SOME THEOREMS CONCERNING PARTITIONS⁽¹⁾

BY
EMIL GROSSWALD

1. Introduction. Let q be an odd prime and, unless another modulus is specifically indicated, let the congruence symbol denote congruences modulo q . Let $\{a\} = \{a_1, a_2, \dots, a_m\}$ be a set of m distinct least positive residues mod q . Such a set $\{a\}$ will be called symmetrical, if with a also $q-a$ belongs to $\{a\}$. Furthermore, $\nu \equiv \{a\}$ shall mean that ν runs only through values congruent (mod q) to some a_j ($j=1, 2, \dots, m$). Define the function $F(x) = \prod_{\nu \equiv \{a\}} (1-x^\nu)^{-1} = \sum_{n=0}^{\infty} p_n(q)x^n$. Then $p_n(q)$ represents the number of partitions of the integer n into summands congruent to elements of $\{a\}$. Similarly, let $H(x) = \prod_{\nu \equiv \{a\}} (1+x^\nu+x^{2\nu}+\dots+x^{l\nu}) = \sum_{n=0}^{\infty} p_n(q, l)x^n$; then $p_n(q, l)$ represents the number of partitions of the integer n into summands congruent to elements of $\{a\}$, no summand being repeated more than l times. In case $q=5$ and $\{a\}$ is either the set of quadratic residues, or that of quadratic nonresidues, Lehner, using the method of Rademacher [10; 11], obtained convergent series for $p_n(5)$ (see [5]). The first term of these series, being the dominant one, is asymptotically equal to $p_n(5)$. Lehner's result has been generalized by Livingood [6] for any prime $q>3$ and $\{a\} = (a, q-a)$, $a \neq 0$. Recently, Petersson (see especially [8] and [9]), using an entirely different approach, obtained some very interesting results concerning these, and more general, partition numbers. In particular, he has obtained (see [8]) an expression for $p_n(q)$ and also a very precise asymptotic formula. By further exploiting his method he obtained [8] among others, the following result: Let the prime $q>5$ satisfy $q \equiv 1 \pmod{4}$ and denote by $p_n^+(q)$, $p_n^-(q)$ the number of partitions of n into quadratic residues and into quadratic nonresidues (mod q), respectively. Let h be the class number of the real quadratic field $R(q^{1/2})$ and denote by $\epsilon > 1$ its fundamental unit; denote also by $\alpha_5(q)$ the number of representations of q by five squares, representations differing by sign, or order of summands being considered as distinct. Then, as $n \rightarrow \infty$,

$$(1) \quad \frac{p_n^+(q)}{p_n^-(q)} = \epsilon^h \left(1 - \frac{\pi}{6} \left(\frac{3(q-1)}{q} \right)^{1/2} c \cdot \alpha_5(q) n^{-1/2} + O(n^{-1}) \right)$$

holds with $c=1/240$ if $q \equiv 1 \pmod{8}$ and $c=1/560$ if $q \equiv 5 \pmod{8}$. Similarly, if $p_n^+(q, l)$, $p_n^-(q, l)$ are defined as above, with the additional restriction that

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no summand may occur more than l times, then, for $n \rightarrow \infty$,

$$(2) \quad \frac{p_n^+(q, l)}{p_n^-(q, l)} = 1 + \frac{\pi}{6} \left(\frac{3(q-1)l}{q(l+1)} \right)^{1/2} \cdot l c \alpha_5(q) n^{-1/2} + O(n^{-1}),$$

c being defined as in (1). (For comparison it should be observed that, while in general Petersson's notation has been adhered to, here l is used for his $l-1$.)

2. In both the Rademacher and the Petersson method use is made of the modular character of the generating functions. Indeed, if $\{a\}$ is symmetrical and if we set $x = \exp(2\pi i \tau / q)$, then $F(x)$ and $H(x)$ become modular functions (or forms) of τ ; hence, they remain (essentially) invariant if τ is replaced by $T(\tau)$, provided that T is a transformation belonging to some subgroup $\bar{\Gamma}$ of finite index of the modular group Γ . Whenever applicable, both methods lead to exact representations for the partition numbers, which are the coefficients of F and H . In case $\{a\}$ is not symmetric, F and H (as functions of τ) are no more modular and above methods cannot be applied directly. One may observe, however (see [5] and [6]; also [8]) what happens already in the case of modular generating functions when $\tau \rightarrow T(\tau)$, with $T \in \Gamma$, but $T \notin \bar{\Gamma}$. Although the generating function does not stay invariant, it is only transformed into another function, whose behavior for $|x| \rightarrow 1$, resembles closely that of the original function and can be controlled. One may consider the case of asymmetric sets $\{a\}$ as a limiting case, with $\bar{\Gamma}$ reduced to the identity. The results are not quite as satisfactory as in the case of modular functions and, instead of exact expressions, one obtains only asymptotic formulae^(*). Some results for asymmetric $\{a\}$ were already found^(*) by G. Meinardus [7]. His method presents a certain similarity to that used in §§7-9 of this paper (both are essentially saddle point methods), but makes no use of the transformation formulae. He thus obtains the leading term of our (17), in the particular case $m=1$, but without the restriction of q to be a prime. He also obtains the leading term of Petersson's formula (1) and of our formula (19).

In what follows, we proceed in two steps: (a) Using the Hardy-Rademacher approach, the transformation formulae are established for $F(x)$ and $H(x)$ (§§4, 5, 6) by the method of residues; (b) having determined the behavior of the generating functions, their coefficients are obtained (§§8, 9), using a lemma (established in §7) essentially due to Hayman [3]. Finally, some particular cases are considered (§10) and the corollaries are proven (§§11, 12).

3. The results may be summarized in the following theorems.

THEOREM 1. As $x \rightarrow 1$,

(*) See, however [8, p. 27] for Petersson's objection to this use of the term "asymptotic."

(*) The author acknowledges his indebtedness to the referee and to Professor P. T. Bateman, for calling his attention to this important reference.

$$\begin{aligned}
 F(x) &= \prod_{\nu \equiv \{a\}} (1 - x^\nu)^{-1} \\
 &= \prod_{j=1}^m E\left(\frac{n_j}{q}\right) (-q \log x)^d \exp \left\{ \frac{\Lambda}{-\log x} + K(-\log x) + O(\log^2 x) \right\}, \\
 H(x) &= \prod_{\nu \equiv \{a\}; \nu \not\equiv 0 \pmod{l+1}} (1 - x^\nu)^{-1} \\
 &= (l+1)^{-d} \exp \left\{ \frac{\mu}{-\log x} + Kl(-\log x) + O(\log^2 x) \right\},
 \end{aligned}$$

with

$$\begin{aligned}
 d &= \frac{1}{q} \sum_{j=1}^m \left(a_j - \frac{q}{2} \right), \quad K = \frac{1}{24q} \sum_{j=1}^m A_j, \quad A_j = q^2 - 6a_jq + 6a_j^2, \\
 \Lambda &= \frac{m\pi^2}{6q}, \quad \mu = \frac{l\Lambda}{l+1},
 \end{aligned}$$

$\Gamma(x)$ the gamma-function and $E(x) = \Gamma(x)(2\pi)^{-1/2}$.

