LINEAR OPERATORS ON QUASI-CONTINUOUS FUNCTIONS

BY
RALPH E. LANE

1. Introduction. In this paper we study a class of linear transformations for each of which the transform of a function is a function. This special class of transformations has such applications as the smoothing of experimental data, the prediction of outputs of a physical system for various inputs, and the estimation of the velocities and accelerations of an object from observations of its positions at various times.

By the statement that f is a function, we imply that if t is a real number, then $f(t)$ is a number. By the statement that f is quasi-continuous⁽¹⁾, we mean that f is a function such that if t is a real number then the limits $f(t-)$ and $f(t+)$ exist. Some lemmas on quasi-continuous functions appear in §2.

DEFINITION 1.1. The statement that T is a Q operator over the interval $[a, b]$ means that T is a transformation such that

(i) if y is quasi-continuous, then Ty is a function; if $g = Ty$ and s is a real number, then we denote the number $g(s)$ by $Ty(s)$,

(ii) if y_1 is quasi-continuous and y_2 is quasi-continuous, then $T(y_1 + y_2) = Ty_1 + Ty_2$,

(iii) if y is quasi-continuous and k is a number, then $T(ky) = k(Ty)$,

(iv) if y is quasi-continuous, c is a real number, and $z(t) = y(t+c)$ for each real number t , then $Tz(s) = Ty(s+c)$ for each real number s , and

(v) if s is a real number, then there is a positive number B_s such that if y is quasi-continuous and $M > |y(s-t)|$ for each number t in $[a, b]$, then $|Ty(s)| \leq MB_s$; by the norm, $|T(s)|$, of T at s we mean the greatest lower bound of the set of all such numbers B_s .

It will be observed that if T is a Q operator over the interval $[a, b]$ and $T_0y = Ty(0)$ for each quasi-continuous function y , then T_0 is a bounded linear transformation from the set of all quasi-continuous functions to the set of all numbers (i.e., T_0 is a bounded linear functional operation as defined in [2] and [3]). We give the following example of a Q operator.

EXAMPLE 1.1. Suppose that if y is quasi-continuous and s is a real number, then

$$Ty(s) = [-y(s-2) - 3y(s-1) + 76y(s) + 76y(s+1) - 3y(s+2) - y(s+3)]/144.$$

Presented to the Society April 21, 1956; received by the editors August 16, 1956.

⁽¹⁾ Except for the use of the word "quasi-continuous," we use the terminology and notation of [1]. In particular, "integral" is defined as in [1].

It follows that T is a Q operator over the interval $[-3, 2]$ and that $|T(s)| = 10/9$ for each real number s . Moreover, if y is a polynomial of degree 3 or less, then $Ty(s) = \int_s^{s+1} y(t) dt$. This operator is designed for use with experimental data, for which the values used for y in the formula may include errors of observation. In effect, the operator smooths the raw data, interpolates, and gives the integral of the smoothed and interpolated data; it is derived from Jenkins' modified osculatory interpolation formula [4].

In §3 we show that if T is a Q operator over the interval $[a, b]$, then Ty is quasi-continuous, of bounded variation, or continuous, according as y is quasi-continuous, of bounded variation, or continuous. In §4 we show that if T is a Q operator and y is a quasi-continuous function then Ty is the sum of two integrals.

We adopt the following notation. If y is quasi-continuous, then y_L and y_R denote the functions such that $y_L(t) = y(t-)$ and $y_R(t) = y(t+)$ for each real number t .

DEFINITION 1.2. The statement that T is a Q_1 operator over the interval $[a, b]$ means that T is a Q operator over $[a, b]$ such that if y is quasi-continuous then $Ty(s+) - Ty(s-) = 2[Ty_R(s) - Ty_L(s)]$ for each real number s ; i.e., if $Ty = x$, $T(y_L) = u$ and $T(y_R) = v$, then $x(s+) - x(s-) = 2[v(s) - u(s)]$ for each real number s .

In §5 we show that a Q_1 operator T is a Q operator such that if y is a quasi-continuous function then Ty is an integral. In §6 we find conditions sufficient to assure that a Q_1 operator T has various properties which may be desirable in applications. For example, we find a condition sufficient to assure that if y is quasi-continuous then Ty has a derivative, and we exhibit a Q_1 operator T' such that $T'y$ is the derivative of Ty . In §7 we give a family of Q_1 operators, one of which is a limit of the "most powerful" smoothing operators given in [4].

2. Lemmas concerning quasi-continuous functions. The following results will be used later in this paper.

LEMMA 2.1. *For the function f to be quasi-continuous, it is necessary and sufficient that if $[a, b]$ is an interval and $\epsilon > 0$ then there is a step-function s such that $|f(t) - s(t)| < \epsilon$ for each number t in $[a, b]$.*

For a proof, see Lemma 4.1b of [1]; see [5] also.

LEMMA 2.2. *For the function f to be quasi-continuous it is necessary and sufficient that if $[a, b]$ is an interval and $\epsilon > 0$ then there is a subdivision $t_0, t_1, \dots, t_n$ of $[a, b]$ such that if p and q are in one of the segments (t_i, t_{i+1}) then $|f(p) - f(q)| < \epsilon$.*

Proof is omitted.

LEMMA 2.3. *If $f_1, f_2, f_3, \dots$ is a sequence of quasi-continuous functions which converges uniformly to a function f , then f is quasi-continuous. Moreover,*

if $f_n(t+) = f_n(t)$ for each positive integer n and each real number t , then $f_R = f$; likewise, if $f_n(t-) = f_n(t)$ for each positive integer n and each real number t , then $f_L = f$.

Proof is omitted.

LEMMA 2.4. If $f_1, f_2, \dots$, is a sequence of functions which converges uniformly to a function f , and $[a, b]$ is an interval, and $V > 0$, and $V_a^b(f_n) \leq V$ for each positive integer n , then $V_a^b(f) \leq V$.

For a proof, see Lemma 4.2a of [1].

LEMMA 2.5. If f is a quasi-continuous function, then there are a quasi-continuous function g and a quasi-continuous function h such that

- (i) $g_L = g$ and $h_R = h$,
- (ii) $g + h = f$, and
- (iii) if $[a, b]$ is an interval and $|f(t)| \leq M < M_1$ for each number t in $[a, b]$, then $|g(t)| \leq 1.5M_1$ and $|h(t)| \leq 1.5M_1$ for each number t in $[a, b]$.

Proof. We introduce the following notation. Suppose that $[a, b]$ is an interval, z is quasi-continuous, and D is a subdivision $t_0, t_1, \dots, t_m$ of $[a, b]$. Then $L_D z$ and $R_D z$ denote the pair of functions such that $2L_D z(a) = z(a)$, $2R_D z(b) = z(b)$, and for $i = 0, 1, \dots, m-1$,

$$2(t_{i+1} - t_i)L_D z(t) = (t_{i+1} - t)[2z(t_i+) - z(t_i)] + (t - t_i)z(t_{i+1}) \text{ if } t_i < t \leq t_{i+1},$$

and

$$2(t_{i+1} - t_i)R_D z(t) = (t_{i+1} - t)z(t_i) + (t - t_i)[2z(t_{i+1}-) - z(t_{i+1})] \text{ if } t_i \leq t < t_{i+1}.$$

We make the following three observations. First, if $|z(t)| \leq M$ for each number t in $[a, b]$, then $|L_D z(t)| \leq 1.5M$ and $|R_D z(t)| \leq 1.5M$ for $a \leq t \leq b$. Second, if $\epsilon > 0$ and it is true that $|z(p) - z(q)| \leq \epsilon$ if p and q are in one of the segments (t_i, t_{i+1}) , then $|z(t) - L_D z(t) - R_D z(t)| \leq \epsilon$ for $a \leq t \leq b$. Finally, $L_D z(t-) = L_D z(t)$ if $a < t \leq b$, and $R_D z(t+) = R_D z(t)$ if $a \leq t < b$.

