ADDENDUM TO INVARIANT MEANS AND CONES WITH VECTOR INTERIORS

BY

R. J. SILVERMAN AND TI YEN

1. Introduction. In the preceding paper [4] it is proved that the monotone extension property (Hahn-Banach extension property) of a pair $[\bar{G}, V]$, where $\bar{G}$ is a semi-group and V is a boundedly complete vector lattice whose positive cone is sharp and has a vector interior point, is equivalent to the property that there exists an invariant mean definable on the space of bounded real valued functions on $\bar{G}$. It is the purpose of this paper to remove the interior point restriction on V .

The notations and definitions are the same as in [4] except the following slight modification. A semi-group $\bar{G}$ has the *monotone (Hahn-Banach) extension property* if and only if $[\bar{G}, V]$ has the monotone (Hahn-Banach) extension property [2] for every boundedly complete vector lattice V whose positive cone is sharp.

2. THEOREM. *If $\bar{G}$ is a semi-group the following statements are equivalent:*

- (1) $\bar{G}$ has the *monotone extension property*.
- (2) $\bar{G}$ has the *Hahn-Banach extension property*.
- (3) $\bar{G}$ has an *invariant mean*.

The equivalence of (1) and (2) has been proved in [2], that (1) and (2) imply (3) has been proved in [3]. One needs, therefore, only consider the proof that (3) implies (1).

CASE A. V has reproducing positive cone K . Consider a collection $[Y, X, C, f, G]$ associated with V and K as in [4, Definition 1], where G is a representation of $\bar{G}$ on Y , and $\bar{G}$ has an invariant mean. Let Γ denote the set of convex combinations of elements $g \in G$ and the identity, and let $\sigma, \tau, \dots$ denote the elements of Γ . It is sufficient to consider the case that Y is spanned by X and $\{\sigma y_0 \mid \sigma \in \Gamma\}$, where y_0 is a fixed element in Y not in X . Since $(y_0 + X) \cap C \neq \emptyset$, it may be assumed that $y_0 > 0$.

Let $A = \{x \in X \mid x \leq y_0\}$, $B = \{x \in X \mid x \geq y_0\}$. The sets A and B are not empty since $(y_0 + X) \cap C \neq \emptyset$ and $(-y_0 + X) \cap C \neq \emptyset$. Consider a fixed $b \in B$. Define $W = \{w \in V \mid w \geq f(b)\}$. If $w \in W$, then $w \geq f(b) \geq 0$. For each $w \in W$ define

$$V_w = \{v \in V \mid tw \geq \sup(v, -v) \text{ for some } t > 0\},$$

$$X_w = \{x \in X \mid f(x) \in V_w\},$$

$$Y_w = \text{the subspace spanned by } X_w \text{ and } \{\sigma y_0 \mid \sigma \in \Gamma\}.$$

Received by the editors December 1, 1956.

Then

LEMMA. (1) V_w is a linear subspace of V .

(2) X_w is a linear subspace of X .

(3) The vector w is a vector interior point of the positive cone $K_w = K \cap V_w$ in V_w and K_w is sharp.

(4) V_w is a boundedly complete vector lattice.

(5) For each $x \in X$, there exists $w \in W$ such that $x \in X_w$, hence $\bigcup_{w \in W} X_w = X$, $\bigcup_{w \in W} Y_w = Y$.

(6) $gX_w \subset X_w$, $gC_w \subset C_w$ ($C_w = C \cap Y_w$), $w \in W$, $g \in G$, and $(y + X_w) \cap C_w \neq \emptyset$, $y \in Y_w$.

(7) If $x = s\sigma y_0 - t\tau y_0 \in X$, where $s, t \geq 0$, then $x \in X_w$ for all $w \in W$.

(8) The system W forms a directed system under $\geq$. If $w \geq w'$, then $X_w \supseteq X_{w'}$, $Y_w \supseteq Y_{w'}$, $C_w \supseteq C_{w'}$, $V_w \supseteq V_{w'}$, $K_w \supseteq K_{w'}$.