THEOREM 2. As $n \rightarrow \infty$

$$p_n(q) = \frac{\omega}{q} \left(\frac{q\Lambda^{1/2}}{\lambda_n} \right)^{d+1} \exp \left(-\frac{d(d+2)}{4\Lambda^{1/2}\lambda_n} \right) I_1(2\Lambda^{1/2}\lambda_n)(1 + O(n^{-1})),$$

$$p_n(q, l) = (l+1)^d (\mu^{1/2}/\nu_n) I_1(2\mu^{1/2}\nu_n)(1 + O(n^{-1})),$$

with $\lambda_n = (n-K)^{1/2}$, $\nu_n = (n+Kl)^{1/2}$, $I_1(x)$ the Bessel function and

$$\omega = \prod_{j=1}^m E(a_j/q).$$

COROLLARY 1 (PETERSSON). If $q \equiv 1 \pmod{4}$ and $n \rightarrow \infty$, then (1) and (2) hold.

COROLLARY 2. If $q \equiv 3 \pmod{4}$ and $n \rightarrow \infty$, then

$$\begin{aligned}
 (3) \quad \frac{p_n^+(q)}{p_n^-(q)} &= \prod_{a \bmod q} \left\{ \Gamma\left(\frac{a}{q}\right) \right\}^{(a/q)} \left(\frac{12n}{\pi^2 q(q-1)} \right)^{h/w} \\
 &\quad \cdot \left(1 + \frac{1}{\pi} \left(\frac{6q}{q-1} \right)^{1/2} \frac{h}{w} n^{-1/2} + O(n^{-1}) \right),
 \end{aligned}$$

$$(4) \quad \frac{p_n^+(q, l)}{p_n^-(q, l)} = (l+1)^h (1 + O(n^{-1})).$$

Here the exponent (a/q) stands for the Legendre symbol, while h is the class number of the field $R((-q)^{1/2})$ and w the number of its roots of unity.

Finally, we may remark that Corollary 1 shows that (1) holds for $q=5$ (not covered by Petersson's theorem); however this (and much more) follows already from Lehner's result [5], simply by taking into account two or more terms of the asymptotic expansion of the Bessel function $I_1(t)$.

4. Proof of Theorem 1. In order to determine the behaviour of $F(x)$ and $H(x)$ near $x=1$ we shall use a simplified version of the method of Hardy and Rademacher. The method itself has already been presented several times in great detail (see [2; 10; 11; 5]); therefore, it will be sufficient only to outline the procedure.

For $a \neq 0$ define $F_a(x) = \prod_{\nu \equiv a} (1 - x^\nu)^{-1}$. Then

$$G_a(x) = \log F_a(x) = - \sum_{\nu \equiv a} \log (1 - x^\nu) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} n^{-1} x^{(qm+a)n}.$$

Setting $x = e^{-2\pi z}$, $\Re z > 0$, and applying Mellin's theorem, one obtains

$$\begin{aligned} G_a(x) &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} n^{-1} \exp \{ -2\pi z(qm + a)n \} \\ &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (2\pi i)^{-1} \int_{(\alpha)} \Gamma(s) ds / (2\pi z)^s (qm + a)^s n^{s+1} \\ &= \frac{1}{2\pi i} \int_{(\alpha)} \frac{\Gamma(s) \zeta(s, a/q) \zeta(s+1)}{(2\pi)^s z^s q^s} ds. \end{aligned}$$

Here $\zeta(s, a/q)$ and $\zeta(s+1)$ are Hurwitz's and Riemann's zeta functions, respectively, and the integrals are taken along a parallel to the imaginary axis, of abscissa say $\alpha=3/2$. Using the functional equation of the Hurwitz zeta function,

$$\begin{aligned} (5) \quad \zeta(s, a/q) &= 2\Gamma(1-s)(2\pi q)^{s-1} \left\{ \sin \frac{\pi s}{2} \sum_{\nu=1}^q \cos \frac{2\pi a\nu}{q} \zeta\left(1-s, \frac{\nu}{q}\right) \right. \\ &\quad \left. + \cos \frac{\pi s}{2} \sum_{\nu=1}^q \sin \frac{2\pi a\nu}{q} \zeta\left(1-s, \frac{\nu}{q}\right) \right\} \end{aligned}$$

one obtains

$$\begin{aligned} G_a(x) &= \frac{1}{4\pi i q} \int_{(\alpha)} \left\{ \sum_{\nu=1}^q \frac{\cos(2\pi a\nu/q)}{\cos(\pi s/2)} z^{-s} \zeta\left(1-s, \frac{\nu}{q}\right) \zeta(s+1) \right. \\ &\quad \left. + \sum_{\nu=1}^q \frac{\sin(2\pi a\nu/q)}{\sin(\pi s/2)} z^{-s} \zeta\left(1-s, \frac{\nu}{q}\right) \zeta(s+1) \right\} ds. \end{aligned}$$

Changing s into $-s$,

$$G_a(x) = \frac{1}{4\pi iq} \int_{(-\alpha)} \left\{ \sum_{\nu=1}^q \frac{\cos(2\pi a\nu/q)}{\cos(\pi s/2)} z^s \zeta\left(1+s, \frac{\nu}{q}\right) \zeta(1-s) - \sum_{\nu=1}^q \frac{\sin(2\pi a\nu/q)}{\sin(\pi s/2)} z^s \zeta\left(1+s, \frac{\nu}{q}\right) \zeta(1-s) \right\} ds.$$