Now suppose that $[a, b]$ is an interval and $|f(t)| \leq M < M_1$ if $a \leq t \leq b$. Let ϵ denote a positive number less than $M_1 - M$. We now construct a sequence $f_1, f_2, f_3, \dots$ of functions and a sequence $D_1, D_2, D_3, \dots$ of subdivisions of $[a, b]$ in the following manner. Let f_1 denote the function f , and let D_1 denote a subdivision $t_0, t_1, \dots, t_m$ of $[a, b]$ such that if p and q are in one of the segments (t_i, t_{i+1}) then $|f_1(p) - f_1(q)| < \epsilon/2$. For each positive integer n , let $f_{n+1} = f_n - L_{D_n} f_n - R_{D_n} f_n$, where D_n denotes a subdivision $t_0, t_1, \dots, t_m$ of $[a, b]$ such that if p and q are in one of the segments (t_i, t_{i+1}) then $|f_n(p) - f_n(q)| \leq \epsilon/2^n$. Suppose that $a \leq t \leq b$. Then $|f_1(t)| \leq M$, and therefore $|L_{D_1} f_1(t)| \leq 1.5M$ and $|R_{D_1} f_1(t)| \leq 1.5M$; moreover, $|f_2(t)| = |f_1(t) - L_{D_1} f_1(t) - R_{D_1} f_1(t)| \leq \epsilon/2$. By induction, if n is an integer greater than 1, and $a \leq t \leq b$, then $|L_{D_n} f_n(t)| \leq 1.5\epsilon/2^{n-1}$, $|R_{D_n} f_n(t)| \leq 1.5\epsilon/2^{n-1}$, and

$$\begin{aligned} \left| f_1(t) - \sum_{p=1}^n L_{D_p} f_p(t) - \sum_{p=1}^n R_{D_p} f_p(t) \right| \\ = |f_{n+1}(t)| = |f_n(t) - L_{D_n} f_n(t) - R_{D_n} f_n(t)| \leq \epsilon/2^n. \end{aligned}$$

But $\sum_{p=1}^{\infty} L_{D_p} f_p$ converges uniformly in $[a, b]$ to a function g ; and if $a \leq t \leq b$, then $|g(t)| \leq \sum_{p=1}^{\infty} |L_{D_p} f_p(t)| \leq 1.5(M + \epsilon) < 1.5M_1$. Similarly, $\sum_{p=1}^{\infty} R_{D_p} f_p$ converges uniformly in $[a, b]$ to a function h , and if $a \leq t \leq b$ then $|h(t)| < 1.5M_1$. Moreover, if $a \leq t \leq b$ then $g(t) + h(t) = f(t)$. From Lemma 2.3, it follows that if $a < t \leq b$ then $h(t-)$ exists and $g(t-) = g(t)$, and that if $a \leq t < b$ then $g(t+)$ exists and $h(t+) = h(t)$. This completes the proof.

LEMMA 2.6. *Suppose that $[t_0, t_1]$ is an interval, $\epsilon > 0$, and f is a function such that*

- (i) *if $t_0 \leq t \leq t_1$, then the derivative $f'(t)$ exists, and*
- (ii) *if s_1 and s_2 are in $[t_0, t_1]$, then $|f'(s_1) - f'(s_2)| < \epsilon$. Then*

$$|(f(t_1) - f(t_0))/(t_1 - t_0) - f'(t)| < \epsilon(2)^{1/2}$$

for each number t in $[t_0, t_1]$.

Proof is omitted, since this lemma can be obtained by applying the theorem of the mean to the real part and the imaginary part of f .

LEMMA 2.7. *If f has a quasi-continuous derivative f' , then f' is continuous.*

Proof is omitted, since the lemma follows readily from well-known results and can be derived from Lemma 2.6.

LEMMA 2.8. *Suppose that $[a, b]$ is an interval, y is a function which is bounded in $[a, b]$, and x is a function whose derivative, x' , is continuous in $[a, b]$. If $\int_a^b y(t) dx(t) = I$ or $\int_a^b y(t) x'(t) dt = I$, then $\int_a^b y(t) dx(t) = \int_a^b y(t) x'(t) dt$.*

Proof is omitted, since this lemma follows with little difficulty from Lemma 2.6.

LEMMA 2.9. *Suppose that f is a function whose derivative, f' , is of bounded variation in the interval $[a, b]$, h is a real number other than zero, and $[c, d]$ is an interval such that if t is in $[c, d]$ then t and $t+h$ are in $[a, b]$. If $g(t) = [f(t+h) - f(t)]/h$ for each number t in $[c, d]$, then g is of bounded variation in $[c, d]$, and $V_c^d(g) \leq V_a^b(f')$.*

Proof. Suppose that $t_0, t_1, \dots, t_n$ is a subdivision of $[c, d]$, and let S denote the sum $\sum_{i=0}^{n-1} |g(t_{i+1}) - g(t_i)|$. Now

$$\begin{aligned} h[g(t_{i+1}) - g(t_i)] &= \int_0^h 1 df(t_{i+1} + t) - \int_0^h 1 df(t_i + t) \\ &= \int_0^h [f'(t_{i+1} + t) - f'(t_i + t)] dt. \end{aligned}$$

It follows that if $h > 0$, then

$$h |g(t_{i+1}) - g(t_i)| \leq \int_0^h |f'(t_{i+1} + t) - f'(t_i + t)| dt$$

and

$$hS \leq \int_0^h \sum_{i=0}^{n-1} |f'(t_{i+1} + t) - f'(t_i + t)| dt \leq \int_0^h V_a^b(f') dt = hV_a^b(f'),$$

so that $S \leq V_a^b(f')$. By a similar argument, if $h < 0$, then $S \leq V_a^b(f')$. Hence g is of bounded variation in $[c, d]$, and $V_c^d(g) \leq V_a^b(f')$. This completes the proof.

3. Some properties of Q operators. In this section we suppose that there are given an interval $[a, b]$ and a Q operator T over $[a, b]$.

LEMMA 3.1a. *Suppose that s is a real number and y and z are quasi-continuous functions such that $y(s-t) = z(s-t)$ if $a \leq t \leq b$. Then $Ty(s) = Tz(s)$.*

Proof. Let $w = y - z$. By (ii) and (iii) of Definition 1.1, $Ty(s) - Tz(s) = Tw(s)$. But $w(s-t) = 0$ if $a \leq t \leq b$; and by (v) of Definition 1.1, if $\epsilon > 0$ then $|Tw(s)| \leq \epsilon |T(s)|$; hence $Tw(s) = 0$, or $Ty(s) = Tz(s)$. This completes the proof.

