2(X_j, X_k).$$

So

(c) $$\Phi_3(X_j, NX_j) \vdash CTCB\Phi_1(X_j, NX_j)Q\Phi_2(X_j, NX_j).$$

If T or Q contains occurrences of X_k, we can appeal to Theorem 13.6 to infer (c) from the preceding formula. Also by Theorem 13.8

$$\vdash \Phi_1(X_j, NX_j) \equiv P,$$
$$\vdash \Phi_2(X_j, NX_j) \equiv R,$$
$$\vdash \Phi_3(X_j, NX_j) \equiv S.$$

Thus by (c), $S \vdash CTCBPQR$.

Case 3. For each j, $d_j \leqq 0$. Then we can proceed as in Cases 1 and 2 if we replace x_j by $1-x_k$ and X_j by NX_k throughout; we conclude by appealing to Theorem 13.7.

THEOREM 13.10. *Let P, Q, R, S, and T be in $PF(f)$, $PF(g)$, $PF(1-f+g)$, $PF(2-f)$, and $PF(g+1)$ respectively. Then*
(13.2) $S \vdash CTCCPQR$.

Proof. Take U to be in PF$(1-f)$. Then $\vdash P \equiv NU$ by Theorem 13.2. Also $S \vdash CTCBUQR$ by Theorem 13.9. So (13.2) follows.

THEOREM 13.11. *Let P, Q, and R be in $PF(f)$, $PF(g)$, and $PF(f+g)$ respectively. Then*

(13.3) $$\vdash CRBPQ.$$

Proof by induction on $\sigma(f+g)$. First let $\sigma(f+g)=0$. Then $f+g\equiv\beta$, where β is an integer. If $\beta \leqq 0$, then $\vdash NR$ by (2.10) and (3.4). So $\vdash CRBPQ$ by (3.53). Now let $\beta \geqq 1$. Then $g=\beta-f$. Take S in PF$(1-f)$. Then $\vdash P\equiv NS$ by Theorem 13.2. As $\vdash BNSS$ by (3.1), we have $\vdash BPS$. But $\vdash CSQ$ by Theorem 13.4. So $\vdash BPQ$ by (3.23), whence $\vdash CRBPQ$ by A1.

Assume the theorem proved if $\sigma(f+g)<\alpha$. Let $\sigma(f+g)=\alpha$. Let

(a) $$f = a + \sum_{j=1}^{n} b_j x_j,$$

(b) $$g = c + \sum_{j=1}^{n} d_j x_j.$$

Case 1. There is a j such that $b_j+d_j>0$. Then either $b_j>0$ or $d_j>0$. Because of the commutativity of B, we can interchange P and Q if desired without affecting (13.3). So there is no loss of generality in assuming that $d_j>0$. Choose U, V, W, Y, and Z in PF$(g-x_j)$, PF$(g+1-x_j)$, PF$(2-f)$, PF$(f+g-x_j)$, and PF$(f+g+1-x_j)$ respectively. By Theorem 13.1(a),

$$\vdash Q \equiv LBUX_jV, \qquad \vdash R \equiv LBYX_jZ.$$

By Theorem 13.1(b),

$$\vdash ANUV, \qquad \vdash ANYZ.$$

By Theorem 13.3,

$$\vdash APW.$$

By Theorem 13.10,

$$W \vdash CZCCPYV.$$

By the hypothesis of the induction

$$\vdash CYBPU, \qquad \vdash CZBPV.$$

Then $\vdash CRBPQ$ by Theorem 10.7.

Case 2. There is a j such that $b_j+d_j<0$. Proceed as in Case 1.

THEOREM 13.12. *Let P, Q, and R be in $PF(f)$, $PF(g)$, and $PF(1-f+g)$ respectively. Then*

(13.4) $$\vdash CRCPQ.$$

Proof. Take S in $\mathrm{PF}(1-f)$. Then $\vdash P\equiv NS$ by Theorem 13.2, and $\vdash CRBSQ$ by Theorem 13.11.

