CHARACTERIZATIONS OF GENERALIZED UNISERIAL ALGEBRAS⁽¹⁾

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1. **Introduction.** Let A be a finite dimensional algebra with unit element over a field K . In recent years a number of classes of algebras in which the radical may not be zero have been studied. One of the main classifications has been the Frobenius type algebras which can be defined in terms of dualities between left and right ideals of the algebra. Another main group has been the uniserial type algebras which can be defined in terms of uniqueness conditions placed upon the composition series of the primitive ideals of the algebra. It is known that uniserial algebras can be characterized as those algebras all of whose residue class algebras are of a certain Frobenius type. The purpose of this paper is to give an extension of this result and to give similar characterizations of generalized uniserial algebras.

2. **Definitions and notation.** An idempotent is a nonzero element e such that $e^2 = e$. A primitive idempotent is one which cannot be written as a sum of two orthogonal idempotents. If e is a primitive idempotent, then the left ideal Ae is called a primitive left ideal and the right ideal eA is called a primitive right ideal. If a primitive left ideal Ae is dual to some primitive right ideal fA , then Ae and fA are called *dominant* ideals. If Ae and Af are primitive left ideals and if Af is A -isomorphic, as a left A -module, to some subideal of Ae , then Af is said to be subordinate to Ae . If Ae and Af are primitive left ideals and if there exists a set $L_1, \dots, L_n$ of subideals of Ae such that Af is A -isomorphic to a subideal of the direct sum $\sum_{i=1}^n L_i$, then Af is said to be weakly subordinate to Ae . Let $Ae_1, \dots, Ae_m$ be a collection of primitive ideals and let $L_1, \dots, L_n$ be a set of ideals such that each L_i is a subideal of some Ae_j . If a primitive left ideal Af is A -isomorphic to a subideal of $\sum_{i=1}^n L_i$ then Af is said to be weakly subordinate to the set $\{Ae_j\}_{j=1}^m$. Note that if Af is subordinate to Ae then it is weakly subordinate to Ae , and if Af is weakly subordinate to Ae then it is weakly subordinate to any set of

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ideals which contain Ae or an ideal A -isomorphic to Ae . Subordinate and weakly subordinate right ideals are defined in a similar manner.

The main classes of Frobenius type algebras are:

1. QF-3 algebras: Every primitive ideal (left or right) is either dominant or is weakly subordinate to a set of dominant ideals. (It has been proved by Thrall [8, Theorem 5] that this is equivalent to the assumption that A has a unique minimal faithful representation) [8].

2. QF-3* algebras: Every primitive ideal is either dominant or is weakly subordinate to some dominant ideal.

3. QF-2 algebras: Every primitive ideal is either dominant or is subordinate to some dominant ideal. (Thrall proved [8, Theorem 1] that this is equivalent to assuming that every primitive ideal has a unique minimal subideal) [8].

4. Quasi-Frobenius algebras: Every primitive ideal is dominant [3; 4].

5. Frobenius algebras: The algebra A considered as a left A -module is dual to A considered as a right A -module [3; 4].

6. Weakly symmetric algebras: For every primitive idempotent e , the left ideal Ae is dual to the right ideal eA [6].

These classes have been defined in descending order of generality. Thus, any one of the classes is properly contained in each of the classes defined before it. For proofs of these inclusions and for further properties of these classes of algebras see the references cited after each definition.

The main classes of uniserial type algebras are:

1. Generalized uniserial algebras: Every primitive ideal has only one composition series [5].

2. Uniserial algebras: Every primitive ideal has only one composition series and the algebra is the direct sum of two-sided ideals each of which is a primary algebra. (A primary algebra is one whose residue class algebra modulo the radical is simple) [2; 3; 4].

It is known [2, Theorem 2; 5, Lemma 2] that each of the following gives a characterization of a uniserial algebra A :

a. For every two-sided ideal Z of A , A/Z is a Frobenius algebra.

b. For every two-sided ideal Z of A , A/Z is a weakly symmetric algebra.