THEOREM 3.1. *If s is a real number, then $|T(s)| = |T(0)|$.*

Proof. Suppose that s is a real number. Now if y is quasi-continuous and $z(t) = y(t-s)$ for each real number t , then $z(s-t) = y(-t)$ for each number t in $[a, b]$; by Lemma 3.1a and (iv) of Definition 1.1, $Tz(s) = Ty(0)$. It follows from (v) of Definition 1.1 that $|T(s)| = |T(0)|$. This completes the proof.

REMARK. In view of Theorem 3.1, we shall hereafter refer to the norm of T as $|T|$; i.e., if s is a real number, then $|T| = |T(s)|$.

THEOREM 3.2. *If $a < c < b$, then there are a Q operator T_1 over $[a, c]$ and a Q operator T_2 over $[c, b]$ such that*

- (i) *if y is quasi-continuous then $Ty = T_1y + T_2y$, and*
- (ii) *$|T_1| + |T_2| = |T|$.*

Proof. If y is quasi-continuous and s is a real number, then we define numbers $T_1y(s)$ and $T_2y(s)$ in the following manner. Let $u(s-t) = 0$ if $t > c$ and $u(s-t) = y(s-t)$ if $t \leq c$ and let $T_1y(s) = Tu(s)$. Let $v(s-t) = y(s-t)$ if $t > c$ and $v(s-t) = 0$ if $t \leq c$; and let $T_2y(s) = Tv(s)$. By Definition 1.1, T_1 is a Q operator over $[a, c]$, and T_2 is a Q operator over $[c, b]$; moreover, if y is quasi-continuous, then $T_1y + T_2y = Ty$.

We now show that $|T_1| + |T_2| \geq |T|$. Suppose that $\epsilon > 0$, s is a real number and y is a quasi-continuous function such that $|Ty(s)| > |T| - \epsilon$ and $|y(s-t)| \leq 1$ for each number t in $[a, b]$. Now $|T_1| + |T_2| \geq |T_1y(s)| + |T_2y(s)| \geq |T_1y(s) + T_2y(s)| = |Ty(s)| > |T| - \epsilon$. Hence $|T_1| + |T_2| \geq |T|$.

Finally we show that $|T_1| + |T_2| \leq |T|$. Suppose that $\epsilon > 0$, s is a real

number, and z is a quasi-continuous function such that $T_1 z(s) = |T_1 z(s)| > |T_1| - \epsilon$, $T_2 z(s) = |T_2 z(s)| > |T_2| - \epsilon$, and $|z(s-t)| \leq 1$ if t is in $[a, b]$. Now $|T| \geq |Tz(s)| = |T_1 z(s) + T_2 z(s)| = T_1 z(s) + T_2 z(s) > |T_1| + |T_2| - 2\epsilon$. Hence $|T| \geq |T_1| + |T_2|$. This completes the proof.

LEMMA 3.3a. *There is a number sequence c_1, c_2, c_3 such that if s is a real number and y is a step-function such that $y(s-p) = y(s-q)$ for each pair p, q of numbers between a and b , then $Ty(s) = c_1 y(s-b) + c_2 y(s-b+) + c_3 y(s-a)$; moreover, $|T| \geq |c_1| + |c_2| + |c_3|$.*

Proof. Let functions u, v, w be defined as follows:

$$u(-t) = 1 \text{ if } t \geq b \text{ and } u(-t) = 0 \text{ if } t < b;$$

$$v(-t) = 1 \text{ if } a < t < b \text{ and } v(-t) = 0 \text{ if } t \geq b \text{ or } t \leq a; \text{ and}$$

$$w(-t) = 1 \text{ if } t \leq a \text{ and } w(-t) = 0 \text{ if } t > a.$$

Let $c_1 = Tu(0)$, $c_2 = Tv(0)$, and $c_3 = Tw(0)$. Now if s is a real number, $a \leq t \leq b$, and y is a step-function such that $y(s-p) = y(s-q)$ for each pair p, q of numbers between a and b , then $y(s-t) = y(s-b)u(-t) + y(s-b+)v(-t) + y(s-a)w(-t)$, and therefore $Ty(s) = c_1 y(s-b) + c_2 y(s-b+) + c_3 y(s-a)$.

Now let d_1, d_2, d_3 denote a number sequence such that $|d_1| = |d_2| = |d_3| = 1$ and $c_1 d_1 \geq 0$, $c_2 d_2 \geq 0$, and $c_3 d_3 \geq 0$. For each real number t , let $z(t) = d_1 u(t) + d_2 v(t) + d_3 w(t)$. Now if $a \leq t \leq b$, then $|z(-t)| = 1$, so that $|Tz(0)| \leq |T|$; but $Tz(0) = c_1 d_1 + c_2 d_2 + c_3 d_3 = |c_1| + |c_2| + |c_3|$; so $|T| \geq |c_1| + |c_2| + |c_3|$. This completes the proof.

LEMMA 3.3b. *Suppose that $t_0, t_1, \dots, t_{2n}$ is a subdivision of $[a, b]$. There is a number sequence $c_0, c_1, \dots, c_{2n}$ such that if s is a real number and y is a step-function such that $y(s-p) = y(s-q)$ for each pair p, q of numbers between t_{2i} and t_{2i+2} , $i = 0, 1, \dots, n-1$, then $Ty(s) = \sum_{i=0}^{2n} c_i y(s-t_i)$; moreover, $|T| \geq \sum_{i=0}^{2n} |c_i|$.*

Proof is omitted, since this lemma follows from Theorem 3.2 and Lemma 3.3a.

LEMMA 3.3c. *If y is a step-function and $[c, d]$ is an interval, then Ty is of bounded variation in $[c, d]$, and $V_c^d(Ty) \leq |T| \cdot V_c^d(y)$.*

Proof. Let $s_0, s_1, \dots, s_m$ denote a subdivision of $[c, d]$. Let $t_0, t_1, \dots, t_{2n}$ denote a subdivision of $[a, b]$ such that if j is one of the integers $0, 1, \dots, m$, and i is one of the integers $0, 1, \dots, n-1$, and p and q are numbers between t_{2i} and t_{2i+2} , then $y(s_j-p) = y(s_j-q)$. By Lemma 3.3b, there is a number sequence $c_0, c_1, \dots, c_{2n}$ such that $Ty(s_j) = \sum_{i=0}^{2n} c_i y(s_j-t_i)$ for $j=0, 1, \dots, m$, and $|T| \geq \sum_{i=0}^{2n} |c_i|$. Now if j is one of the integers $0, 1, \dots, m-1$, then $Ty(s_{j+1}) - Ty(s_j) = \sum_{i=0}^{2n} c_i [y(s_{j+1}-t_i) - y(s_j-t_i)]$. Hence

$$\begin{aligned}
\sum_{j=0}^{m-1} |Ty(s_{j+1}) - Ty(s_j)| &= \sum_{j=0}^{m-1} \left| \sum_{i=0}^{2n} c_i [y(s_{j+1} - t_i) - y(s_j - t_i)] \right| \\
&\leq \sum_{j=0}^{m-1} \sum_{i=0}^{2n} |c_i| \cdot |y(s_{j+1} - t_i) - y(s_j - t_i)| \\
&\leq \sum_{i=0}^{2n} |c_i| \cdot V_{c-t_i}^{d-t_i}(y) \\
&\leq |T| \cdot V_{c-b}^{d-a}(y).
\end{aligned}$$

The lemma now follows at once.