We may shift the line of integration back to the abscissa α , by taking into account the residues of the poles between the two lines of integration. It follows that $G_a(x) = (4\pi iq)^{-1} \int_{(\alpha)} (I_1(s) + I_2(s)) ds - 2\pi i(R_1 + R_2)$, where

$$I_1(s) = \frac{1}{4\pi iq} \sum_{\nu=1}^q \frac{\cos(2\pi a\nu/q)}{\cos(\pi s/2)} z^s \zeta\left(1+s, \frac{\nu}{q}\right) \zeta(1-s), I_2(s)$$

is obtained from $-I_1(s)$, by replacing all cosines by sines, and R_i ($i=1, 2$) are the sums of the residues of the I_i 's between the abscissae $-3/2$ and $3/2$. The integral can be estimated as follows: The functional equation of the Riemann zeta-function is applied to $\zeta(1-s)$, and (5) is used to replace

$$\sum_{\nu=1}^q \left\{ \cos \frac{2\pi a\nu}{q} \zeta\left(1+s, \frac{\nu}{q}\right) - \sin \frac{2\pi a\nu}{q} \zeta\left(1+s, \frac{\nu}{q}\right) \cotg \frac{\pi s}{2} \right\}$$

by $-\zeta(-s, a/q)(2\pi q)^{1+s}/2 \sin(\pi s/2)\Gamma(1+s)$; then the integral becomes

$$\begin{aligned} & -\frac{1}{2i} \int_{(\alpha)} (qz)^s \Gamma(s) \zeta(s) \zeta\left(-s, \frac{a}{q}\right) \frac{\operatorname{cosec} \pi s/2}{\Gamma(1+s)} ds \\ & = -\frac{1}{2i} \int_{(\alpha)} \frac{(qz)^s}{s \sin \pi s/2} \zeta(s) \zeta\left(-s, \frac{a}{q}\right) ds = \psi(z). \end{aligned}$$

As $\psi'(z) = 2^{-1} i \int_{(\alpha)} (qz)^{s-1} \operatorname{cosec}(\pi s/2) \zeta(s) \zeta(-s, a/q) ds$ is absolutely convergent for $1 < \alpha < 2$, it follows that $\psi(z)$ is an analytic function of z , vanishing for $z=0$. Moving the line of integration beyond $\alpha=2$ and taking into account the residue at $s=2$, one obtains

$$(6) \quad \psi(z) = \frac{q^2 \pi^2}{6} \phi_4'(a/q) z^2 + \sum_{\nu=3}^{\infty} a_{\nu} z^{\nu}.$$

From the analyticity of $\psi(z)$ follows the convergence of the second member for sufficiently small z . Here $\phi_4(t)$ stands for the fourth Bernoulli polynomial and all coefficients a_{ν} could be determined explicitly, but are not needed here. The computation of the residues uses mainly (5); most of the details can be found in [5] and [6]. $I_1(s)$ has simple poles at $s=-1$ and $s=1$ and an apparently double pole at $s=0$; let the corresponding residues be R_{11} , R_{12} , R_{13} . For $s=-1$, $\sum_{\nu=1}^q \cos(2\pi a\nu/q) \zeta(0, \nu/q) = -1/2$, so that $R_{11} = -1/24iqz$. For $s=0$,

$$\lim_{s \rightarrow 0} \sum_{\nu=1}^q \cos \frac{2\pi a \nu}{q} \zeta \left(1 + s, \frac{\nu}{q} \right) \\ = -2q \left\{ \left(\frac{1}{2} - \frac{a}{q} \right) (\log 2\pi q + \gamma) - \log \Gamma \left(\frac{a}{q} \right) + \frac{1}{2} \log 2\pi \right\}$$

is regular; hence, $s=0$ is actually a simple pole. Dividing previous sum by $4\pi i q$ one obtains $-R_{12}$. For $s=1$,

$$\sum_{\nu=1}^q \cos \frac{2\pi a \nu}{q} \zeta \left(2, \frac{\nu}{q} \right) = -2\pi^2 q^2 \zeta \left(-1, \frac{a}{q} \right) = 2\pi^2 q^2 (a^2/2q^2 - a/2q + 1/12),$$

so that the residue becomes $R_{13} = Az/24qi$ with $A = 6a^2 - 6aq + q^2 \cdot I_2(s)$ has only the (apparently triple) pole at $s=0$. However,

$$\lim_{s \rightarrow 0} \sum_{\nu=1}^q \sin (2\pi a \nu / q) \zeta(1 + s, \nu/q) = \pi q (1/2 - a/q)$$

is regular, so that $s=0$ is actually a double pole. Its residue is found to be $R_2 = (2\pi i)^{-1} (1/2 - a/q) (\log z - \gamma)$. Here, as in R_{12} , γ stands for the Euler constant. Combining these results, $G_a(x) = -2\pi i (R_{11} + R_{12} + R_{13} + R_2) + O(z^2) = \pi/12 qz - \pi Az/12q - \{ (1/2 - a/q) \log(2\pi qz) - \log \Gamma(a/q) + 2^{-1} \log 2\pi \} + O(z^2)$. Returning now to $F_a(x) = \exp \{ G_a(x) \}$ and to the variable x , we have

$$F_a(x) = (2\pi)^{-1} \Gamma(a/q) (-q \log x)^{(a/q-1/2)} \exp \{ -\pi^2 (6q \log x)^{-1} \\ + A \log x / (24q) + O(\log^2 x) \}.$$

As $F(x) = \prod_{j=1}^m F_{a_j}(x)$, it follows that

$$(7) \quad F(x) = (2\pi)^{-m/2} \prod_{j=1}^m \Gamma \left(\frac{a_j}{q} \right) (-q \log x)^d \exp \{ -\Lambda / \log x + K \log x + O(\log^2 x) \}$$

with Λ , d and K as defined in §3. Observing that

$$H_a(x) = \prod_{\nu \equiv a} (1 + x^\nu + \cdots + x^{l\nu}) = F_a(x) / F_a(x^{l+1}),$$

it follows from (7) and $H(x) = \prod_{j=1}^m H_{a_j}(x)$ that

$$(8) \quad H(x) = (l+1)^{-d} \exp \{ -\mu / \log x - Kl \log x + O(\log^2 x) \},$$

with $\mu = l\Lambda / (l+1)$. For $a \equiv 0$, $F_0(x) = \prod_{\nu \equiv 0} (1 - x^\nu)^{-1} = \prod_{\nu=1}^\infty (1 - x^{q\nu})^{-1}$ and it is well known (it follows, e.g. from Lemma 4.31 in [2, p. 93]) that for $x \rightarrow 1$,

$$\prod_{n=1}^\infty (1 - x^n)^{-1} = (-\log x / 2\pi)^{1/2} \exp \{ -\pi^2 (6 \log x)^{-1} + (\log x) / 24 + O(\log^2 x) \}.$$

Consequently, it follows that (7), hence also (8) holds even if one of the residues in $\{a\}$ satisfies $a \equiv 0$, provided one sets not $a=0$, but $a=q$, whence the condition of *least positive* residues. This finishes the proof of Theorem 1.

