A FINITE RECURSION FORMULA FOR THE COEFFICIENTS IN ASYMPTOTIC EXPANSIONS¹

BY
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1. Introduction. In recent years the asymptotic behavior in the neighborhood of infinity of entire functions, $G(z)$, which are defined by their power series developments

$$G(z) = \sum_{n=0}^{\infty} g(n)z^n,$$

have been studied by W. B. Ford [2], E. M. Wright [9; 10; 11], H. K. Hughes [3; 4], and J. H. B. Kemperman [6]. To apply their theory it is required that the coefficient $g(n)$ may be considered as a function $g(w)$ of a complex variable and as such admits an asymptotic factorial expansion in every right half-plane. From their investigations it follows that the constants occurring in this asymptotic expansion of $g(w)$ are precisely the constants which occur in the asymptotic expansion of $G(z)$.

It is known from a lemma which appears in [2; 4; 11] that the function

$$(1.1) \quad {}_p g_q(w) = \prod_{i=1}^p \Gamma(w + \sigma_i) \bigg/ \sum_{j=0}^q \Gamma(w + \rho_j)$$

admits the required factorial expansion. Here p and q are non-negative integers with $p \leq q$; the σ_i and ρ_j denote arbitrary complex parameters. The complete determination of the asymptotic expansion for large $|z|$ of the important class of functions

$${}_p G_q(z) = \sum_{n=0}^{\infty} {}_p g_q(n)z^n$$

is thus dependent upon a procedure for computing the constants occurring in the expansion of ${}_p g_q(w)$. In an earlier paper [7] the fact that ${}_p g_q(w)$ satisfies a first order difference equation was used to obtain an inductive formula for the coefficients depending on all the previously calculated coefficients.

This paper deals with a simpler formula for calculating these coefficients. For this purpose we introduce a function $\phi(t)$ defined as an inverse transform

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of ${}_p g_q(w)$ and use the difference equation to obtain a differential equation of Fuchsian type satisfied by $\phi(t)$. The expansion of $\phi(t)$ about one of the regular singular points of the differential equation will be determined and shown to generate the constants in question. By this procedure we shall obtain a finite recursion formula of fixed length q for the coefficients.

2. The factorial expansion. The lemma cited in §1 is now stated as a theorem. The proof which is based on Stirling's formula is omitted.

THEOREM 2.1. *Let h be any real number and let M be any non-negative integer. Then in the half-plane $\operatorname{Re}(w) > -h$ with $|w|$ large, ${}_p g_q(w)$ admits the asymptotic development*

$$(2.1) \quad {}_p g_q(w) = ((2\pi)^{1/2})^{1-\alpha} \alpha^{\alpha w + \beta - 1/2} \cdot \left[\sum_{m=0}^M \frac{c_m}{\Gamma(\alpha w + \beta + m)} + O\left(\frac{1}{\Gamma(\alpha w + \beta + M + 1)}\right) \right],$$

where $c_0 = 1$ and

$$(2.2) \quad \alpha = q + 1 - p; \quad \beta = \sum_{j=0}^q \rho_j - \sum_{i=0}^p \sigma_i + (1 - \alpha)/2.$$

In the subsequent investigation it is convenient to introduce the auxiliary function

$$(2.3) \quad H(x) = ((2\pi)^{1/2})^{\alpha-1} \alpha^{-x-q-5/2} \Gamma(x) {}_p g_q\left(\frac{x - \beta + q + 3}{\alpha}\right),$$

where x is a complex argument. From the multiplication theorem of Gauss [8, p. 240] it follows that $H(x)$ may be written in the form

$$(2.4) \quad H(x) = \alpha^{-q-3} {}_p g_q\left(\frac{x - \beta + q + 3}{\alpha}\right) \prod_{r=0}^{\alpha-1} \Gamma\left(\frac{x + r}{\alpha}\right).$$