THEOREM 3.3. *If y is quasi-continuous, then Ty is quasi-continuous; if y is of bounded variation, then Ty is of bounded variation; if y is continuous, then Ty is continuous.*

Proof. Suppose first that y is quasi-continuous. Let $y_1, y_2, y_3, \dots$ denote a sequence of step-functions which converges uniformly to y . By (v) of Definition 1.1, $Ty_1, Ty_2, Ty_3, \dots$ converges uniformly to Ty . Now if $[c, d]$ is an interval and n is a positive integer, then by Lemma 3.3c, Ty_n is of bounded variation in $[c, d]$. Hence each of the functions Ty_i is quasi-continuous; by Lemma 2.3, Ty is quasi-continuous.

Suppose now that y is of bounded variation, and that $[c, d]$ is an interval, and $V \geq V_{c-b}^{d-a}(y)$. Let $y_1, y_2, y_3, \dots$ denote a sequence of step-functions converging uniformly to y such that $V_{c-b}^{d-a}(y_n) \leq V$, $n=1, 2, 3, \dots$. Then $Ty_1, Ty_2, Ty_3, \dots$ is a sequence of functions converging to Ty uniformly, and $V_c^d(Ty_n) \leq |T| V$ for $n=1, 2, 3, \dots$. By Lemma 2.4, $V_c^d(Ty) \leq |T| V$; hence Ty is of bounded variation.

Suppose, finally, that y is continuous and that $[c, d]$ is an interval. Let ϵ denote a positive number, and let δ denote a positive number such that if $c-b \leq p < q \leq d-a$ and $q-p < \delta$, then $|y(p) - y(q)| < \epsilon$. Now suppose that $c \leq s_1 < s_2 \leq d$ and $s_2 - s_1 < \delta$. If $a \leq t \leq b$, then $|y(s_1 - t) - y(s_2 - t)| < \epsilon$; so $|Ty(s_1) - Ty(s_2)| < \epsilon |T|$. Hence Ty is continuous. This completes the proof.

COROLLARY 3.3a. *If T_1 is a Q operator over the interval $[a_1, b_1]$ and T_2 is a Q operator over the interval $[a_2, b_2]$, and $T_3y = T_1(T_2y)$ for each quasi-continuous function y , then T_3 is a Q operator over the interval $[a_1+a_2, b_1+b_2]$, and $|T_3| \leq |T_1| \cdot |T_2|$.*

Proof. By Theorem 3.3, if y is quasi-continuous, then so is T_2y ; therefore T_3y is a function. It can readily be verified that T_3 has the properties listed as (ii), (iii), and (iv) in Definition 1.1. Let s denote a real number, suppose that y is quasi-continuous, and that $M \geq |y(s-t)|$ if $a_1+a_2 \leq t \leq b_1+b_2$. Now if $a_1 \leq s_1 \leq b_1$ and $a_2 \leq t \leq b_2$, then $a_1+a_2 \leq s_1+t \leq b_1+b_2$; hence $|y(s-s_1-t)| \leq M$ if $a_1 \leq s_1 \leq b_1$ and $a_2 \leq t \leq b_2$, so that $|T_2y(s-s_1)| \leq |T_2| M$ if $a_1 \leq s_1 \leq b_1$, and therefore $|T_3y(s)| \leq |T_1| \cdot |T_2| M$. Hence T_3 is a Q operator over $[a_1+a_2,$

$b_1 + b_2$], and $|T_3| \leq |T_1| \cdot |T_2|$. This completes the proof.

EXAMPLE 3.1. We give an example of Corollary 3.3a for which $|T_3| < |T_1| \cdot |T_2|$. Suppose that if y is quasi-continuous and s is a real number, then $T_1 y(s) = [y(s-) + y(s)]/2$ and $T_2 y(s) = [y(s) - y(s+)]/2$. Then $T_3 y(s) = [y(s) - y(s+)]/4$. Hence $|T_1| = 1$, $|T_2| = 1$, and $|T_3| = 1/2$.

4. Integral representation of Q operators. In this section we suppose that there are given an interval $[a, b]$ and a Q operator over $[a, b]$. We introduce the following notation:

$$J(t) = 0 \text{ if } t < 0 \text{ and } J(t) = 1 \text{ if } t \geq 0,$$

$$L(s) = TJ_L(s) \text{ and } R(s) = TJ_R(s) = TJ(s) \text{ for each real number } s,$$

$$x_1(t) = 2L(t) - L(t-) \text{ and } x_2(t) = 2R(t) - R(t+) \text{ for each real number } t.$$

We note in passing that $L(t) = [x_1(t-) + x_1(t)]/2$ and $R(t) = [x_2(t) + x_2(t+)]/2$.

LEMMA 4.1a. $R(s) = L(s) = L(a) = 0$ if $s < a$; and $R(s) = L(s) = R(b)$ if $s > b$. Moreover, $R(s-) = L(s-)$ and $R(s+) = L(s+)$ for each real number s .

Proof. If $s < a$ and $a \leq t \leq b$, then $s - t < 0$, so that $J_L(s - t) = J_R(s - t) = J_L(a - t) = 0$, and therefore $L(s) = R(s) = L(a) = 0$. If $s > b$ and $a \leq t \leq b$, then $s - t > 0$, so that $J_L(s - t) = J_R(s - t) = J_R(b - t) = 1$, and therefore $L(s) = R(s) = R(b)$.

Suppose now that s is a real number. Let $s_1, s_2, s_3, \dots$ denote a decreasing sequence which converges to s . Let $a_1, a_2, a_3, \dots$ denote a number sequence such that if p is a positive integer then $|a_p| = 1$ and $a_p [R(s_p) - L(s_p)] \geq 0$. For each positive integer n , let f_n denote the step-function such that if t is a real number then $f_n(t) = \sum_{p=1}^n a_p [J_R(s_p + t) - J_L(s_p + t)]$. Then $Tf_n(0) = \sum_{p=1}^n |R(s_p) - L(s_p)|$. But if $a \leq t \leq b$, then $|f_n(-t)| \leq 1$, whence $|Tf_n(0)| \leq |T|$. Hence $\sum_{p=1}^n |R(s_p) - L(s_p)|$ converges, and therefore $|R(s_p) - L(s_p)| \rightarrow 0$ as $p \rightarrow +\infty$. It follows that $R(s+) = L(s+)$. By a similar argument, if s is a real number, then $R(s-) = L(s-)$. This completes the proof.