THEOREM 13.13. *If α is a positive integer and P and Q are in $PF(1+f)$ and $PF(1+\alpha f)$ respectively, then $\vdash CQP$.*

Proof by induction on α. If $\alpha=1$, use Theorem 13.1(a). So assume the theorem for α, and let P and Q be in $\mathrm{PF}(1+f)$ and $\mathrm{PF}(1+(\alpha+1)f)$ respectively. Choose R and S in $\mathrm{PF}(1+\alpha f)$ and $\mathrm{PF}(1-\alpha f)$ respectively. By Theorem 13.3,

(a) $$\vdash ARS.$$

By Theorem 13.12,

(b) $$\vdash CSCQP.$$

By our induction hypothesis, $\vdash CRP$. But $\vdash CPCQP$ by A1, so that

(c) $$\vdash CRCQP.$$

Then we conclude $\vdash CQP$ by (a), (b), (c), and (2.20).

THEOREM 13.14. *If m is a positive integer, P_i is in $PF(1+f_i)$ $(1\leqq i\leqq m)$ and Q is in $PF(1+\sum_{i=1}^{m} f_i)$, then*

(13.5) $$P_1, \cdots, P_m \vdash Q.$$

Proof by induction on m. If $m=1$, use Theorem 13.1(a). So assume the theorem for m. Let P_i be in PF$(1+f_i)$ $(1 \leqq i \leqq m+1)$, and Q be in PF$(1+\sum_{i=1}^{m+1} f_i)$. Choose R in PF$(1+\sum_{i=1}^{m} f_i)$. By the hypothesis of the induction,

(a) $$P_1, \cdots, P_m \vdash R.$$

By Theorem 13.12,

(b) $$\vdash CRCP_{m+1}Q.$$

By (a) and (b), we readily infer (13.5).

THEOREM 13.15. *If P is in $PF(1+x_j)$, then $\vdash P$.*

Proof. If P is in PF$(1+x_j)$, then there must be a Q and R, in PF(1) and PF(2) respectively, such that $\vdash P \equiv LBQX_jR$. But $\vdash Q$ and $\vdash R$ by (2.10). Then $\vdash P$ by (3.32) and (3.36).

THEOREM 13.16. *If P is in $PF(2-x_j)$, then $\vdash P$.*

Proof is similar to that of Theorem 13.15.

THEOREM 13.17. *Let m be a positive integer. Let f_i and g be as in* (11.1) *and* (11.2) *with integer coefficients. Suppose that there are sets of rational values of the x's for which*

(13.6) $$f_i \geqq 0 \qquad (1 \leqq i \leqq m),$$

(13.7) $$x_j \geqq 0 \qquad (1 \leqq j \leqq n),$$

(13.8) $$1 - x_j \geqq 0 \qquad (1 \leqq j \leqq n).$$

Suppose that whenever (13.6), (13.7) *and* (13.8) *hold and the x's are rational, then $g \geqq 0$. Let P_i be in $PF(1+f_i)$ $(1 \leqq i \leqq m)$, and Q be in $PF(1+g)$. Then*

(13.9) $$P_1, \cdots, P_m \vdash Q.$$

Proof. Let R_j be in PF$(1+x_j)$ and S_j be in PF$(2-x_j)$. Then by Theorem 13.15 and Theorem 13.16

(a) $$\vdash R_j \qquad (1 \leqq j \leqq n),$$

(b) $$\vdash S_j \qquad (1 \leqq j \leqq n).$$

By Theorem 11.3, there are non-negative rationals $\lambda_1, \cdots, \lambda_{m+2n}, \mu$ such that

$$g = \mu + \sum_{i=1}^{m} \lambda_i f_i + \sum_{j=1}^{n} \lambda_{m+j} x_j + \sum_{j=1}^{n} \lambda_{m+n+j}(1 - x_j).$$

Multiplying through by the LCM of the denominators of the λ's and μ, we find non-negative integers $L_1, \cdots, L_{m+2n}, M$, and a positive integer K such that

(c) $$Kg = M + \sum_{i=1}^{m} L_i f_i + \sum_{j=1}^{n} L_{m+j} x_j + \sum_{j=1}^{n} L_{m+n+j}(1 - x_j).$$

Take T and U in $\mathrm{PF}(1+Kg-M)$ and $\mathrm{PF}(1+Kg)$ respectively. By Theorem 13.14, using each P_i L_i times, each R_j L_{m+j} times, and each S_j L_{m+n+j} times, we conclude by (c)

$$P_1, \cdots, P_m, R_1, \cdots, R_n, S_1, \cdots, S_n \vdash T,$$

so that by (a) and (b)

(d) $$P_1, \cdots, P_M \vdash T.$$

By Theorem 13.4,

(e) $$\vdash CTU.$$

By Theorem 13.13

(f) $$\vdash CUQ.$$

Then we infer (13.9), by (d), (e), and (f).