We note that as long as x is confined to an arbitrary right half-plane,

$$\Gamma(x)/\Gamma(x+a) \sim x^{-a} \quad \text{for } |x| \text{ large.}$$

This is true for any complex number a by Stirling's formula. It therefore follows from (2.4), (1.1) and (2.2) that if x is restricted to any right half-plane then

$$(2.5) \quad H(x) \sim x^{-q-3} \quad \text{for } |x| \text{ large.}$$

Moreover, it follows from (2.3) and Theorem 2.1 that in any right half-plane $\operatorname{Re}(x) > -h$, $H(x)$ has the asymptotic development

$$(2.6) \quad H(x) = \sum_{m=0}^M \frac{c_m \Gamma(x)}{\Gamma(x + q + 3 + m)} + O\left(\frac{\Gamma(x)}{\Gamma(x + q + 4 + M)}\right).$$

3. **Definition of $\phi(t)$.** We introduce the notation

$$(3.1) \quad \lambda_0 = \max \left\{ 0, \max_i \operatorname{Re} (\beta - q - 3 - \alpha \sigma_i) \right\},$$

$$\lambda = \text{any real number greater than } \lambda_0.$$

It follows from (2.3) and (1.1) that $H(x)$ is single-valued and regular in the half-plane $\operatorname{Re}(x) > \lambda_0$. From (2.5) we have $H(x) = O(x^{-q-3})$ for x in any right half-plane and $|x|$ large. Hence for real values ξ the inversion integral

$$U(\xi) = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} e^{\xi x} H(x) dx$$

converges. It follows from a well-known theorem [1, p. 126] that $H(x)$ is represented by the Laplacian integral

$$H(x) = \int_0^\infty e^{-\xi x} U(\xi) d\xi,$$

provided $\operatorname{Re}(x) > \lambda_0$. Making the transformation $\xi = \log t^{-1}$, $t = e^{-\xi}$, these two relations become

$$(3.2) \quad \phi(t) = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} t^{-x} H(x) dx, \quad 0 < t \leq 1,$$

and

$$(3.3) \quad H(x) = \int_0^1 t^{x-1} \phi(t) dt, \quad \operatorname{Re}(x) > \lambda_0,$$

where we have replaced $U(\log t^{-1})$ by $\phi(t)$.

We now derive upper bounds for $\phi(t)$ at both $t=1$ and $t=0+$.

THEOREM 3.1. *The function $\phi(t)$ defined by (3.2) is $O(t^{-\lambda})$ as $t \rightarrow 0+$, where λ is defined by (3.1). Moreover, $\phi(t) = O((1-t)^{q+2})$ near $t=1$.*

Proof. Let v be real and set $x = \lambda + iv$, then

$$\phi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} t^{-\lambda - iv} H(\lambda + iv) dv.$$

From (2.5) there exists a constant M such that

$$(3.4) \quad |H(\lambda + iv)| < M(\lambda^2 + v^2)^{-(q+3)/2}$$

for $|\lambda + iv|$ large enough, $\lambda > \lambda_0$. Since $H(\lambda + iv)$ has no poles for $\lambda > \lambda_0$ it follows that (3.4) holds, for $\lambda \geq \lambda_0 + \epsilon$ (for any $\epsilon > 0$) with M independent of λ and v . Thus

$$|\phi(t)| < \frac{M}{\pi} t^{-\lambda} \int_0^{\infty} (\lambda^2 + v^2)^{-(q+3)/2} dv.$$

We set $v^2 = \lambda^2 u$ and obtain

$$|\phi(t)| < \frac{M}{2\pi} \lambda^{-q-2} t^{-\lambda} \int_0^{\infty} (u+1)^{-(q+3)/2} u^{-1/2} du,$$

from which the first assertion of the theorem follows since the integral converges. If we choose $\lambda = (1-t)^{-1}$ in the latter inequality we see that near $t=1$

$$\phi(t) = O((1-t)^{q+2})$$

since $t^{-\lambda}$ is bounded for this special choice of λ .

