ON THE ZEROS OF THE DERIVATIVES OF SOME ENTIRE FUNCTIONS[1]

BY

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1. **Introduction.** 1.1. *Objective.* Let f be an entire function. Let $\mathfrak{L}_f$ be the set of points z such that to each disk D centered at z there corresponds a sequence of integers $\{n_k\}$ and a sequence of points $\{z_k\}$, $z_k \in D$, such that $f^{(n_k)}(z_k) = 0$. In [6] Pólya determined $\mathfrak{L}_f$ for functions f of finite order at least 2 having only a finite set of zeros. The object of this paper is to extend the results by relaxing the restriction on the set of zeros of f.

1.2. *Notations.* Throughout the paper the following notations will be used: $\boldsymbol{C}$ is the complex plane.

$$D(z, r) = \{u: |u - z| < r\}; \qquad C(z, r) = \{u: |u - z| = r\}.$$

If $c = |c|e^{i\gamma}$, $0 \leqq \gamma < 2\pi$, $|c| > 0$, then

$$\alpha_k(c) = \exp[-\gamma i/q + 2\pi ki/q], \qquad k = 0, 1, \cdots, q - 1, \tag{1.1}$$

and $\beta_k(c) = |c|^{-1/q}\alpha_k(c)$. Also,

$$A_k(c, \rho) = \{u: |u| > 0, |\arg[\alpha_k(c)/u]| < \rho\}$$

and $E(c, \rho) = \bigcup_{k=0}^{q-1} A_k(c, \rho)$. If $w \in \boldsymbol{C}$, $F(z) = w$, then $F^*(z) = w^*$, where w^* is the complex conjugate of w. If F is a bounded, real-valued function,

$$M(F, z, r) = \operatorname*{Sup}_{u \in D(z,r)} F(u).$$

1.3. *Results.* Suppose $f(z) = \phi(z)\exp(cz^q + dz^{q-1})$, $q \geqq 2$, where

$$M(\log|\phi|, 0, r) = o(r^{q+1}).$$

Let $\mathfrak{R}_f$ be the set made up of the q rays emanating from the point $-d/(qc)$ and passing through $-d/(qc) + \alpha_k(c)e^{i\pi/q}$, $k = 0, 1, \cdots, q-1$. Pólya proved that $\mathfrak{L}_f = \mathfrak{R}_f$ if f has a finite set of zeros. By application of his result to functions approximating to an f having an infinite set of zeros, it soon becomes clear that only in the directions $\alpha_k(c)$ do the zeros of f influence $\mathfrak{L}_f$. For functions f of the classes $\mathfrak{F}$ and $\mathfrak{G}$ defined below it is again true that

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$\mathcal{L}_f = \mathcal{R}_f$. The theorem to this effect is given in §4.

DEFINITION 1. $f \in \mathfrak{F}$ *if and only if* f *is entire and there exist* c *and* d *in* $\mathbf{C}$, $c \neq 0$, *an integer* q, $q \geqq 2$, *and* $\rho > 0$ *such that*

$$\max_{|z|=r} \log \left| f(z)e^{-cz^q} \right| = o(r^q) \qquad \text{as } r \to \infty, \tag{1.2}$$

$$\log \left| f(z)e^{-cz^q - dz^{q-1}} \right| = o(|z|^{q-1}), \qquad z \in E(c, \rho). \tag{1.3}$$

Note that (1.3) requires f to have a finite set of zeros in $E(c, \rho)$.

DEFINITION 2. $f \in \mathfrak{G}$ *if and only if* f *is entire and there exist* c *and* d *in* $\mathbf{C}$, $c \neq 0$, *and* $\rho > 0$ *such that*

$$\max_{|z|=r} \log \left| f(z)e^{-cz^2} \right| = o(r^2) \qquad \text{as } r \to \infty, \tag{1.4}$$

$$\log \left| f(z)e^{-cz^2} \right| = o(|z|^2) \text{ as } z \to \infty,\ z \in E(c, \rho), \tag{1.5}$$

$$\log \left| \phi(z)/\phi(w_z) \right| = o(|z|), \text{ as } z \to \infty,\ z \in E(c, \rho), \tag{1.6}$$

where $\phi(z) = f(z) \exp[-cz^2 - dz]$ *and* w_z *is the reflection of* z *in the line through the origin and the points* $\alpha_k(c)e^{i\pi/2}$, $k = 0, 1$.