5. In what follows, it will be necessary to have some information concerning the behavior of $F(x)$ and $H(x)$ in the neighborhood of an arbitrary rational point on the circle of radius $|x|=r<1$, i.e. a point of argument $2\pi h/k$. Let $x=re^{i\theta}=\exp\{(-2\pi z+2\pi i h)/k\}$, $z=-k(\log r+2\pi i \phi)/2\pi$ so that $\phi=\theta/2\pi-h/k$. Proceeding essentially as in §4 (see also [6]) one finds that for $q|k$, $G_a(x)=\sum_{j=1}^4\{\int_{(\alpha)} I_j(s)ds-2\pi i R_j\}$. Here

$$I_1(s) = \frac{1}{4\pi i k^2} \sum_{\rho, \mu, \nu=1}^k \cos \frac{2\pi \rho \mu}{k} \cos \frac{2\pi \rho h' \nu}{k} \zeta\left(1-s, \frac{\mu}{k}\right) \zeta\left(1+s, \frac{\nu}{k}\right) z^s \sec \frac{\pi s}{2},$$

$$I_2(s) = -\frac{1}{4\pi i k^2} \sum_{\rho, \mu, \nu=1}^k \cos \frac{2\pi \rho \mu}{k} \sin \frac{2\pi \rho h' \nu}{k} \zeta\left(1-s, \frac{\mu}{k}\right) \zeta\left(1+s, \frac{\nu}{k}\right) z^s$$

$$\cdot \operatorname{cosec} \frac{\pi s}{2};$$

$I_3(s)$ is obtained from $iI_2(s)$, by changing each trigonometric function into its co-function and $I_4(s)$ is obtained in the same way from $iI_1(s)$; also, R_j stands for the sum of the residues of $I_j(s)$ ($j=1, 2, 3, 4$) inside the strip $(-3/2, 3/2)$, while h' and ρ are defined, respectively, by $hh' \equiv -1 \pmod{k}$ and $\rho \equiv -ha \equiv b \pmod{q}$, $0 < h', \rho < k$, $0 < b < q$. One may remark that $I_1(s)$, $I_4(s)$, hence also R_1, R_4 are even functions of a , while I_2, I_3, R_2 and R_3 are odd functions. The integrals, estimated as in §4, are $O(z^2)$. For the residues one obtains

$$R_1 = (Az - B/z)/24ikq, \quad R_2 = i\phi_a(h, k)/\pi, \quad R_3 = C_a(h, k)z, \quad R_4 = -\sigma_a(h, k)/2.$$

Here A is defined as in §3, $B=q^2-6qb+6b^2$, while

$$\sigma_a = \sum ((\lambda/vk))((h\lambda/k)),$$

$$(9) \quad C_a = -\frac{1}{12\pi k^2} \sum \frac{\Gamma'(u)}{\Gamma(u)} (6\lambda^2 - 6kv\lambda + k^2v^2),$$

$$\phi_a = \sum ((\lambda/vk)) \left\{ ((\lambda h/k)) (\log 2\pi k + \gamma) - \log \Gamma(u) + \frac{1}{2} \log 2\pi \right\}$$

with $v=1$, $u=1/2 - ((\lambda h/k))$, the summations being extended over all values of λ satisfying $\lambda \equiv a$, $1 \leq \lambda < vk$, $h\lambda \not\equiv 0 \pmod{k}$ and with $((y))$ defined as usual to be zero for integral y , $((y)) = y - [y] - 1/2$ otherwise. In case $q \nmid k$, the procedure has to be modified slightly, as in [5] and [6] and leads to $R_1 = (Az - z^{-1})/24ikq$, while the other residues remain formally the same, provided that now one sets $v=q$ in (9), instead of $v=1$, as previously done. If we also define B as before, when $q|k$, $B=1$ otherwise, all residues become for-

mally the same and we may omit the case distinction $q|k$, vs. $q \nmid k$. Combining these results, $\log F_a(x) = -2\pi i \sum_{j=1}^4 R_j + O(z^2)$, whence

$$(10) \quad \log F_a(x) = \frac{A}{24q} \log r - \frac{\pi^2 B}{6qk^2} \frac{\log r}{\log^2 r + 4\pi^2 \phi^2} + 2\phi_a - 2\pi C_a k\phi \\ + i \left\{ \left(\frac{A\pi}{12q} + \frac{\pi^3 B}{3qk^2(\log^2 r + 4\pi^2 \phi^2)} \right) \phi + C_a k \log r + \pi \sigma_a \right\} \\ + O(\log^2 r + \phi \log r + \phi^2)$$

follows. In particular, setting $L = (\pi^2/6q) \sum_{a \in \{a\}} B$ one obtains, as a consequence of (10),

$$(11) \quad |F(x)| = \exp \left\{ -\frac{L \log r}{k^2(\log^2 r + 4\pi^2 \phi^2)} \right. \\ \left. + K \log r + 2\Phi - 2\pi C k\phi + O(\log^2 r + \phi \log r + \phi^2) \right\}$$

where $\Phi = \sum_{a \in \{a\}} \phi_a$, $C = \sum_{a \in \{a\}} C_a$. If $k = qs$, $s \geq 1$, then, for $q > 3$, $B = q^2 - 6bq + 6b^2 \leq q^2 t$, $L \leq \pi^2 mqt/6$, $0 < L/k^2 \leq t\Delta/s^2 \leq \Delta t$ with $t = 1 - 6(q-1)/q^2 < 1$. For $q = 3$, $b = 1, 2$, $B = -3$ and $L < 0$. If $q \nmid k$, then $B = 1$, $L = \Delta$, $L/k^2 = \Delta/k^2 \leq \Delta/4$, for $k > 1$. Therefore, if $k > 1$, in all cases $L/k^2 \leq t\Delta$; also, $K > -mq/48$. Concerning $H(x) = F(x)/F(x^{i+1})$ it follows from (10) that

$$(12) \quad |H(x)| \leq \exp \left\{ \frac{-L \log r}{k^2(\log^2 r + 4\pi^2 \phi^2)} - Kl \log r + 2(\Phi(h, k) - \Phi(h_1, k_1)) \right. \\ \left. - 2\pi(C(h, k)k\phi - C(h_1, k_1)k_1\phi_1) \right. \\ \left. + O(\log^2 r + (\phi + \phi_1) \log r + \phi^2 + \phi_1^2) \right\}$$

where $x_1 = x^{i+1}$ and h_1, k_1 and ϕ_1 are defined with respect to x_1 , as h, k and ϕ were defined with respect to x .