4. A differential equation satisfied $\phi(t)$. The reader will readily note that by applying the recurrence relation $\Gamma(x+1) = x\Gamma(x)$ to (2.4) we immediately obtain the difference equation

$$(4.1) \quad P(x)H(x) = Q(x+\alpha)H(x+\alpha),$$

where

$$(4.2) \quad \begin{aligned} P(x) &= \prod_{i=1}^p (x - \beta + q + 3 + \alpha\sigma_i) \prod_{r=0}^{\alpha-1} (x + r), \\ Q(x) &= \prod_{j=0}^q (x - \alpha - \beta + q + 3 + \alpha\rho_j). \end{aligned}$$

Equations (3.2) and (4.1) may be combined to yield a linear differential equation satisfied by $\phi(t)$.

THEOREM 4.1. *Let $\theta = t(d/dt)$, then the function $\phi(t)$ defined by (3.2) is an integral of the linear differential equation*

$$(4.3) \quad L(y) \equiv \{t^\alpha Q(-\theta) - P(-\theta)\}y(t) = 0,$$

where $P(x)$ and $Q(x)$ are defined by (4.2).

Proof. The integral (3.2) which defines $\phi(t)$ may be differentiated $m(m \leq q+1)$ times under the integral sign. To show this it suffices to show that

$$\int_{\lambda-i\infty}^{\lambda+i\infty} (-\theta)^m t^{-x} H(x) dx = \int_{\lambda-i\infty}^{\lambda+i\infty} t^{-x} x^m H(x) dx$$

is uniformly convergent. This is indeed true for $m \leq q+1$ as follows from (2.5). Observing that

$$P(-\theta)t^{-x} = P(x)t^{-x} \quad \text{and} \quad Q(-\theta)t^{-x} = Q(x)t^{-x},$$

we have

$$\begin{aligned}
P(-\theta)\phi(t) &= \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} P(-\theta)t^{-x}H(x)dx \\
&= \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} P(x)t^{-x}H(x)dx \\
&= \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} Q(x+\alpha)t^{-x}H(x+\alpha)dx \\
&= \frac{t^\alpha}{2\pi i} \int_{\lambda+\alpha-i\infty}^{\lambda+\alpha+i\infty} Q(x)t^{-x}H(x)dx \\
&= t^\alpha Q(-\theta)\phi(t)
\end{aligned}$$

which proves the assertion.

In order to write $L(y)=0$ in the usual linear form we shall need the following lemma.

LEMMA 4.1. *Let $\theta = td/dt$ and let B_m ($m=0, 1, \dots, q$) be arbitrary complex parameters. Then the constants A_k in the operator identity*

$$(4.4) \quad \prod_{m=0}^q (\theta + B_m) = \sum_{k=0}^{q+1} A_k t^k \frac{d^k}{dt^k}$$

are given recursively by the formula

$$(4.5) \quad A_k = \frac{1}{k!} \prod_{m=0}^q (k + B_m) - \sum_{j=0}^{k-1} \frac{A_j}{(k-j)!}.$$

In particular,

$$(4.6) \quad A_{q+1} = 1; \quad A_q = \sum_{m=0}^q B_m + q(q+1)/2.$$

Proof. It is clear that such constants exist. Observing that

$$\prod_{m=0}^q (\theta + B_m)t^x = t^x \prod_{m=0}^q (x + B_m)$$

and

$$t^k \frac{d^k}{dt^k} t^x = \frac{\Gamma(x+1)}{\Gamma(x+1-k)} t^x,$$

we have the identity

$$\prod_{m=0}^q (x + B_m) = \sum_{k=0}^{q+1} A_k \frac{\Gamma(x+1)}{\Gamma(x+1-k)},$$

or

$$(4.7) \quad \frac{1}{\Gamma(x+1)} \prod_{m=0}^q (x + B_m) = \sum_{j=0}^{q+1} \frac{A_j}{\Gamma(k+1-j)}.$$

We put $x=k$ in (4.7) and obtain

$$\frac{1}{k!} \prod_{m=0}^q (k + B_m) = \sum_{j=0}^k \frac{A_j}{\Gamma(x-j+1)},$$

from which (4.5) follows. Relations (4.6) follow from a direct operation of $\prod_{m=0}^{m=q} (\theta + B_m)$ on a function $y(t)$.

