DERIVATIVE MANIFOLDS AND TAYLOR SERIES IN THE MEAN⁽¹⁾

BY
D. S. GREENSTEIN

1. **Introduction.** Let $f(x) \in C^\infty(-\infty, \infty)$ and let there be some $p \geq 1$ such that $f^{(n)}(x) \in L^p(-\infty, \infty)$ ($n \geq 0$). It has been shown [1, p. 38] that the closed linear manifold $D[f]$ of L^p spanned by $f(x), f'(x), f''(x), \dots$ is contained in the closed linear manifold $T[f]$ spanned by the translates $f(x+h)$. For $p=2$ there is the stronger result [2, p. 130]⁽²⁾

$$\text{l.i.m.}_{h \rightarrow 0} \frac{f^{(n)}(x+h) - f^{(n)}(x)}{h} = f^{(n+1)}(x).$$

It is thus of interest to determine when $D[f] = T[f]$. This question has been studied by Mandelbrojt [1, pp. 39–40], who has found it necessary to consider classes of functions designated $L^p\{M_n\}$, where $\{M_n\}$ is a sequence of positive reals and $L^p\{M_n\}$ consists of all $C^\infty(-\infty, \infty)$ functions such that $\|f^{(n)}(x)\|_p \leq M_n$ ($n \geq 0$). His principal result states that $D[f] = T[f]$ for all $f(x) \in L^p\{M_n\}$ if and only if the class $C\{M_n\}$ is quasi-analytic, $C\{M_n\}$ being the class of all $f(x) \in C(-\infty, \infty)$ such that $|f^{(n)}(x)| \leq k^n M_n$ ($-\infty \leq x \leq \infty$) ($n \geq 0$) for some positive k . If $C\{M_n\}$ is not quasi-analytic, then there is at least one $f(x) \in L^p\{M_n\}$ for which $D[f] \neq T[f]$. Hence, if the derivatives of $f(x)$ are too large in norm, Mandelbrojt's theorem cannot be used to decide whether $D[f] = T[f]$.

For $p=2$ we shall establish the following necessary and sufficient condition on $f(x)$ for $D[f] = T[f]$:

THEOREM 1.1. *Let $f(x)$ be a $C^\infty(-\infty, \infty)$ function such that $f^{(n)}(x) \in L^2(-\infty, \infty)$ ($n \geq 0$) and let $F(x)$ be the Fourier transform of $f(x)$. Then $D[f] = T[f]$ if and only if the distribution function $\psi(t) = \int_{-\infty}^t |F(x)|^2 dx$ is the solution of a determined Hamburger moment problem.*

Thus in L^2 it is not necessary to consider classes of functions. Furthermore, it will be shown that if $D[f] = T[f]$ for an L^2 function, then $D[f^{(n)}] = T[f^{(n)}]$ ($n \geq 0$), which in turn implies an interesting result in the theory of orthogonal polynomials.

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(²) As a corollary of our Theorem 3.1, this result is actually valid for general p .

We shall also consider approximating translates by Taylor series; that is, the possibility of representing $f(x+h)$ by

$$\text{l.i.m.}_{n \rightarrow \infty} \sum_{\nu=0}^{n-1} \frac{h^\nu}{\nu!} f^{(\nu)}(x).$$

It will be shown that the class of L^p functions admitting such a representation is the generalization to L^p of an analytic function class in L^2 studied by Paley and Wiener [3, pp. 3–13] and that the “Taylor series in the mean” can be used to prove a theorem of Paley and Wiener concerning the Fourier transforms of functions in the Paley-Wiener class.

2. Derivative manifolds in L^2 and the Hamburger moment problem. In this chapter we shall restrict ourselves to L^2 . Our principal aim is to prove Theorem 1.1, thus showing the intimate connection between the Hamburger moment problem and the derivative manifold in L^2 .