6. For further use we need the trivial estimates

$$(13) \quad \Phi = O(k \log k), \quad C = O(k^2).$$

For that, set $(\lambda h/k) = y$. From the conditions on λ and h it follows that $|y| \leq 1/2 - 1/k$. One observes that within this range $U(y) = y(\log 2\pi k + \gamma) - \log \Gamma(1/2 - y) + 2^{-1} \log 2\pi$ satisfies $-2^{-1} \log k - \gamma/2 + o(1) < U(y) < 2^{-1} \log k + \log \log k + (\log 2\pi + \gamma/2 - 1) + o(1)$. Hence, $\phi_a = \sum_k (\lambda/vk) U(y) = O(k \log k)$. Similarly, for $1/k \leq u \leq 1 - 1/k$, from $\Gamma'(u)/\Gamma(u) = -1/u - \gamma + u\zeta(2) - u^2\zeta(3) + \dots$ follows that $-k - \gamma < \Gamma'(u)/\Gamma(u) < -\gamma + \zeta(2)$. Also $|6\lambda^2 - 6kv\lambda + v^2k^2| < v^2k^2$, so that $|(\Gamma'(u)/\Gamma(u))(6\lambda^2 - 6kv\lambda + v^2k^2)| < k^3v^2(1 + o(1))$ and $C_a = O(k^{-2}k^4v) = O(k^2)$ and (13) follows. One may observe that in the case of symmetric sets, (13) follows trivially from the previously remarked fact that R_2 and R_3 are odd functions of a , so that $C = \Phi = 0$ for symmetric sets $\{a\}$.

7. In the proof of Theorem 2 we need the following

LEMMA. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be analytic inside the unit circle and define the functions $a(r) = r \cdot d(\log f(r))/dr$ and $b(r) = r \cdot da(r)/dr$ of $r = |x|$. Denote by $\rho = \rho_n$ the unique⁽⁴⁾ root of $a(\rho) = n$ and assume that for $r_0 < r < 1$, functions $\delta(r) > 0$ and $u(r)$ exist, with the following properties: As $n \rightarrow \infty$, one has, for some $\alpha > 0$:

$$(a) \quad \int_{|\theta| \geq \delta(\rho)} |f(\rho e^{i\theta})| d\theta = O(n^{-\alpha} f(\rho) b(\rho)^{-1/2});$$

$$(b) \quad \int_{-\delta(\rho)}^{+\delta(\rho)} \left(f(\rho e^{i\theta}) - f(\rho) \exp \left\{ i\theta a(\rho) - \frac{1}{2} \theta^2 b(\rho) \right\} \right) e^{-in\theta} d\theta \\ = (2\pi)^{1/2} f(\rho) b(\rho)^{-1/2} (u(\rho) + O(n^{-\alpha}));$$

$$(c) \quad \delta(\rho)^2 b(\rho) \geq 2\alpha \log n.$$

Then

$$(14) \quad a_n = \rho^{-n} (2\pi b(\rho))^{-1/2} f(\rho) (1 + u(\rho) + O(n^{-\alpha})).$$

This lemma is essentially Hayman's Theorem 12 in [3]. The sharper estimates of the error terms were needed; hence, generality has been traded for precision.

Proof of the lemma. Integrating around the circle of radius $r < 1$, by Cauchy's theorem,

$$a_n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(re^{i\theta})}{r^{n+1} e^{i(n+1)\theta}} r i e^{i\theta} d\theta;$$

consequently,

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta = \frac{1}{2\pi} \left[\int_{|\theta| > \delta} f(re^{i\theta}) e^{-in\theta} d\theta \right. \\ \left. + \int_{-\delta}^{+\delta} \left\{ f(re^{i\theta}) - f(r) \exp \left\{ i\theta a(r) - \frac{1}{2} \theta^2 b(r) \right\} \right\} e^{-in\theta} d\theta \right. \\ \left. + f(r) \int_{-\delta}^{+\delta} \exp \left\{ i\theta (a(r) - n) - \frac{1}{2} \theta^2 b(r) \right\} d\theta \right].$$

For $r = \rho$ it follows from the definition of ρ that the last integral becomes $\int_{-\delta}^{+\delta} e^{-\theta^2 b/2} d\theta = (2/b(\rho))^{1/2} \int_{-D}^{+D} e^{-x^2} dx = (2/b(\rho))^{1/2} (\pi^{1/2} - \epsilon)$ with $D = \delta(\rho) (b(\rho)/2)^{1/2}$ and $\epsilon = \int_D^{\infty} e^{-x^2} dx < e^{-D^2}/2D$. The conclusion now follows by assumptions (a), (b) and (c) of the lemma and observing that, by (c), $0 < \epsilon < n^{-\alpha} (8\alpha \log n)^{-1/2} = o(n^{-\alpha})$.

8. **Proof of Theorem 2.** Observing that the term $O(\log^2 x)$ in (7) originates from (6), taking into account the convergence of the series in $\psi(z)$ and using the definitions of the lemma, one obtains for $F(x)$, that $a(r) = \Lambda/\log^2 r + d/\log r$

⁽⁴⁾ For the uniqueness see [3, p. 67].

$+K+O(r \log r)$; $b(r) = -2\Lambda/\log^3 r - d/\log^2 r + O(1)$ and that $\rho = \rho_n$, the root of $a(\rho) = n$, satisfies:

$$\begin{aligned} (\log \rho)^{-1} &= -\left(\frac{n}{\Lambda}\right)^{1/2} \left(1 + \frac{1}{2} d(\Lambda n)^{-1/2} + \frac{1}{2} \left(\frac{d^2}{4\Lambda} - K\right) n^{-1} \right. \\ &\quad \left. - \frac{1}{8} \left(\frac{d^2}{4\Lambda} - K\right)^2 n^{-2} + O(n^{-3})\right), \\ (15) \quad \log \rho &= -\left(\frac{\Lambda}{n}\right)^{1/2} \left(1 - \frac{1}{2} d(\Lambda n)^{-1/2} + \frac{1}{2} \left(\frac{d^2}{4\Lambda} + K\right) n^{-1} \right. \\ &\quad \left. + \frac{1}{2} d\Lambda^{-1/2} \left(\frac{d^2}{4\Lambda} - K\right)^2 n^{-3/2} + O(n^{-2})\right). \end{aligned}$$

Define $\delta = \delta(r) = (-\log r)^{3/2} \psi(r)$ with $\psi(r) = ((-4/\Lambda) \log(-\log r))^{1/2}$. Then $\delta(r)^2 b(r) = -8 \log(-\log r)(1 + O(\log r))$ and, for $r = \rho$, using (15), $\delta(\rho)^2 b(\rho) = 4 \log n(1 + o(1)) > 2 \log n$; hence, assumption (c) of the lemma holds with $\alpha = 1$. Next, let $D(\rho, \theta) = (F(\rho e^{i\theta}) - F(\rho) \exp\{i\theta a(\rho) - 2^{-1}\theta^2 b(\rho)\})e^{-in\theta}$. Using (7), previous values of $a(r)$, $b(r)$ and (15), expanding the exponentials, simplifying, taking into account that $|\theta| \leq \delta(\rho)$ and regrouping terms, one obtains $D(\rho, \theta) = F_1(\theta) + F_2(\theta)$, with

$$F_1(\theta) = F(\rho) \exp\left(-\frac{1}{2} b(\rho)\theta^2\right) \left(\frac{-i\Lambda\theta^3}{\log^2 \rho (\log^2 \rho + \theta^2)} + \dots\right),$$

an odd function of θ and

$$F_2(\theta) = F(\rho) \exp\left(-\frac{1}{2} b(\rho)\theta^2\right) \left\{\frac{-\Lambda\theta^4}{\log^5 \rho} - \frac{\Lambda^2\theta^6}{2\log^8 \rho} + O\left(\log^2 \rho + \frac{\theta^{12}}{\log^{16} \rho}\right)\right\}.$$