The classes $\mathfrak{F}$ and $\mathfrak{G}$ intersect. $\mathfrak{F}$ includes the class treated by Pólya.

In §6 there is given a class with yet weaker conditions on the zeros of f for which $\mathcal{R}_f$ may be a proper subset of $\mathcal{L}_f$.

1.4. *Methods.* The proof here is fundamentally the same as that used by Pólya. But the details are somewhat simpler. The method of proof is: (1) find the asymptotic behavior of $f^{(n)}(z)$ in certain sectors using a modification of a generalization of Stirling's formula due to Hayman [2, p. 69], (2) apply a theorem of Ganelius [1, p. 33] which gives an estimate from below on the number of zeros in certain neighborhoods.

1.5. *Related work.* Wyman and Moser have developed, in [4; 5], asymptotic series expansions for $f^{(n)}(z)$ where f is the exponential of a polynomial. Theorem 1 of this paper gives only the first term of the asymptotic series.

Results similar to that of Ganelius have been given by Kay in [3].

General surveys of the study of the zeros of the sequence of derivatives are available in [7; 10].

2. A generalization of Stirling's formula.

2.1. *Definitions.* For convenience let $D(0, \infty) = \mathbf{C}$. Suppose f is holomorphic in $D(0, R)$, $0 < R \leqq \infty$. Associate with f the functions a_f and b_f defined by

$$a_f(z) = zf'(z)/f(z), \tag{2.1}$$

$$b_f(z) = za_f'(z). \tag{2.2}$$

DEFINITION 3. *The class* $\mathfrak{Z}_R$ *consists of those functions* f, *holomorphic in* $D(0, R)$, *with the following properties*:

(a) *There exist numbers K_f and R_f, $0<R_f<R$, and for each $r \geqq R_f$ a nonvoid set $I_f(r)$ such that $z \in I_f(r)$ implies $|z|=r$ and*

$$[\operatorname{Im} a_f(z)]^2/[\operatorname{Re} b_f(z)] \leqq K_f. \tag{2.3}$$

(b) *There exists a real-valued function δ_f defined on the interval (R_f, R) such that $0<\delta_f(r)<\pi$ and*

$$f(ze^{it}) = [1 + o(1)]f(z) \exp [it\, a_f(z) - (t^2/2)b_f(z)] \tag{2.4}$$

as $r \to R$, $z \in I_f(r)$, uniformly for $|t| \leqq \delta_f(r)$, while

$$f(ze^{it}) = o[f(z)[b_f(z)]^{-1/2}] \tag{2.5}$$

as $r \to R$, $z \in I_f(r)$, uniformly for $\delta_f(r) \leqq |t| \leqq \pi$.

(c) *There exists a number M_f such that*

$$|b_f(z)/\operatorname{Re} b_f(z)| \leqq M_f, \qquad z \in I_f(r),\ R_f \leqq r < R. \tag{2.6}$$

Furthermore,

$$|b_f(z)| \to \infty \qquad \text{as } r \to R,\ z \in I_f(r). \tag{2.7}$$

2.2. THEOREM 1. *Let $f \in \mathcal{Z}_R$. Then, as $r \to R$,*

$$\frac{f^{(n)}(0)}{n!} z^n = \frac{f(z)}{[2\pi b_f(z)]^{1/2}} \left\{ \exp\left[-\frac{(a_f(z) - n)^2}{2b_f(z)} \right] + o(1) \right\}, \tag{2.8}$$

if $z \in I_f(r)$, uniformly for all integers n.

Moreover, if $\mathcal{W}$ is a subclass of $\mathcal{Z}_R$ such that there is a number R_0 satisfying $R_f \leqq R_0$, $f \in \mathcal{W}$, and such that (2.3), (2.4), (2.5), (2.6), *and* (2.7) *hold uniformly for all f in $\mathcal{W}$, then* (2.8) *holds uniformly for all f in $\mathcal{W}$.*

This theorem generalizes that of Hayman [2, p. 69]. The difference is primarily that Hayman requires that $f(z)$ be real if z is real and that $I_f(r) = \{r\}$.

The following lemma anticipates our needs in proving Theorem 1.