To characterize $D[f]$ and $T[f]$ we work with the Fourier transform

$$F(x) = \text{l.i.m.}_{n \rightarrow \infty} (2\pi)^{-1/2} \int_{-n}^n f(t) e^{-ixt} dt$$

of $f(x)$. Let A be the (measurable) set of all x for which $F(x) \neq 0$. Then by a theorem of Bochner and Chandrasekharan [2, p. 149], $T[f]$ consists of all $g(x) \in L^2(-\infty, \infty)$ whose Fourier transforms vanish almost everywhere on the complement of A . (As a special case of the theorem, $T[f] = L^2$ if and only if the complement of A has measure zero. This result is due to Wiener [4, p. 100].)

Since the Fourier transform of $f^{(n)}(x)$ is equal to $(ix)^n F(x)$, $D[f]$ consists of all L^2 functions whose Fourier transforms belong to the closed linear manifold spanned by $F(x)$, $xF(x)$, $x^2F(x)$, $\dots$. It therefore follows that $D[f] = T[f]$ if and only if $F(x)$, $xF(x)$, $x^2F(x)$, $\dots$ span $L^2(A)$.

We next introduce the distribution function

$$\psi(t) = \int_{-\infty}^t |F(x)|^2 dx.$$

Since

$$(2.1) \quad \int_{-\infty}^{\infty} t^{2n} d\psi(t) = \int_{-\infty}^{\infty} |x^n F(x)|^2 dx < \infty,$$

$\psi(t)$ has moments of all orders. Conversely, if $\psi(t)$ is an absolutely continuous distribution having moments of all orders and $F(t)$ is any measurable function such that $|F(t)|^2 = \psi'(t)$ almost everywhere, then

$$(2.2) \quad f(x) = \text{l.i.m.}_{n \rightarrow \infty} \frac{1}{(2\pi)^{1/2}} \int_{-n}^n F(t) e^{itz} dt$$

is equivalent to a $C^\infty(-\infty, \infty)$ function all of whose derivatives belong to L^2 .

Consider now the Hilbert space L^2_ψ consisting of all ψ -measurable $g(t)$ such that

$$\|g(t)\|_\psi^2 = \int_{-\infty}^{\infty} |g(t)|^2 d\psi(t) < \infty.$$

It follows from (2.1) that

$$(2.3) \quad \|t^n\|_\psi = \|x^n F(x)\|.$$

We are thus led to consider the following isometry of $L^2(A)$ onto L^2_ψ

$$(2.4) \quad \begin{aligned} g(x)(\epsilon L^2(A)) &\leftrightarrow h(t)(\epsilon L^2_\psi), \\ h(t) &= g(t)/F(t), \end{aligned}$$

$$(2.5) \quad g(x) = F(x)h(x).$$

(Note that the points where $h(t)$ might be undefined by (2.4) are of ψ -measure zero.) Hence, $D[f] = T[f]$ if and only if $1, t, t^2, \dots$ are complete in L^2_ψ .

The question of completeness of polynomials in L^2_ψ is completely resolved by the following theorem of M. Riesz [5; 6, p. 62]:

THEOREM 2.1. *Let $\psi(t)$ be a distribution function having moments of all orders. Then $1, t, t^2, \dots$ span L^2_ψ if and only if one of the following conditions holds:*

- (1) $\psi(t)$ is the solution of a determined Hamburger moment problem.
- (2) The Hamburger moment problem of which $\psi(t)$ is a solution is indeterminate, but $\psi(t)$ is one of R. Nevanlinna's [6, p. 60; 7] extremal solutions.

We need not consider here the definition of "extremal solution"; for our purpose we need only remark that such distribution functions are known [6, p. 60; 7] to have discontinuous spectra. Since our $\psi(t)$ is absolutely continuous, the validity of Theorem 1.1 follows.

Theorem 1.1 allows us to construct functions for which $D[f] \neq T[f]$. If $\psi(t)$ is an absolutely continuous solution of an indeterminate Hamburger moment problem⁽³⁾ and $F(t)$ is a measurable function such that $|F(t)|^2 = \psi'(t)$ almost everywhere, then such an $f(x)$ may be obtained from $F(t)$ via (2.2). An example of such a ψ' is $\exp(-|t|^{1/2})$ [6, p. 22]. Hence for

$$f(x) = \int_0^\infty \exp(-t^{1/2}/2) \cos xtdt$$

we have $D[f] \neq T[f]$.