The main theorems of this paper give a further characterization of uniserial algebras and similar characterizations for generalized uniserial algebras. Theorem 1 states that an algebra is generalized uniserial if and only if for every two-sided ideal Z of A , A/Z is a QF-2 algebra. By use of Theorem 1, a further characterization of uniserial algebras is obtained. This is given by Theorem 2 which states that an algebra A is uniserial if for every two-sided ideal Z of A , A/Z is quasi-Frobenius. This result has also been obtained by Osima [7; Theorem 12] by using basic algebras. An example will be given to show that the requirement that each A/Z be QF-2 does not imply that A is uniserial. Theorem 3 extends Theorem 1, showing that it is sufficient to as-

sume that each A/Z is QF-3* in order to imply that A is a generalized uniserial algebra.

3. Generalized uniserial algebras.

LEMMA 1. *If L is a primitive left ideal of A and L' is a subideal of L contained in every composition series of L , then $L'A$ is the minimal two-sided ideal containing L' and the following equations hold: $L \cap L'A = L'L = L'$.*

Proof. Since $A(L'A)A = (AL')(AA) \subset L'A$, it follows that $L'A$ is a two-sided ideal of A . If S is any two-sided of A such that $L' \subset S$ then $L'A \subset SA \subset S$. Thus, $L'A$ is the minimal two-sided ideal containing L' . Since L is a primitive ideal, i.e., $L = Ae$, the algebra A , considered as a left A -module, can be written as a direct sum of left ideals, $A = L \dot{+} M$, where $M = A(1 - e)$. Thus, $L'A = L'L \dot{+} L'M$, where $L'L \subset L$ and $L'M \subset M$. Therefore, $L'A \cap L = L'L$.

Since L' is contained in every composition series of L , it is contained in every refinement of the Loewy series $L \supset NL \supset N^2L \supset \cdots \supset N^sL = 0$, where N is the radical of A . This implies that L' is one of the terms of the Loewy series [1; pp. 102-104]. Thus, for some integer k , $N^kL = L'$ and so $L'L = (N^kL)L = N^k(LL) = N^kL = L'$.

A similar lemma holds for primitive right ideals, and the proof would be analogous to that given above. (Lemmas will be stated and proved only for left ideals but will be applied if either the lemma for left ideals or the corresponding one for right ideals is needed.)

LEMMA 2. *If A is a generalized uniserial algebra and Z is a two-sided ideal of A , then A/Z is a generalized uniserial algebra.*

Proof. Consider the residue class algebra A/Z . Let L' be any primitive left ideal of A/Z . Let L be the set of all elements of A whose residue classes mod Z are elements of L' in A/Z . Then L is a left ideal of A and a primitive ideal, $L = Ae$. Since A is generalized uniserial, L has a unique composition series: $L = L_m \supset L_{m-1} \supset \cdots \supset L_1 \supset L_0 = 0$. But $L \cap Z = Ze$ and Ze is a subideal of L , and hence for some n , $L_n = Ze$. Moreover, $L/Ze = L'$. For i such that $n < i \leq m$ define L'_i to be L_i/Ze . Then, each L'_i is a subideal of L' and the $\{L'_i\}_{i=n+1}^m$ form a composition series for L' . Let M' be any subideal of L' . Let M be the set of elements of A whose residue classes are elements of M' . Thus, M is a left ideal of A and is a subideal of L . But M must be one of the L_i since L has only one composition series. However, this implies that M' is one of the L'_i defined above and therefore, L' has a unique composition series.

Similarly, it can be shown that any primitive right ideal of A/Z has only one composition series. Thus A/Z is a generalized uniserial algebra.

THEOREM 1. *An algebra A is a generalized uniserial algebra if and only if for every two-sided ideal Z of A , the residue class algebra A/Z is a QF-2 algebra.*

Proof. 1. Assume that A is generalized uniserial. Let Z be any two-sided ideal of A . By Lemma 2, A/Z is generalized uniserial and hence every primitive ideal of A/Z has a unique composition series. This implies that every primitive ideal of A/Z has a unique minimal subideal and thus, A/Z is a QF-2 algebra.