2.3. LEMMA 1. *Let δ be a positive number. Let a and b be complex numbers with* $\operatorname{Re} b>0$. *Let ω be a continuous, complex-valued function on the interval $[-\delta, \delta]$. Let $\Omega = \operatorname{Max} |\omega(t)|$. Set $a=a_1+ia_2$, $b=b_1+ib_2$, and $b_1=(B_1)^2$, $B_1>0$. Then*

$$\left| \int_{-\delta}^{\delta} \omega(t) e^{at-bt^2} dt \right| \leqq \Omega \pi^{1/2} B_1^{-1} e^{a_1^2(4b_1)^{-1}}. \tag{2.9}$$

Furthermore, if $B_1\delta - a_1(2B_1)^{-1}>0$, then

$$\left| \int_{\delta}^{\infty} e^{at-bt^2} dt \right| \leqq B_1^{-1} e^{-B_1^2\delta^2 + a_1\delta}. \tag{2.10}$$

Proof. By obvious steps,

$$\left|\int_{-\delta}^{\delta}\omega(t)e^{at-bt^2}dt\right| \leqq \Omega\int_{-\delta}^{\delta}e^{a_1t-b_1t^2}dt$$

$$= \Omega e^{a_1^2(4B_1^2)^{-1}}\int_{-\delta}^{\delta}e^{-(B_1t-a_1/2B_1)^2}dt.$$

The dominant becomes larger if $\int_{-\infty}^{\infty}$ replaces $\int_{-\delta}^{\delta}$. But

$$\int_{-\infty}^{\infty}\exp\left[-(B_1t - a_1/2B_1)^2\right]dt = (1/B_1)\int_{-\infty}^{\infty}\exp(-x^2)dx = \pi^{1/2}(1/B_1).$$

Thus we have (2.9).

In like manner,

$$\left|\int_{\delta}^{\infty}e^{at-bt^2}dt\right| \leqq B_1^{-1}e^{a_1^2(4B_1^2)^{-1}}\int_{p}^{\infty}e^{-x^2}dx$$

where $p = B_1\delta - a_1(2B_1)^{-1}$. But, as one shows easily by a change of variable, $p>0$ implies $\int_p^{\infty}\exp(-x^2)dx < \exp(-p^2)$. Thus we have (2.10).

2.4. **Proof of Theorem 1.** For convenience, we omit the subscript f. From Cauchy's formula,

$$\frac{f^{(n)}(0)}{n!}z^n = \frac{1}{2\pi}\int_{-\delta(r)}^{2\pi-\delta(r)}f(ze^{it})e^{-nit}dt.$$

With the same integrand, set $I_1 = (1/2\pi)\int_{-\delta(r)}^{\delta(r)}$ and $I_2 = (1/2\pi)\int_{\delta(r)}^{2\pi-\delta(r)}$. From (2.5), $I_2 = f(z)\left[2\pi b(z)\right]^{-1/2}o(1)$ as $r\to R$, if $z\in I(r)$, uniformly with respect to n.

From (2.4),

$$I_1 = \frac{f(z)}{2\pi}\int_{-\delta(r)}^{\delta(r)}\left[1 + \omega(t)\right]\exp\left[i(a(z) - n)t - b(z)t^2/2\right]dt$$

where $\omega(t) = o(1)$. Call the exponential in the integrand $E(t)$. Set

$$I_1 = \frac{f(z)}{2\pi}\left\{\int_{-\infty}^{\infty}E(t)dt + \int_{-\delta(r)}^{\delta(r)}\omega(t)E(t)dt - \left[\int_{-\infty}^{-\delta(r)} + \int_{\delta(r)}^{\infty}\right]E(t)dt\right\}.$$

To apply Lemma 1 to the various terms we need some deductions from (2.4) and (2.5).

Taking in turn $t = \delta(r)$, $t = -\delta(r)$ in (2.4) and (2.5), one infers that

$$\exp\left[\delta(r)\ \mathrm{Im}\ a(z) - \frac{1}{2}\delta^2(r)\ \mathrm{Re}\ b(z)\right] = o(\left[b(z)\right]^{-1/2})$$

and

$$\exp\left[-\delta(r)\operatorname{Im} a(z) - \frac{1}{2}\delta^2(r)\operatorname{Re} b(z)\right] = o([b(z)]^{-1/2}).$$

Out of these relations and (2.7) it follows that

$$\frac{1}{2}\delta^2(r)\operatorname{Re} b(z) - \delta(r)\left|\operatorname{Im} a(z)\right| \to \infty \text{ as } r \to R.$$

In particular, $\operatorname{Re} b(z)$ is positive for r sufficiently large.