THEOREM 13.18. *Let m and the f_i be as in Theorem* 13.17, *but suppose that there is no set of rational values of the x's for which* (13.6), (13.7), *and* (13.8) *all hold. Let P_i be in* $\mathrm{PF}(1+f_i)$ $(1 \leqq i \leqq m)$, *and let Q be any statement whatever. Then*

(13.10) $$P_1, \cdots, P_m \vdash Q.$$

Proof. Let α be the least integer for which there is no set of rational values of the x's satisfying (13.7), (13.8) and

$$f_i \geqq 0 \qquad (1 \leqq i \leqq \alpha + 1).$$

Then there is a rational set of x's satisfying (13.7), (13.8), and

(a) $$f_i \geqq 0 \qquad (1 \leqq i \leqq \alpha).$$

Also, for each such set of x's, $-f_{\alpha+1} > 0$. Then by Theorem 11.2 there is a positive rational μ such that $-f_{\alpha+1} \geqq \mu$ whenever the x's are rational and satisfy (13.7), (13.8), and (a). Let $\mu = M/K$, where M and K are positive integers. Then $-M-Kf_{\alpha+1} \geqq 0$ whenever $-f_{\alpha+1} \geqq \mu$; that is, whenever the x's are rational and satisfy (13.7), (13.8), and (a). Choose R and S in $\mathrm{PF}(1-M-Kf_{\alpha+1})$ and $\mathrm{PF}(1+Kf_{\alpha+1})$ respectively. By Theorem 13.17

(b) $$P_1, \cdots, P_\alpha \vdash R,$$

by Theorem 13.14, using $P_{\alpha+1}$ K times,

(c) $$P_{\alpha+1} \vdash S,$$

and by Theorem 13.5

(d) $$\vdash CRCSQ.$$

Then we infer (13.10) by (b), (c), and (d).

THEOREM 13.19. *Let m be a positive integer. Let f_i and g be as in* (11.1) *and* (11.2) *with integer coefficients. Suppose that $g \geqq 0$ whenever the x's are rationals in the range $0 \leqq x \leqq 1$ such that each $f_i \geqq 0$. Let P_i be in $PF(1+f_i)$ $(1 \leqq i \leqq m)$ and Q be in $PF(1+g)$. Then*

$$P_1, \cdots, P_m \vdash Q. \tag{13.11}$$

Proof. If there are sets of rational x's in the range $0 \leqq x \leqq 1$ for which each $f_i \geqq 0$, use Theorem 13.17. Otherwise, use Theorem 13.18.

DEFINITION. If P is a statement, and f is a polynomial, we define VfP as follows. Choose P_1, P_2, and P_3 in PF(f), PF($2-f$), and PF($1+f$) respectively. Then we set

$$VfP = LLEPP_1P_2P_3. \tag{13.12}$$

By Theorem 13.1(a), the exact choice of P_1, P_2, and P_3 is immaterial.

THEOREM 13.20. *Let P be a statement formula of $X_1, X_2, \cdots$. Then there is a non-negative integer p, there are PF's $\overline{P}_1, \cdots, \overline{P}_p, P_1^*, \cdots, P_p^*$, and there are polynomials f_i $(1 \leqq i \leqq 2^p)$, with the following properties*:

$$\vdash A\overline{P}_iP_i^* \qquad (1 \leqq i \leqq p). \tag{13.13}$$

If $j_1, \cdots, j_m$ constitute some subset (possibly empty) of the positive integers $\leqq p$, and $j_{m+1}, \cdots, j_p$ constitute the remaining positive integers $\leqq p$ (if any), then there is a k $(1 \leqq k \leqq 2^p)$ such that

$$\overline{P}_{j_1}, \cdots, \overline{P}_{j_m}, P^*_{j_{m+1}}, \cdots, P^*_{j_p} \vdash Vf_kP. \tag{13.14}$$