LEMMA 4.1b. If s is a real number, then

$$L(s) = \int_a^{b+} J_L(s - t) dx_1(t)$$

and

$$R(s) = \int_{a-}^b J_R(s - t) dx_2(t).$$

Proof. By Lemma 4.1a, $x_1(t) = x_2(t) = x_1(a) = 0$ if $t < a$; and $x_1(t) = x_2(t) = x_2(b)$ if $t > b$. Moreover, by Theorem 3.3, R and L are of bounded variation, and therefore x_1 and x_2 are of bounded variation. Suppose that s is a real number and p is a positive number such that $p > s - b$ and $p > a - s$. Then $\int_a^{b+} J_L(s - t) dx_1(t) = \int_{a-p}^{b+p} J_L(s - t) dx_1(t)$. By the integration by parts formula,

$$\int_{a-p}^{b+p} J_L(s-t) dx_1(t) = J_L(s-b-p)x_1(b+p) - J_L(s-a+p)x_1(a-p) \\ - \int_{a-p}^{b+p} x_1(t) dJ_L(s-t),$$

or

$$\int_a^{b+} J_L(s-t) dx_1(t) = [x_1(s-) + x_1(s)]/2 = L(s).$$

By a similar argument,

$$\int_{a-}^b J_R(s-t) dx_2(t) = [x_2(s) + x_2(s+)]/2 = R(s).$$

THEOREM 4.1. *Suppose that y is quasi-continuous and s is a real number. Let g and h denote quasi-continuous functions such that $g_L = g$, $h_R = h$, and $g+h = y$. Then*

$$Ty(s) = \int_a^{b+} g(s-t) dx_1(t) + \int_{a-}^b h(s-t) dx_2(t).$$

Proof. If g is a step-function, it follows from Lemma 4.1b that $Tg(s) = \int_a^{b+} g(s-t) dx_1(t)$; if g is not a step-function, it follows from Lemmas 2.3 and 4.1b of the present paper and Lemma 4.1a of [1] that $Tg(s) = \int_a^{b+} g(s-t) dx_1(t)$. Similarly, $Th(s) = \int_{a-}^b h(s-t) dx_2(t)$. The theorem now follows from (ii) of Definition 1.1.

REMARK 4.1. Upon comparing Theorem 4.1 with (v) of Definition 1.1, one might suppose that $x_1(b+) = x_1(b)$ and $x_2(a-) = x_2(a)$. To show that this need not be so, we give the following example. For each quasi-continuous function y and real number s , let $Ty(s) = [y(s) + y(s-1)]/2$, so that T is a Q operator over $[0, 1]$. For this example,

$$x_1(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1/2 & \text{if } 0 < t \leq 1, \\ 1 & \text{if } t > 1. \end{cases} \quad x_2(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1/2 & \text{if } 0 \leq t < 1, \\ 1 & \text{if } t \geq 1. \end{cases}$$

THEOREM 4.2. *Suppose that $[c, d]$ is an interval and u_1 and u_2 are functions of bounded variation such that*

- (i) $u_1(t) = u_2(t) = u_1(c) = 0$ if $t < c$,
- (ii) $u_1(t) = u_2(t) = u_2(d)$ if $t > d$, and
- (iii) $u_1(t-) = u_2(t-) = u_1(t+) = u_2(t+)$ for each real number t .

If s is a real number and g and h are quasi-continuous functions such that $g_L = g$ and $h_R = h$, let $Ug(s)$, $Uh(s)$, and $Uy(s)$, where $y = g+h$, denote the numbers $\int_c^{a+} g(s-t) du_1(t)$, $\int_{c-}^d h(s-t) du_2(t)$, and $Ug(s) + Uh(s)$, respectively. Then U is a

Q operator over $[c, d]$, and $|U| \leq 1.5[V_c^{d+}(u_1) + V_c^d(u_2)]$.

Proof. We show first that if y is quasi-continuous, then there is just one function which is Uy . Suppose that y is quasi-continuous and that g, h, p, q is a sequence of quasi-continuous functions such that $g_L = g$, $h_R = h$, $p_L = p$, $q_R = q$, $g + h = y$, and $p + q = y$. Suppose that $f = g - p$; then $f_L = f$. But $f = q - h$, and therefore $f_R = f$; so f is continuous. Now if s is a real number, then $\int_{-\infty}^+ f(s-t) du_1(t) - \int_{-\infty}^+ f(s-t) du_2(t) = \int_{-\infty}^+ f(s-t) d[u_1(t) - u_2(t)]$. Since f is continuous, $u_1 - u_2$ is of bounded variation, and $u_1 - u_2$ is zero except for a countable subset of $[c, d]$, it follows that $\int_{-\infty}^+ f(s-t) d[u_1(t) - u_2(t)] = 0$, so that $\int_c^{d+} f(s-t) du_1(t) = \int_c^{d+} f(s-t) du_2(t)$, or $\int_c^{d+} g(s-t) du_1(t) - \int_c^{d+} p(s-t) du_1(t) = \int_c^{d+} q(s-t) du_2(t) - \int_c^{d+} h(s-t) du_2(t)$, or $Ug(s) - Up(s) = Uq(s) - Uh(s)$. Hence $Ug(s) + Uh(s) = Up(s) + Uq(s)$; i.e., there is just one function which is Uy . Moreover, if $|y(s-t)| \leq M < M_1$ for each number t in $[c, d]$, then by Lemma 2.5 the functions p and q may be chosen so that $|p(s-t)| \leq 1.5M_1$ and $|q(s-t)| \leq 1.5M_1$ for each number t in $[c, d]$; so by Theorem 2.1 of [1], $|Uy(s)| \leq 1.5M_1[V_c^{d+}(u_1) + V_c^d(u_2)]$. The theorem now follows from Definition 1.1 and the properties of integrals (cf. Theorem 2.1 of [1]).

REMARK 4.2. Compare Theorem 4.1 with the representation given in [2]; for an expression for the norm $|T|$, see [6].

5. Q_1 operators. In this section, we show that if T is a Q_1 operator and y is a quasi-continuous function, then Ty is an integral; and we obtain a condition sufficient for the product of two Q_1 operators to be a Q_1 operator.

THEOREM 5.1. *For T to be a Q_1 operator over the interval $[a, b]$ it is necessary and sufficient that there is a function x such that*

- (i) x is of bounded variation, $x(t) = 0$ if $t \leq a$, $x(t) = x(b)$ if $t > b$, and
- (ii) if y is quasi-continuous and s is a real number, then

$$Ty(s) = \int_a^b y(s-t) dx(t).$$

Proof. A. Suppose that T is a Q_1 operator over $[a, b]$, and let R, L, x_1 , and x_2 be defined as in §4. By Definition 1.2, $R(s+) - R(s-) = 2[R(s) - L(s)]$ for each real number s ; and by Lemma 4.1a, $R(s-) = L(s-)$, so that $2R(s) - R(s+) = 2L(s) - L(s-)$, or $x_2(s) = x_1(s)$ for each real number s . It follows from Theorem 3.3 that x_1 is of bounded variation, from Lemma 4.1a that $x_1(t) = 0$ if $t \leq a$ and $x_1(t) = x_1(b)$ if $t > b$, and from Theorem 4.1 that $Ty(s) = \int_a^b y(s-t) dx_1(t)$ for each quasi-continuous function y and real number s .

B. Suppose that x is a function such that (i) and (ii) of the theorem are true. By Theorem 4.2, T is a Q operator over $[a, b]$. Let L and R be defined as in §4. Now if s is a real number, then $R(s) = [x(s) + x(s+)]/2$ and $L(s) = [x(s-) + x(s)]/2$, whence $2R(s) - R(s+) = 2L(s) - L(s-) = x(s)$; since $R(s+) = L(s+)$ and $R(s-) = L(s-)$, it follows that $R(s+) - R(s-) = 2[R(s) - L(s)]$ and $L(s+) - L(s-) = 2[R(s) - L(s)]$. Now if y is a step-function and s is a real number, then there are a number sequence $a_1, a_2, \dots, a_{2n}$ and a real-number sequence $t_1, t_2, \dots, t_n$ such that

$$y(s-t) = \sum_{p=1}^n [a_{2p-1}J_R(s-t_p-t) + a_{2p}J_L(s-t_p-t)] \text{ if } a \leq t \leq b;$$

hence $Ty(s) = \sum_{p=1}^n [a_{2p-1}R(s-t_p) + a_{2p}L(s-t_p)]$, so that $Ty(s+) - Ty(s-) = 2[Ty_R(s) - Ty_L(s)]$. It now follows from Lemmas 2.1, 3.3c, and 2.3 that if y is quasi-continuous and s is a real number, then $Ty(s+) - Ty(s-) = 2[Ty_R(s) - Ty_L(s)]$. Hence T is a Q_1 operator over $[a, b]$. This completes the proof.