⁽³⁾ It is of interest to note that Hamburger [6, p. 61; 9] has shown that every indeterminate Hamburger moment problem has absolutely continuous solutions.

Theorem 1.1 may also be used to show that if $D[f] = T[f]$, then derivatives of all orders are not necessary to approximate translates. In fact, we have:

THEOREM 2.2. *Let $f(x) \in C^\infty(-\infty, \infty)$ and let $f^{(n)}(x) \in L^2(-\infty, \infty)$ ($n \geq 0$). Then if $D[f] = T[f]$, $D[f^{(n)}] = T[f]$ ($n \geq 0$). In other words, for each non-negative integer n and each real h , $f(x+h)$ is the limit in norm of finite linear combinations of the form*

$$C_n f^{(n)}(x) + \cdots + C_{n+N} f^{(n+N)}(x).$$

To prove Theorem 2.2, we first show that $D[f] = T[f]$ implies that $f(x) \in D[f']$. Taking Fourier transforms and using the isometry (2.4) of $L^2(A)$ onto L^2_ψ , this is seen to be equivalent to showing that 1 belongs to the closed linear manifold of L^2 spanned by $t, t^2, \dots$. To show this, we evaluate

$$\rho_n = \min_{C_1, \dots, C_n} \|1 - C_1 t - \cdots - C_n t^n\|_\psi^2.$$

If $\omega_n(t)$ is the orthonormal polynomial of degree n (i.e., $\omega_n(t)$ is of degree n and $\int_{-\infty}^{\infty} \omega_m(t) \omega_n(t) d\psi(t) = \delta_{mn}$) and if

$$K_n(t, 0) = \sum_{k=0}^n \omega_k(0) \omega_k(t),$$

then for any polynomial $P(t)$ of degree not exceeding n

$$P_n(0) = \int_{-\infty}^{\infty} P_n(t) K_n(t, 0) d\psi(t).$$

Applying the Schwarz inequality,

$$|P_n(0)|^2 \leq \|P_n(t)\|_\psi^2 \sum_{k=0}^n \omega_k^2(0).$$

Hence, it is seen that $\rho_n = 1 / \sum_{k=0}^n \omega_k^2(0)$, the minimum being attained when the C_k are the coefficients of

$$\frac{K_n(t, 0)}{K_n(0, 0)} - 1.$$

But in the theory of the Hamburger moment problem [6, p. 44], it is shown that $1 / \sum_{k=0}^{\infty} \omega_k^2(0)$ is the minimum mass at $t=0$ for any distribution having the same moments as $\psi(t)$. Hence, since $\psi(t)$ is absolutely continuous and determined by its moments, $\lim_{n \rightarrow \infty} \rho_n = 0$.

Thus $D[f'] = T[f']$. But $T[f] = T[f']$, since $\{x | F(x) \neq 0\}$ and $\{x | xF(x) \neq 0\}$ differ by at most a set of measure zero. Hence, we have

$$D[f] = T[f] = T[f'] = D[f''] = \cdots = D[f^{(n)}] = \cdots$$

Theorem 2.2 is considerably stronger than the following theorem of Mandelbrojt [1, pp. 39–40], which only states that the first derivative is not needed for approximating translates provided $f(x)$ belongs to a class of functions $\{g(x)\}$ for each of which $D[g] = T[g]$:

THEOREM 2.3. *Let $f(x) \in L^p\{M_n\}$ and let $C\{M_n\}$ be quasi-analytic. Then for each real h , $f(x+h)$ is the limit in norm of finite linear combinations of the form*

$$f(x) + C_2 f''(x) + \cdots + C_N f^{(N)}(x).$$

It is interesting to note the following immediate consequence of Theorem 2.2:

THEOREM 2.4. *Let $F(x)$ be a measurable function which is positive almost everywhere on the interval (a, b) , where a may equal $-\infty$ and b may equal $+\infty$ and let*

$$\int_a^b x^{2n} F(x) dx < \infty \quad (n \geq 0).$$

Then if $(F(x))^{1/2}$, $x(F(x))^{1/2}$, $x^2(F(x))^{1/2}$, $\cdots$ are complete in $L^2(a, b)$, it follows that for each positive integer n , $x^n(F(x))^{1/2}$, $x^{n+1}(F(x))^{1/2}$, $x^{n+2}(F(x))^{1/2}$, $\cdots$ are also complete in $L^2(a, b)$.