2. Assume that for every two-sided ideal Z of A , A/Z is a QF-2 algebra. Let L be any primitive left ideal of A and let $L = L_n \supset L_{n-1} \supset \cdots \supset L_1 \supset L_0 = 0$ be any composition series of L . Since A is QF-2, L has a unique minimal subideal. Thus, L_1 is this unique minimal subideal and is contained in every composition series of L . Proceeding by induction, assume that L_{k-1} is contained in every composition series of L . Denote L_{k-1} by L' and $L'A$ by Z . Since $A = L \dot{+} M$, it follows that $A/Z = L/L \cap Z \dot{+} M/M \cap Z$. Applying Lemma 1, it follows that $L \cap Z = L'$ and so $L/L \cap Z = L/L'$. If L/L' is zero then L has a unique composition series. Otherwise, L/L' is a primitive ideal of A/Z , thus $L/L' = (A/Z)e'$ where e' is the residue class of e in A/Z . But since A/Z is QF-2, L/L' has a unique minimal subideal L'_0 . Let $L^* = \{x \in L \mid x + L' \in L'_0\}$. Then L^* is a left ideal of A and is a subideal of L and moreover, L_{k-1} is a maximal subideal of L^* . Thus, both L'_0 and L_k/L_{k-1} are minimal subideals of L/L_{k-1} and thus are the same. This implies that $L^* = L_k$ and so L_k is contained in every composition series of L . Thus, by induction, each L_n is contained in every composition series of L and hence, L has only one composition series.

Similarly, it can be shown that each primitive right ideal has only one composition series. Thus, A is a generalized uniserial algebra.

4. Uniserial algebras.

LEMMA 3. *Let L be a primitive left ideal and L' a subideal contained in every composition series of L . If L is A -isomorphic to some left ideal M and if M' is the image of L' under the isomorphism then $L'M = M'$, $M' \subset L'A$ and $M'A \subset L'A$.*

Proof. Let ϕ be the A -isomorphism from L onto M . Thus, $\phi(L) = M$ and $\phi(L') = M'$. By Lemma 1, $L'L = L'$. Thus, $L'M = L'\phi(L) = \phi(L'L) = \phi(L') = M'$. Thus, $M' = L'M \subset L'A$ and so $M'A \subset L'AA = L'A$.

LEMMA 4. *If L is a primitive left ideal with a unique minimal subideal L' then $L'A$ is the union of the minimal subideals of all ideals which are A -isomorphic to L .*

Proof. Let v be a nonzero element of $L'A$. Then $v = yz$, where $y \in L'$ and $z \in A$. Since $L = Ae$ it follows that $L'e = L'$ and $ye = y$. Thus $v = (ye)z = y(ez)$. Let $w = ez$ and consider the mapping ϕ from Ae to Aw defined by $\phi(x) = xw$. Then ϕ is a homomorphism onto Aw and its kernel is either zero or contains L' the unique minimal subideal of L . If L' were in the kernel it would follow that $L'w = 0$ and this would imply that $w = 0$. However, since v is not zero, w is not zero and so L' cannot be in the kernel of ϕ . Thus, ϕ is an A -isomor-

phism and $\phi(L')$ is the unique minimal subideal of Aw . But $v \in L'w = \phi(L')$. Thus, v is an element of the minimal subideal of Aw . Hence $L'A$ is contained in the union of all minimal subideals of ideals which are A -isomorphic to L . Lemma 3 shows that $L'A$ contains all such minimal ideals, and so $L'A$ is the union of the minimal subideals of all ideals which are A -isomorphic to L .

LEMMA 5. *If L is a primitive left ideal dual to a primitive right ideal R and if L' and R' are the unique minimal subideals of L and R respectively, then $L'A = AR'$.*

Proof. Since R' is the unique minimal subideal of R there exists an integer r such that $R' = RN^r$, where N is the radical of A . Since R and L are dual it follows that $L' = N^rL$. Thus, $R'L = (RN^r)L = R(N^rL) = RL'$. Since R' is a right ideal $R'L \subset R'$ and since L' is a left ideal $RL' \subset L'$. Thus, $R'L \subset R' \cap L'$. Since L and R are dual, if y is an element of R such that $yL = 0$ then $y = 0$. Thus, since $R' \neq 0$ it follows that $R'L \neq 0$ and hence, $R' \cap L' \neq 0$. Let z be a nonzero element of $R' \cap L'$. Then $Az \subset AL' = L'$ and Az is a nonzero left ideal. But since L' is minimal, this implies that $Az = L'$. Similarly, $zA \neq 0$, $zA \subset R'A = R'$ and hence $zA = R'$. Thus, $L'A = (Az)A = A(zA) = AR'$.

LEMMA 6. *If Ae is a primitive left ideal and M is a left A -module with a unique maximal submodule M_1 and $M/M_1 \cong Ae/Ne$ then there exists a homomorphism from Ae onto M .*

Proof. See Nakayama [3, Lemma 1, p. 613].