THEOREM 4.2. *The differential equation $L(y)=0$, of which $\phi(t)$ is an integral, may be written in the form*

$$(4.8) \quad L(y) \equiv \sum_{k=\alpha}^{q+1} t^{k-\alpha} (E_k t^\alpha - C_k) y^{(k)}(t) + \sum_{k=0}^{\alpha-1} E_k t^k y^{(k)}(t) = 0,$$

where

$$(4.9) \quad E_k = \frac{1}{k!} \prod_{m=0}^q (\beta + k - p - 2 - \alpha \rho_m) - \sum_{j=0}^{k-1} \frac{E_j}{(k-j)!} \quad (0 \leq k \leq q+1),$$

and

$$(4.10) \quad C_k = \frac{1}{k!} \prod_{m=0}^{p-1} (\beta + k - q - 3 - \alpha \sigma_{m+1}) \prod_{m=p}^q (k + p - m) - \sum_{j=\alpha}^{k-1} \frac{C_j}{(k-j)!} \\ (\alpha \leq k \leq q+1).$$

In particular,

$$(4.11) \quad E_{q+1} = 1; \quad E_q = (q+1)(\beta - p - 2) - \alpha \sum_{m=0}^q \rho_m + q(q+1)/2,$$

and

$$(4.12) \quad C_{q+1} = 1; \quad C_q = p(\beta - q - 3) - \alpha \sum_{m=1}^p \sigma_m + \alpha(1 - \alpha)/2 + q(q+1)/2.$$

Here α and β are the quantities defined by (2.2).

Proof. The differential equation $L(y)=0$ may be written in the form

$$(4.13) \quad t^\alpha \prod_{m=0}^q (\theta + \beta - p - 2 - \alpha \rho_m) y(t) \\ = \prod_{m=0}^{p-1} (\theta + \beta - q - 3 - \alpha \sigma_{m+1}) \prod_{m=p}^q (\theta + p - m) y(t).$$

In Lemma 4.1 put $B_m = \beta - p - 2 - \alpha \rho_m$ and obtain the following representa-

tion for the left side of (4.13):

$$(4.14) \quad t^\alpha \sum_{k=0}^{q+1} E_k t^k \frac{d^k}{dt^k} y(t),$$

where E_k ($k=0, 1, \dots, q+1$) is given by (4.9). If we put $B_m = \beta - q - 3 - \alpha\sigma_{m+1}$ for $m=0, 1, \dots, p-1$ and $B_m = p-m$ for $m=p, \dots, q$, then from Lemma 4.1 the right side of (4.13) takes the form

$$(4.15) \quad \sum_{k=0}^{q+1} C_k t^k \frac{d^k}{dt^k} y(t),$$

where

$$C_k = \frac{1}{k!} \prod_{m=0}^{p-1} (\beta + k - q - 3 - \alpha\sigma_{m+1}) \prod_{m=p}^q (k + p - m) - \sum_{j=0}^{k-1} \frac{C_j}{(k-j)!} \quad (0 \leq k \leq q+1).$$

By recursion $C_k=0$ for $k=0, 1, \dots, \alpha-1$ and hence this latter formula reduces to (4.10). From (4.14) and (4.15) we have

$$t^\alpha \sum_{k=0}^{q+1} E_k t^k y^{(k)}(t) = \sum_{k=\alpha}^{q+1} C_k t^k y^{(k)}(t)$$

which yields (4.8). Finally, relations (4.11) and (4.12) are a consequence of (4.6) with the above choices of B_m .

5. Properties of the differential equation. Let the variable t in the differential equation $L(y(t))=0$ be extended to the complex plane. From (4.8) we see that $L(y)=0$ is of Fuchsian type [5]. If $p=0$ the regular singular points of $L(y)=0$ are $t=\infty$ and the α roots of unity for which $t^\alpha=1$. If $p \geq 1$ the regular singular points are $t=0$, $t=\infty$ and the α roots of unity for which $t^\alpha=1$. In any case, there are no singularities for t real with $0 < t < 1$.