Theorem 2.4 is believed to be new. It would be interesting to see a proof which does not depend on Theorem 2.1.

3. Taylor series in the mean. We shall now consider the use of Taylor series to approximate translates; i.e., given $f(x) \in C^\infty(-\infty, \infty)$ such that $f^{(n)}(x) \in L^p(-\infty, \infty)$ ($n \geq 0$), we wish to represent $f(x+h)$ by

$$(3.1) \quad (\text{l.i.m.})_p \sum_{\nu=0}^{n-1} \frac{h^\nu}{\nu!} f^{(\nu)}(x).$$

The expression (3.1) will be called the " L^p -Taylor series" of $f(x)$ (h being restricted to real values for the present).

To study the convergence of L^p -Taylor series, we shall first establish the following analog of Taylor's theorem:

THEOREM 3.1. *Let $f(x) \in C^n(-\infty, \infty)$ and let $f^{(\nu)}(x) \in L^p(-\infty, \infty)$ ($0 \leq \nu \leq n$). Then the remainder*

$$r_{n,h}(x) = f(x+h) - \sum_{\nu=0}^{n-1} \frac{h^\nu}{\nu!} f^{(\nu)}(x)$$

admits the following estimates:

$$(3.2) \quad \|r_{n,h}(x)\|_p \leq \frac{|h|^n}{n!} \|f^{(n)}(x)\|_p,$$

$$(3.3) \quad |r_{n,h}(x)| \leq \begin{cases} \frac{|h|^{n-1}}{(n-1)!} \|f^{(n)}(x)\|_p [|h| / (1 + q(n-1))]^{1/q} & \text{if } p > 1, \\ \frac{|h|^{n-1}}{(n-1)!} \|f^{(n)}(x)\|_p & \text{if } p = 1, \end{cases}$$

where $q = p/(p-1)$.

The proof of Theorem 3.1 depends on the well-known formula

$$(3.4) \quad r_{n,h}(x) = \int_0^h \frac{(h-t)^{n-1}}{(n-1)!} f^{(n)}(x+t) dt.$$

Applying the generalized Minkowski inequality [9, Theorem 202] to (3.4):

$$\|r_{n,h}(x)\|_p \leq \left| \int_0^h \frac{|h-t|^{n-1}}{(n-1)!} \|f^{(n)}(x+t)\|_p dt \right|$$

which immediately yields (3.2).

To prove (3.3) for $p=1$, we note that

$$|r_{n,h}(x)| \leq \frac{|h|^{n-1}}{(n-1)!} \left| \int_0^h |f^{(n)}(x+t)| dt \right| \leq \frac{|h|^{n-1}}{(n-1)!} \int_{-\infty}^{\infty} |f^{(n)}(x+t)| dt.$$

For $p > 1$, we note that

$$\frac{(h-t)^{n-1}}{(n-1)!} \in L^q(0, h)$$

and apply Hölder's inequality.

The right hand side of (3.2) is equal to the norm of the first neglected term of the L^p -Taylor series, a quantity which must tend to zero in order that the series converge. Hence, for the convergence of the L^p -Taylor series, it is both necessary and sufficient that the n th term converge strongly to zero. Furthermore, convergent L^p -Taylor series always converge to the proper limit $f(x+h)$, in contrast to the behavior of ordinary Taylor series.