LEMMA 7. *Let A be a quasi-Frobenius algebra. Let L be a primitive left ideal dual to a primitive right ideal R . Let L' and R' be the unique minimal subideals of L and R respectively. Assume $L' \neq L$ and denote $L'A$ by Z . If A/Z is also quasi-Frobenius then $L/L \cap Z$ is dual to $R/R \cap Z$.*

Proof. By Lemma 1, $L \cap Z = L'L = L'$ and so, $L/L \cap Z = L/L'$. By Lemma 5, $L'A = AR'$ and so, $Z = AR'$. Thus, since Lemma 1 holds for right ideals, $R \cap Z = RR' = R'$ and so, $R/R \cap Z = R/R'$. Since L and R are dual and $L' \neq L$ it follows that $L/L' \neq 0$, $R' \neq R$ and $R/R' \neq 0$. Denote L/L' by L_1 and R/R' by R_1 . It can be shown that L_1 and R_1 are primitive ideals of A/Z . Since it is assumed that A/Z is also quasi-Frobenius, L_1 must be dual to some primitive right ideal R^* of A/Z . If R^* is A -isomorphic to R_1 the lemma is proved since this implies that R_1 is dual to L_1 . Therefore, assume that R^* is not A -isomorphic to R_1 . Then R^* can be considered as an ideal of A itself, since by Lemma 4 for right ideals, Z is the union of all minimal subideals of all ideals of A which are A -isomorphic to R . But as an ideal of A , R^* is dual to some primitive ideal L^* of A . Thus L_1 is A -isomorphic to L^* . Since $L/L' = L_1$, and since L and L^* are both primitive ideals of A and hence have unique maximal subideals, Lemma 6 can be applied to give a homomorphism from L^* onto L . However, this is impossible since L^* is isomorphic to L/L' . Thus, the assumption that R^* is not isomorphic to R_1 is false and the lemma is proved.

THEOREM 2. *If for every two-sided ideal Z of A , A/Z is a quasi-Frobenius algebra, then A is a uniserial algebra.*

Proof. Since every quasi-Frobenius algebra is also a QF-2 algebra, Theorem 1 shows that A is a generalized uniserial algebra. It remains to show that A is the direct sum of two-sided ideals which are themselves primary algebras. By use of the Wedderburn structure theorems it is possible to choose a set of mutually orthogonal primitive idempotents whose sum is the identity element. Let e_{ij} , $i=1, \dots, n$, $j=1, \dots, f(i)$, be the set of idempotents ordered so that Ae_{ij} is A -isomorphic to Ae_{hk} if and only if $i=h$. It will follow that $e_{ij}A$ is A -isomorphic to $e_{hk}A$ if and only if $i=h$. Let $E_i = \sum_{j=1}^{f(i)} e_{ij}$. (For details see [1; 2; 3].)

Fix an integer i and the corresponding integer h such that Ae_{ij} is dual to $e_{hk}A$. Let L denote Ae_{i1} and R denote $e_{h1}A$. Let L' and R' denote the unique minimal subideals of L and R respectively. By Lemma 1, $L'A$ is the minimal two-sided ideal containing L' and by Lemma 5, $L'A = AR'$. Denote this two-sided ideal by Z . It is known [1, Theorem 1.6B, p. 8] that Z can be written as a direct sum of left ideals and by Lemma 4, this sum can be taken to be $\sum_{j=1}^{f(i)} L'_{ij}$, where L'_{ij} is the unique minimal subideal of Ae_{ij} . Thus, Z is contained in AE_i , where $AE_i = \sum_{j=1}^{f(i)} Ae_{ij}$. By a similar argument Z is contained in E_hA , where $E_hA = \sum_{k=1}^{f(h)} e_{hk}A$. Thus, $Z \subset AE_i \cap E_hA$.

Consider the residue class algebra A/Z . Since A/Z is quasi-Frobenius, Lemma 7 can be applied and L/L' and R/R' are dual unless they are zero. If L/L' and R/R' are zero then $Z = AE_i = E_hA$ and Z is a component of A written as the sum of two-sided ideals.