As $F_1(\theta)$ is an odd function, $\int_{-\delta}^{+\delta} D(\rho, \theta) d\theta = \int_{-\delta}^{+\delta} F_2(\theta) d\theta$. The integral can be computed, using repeated integrations by parts and one obtains $\int_{-\delta}^{+\delta} D(\rho, \theta) d\theta = (2\pi/b(\rho))^{1/2} F(\rho) \{- (3/16\Lambda)(-\log \rho) + O(\log^2 \rho)\}$. Hence, by (15), assumption (b) of the lemma holds for $F(x)$ with $u(\rho) = - (3/16\Lambda)(-\log \rho)$ and $\alpha = 1$. Concerning assumption (a), one has from (7) that, for $\theta = \delta(\rho)$, $|F(\rho e^{i\theta})| = F(\rho) \exp\{\Lambda\delta^2/\log \rho (\log^2 \rho + \delta^2) + O(\log^2 \rho)\}$ and, using (15) and the definition of $\delta(r)$,

$$\begin{aligned} |F(\rho e^{i\theta})| &= F(\rho) \exp\left\{-\frac{1}{2} \delta(\rho)^2 b(\rho)(1 + \delta^2/\log^2 \rho)^{-1} + O(\log^2 \rho)\right\} \\ &= F(\rho) \exp\{-\Lambda\psi^2(1 - \psi^2 \log \rho + \psi^4 \log^2 \rho) + O(\log^2 \rho)\} \\ &< F(\rho) \exp\left(-\frac{9}{10} \Lambda\psi^2\right) = F(\rho)(-\log \rho)^{18/5} \\ &= F(\rho)b(\rho)^{-1/2}(2\Lambda)^{1/2}(-\log \rho)^{21/20}(1 + o(1)) \end{aligned}$$

and, by (15),

$$(16) \quad |F(\rho e^{i\theta})| < (2\Lambda/b(\rho))^{1/2} F(\rho) (\Lambda/n)^{21/20}.$$

We now consider the circle of radius $r=\rho$, divided by a Farey dissection of order $N=[n^{3/8}]+1$. If $|\theta| \geq \delta(\rho)$, from $\delta(\rho)=2^{1/2}\Lambda^{1/4}n^{-3/4}(\log n)^{1/2}(1+o(1))$ and $1/(N+1)=n^{-3/8}(1+o(1))$ follows that θ may belong to any one of the three intervals $(\delta(\rho), 2\pi/(N+1))$, $(2\pi/(N+1), 2\pi N/(N+1))$, $(2\pi N/(N+1), 2\pi-\delta(\rho))$. The first and last interval belong to the Farey arc centered at $x=\rho$. In that case, by (7),

$$F(\rho e^{i\theta}) = F(\rho) \exp \left\{ \frac{\Lambda \theta^2}{\log \rho (\log^2 \rho + \theta^2)} + d \log \left(1 + \frac{i\theta}{\log \rho} \right) + O(\log^2 \rho + \theta \log \rho + \theta^2) \right\}.$$

Using (16), it follows for $|\theta| > \delta(\rho)$, that

$$|F(\rho e^{i\theta})/F(\rho e^{i\delta})| = \exp \left\{ \frac{\Lambda}{\log \rho} \frac{\log^2 \rho (\theta^2 - \delta^2)}{(\log^2 \rho + \theta^2)(\log^2 \rho + \delta^2)} + d \log \left| 1 + \frac{i\theta}{\log \rho} \right| + O(\log^2 \rho + \theta^2) \right\}.$$

We distinguish between the two cases:

(i) $\delta < |\theta| < -\log \rho$, and (ii) $-\log \rho < |\theta| < 2\pi/(N+1)$.

$$(i) \quad |F(\rho e^{i\theta})/F(\rho e^{i\delta})| = \exp \left\{ \frac{-\Lambda(-\log \rho)(\theta^2 - \delta^2)}{(\log^2 \rho + \theta^2)(\log^2 \rho + \delta^2)} + O(1) \right\};$$

hence, $|F(\rho e^{i\theta})/F(\rho e^{i\delta})| = O(1)$, because of $|\theta| > \delta(\rho)$.

(ii) $|\theta| > -\log \rho$ implies $\theta^2 - \delta^2 > \log^2 \rho + \log^3 \rho$.

$\psi^2(\rho) = \log^2 \rho(1 + o(1))$ and

$$\begin{aligned} \left| \frac{F(\rho e^{i\theta})}{F(\rho e^{i\delta})} \right| &< \exp \left\{ \frac{\Lambda \log^3 \rho}{(\log^2 \rho + \theta^2)(\log^2 \rho + \delta^2)} - \frac{d}{4} \log(-\log \rho) + O(1) \right\} \\ &< \exp \left\{ \frac{\Lambda \log \rho}{2\theta^2} \left(1 - \frac{\delta^2}{\log^2 \rho} \right) - \frac{d}{4} \log(-\log \rho) + O(1) \right\} \\ &< \exp \left\{ \frac{-\Lambda}{2} \frac{(-\log \rho)}{4\pi^2} n^{3/4}(1 + o(1)) \right\} \\ &= \exp \left\{ \frac{-\Lambda^{3/2}}{8\pi^2} n^{1/4}(1 + o(1)) \right\} \rightarrow 0. \end{aligned}$$