We are now able to compute the radius of convergence of the L^p -Taylor series. Using the well-known root criterion of Cauchy, it is seen that for the series to converge, the relation

$$|h| \limsup_{n \rightarrow \infty} \left[\frac{\|f^{(n)}(x)\|_p}{n!} \right]^{1/n} \leq 1$$

must hold. Hence, the following theorem is true:

THEOREM 3.2. *Let $f(x) \in C^\infty(-\infty, \infty)$ and let $f^{(n)}(x) \in L^p(-\infty, \infty)$ ($n \geq 0$). Then for*

$$-\rho < |h| < \rho = \left(\limsup \left[\frac{\|f^{(n)}(x)\|_p}{n!} \right]^{1/n} \right)^{-1},$$

$$f^{(k)}(x+h) = (\lim)_{n \rightarrow \infty} \sum_{\nu=0}^{n-1} \frac{h^\nu}{\nu!} f^{(k+\nu)}(x) \quad (k \geq 0).$$

For $|h| > \rho$, the L^p -Taylor series for $f(x)$ and its derivatives do not converge.

Thus if $\rho > 0$, $f^{(n)}(x+h) \in D[f]$ for all n and for $|h| < \rho$. We shall now show that the restriction on h is not needed; i.e., the condition $\rho > 0$ implies that $f^{(n)}(x+h) \in D[f]$ for all h and for all n . This follows from the result of Mandelbrojt [1, pp. 39–40] cited in Chapter 1, since the class $C\{\|f^{(n)}(x)\|_p\}$ is regular in the strip $-\rho < \operatorname{Im} z < \rho$. It is also possible to give a proof based on Theorem 3.2. Consider any h_0 smaller than ρ in absolute value. Since L^p norms are invariant under translation, the L^p -Taylor series of $f(x+h_0)$ also has ρ as its radius of convergence. Hence by a finite number of “continuations in the mean,” every translate of $f^{(n)}(x)$ can be shown to be in $D[f]$.

It might be thought possible to obtain a larger radius of convergence for the L^p -Taylor series by considering weak L^p convergence instead of strong L^p convergence. This is not so. If the L^p -Taylor series converges weakly, then its partial sums, and hence its terms, must be bounded in norm. But for $|h| > \rho$ the terms are not bounded in norm. Hence the weak and strong radii are equal.

We shall now determine what functions may be represented by L^p -Taylor series. From the estimate (3.3), it is seen that if $\rho > 0$, then the ordinary Taylor series for $f(x)$ about each real x_0 converges to $f(x)$ for $x_0 - \rho < x < x_0 + \rho$. Hence $f(x)$ may be extended to an analytic function $f(z)$ which is regular in the strip S_ρ : $-\rho < \operatorname{Im} z < \rho$.

Hence we may extend the definition of L^p -Taylor series to complex h . For any h such that $|h| < \rho$, the series

$$\sum_{n=0}^{\infty} \frac{|h|^n}{n!} \|f^{(n)}(x)\|_p$$

converges, so that the limit (3.1) exists for such h and is necessarily equal to $f(x+h)$. More generally, if $h = \xi + i\eta \in S_\rho$

$$f(x+h) = (\text{l.i.m.})_p \sum_{\nu=0}^{n-1} \frac{(i\eta)^\nu}{\nu!} f^{(\nu)}(x+\xi)$$

and

$$\|f(x+h)\|_p \leq \sum_{n=0}^{\infty} \frac{|\eta|^n}{n!} \|f^{(n)}(x)\|_p.$$

Thus we have:

THEOREM 3.3. *If the L^p -Taylor series of $f(x)$ has radius of convergence $\rho > 0$, then $f(x)$ may be extended to an analytic function $f(z)$ which is regular in the strip S_ρ . Furthermore, for each positive σ which is less than ρ , $\sup_{h \in S_\sigma} \|f(x+h)\|_p < \infty$.*

We thus find it convenient to make the following definitions:

DEFINITION 3.1. The function class $W_p(\sigma)$ consists of all analytic functions $f(z)$ which are regular in S_σ and are such that $f(x+h) \in L^p(-\infty, \infty)$ for each h in S_σ , $\|f(x+h)\|_p$ being bounded in each S_λ for which $\lambda < \sigma$.