REMARK 5.1. If T is a Q_1 operator over an interval $[a, b]$ then there is just one function x such that

(i) $x(a) = 0$ and

(ii) if y is quasi-continuous and s is a real number, then $Ty(s) = \int_{-\infty}^{+\infty} y(s-t) dx(t)$. Moreover, x is of bounded variation.

THEOREM 5.2. Suppose that U is a Q_1 operator over the interval $[a, b]$, V is a Q_1 operator over the interval $[c, d]$, and $Ty = U(Vy)$ for each quasi-continuous function y . Let u and v denote the functions such that $u(a) = v(c) = 0$ and $Uy(s) = \int_{-\infty}^{+\infty} y(s-t) du(t)$ and $Vy(s) = \int_{-\infty}^{+\infty} y(s-t) dv(t)$ if y is quasi-continuous and s is a real number. If $u_L = u_R$ or $v_L = v_R$, then T is a Q_1 operator over $[a+c, b+d]$.

Proof. By Corollary 3.3a, T is a Q operator over $[a+c, b+d]$. Suppose that $u_L = u_R$ or $v_L = v_R$. Let R and L denote the functions TJ_R and TJ_L , respectively. Now $VJ_L = (v_L + v)/2$ and $VJ_R = (v + v_R)/2$; hence if s is a real number, then $2L(s) = \int_a^b [v_L(s-t) + v(s-t)] du(t)$ and $2R(s) = \int_a^b [v(s-t) + v_R(s-t)] du(t)$, so that $2[R(s) - L(s)] = \int_a^b [v_R(s-t) - v_L(s-t)] du(t)$. Since v is of bounded variation, so is $v_R - v_L$; and if $v_R \neq v_L$, then there is a countable real-number set K such that $v_R(t) \neq v_L(t)$ if and only if t is in K . Since u is of bounded variation, it follows from Theorem 3.1 of [1] and Lemma 4.2a of [1] that

$$\begin{aligned} 4[R(s) - L(s)] &= \sum_{(t)} [v_R(s-t) - v_L(s-t)][u(t) - u(t-)] \\ &\quad + \sum_{(t)} [v_R(s-t) - v_L(s-t)][u(t+) - u(t)] \\ &= \sum_{(t)} [v_R(s-t) - v_L(s-t)][u_R(t) - u_L(t)]. \end{aligned}$$

But by hypothesis, $v_R = v_L$ or $u_R = u_L$; hence $R = L$.

Now $2L = U(v_L + v)$, where U is a Q_1 operator; hence if s is a real number, then $2[L(s+) - L(s-)] = 4[Uv_R(s) - Uv_L(s)]$. But $Uv_R - Uv_L = U(v + v_R) - U(v_L + v) = 2(R - L) = 0$. Hence if s is a real number then $L(s+) - L(s-) = 0$; and by Lemma 4.1a, $R(s+) = L(s+)$ and $R(s-) = L(s-)$; hence $R(s+) - R(s-) = 2[R(s) - L(s)] = 0$. By the argument in part B of the proof of Theorem 5.1, it follows that T is a Q_1 operator over $[a+c, b+d]$. This completes the proof.

6. Q_1 operators having specified properties. In this section we suppose that T is a Q_1 operator over an interval $[a, b]$, and that x is the function of bounded variation such that $x(t)=0$ if $t \leq a$, $x(t)=x(b)$ if $t > b$, and $Ty(s) = \int_a^b y(s-t)dx(t)$ if y is quasi-continuous and s is a real number. We shall obtain conditions on x which are sufficient to assure that T has various ones of the properties described in the following definition.

DEFINITION 6.1. (i) The statement that T is symmetric means that if y is quasi-continuous and $z(t)=y(-t)$ for each real number t , then $Tz(s) = Ty(-s)$ for each real number s .

(ii) The statement that T has property B means that if y is quasi-continuous, then Ty is of bounded variation.

(iii) The statement that T has property C means that if y is quasi-continuous, then Ty is continuous.

(iv) The statement that T has property D means that (a) if y is quasi-continuous, then Ty has a derivative, and (b) there is a positive number N such that if y is quasi-continuous, s is a real number, and $M \geq |y(s-t)|$ for each number t in $[a, b]$, and $g = Ty$, then $|g'(s)| \leq MN$.

(v) If K is a set of quasi-continuous functions, then the statement that the members of K are invariant under T means that if y is in K then $Ty=y$.

THEOREM 6.1. For T to be symmetric, it is necessary and sufficient that $x(-t) - x(0) = x(0) - x(t)$ for each real number t .

Proof. A. Suppose that T is symmetric. If y is quasi-continuous, and $z(t) = y(-t)$ for each real number t , then $Tz(0) = Ty(0)$, or $\int_{-\infty}^{+\infty} z(-t)dx(t) = \int_{-\infty}^{+\infty} y(-t)dx(t)$, or $\int_{-\infty}^{+\infty} [y(t) - y(-t)]dx(t) = 0$. Hence if f is an odd quasi-continuous function, then $\int_{-\infty}^{+\infty} f(t)dx(t) = 0$. Suppose that if $p > 0$ then

$$f_p(t) = \begin{cases} 0 & \text{if } t < -p, \text{ or } t = 0 \text{ or } t > p, \\ -1 & \text{if } -p \leq t < 0, \\ 1 & \text{if } 0 < t \leq p. \end{cases}$$

Since f_p is an odd function, it follows that $\int_{-\infty}^{+\infty} f_p(t)dx(t) = 0$. Integration by parts gives the equation $\int_{-\infty}^{+\infty} x(t)df_p(t) = 0$. By Theorem 3.1 of [1],

$$[x(-p-) - x(0)] + [x(-p) - x(0)] - [x(0-) - x(0)] - [x(0+) - x(0)] \\ + [x(p) - x(0)] + [x(p+) - x(0)] = 0$$

for each positive number p . Consequently, upon considering a sequence of positive numbers $p_1, p_2, p_3, \dots$ which converges to zero, we conclude that $x(0-) - x(0) = x(0) - x(0+)$. By a similar argument, if $p > 0$, then $x(-p-) - x(0) = x(0) - x(p+)$; hence $x(-p) - x(0) = x(0) - x(p)$.

B. Suppose that $x(-t) - x(0) = x(0) - x(t)$ for each real number t . Suppose that y is quasi-continuous, $z(t) = y(-t)$ for each real number t , and s is a real number. Then

$$\begin{aligned}
Tz(s) &= \int_{-\infty}^{+\infty} z(s-t)dx(t) = \int_{+\infty}^{-\infty} z(s+t)dx(-t) \\
&= \int_{+\infty}^{-\infty} y(-s-t)d[x(-t) - x(0)] \\
&= - \int_{+\infty}^{-\infty} y(-s-t)d[x(t) - x(0)] = \int_{-\infty}^{+\infty} y(-s-t)dx(t) = Ty(-s).
\end{aligned}$$

This completes the proof.