(9) For every V_w , K_w and associated $[Y_w, X_w, C_w, f_w, G]$, where f_w is the restriction of f to X_w , there exists a monotone invariant extension $F_w: Y_w \rightarrow V_w$ of f_w .

Proof of lemma. Ad (1). If $v, v' \in V_w$, then there exist $t, t' > 0$ such that $tw > \sup(v, -v)$ and $t'w > \sup(v', -v')$. Then $(t+t')w > \sup(v+v', -v-v')$. Therefore $v+v' \in V_w$. Clearly, $-v \in V_w$ and $tv \in V_w$ for all $t > 0$.

Ad (2). This follows from (1) and the linearity of f .

Ad (3). That w is a vector interior point of K_w follows easily from the facts that $w > 0$ and $tw \pm v \geq 0$ for some $t > 0$. K_w is sharp because K is sharp.

Ad (4). Let S be a subset of V_w bounded from above by v_1 in V_w . Then S has a least upper bound v in V . Take any vector v_2 in S . There exist $t_1, t_2 > 0$ such that $t_1w > \sup(v_1, -v_1)$ and $t_2w > \sup(v_2, -v_2)$. Then $\max(t_1, t_2)w > \sup(v_1, -v_1, v_2, -v_2) \geq \sup(v, -v)$, since $v_1 \geq v$ and $-v_2 \geq -v$. Hence $v \in V_w$.

Ad (5). For each $x \in X$, the vector $w = \sup(f(b), f(x), -f(x))$ is in W and $x \in X_w$.

Ad (6). X_w and C_w are invariant under G because f and C are invariant under G . That every translate of X_w intersects C_w follows from the facts $y_0 \geq 0$, $b - y_0 \geq 0$ and $b \in X_w$.

Ad (7). If $x = (s\sigma - t\tau)y_0 \in X$ then $x \leq s\sigma b$ and $-x \leq t\tau b$. Hence

$$\sup(f(x), -f(x)) \leq \max(s, t)f(b),$$

consequently, $x \in X_w$ for every $w \in W$.

Ad (8). This is clear.

Ad (9). Statement (9) follows from (1)–(8) and [4, Theorem 1].

Let $u_0 = \inf_{w' \in W} \sup_{w \geq w'} F_w(y_0)$. The vector u_0 exists since $0 \leq F_w(y_0) \leq f(b)$, $w \in W$. Define $F: Y \rightarrow V$ by

$$F(x + s\sigma y_0 - t\tau y_0) = f(x) + (s - t)u_0, \quad s, t \geq 0.$$

The function F is well defined, that is, if $x + s\sigma y_0 - t\tau y_0 = x' + s'\sigma' y_0 - t'\tau' y_0$

then $f(x) + (s-t)u_0 = f(x') + (s'-t')u_0$; or equivalently, if $(s\sigma - t\tau)y_0 = x \in X$ then $f(x) = (s-t)u_0$. But in this case $x \in X_w$ for every w . Therefore $f(x) = F_w(s\sigma y_0 - t\tau y_0) = (s-t)F_w(y_0)$ for all $w \in W$. If $s-t=0$ then $f(x)=0=(s-t)u_0$. If $s-t \neq 0$ then $F_w(y_0) = f(x)/(s-t)$ is independent of w . Hence $u_0 = F_w(y_0)$. The function F is obviously invariant under G , and by definition distributive. To prove that F is monotone, let $x + s\sigma y_0 - t\tau y_0 > 0$. Consider $w_0 \in W$, where $x \in X_{w_0}$ and thus $x \in X_w$, $w \geq w_0$.