From these facts we may verify the hypotheses of Lemma 1 for the last three integrals in the expression for I_1. Therefore, in view of (2.3),

$$I_1 = \frac{f(z)}{2\pi}\left\{\int_{-\infty}^{\infty} E(t)dt + o([\operatorname{Re} b(z)]^{-1/2})\right\}$$

uniformly for all integers n. But now

$$\int_{-\infty}^{\infty} E(t)dt = e^{-c^2}[b(z)/2]^{-1/2}\int_L e^{-y^2}dy$$

where $c = [a(z) - n][2b(z)]^{-1/2}$ and L is the parametrized path given by the function $t \to t[b(z)/2]^{1/2} - ic$, t real. (The square root $[b(z)/2]^{1/2}$ is the one having positive real part.) A simple application of Cauchy's theorem gives $\int_L \exp(-y^2)dy = \int_{-\infty}^{\infty}\exp(-x^2)dx$.

Finally, in view of (2.6),

$$I_1 = \frac{f(z)}{[2\pi b(z)]^{1/2}}\left\{\exp\left[-\frac{(a(z) - n)^2}{2b(z)}\right] + o(1)\right\}.$$

On combining this with the earlier estimate on I_2 we have (2.8).

The statement regarding uniformity over a class $\mathcal{W}$ is easy to check.

3. The classes $\mathfrak{F}$ and $\mathcal{G}$.

3.1. *General lemmas.* The first of the following lemmas has been given by Hayman, [2, p. 78], in very slightly different form.

LEMMA 2. *Let f be a function which is holomorphic and has no zeros in the disk $D(w, \rho|w|)$, $0 < \rho \leqq 1$. Let a_f and b_f be defined by* (2.1) *and* (2.2). *If*

$$(3.1)\qquad |b_f(z)| < C|b_f(w)|, \qquad z \in D(w, \rho|w|),$$

then

$$(3.2)\qquad \log f(we^{it}) = \log f(w) + ita_f(w) - t^2 b_f(w)/2 + \eta(w, t)$$

where $|\eta(w, t)| < C|b_f(w)t^3|/\rho$ for $|t| \leqq \rho/2$.

LEMMA 3. *Let f be a function which is holomorphic and has no zeros in $D(w, \rho|w|)$, $w \neq 0$. Let $g_k(z) = (d^k/dz^k)\log f(z)$, $k \geqq 1$, $g_0(z) = \log f(z)$, and $h(z) = \log[f(z)/f(w)]$. If $0 < \tau < \sigma < \rho$, there is a positive number A, depending on σ and τ, such that*

$$(3.3)\qquad |g_k(z)| \leqq k!A^k|w|^{-k}M(\mathrm{Re}\, h, w, \sigma|w|)$$

for $k\geqq 1$ *and* z *in* $D(w, \tau|w|)$.

Furthermore, there are numbers B *and* ω *such that for* $k\geqq 0$ *and* z *in* $D(0, \omega|w|)$

$$(3.4)\qquad g_k(w+z) = g_k(w) + \eta_k(w, z)$$

where

$$|\eta_k(w, z)| < (k+1)!|z|\;|B/w|^{k+1}M(\mathrm{Re}\, h, w, \sigma|w|).$$

Proof. Set $\epsilon=(\tau+\sigma)/2$. Since the functions g_k, $k\geqq 1$, are also derivatives of h, Cauchy's integral yields

$$|g_k(z)| \leqq \frac{k!\epsilon}{(\epsilon-\tau)^{k+1}}|w|^{-k}M(|h|, w, \epsilon|w|)$$

if $z\in D(w, \tau|w|)$. The Borel-Carathéodory inequality,

$$\max_{|z|=r}|\phi(z)| \leqq \frac{2r}{R-r}\max_{|z|=R}\mathrm{Re}\,\phi(z) + \frac{R+r}{R-r}|\phi(0)|, \quad 0<r<R,$$

(see [8, p. 174], e.g.) gives

$$M(|h|, w, \epsilon|w|) \leqq \frac{2\epsilon}{\sigma-\epsilon}M(\mathrm{Re}\, h, w, \sigma|w|).$$

The conclusion (3.3) is now only a matter of naming an appropriate number A.