DEFINITION 3.2. The function class $T_p(\sigma)$ consists of all $f(x) \in C^\infty(-\infty, \infty)$ such that $f^{(n)}(x) \in L^p(-\infty, \infty)$ ($n \geq 0$) whose L^p -Taylor series have radius of convergence at least equal to σ .

According to Theorem 3.3, $T_p(\sigma) \subset W_p(\sigma)$. We shall show that $T_p(\sigma) = W_p(\sigma)$. To prove this we first establish the following generalization of a theorem of Paley and Wiener [3, p. 5].

THEOREM 3.4. *Let $f(z)$ be regular in $\bar{S}_\lambda$, the closure of S_λ . Furthermore, let*

$$\sup_{h \in \bar{S}_\lambda} \|f(x+h)\|_p = M < \infty.$$

Then for each $\zeta \in S_\lambda$

$$(3.5) \quad f^{(n)}(\zeta) = \frac{n!}{2\pi i} \left[\int_{-\infty}^{\infty} \frac{f(x-i\lambda)}{(x-i\lambda-\zeta)^{n+1}} dx - \int_{-\infty}^{\infty} \frac{f(x+i\lambda)}{(x+i\lambda-\zeta)^{n+1}} dx \right] \quad (n \geq 0).$$

To prove Theorem 3.4, we use Cauchy's integral formula, observing that for sufficiently large A

$$\begin{aligned} f^{(n)}(\zeta) &= \frac{n!}{2\pi i} \left[\int_{-A}^A \frac{f(x-i\lambda)}{(x-i\lambda-\zeta)^{n+1}} dx - \int_{-A}^A \frac{f(x+i\lambda)}{(x+i\lambda-\zeta)^{n+1}} dx \right. \\ &\quad \left. + i \int_{-\lambda}^{\lambda} \frac{f(A+iy)}{(A+iy-\zeta)^{n+1}} dy - i \int_{-\lambda}^{\lambda} \frac{f(-A+iy)}{(-A+iy-\zeta)^{n+1}} dy \right] \\ &= \frac{n!}{2\pi i} [I_1(A) - I_2(A) + iI_3(A) - iI_4(A)]. \end{aligned}$$

Thus for sufficiently large B

$$f^{(n)}(\zeta) = \frac{n!}{2\pi i} \int_B^{B+1} [I_1(A) - I_2(A) + iI_3(A) - iI_4(A)] dA.$$

Now

$$\int_B^{B+1} I_3(A) dA = \int_{-\lambda}^{\lambda} dy \int_B^{B+1} \frac{f(A+iy)}{(A+iy-\zeta)^{n+1}} dA.$$

If $p > 1$, we apply Hölder's inequality to obtain

$$\begin{aligned}
& \left| \int_B^{B+1} I_3(A) dA \right| \\
& \leq \int_{-\lambda}^{\lambda} dy \left(\int_B^{B+1} |f(A + iy)|^p dA \right)^{1/p} \left(\int_B^{B+1} \frac{dA}{|A + iy - \zeta|^{q(n+1)}} \right)^{1/q} \\
& \leq \frac{2\lambda M}{|B - \operatorname{Re} \zeta|^{n+1}}.
\end{aligned}$$

For $p=1$, the same inequality is even more easily established. Hence $\int_B^{B+1} I_3(A) dA \rightarrow 0$ as $B \rightarrow \infty$. Similarly, $\int_B^{B+1} I_4(A) dA \rightarrow 0$. Clearly $\int_B^{B+1} I_1(A) dA \rightarrow I_1(\infty)$, $\int_B^{B+1} I_2(A) dA \rightarrow I_2(\infty)$, and the theorem is proven.