Proof by induction on the number of occurrences of symbols in P. First let P have a single symbol. Then it must be X_j. We take $p=0$, and $f_1 = x_j$. Let us take P_1, P_2, and P_3 in PF(x_j), PF($2-x_j$), and PF($1+x_j$) respectively. By Theorem 13.15 and Theorem 13.16

(a) $$\vdash P_2,$$

(b) $$\vdash P_3.$$

Also, by Theorem 13.1(a)

$$\vdash P_1 \equiv LBNCYYX_jCYY.$$

That is

$$\vdash P_1 \equiv LBNCYYPCYY.$$

By (2.10), (3.37), and the commutativity of L

$$\vdash P_1 \equiv BNCYYP.$$

So by (2.10), (3.4), and (3.45)

$$\vdash P_1 \equiv P.$$

Then $\vdash EPP_1$ by Theorem 3.1, whence we get $\vdash Vf_1P$ by (a), (b), (3.36), and (13.12).

Assume the theorem for all P's with fewer than α symbols, and let P have α symbols.

Case 1. P is of the form NQ. Then there are q, $\overline{Q}$'s, Q^*'s, and g's for Q with the stated properties. We take $p=q$, $\overline{P}_i=\overline{Q}_i$, $P_i^*=Q_i^*$, and $f_i=1-g_i$. Now consider any set of j's. We have by the hypothesis of the induction

(c) $$\overline{P}_{j_1}, \cdots, \overline{P}_{j_m}, \overset{*}{P}_{j_{m+1}}, \cdots, \overset{*}{P}_{j_p} \vdash Vg_kQ.$$

Now Q_1, Q_2, and Q_3 are in PF(g_k), PF$(2-g_k)$, and PF$(1+g_k)$ respectively. Take P_1, P_2, P_3 in PF(f_k), PF$(2-f_k)$, and PF$(1+f_k)$ respectively. Since $g_k=1-f_k$, we have

(d) $$\vdash P_2 \equiv Q_3,$$

(e) $$\vdash P_3 \equiv Q_2$$

by Theorem 13.1(a), and

$$\vdash P_1 \equiv NQ_1$$

by Theorem 13.2. As $EQQ_1 \vdash ENQNQ_1$ by (3.40), we have $EQQ_1 \vdash EPP_1$. Then by (d), (e), (3.33), (3.34), and (3.36), $Vg_kQ \vdash Vf_kP$. Then (c) gives (13.14).

Case 2. P is of the form CQR. Then there are q, $\overline{Q}$'s, Q^*'s, and g's for Q with the stated properties, and there are r, $\overline{R}$'s, R^*'s, and h's for R with the stated properties. We take

(f) $$p = q + r + 2^{q+r}.$$

For $1 \leqq l \leqq 2^q$ and $1 \leqq m \leqq 2^r$, take $\overline{S}_{lm}$ and S_{lm}^* to be in PF$(1-g_l+h_m)$ and PF$(1+g_l-h_m)$ respectively. Then

(g) $$\vdash A\overline{S}_{lm}\overset{*}{S}_{lm}$$

by Theorem 13.3. We take the $\overline{P}$'s to consist of the $\overline{Q}$'s, $\overline{R}$'s, and $\overline{S}$'s, and we take the P^*'s to consist of the Q^*'s, R^*'s, and S^*'s. Then (13.13) holds. We choose the f's as follows. Let $j_1, \cdots, j_m$ be some subset of the positive integers $\leqq p$. Among the formulas

(h) $$\overline{P}_{j_1}, \cdots, \overline{P}_{j_m}, \overset{*}{P}_{j_{m+1}}, \cdots, \overset{*}{P}_{j_p}$$

will be a subset $\mathfrak{Q}$ of the $\overline{Q}$'s and Q^*'s, corresponding to which there is an l such that