The point $t=1$ is a regular singularity of $L(y)=0$ for each $p \leq q$. From (4.8) we see that the indicial equation of the differential equation at this point is

$$\mu(\mu-1) \cdots (\mu-q) + \frac{E_q - C_q}{\alpha} \mu(\mu-1) \cdots (\mu-q+1) = 0$$

which has the solutions

$$(5.1) \quad \mu = 0, 1, \dots, q-1 \quad \text{and} \quad \mu = q+2.$$

Here we have used the relation $C_q - E_q = 2\alpha$.

When $p=0$ the point $t=0$ is an ordinary point of $L(y)=0$. If $p \geq 1$ then $t=0$ is a regular singularity of the differential equation and from (4.3) it follows that the indicial equation at the origin is

$$\prod_{k=1}^p (\mu + \beta - q - 3 - \alpha\sigma_k) \prod_{r=0}^{\alpha-1} (\mu - r) = 0$$

which has the solutions

$$(5.2) \quad \mu = 0, 1, \dots, \alpha - 1; \quad \mu = -\beta + q + 3 + \alpha\sigma_k \quad (1 \leq k \leq p).$$

The function $\phi(t)$ has been defined by (3.2) for t real with $0 < t \leq 1$. Since $\phi(t)$ solves $L(y) = 0$ for $0 < t \leq 1$ it follows from the location of the singularities of the differential equation that $\phi(t)$ is analytic for $0 < t < 1$. Hence, $\phi(t)$ may be extended analytically and the extension is regular except for possible branch points at the regular singular points of $L(y) = 0$.

If we let K denote the distance from $t = 1$ to the nearest of the other singularities of $L(y) = 0$, then it follows from the theory of Frobenius [5] and (5.1) that the functions

$$\begin{aligned} y_0(t) &= (1-t)^{q+2} \psi_{0,0}(t) \quad y_k(t) \\ &= (1-t)^{q-k} \sum_{m=0}^k [\log(t-1)]^m \psi_{k,m}(t) \quad (1 \leq k \leq q), \end{aligned}$$

form a fundamental system of solutions of $L(y) = 0$ in the region $|t-1| < K$. Here the functions $\psi_{k,m}(t)$ are regular for $|t-1| < K$ and $\psi_{k,0}(1) \neq 0$ for $k = 0, 1, \dots, q$. Since $\phi(t)$ is an integral of $L(y) = 0$ in the neighborhood of $t = 1$

$$\phi(t) = \sum_{k=0}^q A_k y_k(t), \quad |t-1| < K,$$

where the A_k are constants. From Theorem 3.1 we have that $\phi(t) = O((1-t)^{q+2})$ as $t \rightarrow 1$ through positive values of t . Since $\psi_{k,0}(1) \neq 0$ it follows that $A_k = 0$ for $k = 1, \dots, q$ and hence $\phi(t)$ is a multiple of $y_0(t)$. Hence, by Taylor's theorem,

$$(5.3) \quad \phi(t) = \sum_{m=0}^{\infty} \frac{\phi^{(q+2+m)}(1)}{\Gamma(q+3+m)} (t-1)^{q+2+m}$$

for values of t such that $|t-1| < K$.

The results of this section may be formulated in the following theorem.

THEOREM 5.1. *Let the variable t in the differential equation (4.3) be extended to the complex domain. Then $L(y(t)) = 0$ is of Fuchsian type. If $p = 0$ the regular singular points are $t = \infty$ and the α roots of unity for which $t^\alpha = 1$. If $p > 0$ then $t = 0$ is also a regular singular point of $L(y(t)) = 0$. If K denotes the distance from $t = 1$ to the nearest of the other singularities of the differential equation, then $\phi(t)$ has the Taylor series expansion (5.3) for values of t such that $|t-1| < K$.*

6. Identification of the constants. In this section we shall see that $\phi(t)$ acts as a generating function for the constants c_m . To show this we shall first need the following result.