We are now ready to prove:

THEOREM 3.5. *Under the conditions of Theorem 3.4, $f^{(n)}(x) \in L^p(-\infty, \infty)$ ($n \geq 0$) and*

$$(3.6) \quad \|f^{(n)}(x)\|_p \leq \frac{n!M}{\lambda^n} \quad (n \geq 0).$$

Proof. Applying the generalized Minkowski inequality [9, Theorem 202] to (3.5), we have for $n \geq 1$

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} |f^{(n)}(t)|^p dt \right)^{1/p} & \leq \frac{n!}{2\pi} \left[\int_{-\infty}^{\infty} \frac{dx}{|x - i\lambda|^{n+1}} \left(\int_{-\infty}^{\infty} |f(x + t - i\lambda)|^p dt \right)^{1/p} \right. \\
& \quad \left. + \int_{-\infty}^{\infty} \frac{dx}{|x + i\lambda|^{n+1}} \left(\int_{-\infty}^{\infty} |f(x + t + i\lambda)|^p dt \right)^{1/p} \right] \\
& \leq \frac{n!M}{2\pi} \left[\int_{-\infty}^{\infty} \frac{dx}{|x - i\lambda|^{n+1}} + \int_{-\infty}^{\infty} \frac{dx}{|x + i\lambda|^{n+1}} \right] \\
& = \frac{n!M}{\pi\lambda^{n-1}} \int_{-\infty}^{\infty} \frac{dx}{x^2 + \lambda^2} = \frac{n!M}{\lambda^n}.
\end{aligned}$$

Now we shall prove:

THEOREM 3.6. *The classes $W_p(\sigma)$ and $T_p(\sigma)$ are identical. Given $f(z) \in W_p(\sigma)$ the radius of convergence ρ of the L^p -Taylor series of $f(x)$ is given by*

$$(3.7) \quad \rho = \sup \{ \sigma \mid f(z) \in W_p(\sigma) \}.$$

Proof. By Theorem 3.3, $T_p(\sigma) \subset W_p(\sigma)$. Let $f(z) \in W_p(\sigma)$ and let $|h| < \sigma$. Choose λ so that $|h| < \lambda < \sigma$. Then by (3.6)

$$\frac{|h|^n}{n!} \|f^{(n)}(x)\|_p \leq \frac{|h|^n M}{\lambda^n} \rightarrow 0 \quad (n \rightarrow \infty).$$

Hence $\rho \geq \sigma$ and $W_p(\sigma) = T_p(\sigma)$. Since $f(z) \in W_p(\rho)$, (3.7) is clear.

The class $W_2(\sigma)$ was studied by Paley and Wiener [3, pp. 3-13]. Their

principal theorem concerning this class may be stated as follows:

THEOREM 3.7. *Let $f(z) \in W_2(\sigma)$ and let $F(x)$ be the Fourier transform of $f(x)$. Then for each $h \in S_\sigma$, $e^{ihx}F(x)$ is the Fourier transform of $f(x+h)$ and is absolutely integrable so that*

$$(3.8) \quad f(z) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{izt} F(t) dt \quad (z \in S_\sigma).$$

Conversely, if $e^{ihx}F(x)L^2(-\infty, \infty)$ for each $h \in S_\sigma$, then it is absolutely integrable for each such h and $F(x)$ substituted in (3.8) yields an $f(z) \in W_2(\sigma)$.

We shall give a new proof of the first part of the theorem. By Theorem 3.6, $f(x+h)$ is represented for all $h = \xi + i\eta \in S_\sigma$ by

$$(3.9) \quad \text{l.i.m.}_{n \rightarrow \infty} \sum_{\nu=0}^{n-1} \frac{(i\eta)^\nu}{\nu!} f^{(\nu)}(x + \xi).$$

Since the Fourier transform of $f^{(n)}(x+\xi)$ is equal to $(ix)^n e^{i\xi x} F(x)$, the Fourier transform of (3.9) is seen to equal

$$\lim_{n \rightarrow \infty} \sum_{\nu=0}^{n-1} \frac{(-\eta x)^\nu}{\nu!} e^{i\xi x} F(x) = e^{(-\eta + i\xi)x} F(x) = e^{ihx} F(x).$$

The absolute integrability of $e^{ihx}F(x)$ is easily established by applying the Schwarz inequality