(i) $$\mathfrak{Q} \vdash Vg_lQ.$$

Also among the formulas of (h) there will be a subset $\mathfrak{R}$ of the $\overline{R}$'s and R^*'s, corresponding to which there is an m such that

(j) $$\mathfrak{R} \vdash Vh_mR.$$

By (i), (j), (13.12), (3.33), and (3.34),

(k) $$\mathfrak{Q},\ \mathfrak{R} \vdash W$$

where W is any of EQQ_1, Q_2, Q_3, ERR_1, R_2, or R_3. Then by (3.41), (k) also holds when W is $EPCQ_1R_1$. By Theorem 13.12

(l) $$\vdash C\overline{S}_{lm}CQ_1R_1.$$

By Theorem 13.10

(m) $$Q_2 \vdash CR_3CCQ_1R_1\overline{S}_{lm}.$$

Then by (k) with Q_2, R_3, and $EPCQ_1R_1$ successively for W, we infer

(n) $$\mathfrak{Q},\ \mathfrak{R} \vdash EP\overline{S}_{lm}.$$

We still have to define f_k and prove (13.14).

Subcase 1. $\overline{S}_{lm}$ is among the formulas of (h). In this case, we take $f_k=1$. By (2.10) and (3.48),

$$\overline{S}_{lm} \vdash E\overline{S}_{lm}CYY.$$

Then by (n) and (3.39)

$$\mathfrak{Q},\ \mathfrak{R},\ \overline{S}_{lm} \vdash EPCYY.$$

As $f_k=1$, we have by Theorem 13.1(a), $\vdash P_1 \equiv CYY$, $\vdash P_2 \equiv CYY$, and $\vdash P_3 \equiv CYY$. So we easily conclude by (2.10) and (3.36) that

$$\mathfrak{Q},\ \mathfrak{R},\ \overline{S}_{lm} \vdash Vf_kP,$$

so that (13.14) holds.

Subcase 2. $\overline{S}_{lm}$ is not among the formulas of (h), so that S^*_{lm} must be among the formulas of (h). In this case we take $f_k=1-g_l+h_m$. Then by Theorem 13.1(a), we have $\vdash P_1 \equiv \overline{S}_{lm}$ and $\vdash P_2 \equiv S^*_{lm}$. So by (n) and (3.36),

(o) $$\mathfrak{Q},\ \mathfrak{R},\ S^*_{lm} \vdash LEPP_1P_2.$$

By Theorem 13.14

$$Q_2,\ R_3 \vdash P_3.$$

So by taking W to be Q_2 and R_3 in (k), we conclude from (o) that

$$\mathfrak{Q},\ \mathfrak{R},\ S^*_{lm} \vdash Vf_kP,$$

so that (13.14) holds.

THEOREM 13.21. *If* $\vdash P$, *then* P *takes only the value unity.*

Usual proof.

THEOREM 13.22. *If* P *takes the value unity exclusively, then* $\vdash P$.

Proof. Clearly it suffices to restrict attention to statements P which are statement formulas of $X_1, X_2, \cdots$, since any other formula P can be handled by changing the X_j's to the constituents of P. By Theorem 13.20, there are p, $\overline{P}$'s, P^*'s, and f's such that

(a) $$\vdash A\overline{P}_iP_i^* \qquad (1 \leqq i \leqq p)$$

and for each choice of $j_1, \cdots, j_m$ there is an f_k such that

(b) $$\overline{P}_{j_1}, \cdots, \overline{P}_{j_m}, P^*_{j_{m+1}}, \cdots, P^*_{j_p} \vdash CP_1P,$$

(c) $$\overline{P}_{j_1}, \cdots, \overline{P}_{j_m}, P^*_{j_{m+1}}, \cdots, P^*_{j_p}, P \vdash P_1,$$

where P_1 is in PF(f_k). Let $g_1, \cdots, g_p$ be the polynomials such that $\overline{P}_{j_i}$ is in PF(g_i) $(1 \leqq i \leqq m)$ and $P^*_{j_i}$ is in PF(g_i) $(m+1 \leqq i \leqq p)$. Since P takes the value unity exclusively, we may apply to (c) the same sort of reasoning used in the proof of Theorem 13.21, and conclude by use of Theorem 12.1 that whenever the x's are rationals in the range $0 \leqq x \leqq 1$ such that each $g_i \geqq 1$, then $f_k \geqq 1$. Then by Theorem 13.19,

$$\overline{P}_{j_1}, \cdots, \overline{P}_{j_m}, P^*_{j_{m+1}}, \cdots, P^*_{j_p} \vdash P_1.$$