LEMMA 6.1. *Let m be any non-negative integer and let λ_0 be defined by (3.1). Then*

$$\lim_{t \rightarrow 0} t^{x+m} \phi^{(m)}(t) = 0,$$

provided $\operatorname{Re}(x) > \lambda_0$.

Proof. $\phi(t)$ satisfies $L(y) = 0$ which has $t = 0$ as an ordinary point if $p = 0$ and a regular singular point if $p > 0$. The assertion is trivially true for $p = 0$ since $\lambda_0 \geq 0$. If $p > 0$ it follows from the theory of Frobenius that $\phi(t)$ admits the expansion

$$\phi(t) = \sum_i (\log t)^{k_i} t^{\mu_i} \sum_{n=0}^{\infty} d_{i,n} t^n \quad (d_{i,0} \neq 0),$$

for $|t| < 1$. From (5.2) and (3.1) we have that

$$\operatorname{Re}(\mu_i) \geq -\lambda_0 \text{ for all } i.$$

Consequently, for any non-negative integer m ,

$$\phi^{(m)}(t) = O(t^{-\lambda-m}) \quad (\lambda > \lambda_0)$$

as $t \rightarrow 0$, which proves the assertion.

THEOREM 6.1. *The constants c_m ($m = 0, 1, \dots$) occurring in the asymptotic factorial expansion of ${}_p g_q(w)$ are precisely the constants which occur in the Taylor series expansion of $\phi(t)$ about $t = 1$. In fact,*

$$(6.1) \quad c_m = (-1)^{m+q} \phi^{(q+2+m)}(1) \quad (m \geq 0).$$

Proof. Integration of (3.3) by parts $N+1$ times yields

$$H(x) = \sum_{n=0}^N \frac{(-1)^n \Gamma(x)}{\Gamma(x+n+1)} [\phi^{(n)}(t) t^{x+n}]_0^1 + \frac{(-1)^{N+1} \Gamma(x)}{\Gamma(x+N+1)} \int_0^1 t^{x+N} \phi^{(N+1)}(t) dt,$$

provided $\operatorname{Re}(x) > \lambda_0$. From Lemma 6.1 we have that

$$[\phi^{(n)}(t) t^{x+n}]_0^1 = \phi^{(n)}(1)$$

for $\operatorname{Re}(x) > \lambda_0$. Since $\phi^{(n)}(1) = 0$ for $n = 0, 1, \dots, q+1$, by (5.3), we may set $n = m + q + 2$ and obtain

$$(6.2) \quad H(x) = \sum_{m=0}^M \frac{(-1)^{q+2+m} \Gamma(x)}{\Gamma(x+q+3+m)} \phi^{(q+2+m)}(1) + R_M,$$

provided $\operatorname{Re}(x) > \lambda_0$. Here M is any non-negative integer and

$$R_M = \frac{(-1)^{q+3+M}\Gamma(x)}{\Gamma(x+q+4+M)} \left\{ \phi^{(q+3+M)}(1) - \int_0^1 t^{x+q+3+M} \phi^{(q+4+M)}(t) dt \right\}.$$

Let x tend to infinity along the positive real axis. Then the integral is of $O(1)$ and consequently

$$R_M = O(\Gamma(x)/\Gamma(x+q+4+M)).$$

The relation (6.1) now follows by comparing (6.2) with (2.6). For, the coefficients in (2.6) are already uniquely determined if we require (2.6) only for x real and positive.

7. The recursion formula. We substitute (6.1) into (5.3) and obtain

$$(7.1) \quad \phi(t) = \sum_{m=0}^{\infty} \frac{(-1)^{m+q} c_m}{\Gamma(q+3+m)} (t-1)^{q+2+m} \quad |t-1| < K.$$

Since $\phi(t)$ is an integral of the differential equation $L(y)=0$ we may determine a recursion formula for the constants c_m by substituting (7.1) into (4.8). The formula provides a method of computing the constants since we know that $c_0=1$.