Hence, if θ belongs to the first, or third interval, $F(\rho e^{i\theta}) = O(n^{-21/20} F(\rho) b(\rho)^{-1/2})$

follows from (16). If θ belongs to the second interval, it is on a Farey arc corresponding to h/k , $1 < k \leq N$, $(h, k) = 1$, $1/N \leq h/k \leq (N-1)/N$ and $\phi = \theta/2\pi - h/k$ satisfies (see 2.7 in [10]) $|k\phi| < 1/N$. By the remarks following (11), $|F(\rho e^{i\theta})| < \exp \{ -\Lambda t (\log \rho)^{-1} (1 + o(1)) + 2\Phi - 2C\pi k\phi + O(\log \rho) \}$. Using (13), $|k\phi| < 1/N$ and (15), $|2\Phi - 2C\pi k\phi| = O(k \log k + k^2/N) = O(k \log k) = O(n^{3/8+\eta}) = O\{(-\log \rho)^{-3/4-2\eta}\}$ for any $\eta > 0$. It now follows that for θ on the second interval, $|F(\rho e^{i\theta})| \leq \exp \{ \Lambda t (-\log \rho)^{-1} (1 + o(1)) \} = \exp \{ t(\Lambda n)^{1/2} (1 + o(1)) \}$, while, by (7) and (15), $n^{-1}b(\rho)^{-1/2}F(\rho) = n^{-1} \exp \{ (\Lambda n)^{1/2} (1 + o(1)) \} (2\Lambda)^{-1/2} (\Lambda/n)^{3/4}$. Consequently, for any $|\theta| \geq \delta(\rho)$, $|F(\rho e^{i\theta})| = O(n^{-1}b(\rho)^{-1/2}F(\rho))$, whence it follows that condition (a) of the lemma is satisfied by $F(x)$ with $\alpha=1$. We now apply (14) to $F(x) = \sum_{n=0}^{\infty} p_n(q)x^n$ and obtain, as $n \rightarrow \infty$, that $p_n(q) = \rho^{-n} (2\pi b(\rho))^{-1/2} F(\rho) \cdot (1 - (3/16\Lambda)(-\log \rho) + O(\log^2 \rho))$. Replacing $F(\rho)$ and $b(\rho)$ by their values and using (15), one obtains

$$(17) \quad p_n(q) = (2\pi)^{-1/2(m+1)} (2\Lambda)^{-1/2} q^d \cdot \prod_{j=1}^m \Gamma\left(\frac{a_j}{q}\right) \cdot \left(\frac{\Lambda}{n}\right)^{(d/2+3/4)} e^{2(\Lambda n)^{1/2}} \cdot \left(1 - \left(\left(\frac{d+1}{2}\right)^2 + K\Lambda - \frac{1}{16}\right) (\Lambda n)^{-1/2} + O(n^{-1})\right).$$

As in §3, let $\lambda_n = (n-K)^{1/2}$, $E(x) = (2\pi)^{-1/2} \Gamma(x)$, $\omega = \prod_{j=1}^m E(a_j/q)$ and set $\lambda = \Lambda^{1/2}$. Then, if $I_1(x)$ stands for the Bessel function, (17) can also be written as

$$(17') \quad p_n(q) = \frac{\omega}{q} (q\lambda/\lambda_n)^{d+1} \exp \left\{ -\frac{d(d+2)}{4\lambda\lambda_n} \right\} I_1(2\lambda\lambda_n) (1 + O(n^{-1}))$$

and this proves the first assertion of Theorem 2.

9. For $H(x)$ one proceeds in essentially the same way. With $\mu = l\Lambda/(l+1)$, by (8) and (6), $a(r) = \mu(\log r)^{-2} - Kl + O(\log r)$. The solution $\rho = \rho_n$ of $a(\rho) = n$ satisfies here

$$(15') \quad \begin{aligned} (\log \rho)^{-1} &= -\mu^{-1/2} n^{1/2} \left(1 + \frac{1}{2} K l n^{-1} + O(n^{-3/2}) \right), \\ \log \rho &= -\mu^{1/2} n^{-1/2} \left(1 - \frac{1}{2} K l n^{-1} + O(n^{-3/2}) \right). \end{aligned}$$

Finally, $b(r) = 2\mu(-\log r)^{-3} + O(1)$. Setting $\delta = \delta(r) = (-\log r)^{3/2} \chi(r)$ with $\chi(r) = (-4/\mu \log(-\log r))^{1/2}$, one verifies that $H(x)$ satisfies the assumptions (b) and (c) of the lemma, with $\alpha=1$ and $u(\rho) = -3(-\log \rho)/16\mu$. From (12) and (13) also follows that assumption (a) is satisfied with $\alpha=1$. We may now apply (14) to $H(x) = \sum_{n=0}^{\infty} p_n(q, l)x^n$ and obtain that

$$p_n(q, l) = \rho^{-n} (2\pi b(\rho))^{-1/2} H(\rho) \left(1 - \frac{3}{16\mu} (-\log \rho) + O(\log^2 \rho) \right),$$

or, replacing $H(\rho)$ and $b(\rho)$ by their values and using (15'),

$$(18) \quad p_n(q, l) = \frac{1}{2} \pi^{-1/2} (l+1)^{-d} \mu^{1/4} n^{-3/4} \cdot \exp(2(\mu n)^{1/2}) \left(1 + \left(Kl\mu - \frac{3}{16} \right) \mu^{-1/2} n^{-1/2} + O(n^{-1}) \right).$$

Setting $\nu_n = (n + Kl)^{1/2}$ and $\nu = \mu^{1/2}$, (18) can also be written as

$$(18') \quad p_n(q, l) = (l+1)^{-d} \frac{\nu}{\nu_n} I_1(2\nu\nu_n) (1 + O(n^{-1})).$$

This finishes the proof of Theorem 2.

10. A few particular cases are of interest.

(a) If $\{a\} = \{1, 2, \dots, q\}$ is the complete set of least positive residues (mod q) one obtains the unrestricted partitions p_n . In this case $m=q$, $K=1/24$, $d=1/2$, $\omega = \{2\pi \prod_{a=1}^{q-1} 2 \sin(\pi a/q)\}^{-1/2} = (2\pi q)^{-1/2}$, $\lambda_n = (n - 1/24)^{1/2}$, $\lambda = \pi/6^{1/2}$ and (17') becomes

$$(17'') \quad p_n(q) = p_n = (2\pi^{-1/2}) (6\lambda_n^2)^{-3/4} \cdot \exp\left(\frac{-5(6^{1/2})}{16\pi\lambda_n}\right) I_1(\pi(2/3)^{1/2}\lambda_n) (1 + O(n^{-1})).$$

This is indeed in agreement with the result of Rademacher ((7.3)) in [10] and both expressions can be written in the form

$$p_n = \frac{1}{4(3)^{1/2}n} e^{\pi(2/3)^{1/2}(n)^{1/2}} \left(1 - \left(\frac{6(2)^{1/2}}{\pi} + \frac{\pi}{6 \cdot 2^{1/2}} \right) \frac{3^{1/2}}{12} n^{-1/2} + O(n^{-1}) \right).$$

However, if Rademacher's formula is written in the form of (17''), the error term is $O(e^{-\sigma(n)^{1/2}})$ with constant $\sigma > 0$.

(b) If $d=0$, (17') becomes

$$(17''') \quad p_n(q) = \frac{\omega\lambda}{\lambda_n} I_1(2\lambda\lambda_n) (1 + O(n^{-1})).$$

In particular, $d=0$, holds if $\{a\}$ is symmetrical; in this case, (17''') is easily identified with Petersson's formula (10.8) in [8], except for the order of its error term.