Then by (b),

(d) $$\overline{P}_{j_1}, \cdots, \overline{P}_{j_m}, P^*_{j_{m+1}}, \cdots, P^*_{j_p} \vdash P.$$

Since (d) holds for each choice of $j_1, \cdots, j_m$, we may use Theorem 2.3 and (a) to conclude that $\vdash P$.

14. The case when $\mathfrak{J}$ has M members, $\mathfrak{S} = 1$, and C and N are taken as undefined. It suffices to add a single axiom scheme to those used in the preceding section. To describe this axiom scheme, we make some definitions.

Let i be a non-negative integer, and take $\Phi_i(X_1)$ and $\Psi_i(X_1)$ to be in PF$(1+i-(M-1)x_1)$ and PF$(1-i+(M-1)x_1)$ respectively. Define

(14.1) $$M(P) = \sum_{i=0}^{M-1} L\Phi_i(P)\Psi_i(P).$$

We take $M(P)$ as the sixth axiom scheme.

We note that $M(P)$ takes the value 1 if and only if P is assigned one of the values $\alpha/(M-1)$, where α is an integer with $0 \leqq \alpha \leqq M-1$. As these are the only values in $\mathfrak{J}$, $M(P)$ takes the value unity exclusively.

Let Q be a statement formula of $P_1, \cdots, P_n$, and let Q take the value unity whenever each of the P's is assigned a value $\alpha/(M-1)$. This says that if we assign rational values between 0 and 1 inclusive to the P's, then Q takes the value unity whenever

$$\prod_{j=1}^{n} M(P_j)$$

does. Then by Lemma 1 of [7], we conclude that there is a non-negative integer β such that

$$\text{(a)} \qquad (C\prod_{j=1}^{n} M(P_j))^{\beta}Q$$

takes the value unity whenever we assign rational values between 0 and 1 inclusive to the P's. So by Theorem 13.22, we can derive (a) from A1–A5 by means of Rule C. But since each of $M(P_j)$ is an instance of our sixth axiom scheme, we can deduce Q from (a).

References

1. Jan Łukasiewicz and Alfred Tarski, *Untersuchungen über den Aussagenkalkül*, Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie. Classe III vol. 23 (1930) pp. 30–50.

2. Emil L. Post, *Introduction to a general theory of elementary propositions*, Amer. J. Math. vol. 43 (1921) pp. 163–185.

3. Robert McNaughton, *A theorem about infinite-valued sentential logic*, J. Symbolic Logic vol. 16 (1951) pp. 1–13.

4. J. B. Rosser and A. R. Turquette, *Many-valued logics*, Amsterdam, North-Holland, 1952.

5. D. Hilbert and W. Ackermann, *Grundzüge der Theoretischen Logik*, 2d ed., Berlin, Springer, 1938.

6. T. Motzkin, *Beiträge zur Theorie der Linear Ungleichungen*, Doctoral dissertation, University of Basel, 1933 (Jerusalem, Azriel Printers, 1936). An English translation of this, under the title *The theory of linear inequalities* and with the reference number T-22, was published on March 7, 1952 by the RAND Corp., Santa Monica, California.

7. Alan Rose, *The degree of completeness of the* $\aleph_0$*-valued Lukasiewicz propositional calculus*, J. London Math. Soc. vol. 28 (1953) pp. 176–184.

8. M. Wajsberg, *Beiträge zum Metaaussagenkalkül* I, Monatshefte für Mathematik und Physik vol. 42 (1935) pp. 221–242.

9. A. Tarski, *Logic, semantics, metamathematics*, Oxford University Press, 1956.

University of Nottingham,
Nottingham, England
Cornell University,
Ithaca, N. Y.