Construct a sequence of two-sided ideals $\{Z_s\}$ with the following properties: (1) for each s , $Z_s \subset AE_i \cap E_hA$ and (2) for each s , Z_{s-1} is properly contained in Z_s . Let $Z_1 = Z$, $L_1 = L'$, $R_1 = R'$ and $A_1 = A/Z_1$. Then L/L_1 is dual to R/R_1 in A_1 . Assume that Z_{p-1} , L_{p-1} , R_{p-1} , and A_{p-2} have been defined. Then by Lemma 7, either $Z_{p-1} = AE_i = E_hA$ or L/L_{p-1} is dual to R/R_{p-1} in A/Z_{p-1} . Let $A_{p-1} = A/Z_{p-1}$, let L_p^* be the unique minimal subideal of L/L_{p-1} in A_{p-1} , and let $Z_p^* = L_p^* A_{p-1}$. Let Z_p be the set of all elements of A whose residue classes mod Z_{p-1} are elements of Z_p^* . Thus, Z_p will be a two-sided ideal of A and it follows that $Z_p \subset AE_i \cap E_hA$ and that Z_p properly contains Z_{p-1} , by an argument similar to that used to show that Z_1 satisfied these conditions. But, since A is finite dimensional, the sequence must terminate with some $Z_q = AE_i = E_hA$. Thus $AE_i = E_hA$ and this implies that $i=h$. Hence, AE_i is a direct component of A written as a sum of two-sided ideals. Moreover, AE_i/NE_i is simple, where N is the radical of A and NE_i is the radical of AE_i . Hence AE_i is a primary algebra. Therefore, A is a generalized uniserial algebra and is the sum of two-sided ideals which are themselves primary algebras. Thus, A is a uniserial algebra and the theorem is proved.

Theorem 2 cannot be extended in the sense that it is not sufficient to assume that each A/Z is QF-2 in order to prove that A is a uniserial algebra.

As a counter-example consider the total triangular matrix algebra T_n , $n > 1$, of all n by n matrices over K with zeros above the main diagonal. The algebra T_n is generalized uniserial and thus, by Theorem 1, each A/Z is QF-2. But T_n cannot be written as a sum of two-sided ideals which are themselves primary algebras and thus T_n is not a uniserial algebra.

5. QF-3* algebras.

LEMMA 8. *Let Ae be dual to fA , where e and f are primitive idempotents. If, in addition to a unique minimal subideal L_1 , Ae also has a unique subideal L_2 with L_1 as a maximal subideal and if L_2/L_1 and L_1 are A -isomorphic as left A -modules then both Ae and fA have unique composition series and all of the constituents of the series are A -isomorphic.*

Proof. Choose any composition series for $Ae: Ae = L_n \supset L_{n-1} \supset \cdots \supset L_2 \supset L_1 \supset L_0 = 0$ and any composition series for $fA: fA = R_n \supset R_{n-1} \supset \cdots \supset R_2 \supset R_1 \supset R_0 = 0$. Since any primitive ideal has a unique maximal subideal, the ideals L_{n-1} and R_{n-1} are uniquely determined. Thus, $L_{n-1} = Ne$ and $R_{n-1} = fN$, where N is the radical of A . From the assumed duality between Ae and fA , the uniqueness of L_{n-1} and R_{n-1} implies the uniqueness of R_1 and L_1 respectively. It is assumed in addition that L_2 is uniquely determined. By duality, this implies that R_{n-2} is unique. It is also assumed that L_2/L_1 and L_1 are A -isomorphic and, by duality, this implies that R_n/R_{n-1} and R_{n-1}/R_{n-2} are A -isomorphic. But $R_n = fA$ and $R_{n-1} = fN$ and so, R_{n-1}/R_{n-2} is A -isomorphic to fA/fN and R_{n-2} is the unique maximal subideal of R_{n-1} . Thus, the hypotheses of Lemma 6 stated for right A -modules are satisfied and there exists an A -homomorphism from fA onto R_{n-1} . By comparing the composition lengths of fA and R_{n-1} , it is seen that the kernel of the homomorphism must be the unique minimal subideal R_1 . Thus, fA/R_1 is A -isomorphic to R_{n-1} . Denote fA/R_1 by S and consider any composition series of $S: S = S_n \supset S_{n-1} \supset \cdots \supset S_2 \supset S_1 = 0$. The subideal S_{n-1} is unique in S and is R_{n-1}/R_1 . Since R_{n-2} is unique in R_n it follows that S_{n-2} is unique in S and $S_{n-2} = R_{n-2}/R_1$. But, since R_{n-1} and S are A -isomorphic, the uniqueness of S_{n-2} in S implies the uniqueness of R_{n-3} in R_{n-1} and hence R_{n-3} is unique in R_n . Since the composition length of fA is finite, a finite number of steps shows that fA has a unique composition series. Since Ae is dual to fA , this implies that Ae has a unique composition series.