CASE 1. $s-t > 0$. For each

$$w \geq w' \geq w_0, F_w(x + s\sigma y_0 - t\tau y_0) = f(x) + (s-t)F_w(y_0) \geq 0.$$

Since $(s-t)F_w(y_0) \geq 0$, $f(x) + (s-t) \sup_{w \geq w'} F_w(y_0) \geq 0$, and hence

$$\begin{aligned} f(x) + (s-t) \inf_{w' \geq w_0} \sup_{w \geq w'} F_w(y_0) \\ = f(x) + (s-t)u_0 = F(x + s\sigma y_0 - t\tau y_0) \geq 0. \end{aligned}$$

CASE 2. $t-s > 0$. $x > t\tau y_0 - s\sigma y_0$ and as in Case 1 $f(x) \geq (t-s)u_0$.

All that remains to complete the proof is to remove the restriction that V has reproducing cone.

CASE B. The cone of V is not necessarily reproducing. Let V_1 be the subspace of V spanned by K , $X_1 = \{x \in X \mid f(x) \in V_1\}$, and let Y_1 be the subspace of Y spanned by X_1 and $\{\sigma y_0 \mid \sigma \in \Gamma\}$ —assuming again that Y is spanned by X and $\{\sigma y_0 \mid \sigma \in \Gamma\}$ and that $y_0 > 0$. The subsets A, B of X are contained in X_1 for $B \geq 0$, $B-A \geq 0$ and $A \subset B - (B-A)$. If $x \geq s\sigma y_0 - t\tau y_0$, $x \in X$, then $x \geq -t\tau b$ and $x = (x + t\tau b) - t\tau b \in X_1$. Consequently $C \subset Y_1$. The collection $[Y_1, X_1, C, f_1, G]$ together with V_1 and K satisfies the conditions of Case A, where f_1 is the restriction of f on X_1 . Therefore there is a monotone invariant extension $F_1: Y_1 \rightarrow V_1$ of f_1 . Define $F: Y \rightarrow V$ by $F(x + s\sigma y_0 - t\tau y_0) = f(x) + F_1(s\sigma y_0 - t\tau y_0)$. It is easy to show that F is well defined and, hence, is a desired extension of f .

As an immediate corollary is a generalization of a theorem of Kreĭn and Rutman [1, Theorem 3.1].

COROLLARY. *Let Y be an ordered linear space whose positive cone C has a vector interior point y_0 . Let G be a semi-group of operators on Y such that $g(C) \subset C$ and $gy_0 = \lambda_g y_0$, $g \in G$, where $\lambda_g > 0$. Then if G has an invariant mean, there is a positive distributive functional F on Y with $F(gy) = \lambda_g F(y)$, $g \in G$.*

Proof. Let $\bar{G}$ be the set of operators $\{\bar{g} = g/\lambda_g \mid g \in G\}$. Since $\lambda_{g\theta} = \lambda_g \lambda_{\theta}$, $\bar{G}$ is a homomorphic image of G and y_0 is a fixed point of $\bar{G}$. Let X be the one-dimensional subspace spanned by y_0 . Since y_0 is a vector interior point $(y+X) \cap C \neq \emptyset$ for every $y \in Y$. Hence the function $f: X \rightarrow \text{real numbers}$ defined by $f(y_0) = 1$ can be extended to a monotone function F invariant under $\bar{G}$. Thus $F(gy) = \lambda_g F(\bar{g}y) = \lambda_g F(y)$, $y \in Y$.

That this corollary is a generalization of [1, Theorem 3.1] is assured by [1, Lemma 1.1].

REFERENCES

1. M. G. Kreĭn and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Amer. Math. Soc. Translations, no. 26, 1950.
2. R. J. Silverman, *Invariant linear function*, Trans. Amer. Math. Soc. vol. 81 (1956) pp. 411–424.
3. ———, *Means on semi-groups and the Hahn-Banach extension property*, Trans. Amer. Math. Soc. vol. 83 (1956) pp. 222–237.
4. ———, *Invariant means and cones with vector interiors*, Trans. Amer. Math. Soc. vol. 88 (1958) pp. 75–79.

ILLINOIS INSTITUTE OF TECHNOLOGY,
CHICAGO, ILL.