THEOREM 6.2. *Suppose that if $[c, d]$ is an interval then there is a positive number N such that if $s_0, s_1, \dots, s_m$ is a subdivision of $[c, d]$ and $t_0, t_1, \dots, t_n$ is a subdivision of $[a-d, b-c]$, then*

$$\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |x(s_{i+1} + t_{j+1}) - x(s_{i+1} + t_j) - x(s_i + t_{j+1}) + x(s_i + t_j)| \leq N.$$

Then T has property B.

Proof. Suppose that y is quasi-continuous, $[c, d]$ is an interval, and $M \geq |y(s-t)|$ if s is in $[c, d]$ and t is in $[a, b]$. Let $s_0, s_1, \dots, s_m$ denote a subdivision of $[c, d]$, and suppose that $\epsilon > 0$. Now if s is a real number, then $Ty(s) = \int_{-\infty}^{+\infty} y(s-t)dx(t) = \int_{-\infty}^{+\infty} y(-t)dx(s+t)$; hence if I denotes the sum $\sum_{i=0}^{m-1} |Ty(s_{i+1}) - Ty(s_i)|$, then $I = \sum_{i=0}^{m-1} \left| \int_{-\infty}^{+\infty} y(-t)d[x(s_{i+1}+t) - x(s_i+t)] \right|$. Let $t_0, t_1, \dots, t_n$ denote a subdivision of $[a-d, b-c]$ such that if i is one of the integers $0, 1, \dots, m-1$, then

$$\begin{aligned}
&\left| \int_{-\infty}^{+\infty} y(-t)d[x(s_{i+1}+t) - x(s_i+t)] \right. \\
&\quad \left. - 2^{-1} \sum_{j=0}^{n-1} [y(-t_j) + y(-t_{j+1})][x(s_{i+1} + t_{j+1}) - x(s_i + t_{j+1}) \right. \\
&\quad \left. - x(s_{i+1} + t_j) + x(s_i + t_j)] \right| < \epsilon/m.
\end{aligned}$$

Now $|y(-t)| \leq M$ if $a-d \leq t \leq b-c$; hence $I < MN + \epsilon$, and consequently $V_{\epsilon}^d(Ty) \leq MN$. This completes the proof.

COROLLARY 6.2a. *If x has a derivative which is of bounded variation, then T has property B.*

Proof. Suppose that $[c, d]$ is an interval. If D is a subdivision $s_0, s_1, \dots, s_m$ of $[c, d]$ and E is a subdivision $t_0, t_1, \dots, t_n$ of $[a-d, b-c]$, let $S(D, E)$ denote the sum $\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |x(s_{i+1} + t_{j+1}) - x(s_i + t_{j+1}) - x(s_{i+1} + t_j) + x(s_i + t_j)|$. Suppose that D and E are such subdivisions and that $\epsilon > 0$. By hypothesis, x' is of bounded variation and (by Lemma 2.7) is therefore continuous; moreover, $x'(t) = 0$ if $t < a$ or $t > b$. Hence there is a positive number δ such that if

p and q are real numbers and $|p - q| < \delta$, then $|x'(p) - x'(q)| < \epsilon$. Let $u_0, u_1, \dots, u_r$ denote a refinement E' of E such that $u_{j+1} - u_j < \delta, j = 0, 1, \dots, r-1$. Now

$$S(D, E') = \sum_{i=0}^{m-1} \sum_{j=0}^{r-1} \left| \frac{x(s_{i+1} + u_{j+1}) - x(s_{i+1} + u_j)}{u_{j+1} - u_j} - \frac{x(s_i + u_{j+1}) - x(s_i + u_j)}{u_{j+1} - u_j} \right| (u_{j+1} - u_j);$$

and by Lemma 2.6,

$$\begin{aligned} S(D, E') &< 2[(b - c) - (a - d)]\epsilon(2)^{1/2} \\ &\quad + \sum_{i=0}^{m-1} \sum_{j=0}^{r-1} |x'(s_{i+1} + u_j) - x'(s_i + u_j)| (u_{j+1} - u_j) \\ &< 2[(b - c) - (a - d)]\epsilon(2)^{1/2} + \sum_{j=0}^{r-1} (u_{j+1} - u_j)V_{-\infty}^{+\infty}(x') \\ &< [(b - c) - (a - d)][2\epsilon(2)^{1/2} + V_{-\infty}^{+\infty}(x')]. \end{aligned}$$

But since E' is a refinement of E , it follows that $S(D, E) \leq S(D, E')$. Consequently, $S(D, E) \leq [(b - c) - (a - d)]V_{-\infty}^{+\infty}(x')$. By Theorem 6.2, T has property B. This completes the proof.

THEOREM 6.3. *For T to have property C, it is necessary and sufficient that x is continuous.*

Proof. Suppose that T has property C; then TJ_L is continuous and TJ_R is continuous. Since $TJ_L = (x_L + x)/2$ and $TJ_R = (x + x_R)/2$, it follows that x is continuous. Suppose now that x is continuous and that $\epsilon > 0$. Since $x(t) = 0$ if $t \leq a$ and $x(t) = x(b)$ if $t > b$, there is a positive number δ such that if p and q are real numbers and $|p - q| < \delta$, then $|x(p) - x(q)| < \epsilon$. Now if s_1 and s_2 are two real numbers and $|s_1 - s_2| < \delta$, and $g(t) = x(s_1 + t) - x(s_2 + t)$ for each real number t , then $|g(t)| < \epsilon$ for each real number t , and $V_{-\infty}^{+\infty}(g) \leq 2V_a^b(x)$. Suppose that y is quasi-continuous, s_1 is a real number, and $[c, d]$ is an interval containing s_1 . If s_2 is in $[c, d]$, then $|Ty(s_1) - Ty(s_2)| = |\int_{a-d}^{b-c} y(-t) d[x(s_1 + t) - x(s_2 + t)]|$; so by Lemma 4.2a of [1], $Ty(s_2) \rightarrow Ty(s_1)$ as $s_2 \rightarrow s_1$. That is to say, Ty is continuous, or T has property C. This completes the proof.

THEOREM 6.4. *For T to have property D, it is necessary and sufficient that x has a derivative which is of bounded variation. Moreover, if T has property D, y is quasi-continuous, and $g = Ty$, then g' is continuous, and $g'(s) = \int_a^b y(s - t) dx'(t)$ for each real number s .*

Proof. A. Suppose that T has property D. If y is quasi-continuous and $g = Ty$, let Uy denote the function g' . It follows from Definition 1.1 and (iv) of

Definition 6.1 that U is a Q operator over $[a, b]$. By Theorem 3.3, if y is quasi-continuous, then Uy is quasi-continuous; and by Lemma 2.7, Uy is continuous; i.e., if y is quasi-continuous and $g = Ty$, then g' is continuous. In particular, UJ_R and UJ_L are continuous; so by Lemma 4.1a, $UJ_R = UJ_L$, and it follows that U is a Q_1 operator over $[a, b]$. Let u denote the function UJ . If y is quasi-continuous and s is a real number, then $Uy(s) = \int_a^b y(s-t) du(t)$. But if $R = TJ_R$, then $u = R'$; and since R has a derivative, R is continuous so that $x = R$, whence $x' = R' = u$. Consequently, if y is quasi-continuous and s is a real number, then $Uy(s) = \int_a^b y(s-t) dx'(t)$. Moreover, since U is a Q_1 operator, x' is of bounded variation.