Since fA has only one composition series, so do R_n/R_1 and R_{n-1} . The composition series for R_n/R_1 is composed of the $\{R_i/R_1\}_{i=1}^n$. Since R_{n-1} and R_n/R_1 are A -isomorphic, it follows that corresponding constituents are A -isomorphic. Thus, $(R_{i+1}/R_1)/(R_i/R_1) \cong R_i/R_{i-1}$. But, in addition,

$$(R_{i+1}/R_1)/(R_i/R_1) \cong R_{i+1}/R_i$$

and so, $R_{i+1}/R_i \cong R_i/R_{i-1}$. Thus, all constituents of fA are A -isomorphic. Since Ae and fA are dual, this implies that all of the constituents $\{L_i/L_{i-1}\}_{i=1}^n$ are A -isomorphic and the lemma is proved.

If the condition that L_2 is uniquely determined is dropped from the hypothesis of Lemma 8, but it is still assumed that for some L_2 , L_2/L_1 is A -isomorphic to L_1 , neither conclusion of the lemma need hold. Consider the algebra of all matrices of the form:

$$\begin{pmatrix} a_1 & 0 & 0 & 0 \\ a_3 & a_1 & 0 & 0 \\ a_5 & 0 & a_2 & 0 \\ a_6 & a_5 & a_4 & a_2 \end{pmatrix} \quad a_i \in K.$$

Choose as a basis for the algebra the matrices X_i in which $a_i=1$ and $a_j=0$ for $j \neq i$. The primitive left ideal L with basis $\{X_1, X_3, X_5, X_6\}$ is dual to the primitive right ideal R with basis $\{X_2, X_4, X_5, X_6\}$. The ideal X_6K is the unique minimal subideal L_1 of L . If L_2 is chosen as $X_5K + X_6K$ then L_2/L_1 is A -isomorphic to L_1 . However, L_2 is not unique since it could be chosen as $X_3K + X_6K$. Thus, L does not have a unique composition series and not all of the constituents of L are A -isomorphic.

If the condition that L_2 is uniquely determined is assumed but the requirement that L_2/L_1 be A -isomorphic to L_1 is dropped then neither conclusion of Lemma 8 need hold. Consider the algebra of all matrices of the form:

$$\begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 \\ a_5 & a_2 & 0 & 0 & 0 & 0 \\ a_8 & a_6 & a_3 & 0 & 0 & 0 \\ a_{12} & a_9 & 0 & a_4 & 0 & 0 \\ a_{15} & a_{13} & a_{10} & a_7 & a_1 & 0 \\ a_{17} & a_{16} & a_{14} & a_{11} & a_5 & a_2 \end{pmatrix} \quad a_i \in K.$$

Choose as a basis for the algebra the matrices X_i in which $a_i=1$ and $a_j=0$ for $j \neq i$. Then, the primitive left ideal L with basis $\{X_1, X_5, X_8, X_{12}, X_{15}, X_{17}\}$ is dual to the primitive right ideal R with basis $\{X_2, X_5, X_{11}, X_{14}, X_{16}, X_{17}\}$. The ideal $X_{17}K$ is the unique minimal subideal L_1 of L and $X_{15}K + X_{17}K$ is the unique second term L_2 in any composition series of L . However, L_2/L_1 is not A -isomorphic to L_1 and in addition, neither conclusion of Lemma 8 holds for this algebra.

THEOREM 3. *If for every two-sided ideal Z of A , the residue class algebra A/Z is a QF-3* algebra then A is a generalized uniserial algebra.*

Proof. If every A/Z were QF-2 then Theorem 1 would imply that A is a generalized uniserial algebra. Thus, it suffices to consider the case in which, for some two-sided ideal Z , the residue class algebra A/Z is QF-3* but not QF-2. There is no loss in generality in considering this residue class algebra

to be A itself. Thus, it is assumed that there is some primitive ideal of A which is weakly subordinate to a dominant ideal but is not subordinate to it. Consider the case in which this is a left ideal L^* weakly subordinate to a dominant left ideal L .