The Taylor expansion

$$g_k(w+z) = g_k(w) + \sum_{p=1}^{\infty} g_{k+p}(w)z^p/p!$$

yields $\eta_k(w, z)$ in an obvious way. From (3.3)

$$\left|g_{k+p}(w)\frac{z^p}{p!}\right| \leqq \left|\frac{A}{w}\right|^k M(\mathrm{Re}\, h, w, \sigma|w|)\frac{(k+p)!}{p!}\left|\frac{Az}{w}\right|^p.$$

Since $\sum_{p=1}^{\infty} x^p(k+p)!/p! = k![(1-x)^{-k-1}-1]$ when $|x|<1$, it is an easy matter to name suitable numbers ω and B.

3.2. *Lemmas for entire functions.* We shall deal with entire functions f satisfying one or more of the following:

$$(3.5)\qquad \max_{|z|=r}\log|f(z)| = o(r^q) \qquad \text{as } r\to\infty;$$

in a certain unbounded region G,

$$(3.6) \qquad \log |f(z)| = o(|z|^q) \qquad \text{as } z \to \infty;$$

$$(3.7) \qquad \log |f(z)/f(-z^*)| = o(|z|) \qquad \text{as } z \to \infty \text{ in } G.$$

Note that (3.6) implies a finite set of zeros of f in G.

LEMMA 4. *Let f satisfy* (3.5), (3.6), *and* (3.7) *with $q=2$ and $G=E(1, \rho)$ for some $\rho>0$. Then, for $\sigma<\rho$,*

$$\log\left|\frac{f(w+z)}{f(-w^*+u)}\right| = o(|w|) \qquad \textit{as } w \to \infty \text{ in } E(1, \sigma),$$

uniformly for z and u in $D(0, R)$.

Proof. In the notation of the previous lemma, consider

$$\operatorname{Re}[g_0(w+z) - g_0(-w^*+u)].$$

Use (3.4). The dominants for $\eta_0(w, z)$ and $\eta_0(-w^*, u)$ in combination with (3.5), (3.6), and (3.7) yield the desired conclusion.

LEMMA 5. *Under the hypotheses of Lemma* 4,

$$a_f^*(-w^*) - a_f(w) = o(|w|)$$

as $w \to \infty$ in $E(1, \sigma)$, $\sigma<\rho$.

Proof. If ϕ is a function holomorphic in an open set E, then $\phi^{\star}: z \to -\phi^*(-z^*)$ is holomorphic in the set $E^{\star}$ which is the reflection of E across the imaginary axis. Furthermore, $\phi^{\star\prime} = -\phi^{\prime\star}$. Set $\phi=\log f$. There is a number M such that both ϕ and $\phi^{\star}$ are holomorphic in $E(1, \rho) \cap \{z: |z| > M\}$. Let $w \in E(1, \sigma)$, $\sigma<\rho$, and set $\psi(z)=\phi^{\star}(z)+\phi(z)-\phi^{\star}(w)-\phi(w)$. There is a number τ such that $D(w, 2\tau|w|) \subset E(1, \rho)$. Then

$$|\psi'(w)| \leqq (\tau|w|)^{-1} M(|\psi|, w, \tau|w|)$$

for large $|w|$. Using the Borel-Carathéodory inequality we get the further inequality

$$|\psi'(w)| \leqq \frac{2}{\tau|w|} M(\operatorname{Re}\psi, w, 2\tau|w|).$$

But $\operatorname{Re}\psi(z)=\log|f(z)/f(-z^*)| - \log|f(w)/f(-w^*)|$. Therefore (3.7) implies that $\psi'(w)=o(1)$ as $w \to \infty$ in $E(1, \sigma)$.

Now $a_f^*(-w^*)-a_f(w) = -w\psi'(w)$. The conclusion is immediate.