(c) Setting $d=0$ in (18') one obtains $p_n(q, l) = (\nu/\nu_n) I_1(2\nu\nu_n) (1 + O(n^{-1}))$, again a particular case of (10.8) in [8].

In general, one observes that the error terms $O(n^{-1})$ in (17') and (18') are

actually of order $O(e^{-\sigma(n)^{1/2}})$, $\sigma > 0$, as well in the case of symmetric sets $\{a\}$, when $d=0$, as also in case $\{a\} = \{1, 2, \dots, q\}$, when $d=1/2$. Hence, one may state as a plausible

CONJECTURE. The error terms in (17') and (18') are actually of order $O(e^{-\sigma(n)^{1/2}})$, with $\sigma > 0$.

11. **Proof of Corollary 1.** In case $q \equiv 1 \pmod{4}$ the sets of quadratic residues and of quadratic nonresidues are both symmetrical. Furthermore, $m = (q-1)/2$, $d=0$, $\Lambda = \pi^2(q-1)/12q$,

$$K = K^+ = (1/24q) \sum_{(a/q)=+1} (q^2 - 6aq + 6a^2) = -q(q-1)/24 + (1/4q) \sum_{(a/q)=+1} a^2$$

in the case of quadratic residues and, similarly, $K = K^- = -q(q-1)/24 + (1/4q) \sum_{(a/q)=-1} a^2$ in the case of nonresidues. From (17'') now follows:

$$\begin{aligned} p_n^\pm(q) &= 2^{-(q+1)/4} \left(\frac{q-1}{3q} \right)^{1/4} \left\{ \prod_{(a/q)=\pm 1} \sin \frac{a}{q} \right\}^{-1/2} n^{-3/4} \\ &\quad \cdot \exp \left\{ \frac{\pi}{3} \left(\frac{3(q-1)}{q} \right)^{1/2} n^{1/2} \right\} \\ &\quad \cdot \left(1 - \frac{\pi}{6} \left(\frac{3(q-1)}{q} \right)^{1/2} \left(K^\pm - \frac{9q}{4\pi^2(q-1)} \right) n^{-1/2} + O(n^{-1}) \right), \end{aligned}$$

where either $+$ or $-$ has to be taken throughout both members. Taking the ratio, one observes (see [4]) that $\left\{ \prod_{a \bmod q} (\sin a/q)^{(a/q)} \right\}^{-1/2} = \epsilon^h$, so that

$$\frac{p_n^+(q)}{p_n^-(q)} = \epsilon^h \left(1 - \frac{\pi}{6} \left(\frac{3(q-1)}{q} \right)^{1/2} (K^+ - K^-) n^{-1/2} + O(n^{-1}) \right).$$

However, $K^+ - K^- = (1/4q) \sum_{a \bmod q} (a/q) a^2$. This sum is well known (see [11, vol. 2, pp. 677-678] and [1]) and equals $4c\alpha_5(q)$. This finishes the proof of (1). Similarly, from (18), with the same values of the constants, for $n \rightarrow \infty$,

$$\begin{aligned} p_n^\pm(q, l) &= \left(\frac{l}{l+1} \frac{q-1}{12q} \right)^{1/4} n^{-3/4} \exp \left\{ \frac{\pi}{3} \left(\frac{l}{l+1} \frac{3(q-1)}{q} \right)^{1/2} n^{1/2} \right\} \\ &\quad \cdot \left(1 + \frac{\pi}{6} \left(\frac{3(q-1)}{q} \frac{l}{l+1} \right)^{1/2} \left(K^\pm \cdot l - \frac{9}{4\pi^2} \frac{l+1}{l} \frac{q}{q-1} \right) n^{-1/2} + O(n^{-1}) \right). \end{aligned}$$

Taking ratios,

$$\frac{p_n^+(q, l)}{p_n^-(q, l)} = 1 + \frac{\pi}{6} \left(\frac{3(q-1)}{q} \frac{l}{l+1} \right)^{1/2} \cdot l(K^+ - K^-) n^{-1/2} + O(n^{-1})$$

and, replacing $K^+ - K^-$ by its value $c\alpha_5(q)$, one obtains (2), thus finishing the proof of Corollary 1.

12. **Proof of Corollary 2.** If $q \equiv 3 \pmod{4}$, $q > 3$ and $\{a\}$ stands for the set of quadratic residues, or nonresidues, respectively, $m = (q-1)/2$ and $d^\pm = 1/q \sum_{(a/q)=\pm 1} a - (q-1)/4$, where the signs $+$ or $-$ have to be taken the same in both members and refer to quadratic residues, or nonresidues, respectively. (17) yields

$$\frac{p_n^+(q)}{p_n^-(q)} = \left(\frac{q^2 \Lambda}{n}\right)^{(d^+ - d^-)/2} \prod_{a \bmod q} \left\{ \Gamma\left(\frac{a}{q}\right) \right\}^{(a/q)} \cdot \left(1 - \left(\left(\frac{d^+ + 1}{2}\right)^2 - \left(\frac{d^- + 1}{2}\right)^2 + \Lambda(K^+ - K^-)\right)(\Lambda n)^{-1/2} + O(n^{-1})\right).$$

One has now (see [4]) for $q > 3$, that $d^+ - d^- = (1/q) \sum_{a \bmod q} (a/q)a = -h = -2h/w$ where h is the class number of the quadratic field $R((-q)^{1/2})$, and $w=2$ is the number of its roots of unity. Also,

$$K^\pm = (1/24q) \sum_{(a/q)=\pm 1} (q^2 - 6aq + 6a^2) = q(q-1)/48$$

and

$$(19) \quad \frac{p_n^+(q)}{p_n^-(q)} = \prod_{a \bmod q} \left\{ \Gamma\left(\frac{a}{q}\right) \right\}^{(a/q)} \cdot \left(\frac{12n}{\pi^2 q(q-1)}\right)^{h/w} \left(1 - \frac{1}{\pi} \left(\frac{6q}{(q-1)}\right)^{1/2} \frac{hn^{-1/2}}{w} + O(n^{-1})\right).$$

In exactly the same way, using (18) instead of (17), one obtains

$$\frac{p_n^+(q, l)}{p_n^-(q, l)} = (l+1)^{-d^+ + d^-} (1 + O(n^{-1})) = (l+1)^h (1 + O(n^{-1})).$$

In case $q=3$, (7) and (8) still hold, although the proof has to be changed in some details; also the lemma remains applicable. Hence, one obtains as before (17) and (18). Now, however, $d^+ = 1/3 - 1/2 = -1/6$, $d^- = 2/3 - 1/2 = 1/6$, so that $d^+ - d^- = -1/3$. As the number w of roots of unity in $R((-3)^{1/2})$ is $w=6$ and $h=1$, this finishes the proof of Corollary 2.

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UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PA.