B. Suppose that x has a derivative which is of bounded variation. By Lemma 2.7, x' is continuous; and we observe that $x'(t) = 0$ if $t \leq a$ or $t \geq b$. Suppose that $\epsilon > 0$, and let h denote a real number other than zero such that if $p < q$ and $q - p \leq |h|$ then $|x'(p) - x'(q)| < \epsilon$. For each real number t , let $g_h(t)$ denote the number $[x(t+h) - x(t)]/h$. By Lemma 2.6, $|g_h(t) - x'(t)| < 2\epsilon$ for each real number t ; and by Lemma 2.9, the total variation of g_h is less than or equal to the total variation of x' . Now if y is quasi-continuous and s is a real number, then $[Ty(s+h) - Ty(s)]/h = \int_a^b y(-t) dg_h(s+t)$; and by Lemma 4.2a of [1], $[Ty(s+h) - Ty(s)]/h \rightarrow \int_a^b y(-t) dx'(s+t) = \int_a^b y(s-t) dx'(t)$ as $h \rightarrow 0$. If y is quasi-continuous and s is a real number, let $T'y(s)$ denote the number $\int_a^b y(s-t) dx'(t)$. Then T' is a Q_1 operator over $[a, b]$, such that if y is quasi-continuous and $g = Ty$, then $g' = T'y$. Hence T has property D. This completes the proof.

THEOREM 6.5. *Suppose that n is a positive integer. For all polynomials of degree n or less to be invariant under T , it is necessary and sufficient that $x(b) = 1$ and $\int_a^b t^p dx(t) = 0$ for $p = 1, 2, \dots, n$. If T is symmetric, and n is an even integer, and all polynomials of degree n or less are invariant under T , then all polynomials of degree $n+1$ or less are invariant under T .*

Proof is omitted, since the theorem readily follows if for each polynomial y and real number s we consider the Maclaurin expansion of $y(s-t)$ in powers of t .

7. A family of smoothing operators and differentiating operators. In this section we suppose that n is a positive integer and $[a, b]$ is an interval of unit length. We shall exhibit a Q_1 operator T on $[a, b]$ such that

- (i) all polynomials of degree 2 or less are invariant under T , and
- (ii) if y is quasi-continuous, then Ty has an n th derivative.

Let polynomials u , v , and w be defined as follows. If t is a real number, then

$$2u(t) = 1 + [2t - (a + b)] \sum_{p=0}^n \binom{2p}{p} (t-a)^p (b-t)^p,$$

$$v(t) = (n+2) \binom{2n+3}{n+1} (t-a)^{n+1} (b-t)^{n+1},$$

and

$$w(t) = (n+2) \binom{2n+3}{n+1} (t-a)^{n+2} (b-t)^{n+1}.$$

Let $A = (2n+5)[2ab + (n+1)/(2n+3)]$ and $B = (a+b+A)/2$. Let x denote the function such that $x(t)=0$ if $t < a$, $x(t)=1$ if $t > b$, and $x(t)=u(t)+Bv(t)-Aw(t)$ if $a \leq t \leq b$. If y is quasi-continuous and s is a real number, let $Ty(s)$ denote the number $\int_a^b y(s-t)dx(t)$, and let $T'y(s) = \int_a^b y(s-t)dx'(t)$.

By some rather tedious manipulation, it can be seen that x has a continuous n th derivative which is of bounded variation, and that $x(a)=0$, $x(b)=1$, $\int_a^b t dx(t)=0$, and $\int_a^b t^2 dx(t)=0$, so that polynomials of degree 2 or less are invariant under T . Moreover, if $a = -1/2$, then T is symmetric, so that polynomials of degree 3 or less are invariant under T . If y is quasi-continuous and $g = Ty$, then $g' = T'y$.

In particular, if $n=3$ and $a \leq t \leq b$, then

$$\begin{aligned} x(t) = & 0.5 + 0.5(2t - a - b)[1 + 2(t - a)(b - t) + 6(t - a)^2(b - t)^2 \\ & + 20(t - a)^3(b - t)^3 - 140(22 + 99ab)(t - a)^4(b - t)^4] \\ & + 315(a + b)(t - a)^4(b - t)^4, \end{aligned}$$

and

$$\begin{aligned} x'(t) = & 140(t - a)^3(b - t)^3[(45 + 198ab) + 9(a + b)(a + b - 2t) \\ & - (198 + 891ab)(t - a)(b - t)]. \end{aligned}$$

In particular, if $n=3$, $a = -1/2$, and $-1/2 \leq t \leq 1/2$, then

$$\begin{aligned} x(t) = & 0.5 + t[1 + 2(.25 - t^2) + 6(.25 - t^2)^2 + 20(.25 - t^2)^3 \\ & + 385(.25 - t^2)^4], \end{aligned}$$

and

$$x'(t) = 315(.25 - t^2)^3[11(.25 - t^2) - 2],$$

and T is a limit of the "most powerful" smoothing operators (i.e., operators with minimum smoothing coefficients) described, e.g., in [4]. For this instance the operators T and T' have been used with quite satisfactory results on a digital computer with experimental data, the integrals being approximated by an approximating sum as in [1], with the subdivision $-.50, -.45, -.40, \dots, .45, .50$ of the interval $[-1/2, 1/2]$.

If $n=3$, $a=0$, and $0 \leq t \leq 1$, then

$$\begin{aligned} x(t) = & 0.5 + 0.5(2t - 1)[1 + 2t(1 - t) + 6t^2(1 - t)^2 + 20t^3(1 - t)^3 \\ & - 3080t^4(1 - t)^4] + 315t^4(1 - t)^4, \end{aligned}$$

and

$$x'(t) = 2520t^3(1 - t)^3(3 - 12t + 11t^2);$$

in this instance T is not symmetric, and the application of the operators T and T' to experimental data does not give results as satisfactory as those obtained with the symmetric operator previously described.

REFERENCES

1. R. E. Lane, *The integral of a function with respect to a function*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 59-66.
2. H. S. Kaltenborn, *Linear functional operations on functions having discontinuities of the first kind*, Bull. Amer. Math. Soc. vol. 40 (1934) pp. 702-708.
3. T. H. Hildebrandt, *On bounded linear functional operations*, Trans. Amer. Math. Soc. vol. 36 (1934) pp. 868-875.
4. Morton D. Miller, *Elements of graduation*, Actuarial Society of America, 1946.
5. N. Bourbaki, *Éléments de mathématique*; IX, Première Partie, *Les structures fondamentales de l'analyse*; Livre IV, *Fonctions d'une variable réelle*; Hermann et Cie., Paris, 1949, Chapter 2, pp. 59-77.
6. G. Fichtenholz, *Sur une généralisation de l'intégrale de Stieltjes*, Doklady, Akad. Nauk SSSR vol. 3 (12) (1936) pp. 95-100.

THE UNIVERSITY OF TEXAS,
AUSTIN, TEXAS