LEMMA 6. *Let f be an entire function such that $f(z)=\phi(z)\exp(cz^q)$ where ϕ satisfies* (3.5) *and* (3.6) *with $G=E(c, \rho)$ for some ρ such that $0<\rho<\pi/q$.*

A. *Let f_u be the function $z \to f(z+u)$. Given $R>0$ there exists x_R such that to each x, $x \geqq x_R$, there corresponds a set of q functions ψ_{xk}, defined on $D(0, R)$ and holomorphic there, with the property that $a_{f_u}(\psi_{xk}(u))=x$, $u \in D(0, R)$.*

B. *To each* u *and* k *there corresponds a function* ζ_{uk} *defined on a half-line* $[m, \infty)$ *such that* $\operatorname{Im} a_{f_u}(\zeta_{uk}(r))=0$, $r \geqq m$, *and* $\zeta_{uk}(r)=r\alpha_k(c) \exp [i\theta_{uk}(r)]$, $\theta_{uk}(r)$ *real*, $\theta_{uk}(r)\to 0$ *as* $r\to\infty$. *Moreover* $\psi_{xk}(u)=\zeta_{uk}(|\psi_{xk}(u)|)$.

C. *If* $c>0$, $q=2$, *and* ϕ *also satisfies* (3.7), *then* $\psi_{x0}(u)+\psi^*_{x1}(u)=o(1)$ *as* $x\to\infty$.

D. *If* $\phi(z)=\phi_1(z)\phi_2(z) \exp (dz^{q-1})$ *where* $\log |\phi_2(z)|=o(|z|^{q-1})$ *as* $z\to\infty$ *in* $E(c, \rho)$, *then*

$$\psi_{xk}(u) = t_{xk} - (qu + d/c)(q-1)q^{-2} + o(1) \qquad \text{as } x\to\infty, \tag{3.8}$$

uniformly in $D(0, R)$, *where* t_{xk} *is the number* $\psi_{xk}(0)$ *got by taking* $f(z)=\phi_1(z) \exp (cz^q)$ *in part* A. *In particular, if* $f(z)=\exp (cz^q)$, *then* $t_{xk}=\beta_k(c)(x/q)^{1/q}$. *In general*, $t_{xk}=\beta_k(c)(x/q)^{1/q}(1+o(1))$.

Proof. It suffices to take $c=1$ since the general case is easily derived from this special case. We abbreviate $A_k(1, \rho)$ to $A_k(\rho)$ and $\alpha_k(1)$ to α_k. We regard k as fixed throughout the proof.

Let ϕ_u be the function $z\to\phi(z+u)$. Let R and σ be positive numbers with $\sigma<\rho$. Then ϕ_u satisfies (3.5) and (3.6) with $G=A_k(\sigma)$, uniformly for u in $D(0, R)$. With the aid of (3.3) we find that

$$a_{f_u}(z) = qz^q + o(z^{q-1}), \qquad b_{f_u}(z) = q^2z^q + o(z^{q-1})$$

as $z\to\infty$ in $A_k(\sigma)$, uniformly for u in $D(0, R)$.

Let w be a number in $A_k(\sigma)$ such that $w^q>0$. Then by quite simple considerations one may show that there is a positive number t_ϵ such that $|z^q-w^q|\geqq t_\epsilon|z|^q$ for all z inside $A_k(\sigma)$ but outside $D(w, \epsilon|w|)$.

Let $S_k(s, \sigma)=A_k(\sigma)\cap\{z: |z|>s\}$. Fix ϵ for the moment and choose s so that

$$|a_{f_u}(z) - qz^q| \leqq |z|^q qt_\epsilon/2$$

for all z in $S_k(s, \sigma)$ and all u in $D(0, R)$. Let x be positive and set $w=\alpha_k(x/q)^{1/q}$. Evidently there is a number x_R such that $x\geqq x_R$ implies $|a_{f_u}(z)-x|\geqq qt_\epsilon|z|^q/2$ for all z inside $S_k(s, \sigma)$ but outside $D(w, \epsilon|w|)$ and $|a_{f_u}(z)-qz^q|<|qz^q-x|$ for z on $C(w, \epsilon|w|)$. The disk $D(w, \epsilon|w|)$ is the only part of $S_k(s, \sigma)$ in which a root of $a_{f_u}(z)=x$ can lie. Rouché's theorem guarantees exactly one such root there. Denote it by $\psi_{xk}(u)$. The function $\psi_{xk}: u\to\psi_{xk}(u)$ is defined at least on $D(0, R)$ and is holomorphic. This last property may be deduced from the standard implicit function existence theorem.

The preceding argument is valid no matter how small ϵ may be. Thus

$$\psi_{xk}(u) = \alpha_k(x/q)^{1/q}(1 + o(1))$$

as $x\to\infty$, uniformly for u in $D(0, R)$.