THEOREM 7.1. *Let the constants C_k ($k=\alpha, \alpha+1, \dots, q+1$) and E_k ($k=0, 1, \dots, q+1$) be as defined in the formulation of Theorem 4.2. The constants c_m ($m=0, 1, \dots$), defined by Theorem 2.1, satisfy the following recursion formula of length q :*

$$c_0 = 1$$

and

$$(7.2) \quad c_m = (m+2)!(\alpha m)^{-1} \sum_{i=1}^q a_{mi} c_{m-i},$$

($c_{-1}=c_{-2}=\dots=0$), where

$$(7.3) \quad a_{mi} = (-1)^{i-1} \sum_{k=q-i}^{q+1} \binom{k}{q-i} E_k / (m+q+2-i-k)!$$

if $p \leq i \leq q$, while

$$(7.4) \quad a_{mi} = (-1)^{i-1} \sum_{k=q-i}^{q+1} \left\{ \binom{k}{q-i} E_k - \binom{k-\alpha}{q-i-\alpha} C_k \right\} / (m+q+2-i-k)!$$

if $1 \leq i \leq p-1$. Note that the last formula and, hence, the C_k are of no importance if $p=0$ or $p=1$.

Proof. Let us transform the independent variable in (4.8) by the relation $t = t' + 1$ and then drop the prime. We obtain the differential equation

$$\sum_{k=\alpha}^{q+1} \{E_k(t+1)^k - C_k(t+1)^{k-\alpha}\} y^{(k)}(t) + \sum_{k=0}^{\alpha-1} E_k(t+1)^k y^{(k)}(t) = 0$$

which is satisfied by

$$(7.5) \quad y(t) \equiv \phi(t+1) = \sum_{m=0}^{\infty} \frac{(-1)^{m+q} c_m}{\Gamma(q+3+m)} t^{q+2+m}$$

in the neighborhood of $t=0$. Employing the binomial theorem, we may write the differential equation in the form

$$\sum_{j=0}^{\alpha-1} \sum_{k=j}^{q+1} \binom{k}{j} E_k t^{k-j} y^{(k)}(t) + \sum_{j=\alpha}^{q+1} \sum_{k=j}^{q+1} \left\{ \binom{k}{j} E_k - \binom{k-\alpha}{j-\alpha} C_k \right\} t^{k-j} y^{(k)}(t) = 0.$$

By (7.5) the coefficient of t^{m+2} in $t^{k-j} y^{(k)}(t)$ is equal to

$$^A \quad (-1)^{m+j} c_{m+j-q} / (m+2-j-k)!,$$

mplying that, for each integer m ,

$$\begin{aligned} & \sum_{j=0}^{\alpha-1} (-1)^{m+j} c_{m+j-q} \sum_{k=j}^{q+1} \binom{k}{j} E_k / (m+2+j-k)! \\ & + \sum_{j=\alpha}^{q+1} (-1)^{m+j} c_{m+j-q} \sum_{k=j}^{q+1} \left\{ \binom{k}{j} E_k - \binom{k-\alpha}{j-\alpha} C_k \right\} / (m+2+j-k)! = 0, \end{aligned}$$

provided we make the agreement that $c_m = 0$ for $m < 0$. From (4.11) and (4.12), the coefficients of c_{m+1} and c_m in the latter formula are 0 and

$$(-1)^{m+q} \alpha m / (m+2)!$$

respectively. Hence,

$$\begin{aligned} c_m = \frac{(m+2)!}{\alpha m} & \left[\sum_{j=0}^{\alpha-1} (-1)^{j+q+1} c_{m+j-q} \sum_{k=j}^{q+1} \binom{k}{j} E_k / (m+2+j-k)! \right. \\ & \left. + \sum_{j=\alpha}^{q+1} (-1)^{j+q+1} c_{m+j-q} \sum_{k=j}^{q+1} \left\{ \binom{k}{j} E_k - \binom{k-\alpha}{j-\alpha} C_k \right\} / (m+2+j-k)! \right]. \end{aligned}$$

This formula reduces to (7.2) if we set $i = q - j$ and introduce the notations (7.3) and (7.4).