Choose L^* to be maximal with respect to being weakly subordinate to L , i.e., choose L^* so that there is no other primitive ideal M weakly subordinate to L such that L^* is weakly subordinate to M . Consider the collection of all primitive left ideals which are (1) either A -isomorphic to or subordinate to L and which are (2) ideals to which L^* is weakly subordinate. By use of the Wedderburn structure theorems choose a subset $\{L_i\}_{i=1}^r$ of this collection which is maximal with respect to having the primitive idempotents generating the L_i mutually orthogonal. Thus, the L_i can be taken as components of a decomposition of A into a sum of primitive left ideals. Since the set $\{L_i\}_{i=1}^r$ is chosen maximal no other component of the decomposition is in the collection.

Since each L_i is either a dominant ideal or is subordinate to a dominant ideal, each L_i has a unique minimal subideal L'_i [8, Theorem 1]. By Lemma 1, each $L'_i A$ is a two-sided ideal and by Lemma 4, each $L'_i A$ is the union of the minimal subideals of all ideals which are A -isomorphic to L_i . Let Z be the sum of the $L'_i A$. Since each $L'_i A$ can be written as a direct sum [1, Theorem 1.6B], Z can be written as a direct sum. Since A is QF-3* and since L^* was chosen maximal, Z can be taken as the direct sum of the minimal subideals L'_i of the L_i . Thus, $Z = \sum_{i=1}^r L'_i$.

Consider the residue class algebra A/Z and denote it by A' . Since $A = M + \sum_{i=1}^r L_i$ and $Z = \sum_{i=1}^r L'_i$, it follows that

$$A/Z = M/M \cap Z + \sum_{i=1}^r L_i/L_i \cap Z = M + \sum_{i=1}^r L_i/L'_i.$$

Thus, the primitive ideals of A' are either A -isomorphic to some L_i/L'_i or A -isomorphic to some component of M and hence to some primitive ideal of A itself. Since $L^* \cap Z = 0$, L^* can be considered as an ideal of A' as well as an ideal of A . Since L^* does not have a unique minimal subideal, it cannot be a dominant ideal of A' nor be subordinate to a dominant ideal of A' . Thus, since A' is assumed to be a QF-3* algebra, L^* must be weakly subordinate to some dominant ideal of A' .

Suppose that L^* is weakly subordinate to a primitive ideal M' of A' which is A -isomorphic to one of the primitive ideals of A . Then M' could be considered as an ideal of A and hence L^* would be weakly subordinate to M' as an ideal of A . However, this is impossible since this would imply that M' is A -isomorphic to one of the L_i .

The only possible case remaining is that L^* is weakly subordinate to one of the L_i/L'_i . If this is true then, in addition, L^* is weakly subordinate to L/L' . This implies that L/L' is a dominant ideal of A' and that L/L' has a unique minimal subideal S' . Let S_1 denote L' and let S_2 be the set of all ele-

ments of L whose residue classes are in S' . Then, S_2 is a left ideal of A and is a subideal of L . Thus, S_1 and S_2 are the unique first and second terms in any composition series of L . Since L^* is weakly subordinate to both L and L/L' , it follows that S_1 and S_2/S_1 are A -isomorphic. Thus, the hypotheses of Lemma 8 are satisfied and hence L has a unique composition series and all of the constituents of the series are A -isomorphic. However, this implies that there can be no primitive ideals which are subordinate to or weakly subordinate to L without being dominant. Hence, this case is also impossible.

Thus, it is impossible to have every residue class algebra QF-3* and one of them not QF-2. Therefore, if every A/Z is a QF-3* algebra then every A/Z is a QF-2 algebra and by Theorem 1, A is a generalized uniserial algebra.

The previously known results and Theorem 2 give the following characterizations of uniserial algebras: (a) Every residue class algebra is quasi-Frobenius; (b) Every residue class algebra is Frobenius; (c) Every residue class algebra is weakly symmetric. Theorems 1 and 3 give the following characterizations of generalized uniserial algebras: (a) Every residue class algebra is QF-3*; (b) Every residue class algebra is QF-2. It seems probable that a weaker condition than that of requiring that each residue class algebra be QF-3* may suffice to imply that the algebra itself is generalized uniserial.

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