The problem in part B is to study the equation $\operatorname{Im} a_{f_u}(z)=0$. We know already that the numbers $\psi_{xk}(u)$ satisfy it. Since the function $x\to\psi_{xk}(u)$ is continuous, the image of (x_R, ∞) is connected; thus it meets the circle

$C(0, r)$ if r is large enough. Actually the intersection consists of only one point. One may show this by applying elementary calculus to the function $\theta \to \operatorname{Im} a_{f_u}(re^{i\theta})$, θ real. The conclusions of part B are now immediate from the facts given in this paragraph and the one before it.

The first step in proving C and D is to show that there are numbers σ and r such that

$$|a_{f_u}(w+z) - a_{f_u}(w)| \geqq |zw^{q-1}| \tag{3.9}$$

provided $w \in S_k(r, \sigma)$ and $z \in D(0, \sigma|w|)$.

There is a number r such that the right-hand member of

$$a_{f_u}(w+z) - a_{f_u}(w) - \frac{z}{w} b_{f_u}(w) = \sum_{p=2}^{\infty} a_{f_u}^{(p)}(w) z^p / p!$$

has the dominant $A|z/w|^2|w|^q$ if $w \in S_k(r, \rho/3)$, $z \in D(0, |w|\rho/3)$. But r can be chosen so that, in addition, $|b_{f_u}(w)| \geqq q|w|^q$. The choice

$$\sigma = \min[\rho/3, q/2A]$$

yields (3.9).

Now we finish *C*. We can say equally well that (3.9) holds if w lies in $E(1, \sigma)$ with $|w| > r$. Then, using Lemma 5, we find that

$$|a_{f_u}(-\overset{*}{w} + z) - \overset{*}{a}_{f_u}(w)| \geqq |zw|/2$$

provided $0 < \epsilon \leqq |z| \leqq \sigma|w|$ and $|w|$ is large enough. Taking $w = \psi_{x0}(u)$, we conclude, remembering that x is real, that $\psi_{x1}(u)$ lies in $D(-\psi_{x0}^*(u), \epsilon)$ since we know from the remark four paragraphs above that it lies in the concentric disk of radius $\sigma|\psi_{x0}(u)|$. The conclusion is now immediate.

To finish D one need only prove that

$$a_{f_u}(t_{xk} - (qu + d/c)(q-1)/q^2) = x + o(t_{xk}^{q-1})$$

and use (3.9). The computation is quite straightforward; we omit the details.

LEMMA 7. *Let $f \in \mathfrak{F} \cup \mathfrak{G}$. Let ψ_{nk} be the function of Lemma 6, part* B. *Then to each R there corresponds an A such that*

$$\left|\frac{f^{(n)}(u)}{n!}\right| \leqq \left|\frac{f(u + \psi_{nk}(u))}{[\psi_{nk}(u)]^n}\right| e^{An^{1-1/q}}, \qquad u \in D(0, \mathrm{R}).$$

Proof. From Cauchy's inequality,

$$|f^{(n)}(u)/n!| \leqq |\psi_{nk}(u)|^{-n} M(|f|, u, |\psi_{nk}(u)|).$$

The remaining task is to show that $|f(z)/f(u + \psi_{nk}(u))| \leqq \exp(An^{1-1/q})$ if $z \in C(u, |\psi_{nk}(u)|)$ and $u \in D(0, R)$. This will be carried out in the proof of Theorem 2.

3.3. THEOREM 2. *Let $f\in\mathfrak{F}\cup\mathfrak{G}$. There are q sequences $\{\psi_{nk}\}_{n=N}^{\infty}$, $k=0, 1, \cdots, q-1$, such that ψ_{nk} is a function holomorphic in a region D_{nk} and*

$$(3.10)\qquad \frac{f^{(n)}(u)}{n!}=\frac{f(u+\psi_{nk}(u))}{[\psi_{nk}(u)]^n(2\pi qn)^{1/2}}(1+o(1)),\qquad u\in S_k,$$

uniformly in each compact subset of S_k, where S_k is the sector (or half-plane if $q=2$) given by

$$(3.11)\qquad S_k=\{u: |v|>0, |\arg\alpha_k(c)-\arg v|<\pi/q\}$$

where $v=qu+d/c$.