8. Several examples. As a first illustration let us consider the coefficient function $og_1(w)$. In the special case where $\rho_0 = 1$ $og_1(w)$ reduces to the coefficient function of the Bessel function. Here $p = 0$, $q = 1$, $\alpha = 2$ and $\beta = \rho_0 + \rho_1 - 1/2$; (7.2) reduces to

$$c_m = c_{m-1} (m+2)! (2m)^{-1} \sum_{k=0}^2 E_k / (m+2-k)!, \quad (m \geq 1).$$

From (4.11) we recursively calculate

$$E_0 = (\rho_1 - \rho_0 - 5/2)(\rho_0 - \rho_1 - 5/2); \quad E_1 = -4; \quad E_2 = 1.$$

Hence,

$$\begin{aligned} c_m &= \frac{c_{m-1}}{2m} [(m+1)(m+2) - 4(m+2) - (\rho_1 - \rho_0)^2 + 25/4] \\ &= \frac{c_{m-1}}{8m} [(2m-1)^2 - 4(\rho_1 - \rho_0)^2], \quad (m \geq 1). \end{aligned}$$

Since $c_0 = 1$ it follows that

$$(8.1) \quad c_m = \frac{1}{2^{3m} m!} \prod_{k=1}^m [(2k-1)^2 - 4(\rho_1 - \rho_0)^2] \quad (m \geq 0).$$

When $\rho_0 = 1$ this reduces to the familiar expression for the coefficients in the asymptotic expansion of the Bessel function [8, p. 368].

Theorem 7.1 also yields the explicit expression for the constants in the factorial expansion of ${}_1g_1(w)$. In this case one obtains

$$(8.2) \quad c_m = \frac{1}{m!} \sum_{k=0}^{m-1} (\rho_0 - \sigma_1 + k)(\rho_1 - \sigma_1 + k), \quad (m \geq 0).$$

If we set $\rho_0 = 1$ these reduce to the coefficients which occur in the asymptotic expansion of the confluent hypergeometric function [8, p. 342].

As a final illustration let us consider the coefficient function ${}_2g_2(w)$. In this case $p = q = 2$, $\alpha = 1$ and $\beta = \rho_0 + \rho_1 + \rho_2 - \sigma_1 - \sigma_2$; (7.2) reduces to

$$\begin{aligned} (8.3) \quad c_m &= \frac{(m+2)!}{m} \left[c_{m-1} \sum_{k=1}^3 \{kE_k - C_k\} / (m+3-k)! \right. \\ &\quad \left. - c_{m-2} \sum_{k=0}^3 E_k / (m+2-k)! \right], \quad (m \geq 1). \end{aligned}$$

Here the constants E_k and C_k are given by Theorem 4.2. The recursion formula for c_m thus depends on the last two previously calculated coefficients since $q = 2$. Consequently, we cannot write the general explicit formula for c_m . For specified values of ρ_j and σ_i , however, there is no difficulty as may be illustrated by a numerical example.

Suppose $\rho_0 = 1$, $\rho_1 = 2.2 - 0.5i$, $\rho_2 = 2.2 + 0.5i$, $\sigma_1 = 0.6$ and $\sigma_2 = -0.2$. Then $\beta = 5$ and Theorem 4.2 yields $E_0 = 1$, $E_1 = 0.29$, $E_2 = 0.6$, $E_3 = 1$, $C_1 = 0.48$, $C_2 = 2.6$ and $C_3 = 1$. From (8.3) we calculate

$$c_0 = 1,$$

$$c_1 = 7.61 \quad c_0 = 7.61,$$

$$c_2 = 9.105 \quad c_1 - 8.98 \quad c_0 = 60.30905,$$

$$c_3 = 10.9367 \quad c_2 - 24.4833 \quad c_1 = 473.2816, \text{ etc.}$$

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