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\eta x} |F(x)| dx &= \int_{-\infty}^0 e^{\epsilon x} e^{-(\eta+\epsilon)x} |F(x)| dx + \int_0^{\infty} e^{-\epsilon x} e^{-(\eta-\epsilon)x} |F(x)| dx \\ &\leq \left(\int_{-\infty}^0 e^{2\epsilon x} dx \right)^{1/2} \left(\int_{-\infty}^0 e^{-2(\eta+\epsilon)x} |F(x)|^2 dx \right)^{1/2} \\ &\quad + \left(\int_0^{\infty} e^{-2\epsilon x} dx \right)^{1/2} \left(\int_0^{\infty} e^{-2(\eta-\epsilon)x} |F(x)|^2 dx \right)^{1/2}. \end{aligned}$$

The rest of the theorem is easily verified.

Returning to general p , we shall show that two L^p -Taylor series can be multiplied (in a manner analogous to the multiplication of ordinary Taylor series) to give a new L^p -Taylor series whose radius of convergence is at least equal to the smaller of the radii of convergence of the series being multiplied. In other words, if $f_1(x)$ and $f_2(x)$ both belong to $T_p(\sigma)$, then so does $f_1(x)f_2(x)$.

By Theorem 3.6, $f_1(z)$ and $f_2(z)$ both belong to $W_p(\sigma)$. But any $f(z) \in W_p(\sigma)$ is bounded in each S_λ for which $\lambda < \sigma$. This follows immediately from (3.5), by simple estimation when $p=1$ and by Hölder's inequality when $p>1$. Hence if $B = \sup_{z \in S_\lambda} |f_1(z)|$ and $M = \sup_{h \in S_\lambda} \|f_2(x+h)\|_p$ then

$$\|f_1(x+h)f_2(x+h)\|_p \leq BM \quad (h \in S_\lambda)$$

and $f_1(z)f_2(z) \in W_p(\sigma)$. Since the L^p -Taylor series of $f_1(x)f_2(x)$ can be formed from the L^p -Taylor series of $f_1(x)$ and $f_2(x)$ by the usual Cauchy method of multiplying power series, our assertion concerning the multiplication of L^p -Taylor series is valid. Two of our results are important enough to be numbered as theorems; viz.,

THEOREM 3.8. *If $f(z) \in W_p(\sigma)$, then $f(z)$ is bounded in each S_λ for which $\lambda < \sigma$.*

THEOREM 3.9. *The class $W_p(\sigma)$ is closed with respect to multiplication.*

As an immediate consequence of Theorem 3.8, we have:

THEOREM 3.10. *If $p_1 < p_2$, then $W_{p_1}(\sigma) \subset W_{p_2}(\sigma)$.*

We also have the following more general multiplication theorem:

THEOREM 3.11. *If $f_1(z) \in W_{p_1}(\sigma)$ and $f_2(z) \in W_{p_2}(\sigma)$, then $f_1(z)f_2(z) \in W_p(\sigma)$ for all p such that $p \geq \min(p_1, p_2)$. If $1/p_1 + 1/p_2 \geq 1$, then $f_1(z)f_2(z) \in W_p(\sigma)$ for all $p \geq 1$.*

The first part of the theorem being an obvious consequence of Theorems 3.8 and 3.10, let us consider the case $1/p_1 + 1/p_2 \geq 1$. In this case $p_2 \leq q_1$, the index conjugate to p_1 . Hence $f_2(z) \in W_{q_1}(\sigma)$ and by Hölder's inequality, $f_1(z)f_2(z) \in W_1(\sigma)$.

We close with some examples of functions belonging to various $W_p(\sigma)$. It is readily verified that $e^{-z^p} \in W_p(\sigma)$ for all $p \geq 1$ and for all positive σ . The function $(z - i\sigma_0)^{-1/p_0} \in W_p(\sigma_0)$ for each $p > p_0$. Its L^p -Taylor series has radius of convergence σ_0 for each such p .

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UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PA.
UNIVERSITY OF MICHIGAN,
ANN ARBOR, MICH.