If B is a bounded set, then $B\subset D_{nk}$ for $n\geqq n_B$. Finally, ψ_{nk} is given by (3.8).

3.4. **Proof of Theorem** 2; **completion of proof of Lemma** 7. Suppose, without loss of generality, that $c=1$. Let f_u be the function $z\to f(z+u)$. The first objective is to show that $f_u\in Z_\infty$ (see Definition 3) if $u\in\bigcup_{j=0}^{q-1}S_j$. However, since the proof of Lemma 7 is also to be completed, certain inequalities will be proved for $u\in D(0, R)$.

Throughout the proof k is regarded as a fixed integer.

From part B of Lemma 6 there is a function ζ such that $\operatorname{Im} a_{f_u}(\zeta(r))=0$ and $\zeta(r)=\alpha_k(1)re^{i\theta(r)}$ with $\theta(r)$ real and $\lim_{r\to\infty}\theta(r)=0$. Let $I(r)=\{\zeta(r)\}$ and $\delta(r)=r^{-2q/5}$. Now (2.3) is satisfied for f_u with $K=0$.

From Lemma 2 we see that (2.4) is satisfied. Moreover, there is a positive number ϵ_1 such that (2.5) holds for $\delta(r)\leqq|\theta|\leqq\epsilon_1$. Both relations are uniform for u in $D(0, R)$.

Let $s=re^{i2\pi k/q}$.

The first and second derivatives of the function $\theta\to-\log|f_u(se^{i\theta})|$, θ real, are $\operatorname{Im} a_{f_u}(se^{i\theta})$ and $\operatorname{Re} b_{f_u}(se^{i\theta})$, respectively. Therefore $|f_u(se^{i\theta})|\leqq|f_u(\zeta(r))|$ for $|\theta|\leqq\rho$. Then in checking (2.5) for $\epsilon_1\leqq|\theta|\leqq\pi$ we may replace $f_u(\zeta(r))$ by $f_u(s)$.

Consider $|f_u(z)/f_u(s)|$ on that part of $C(0, r)$ outside $E(1, \epsilon)$ for any $\epsilon>0$. One sees readily that there is a positive number γ_1 such that $\operatorname{Re}[z^q-s^q]\leqq-\gamma_1r^q$. It is an easy step to show that to ϵ there corresponds $\gamma>0$ such that

$$(3.12)\qquad \log|f_u(z)/f_u(s)|\leqq-\gamma r^q$$

if $|z|=r$ and $z\notin E(1, \epsilon)$, uniformly for u in $D(0, R)$.

The following two inequalities are proved in succeeding paragraphs. Let R be a positive number and H a compact subset of $A_k(1, \pi/q)$. Then there exists a positive number A such that

$$(3.13)\qquad |f_u(z)/f_u(\zeta(r))|\leqq e^{Ar^{q-1}},\qquad |z|=r,$$

if $u\in D(0, R)$ and $z\in\bigcup_{j\neq k}A_j(1, \epsilon)$. Also, there are positive numbers γ and ϵ_2 such that

$$(3.14)\qquad |f_u(z)/f_u(\zeta(r))|\leqq e^{-\gamma r^{q-1}}$$

if $qu+d\in H$ and $z\in\bigcup_{j\neq k} A_j(1, \epsilon_2)$.

At this point we consider separately the cases $f\in\mathfrak{F}$ and $f\in\mathfrak{G}$.

Let $f\in\mathfrak{F}$. Set $d_u=qu+d$ and $|d_u|e^{i\tau}=d_ue^{-i2\pi k/q}$. If $z=se^{i\theta}$, then

$$\mathrm{Re}\,[d_u(z^{q-1}-s^{q-1})] = |d_u|r^{q-1}[\cos(\tau+(q-1)\theta)-\cos\tau].$$

Set $\theta=2\pi j/q+2\beta/(q-1)$, $1\leqq j\leqq q-1$. Then

$$\cos(\tau+(q-1)\theta)-\cos\tau = -2\sin(-\pi j/q+\beta+\tau)\sin(-\pi j/q+\beta).$$

Since $qu+d\in A_k(1, \pi/q)$, $|\tau|<\pi/q$; consequently $-\pi<-\pi j/q+\tau<0$ and, if $|\beta|$ is sufficiently small, $\cos(\tau+(q-1)\theta)-\cos\tau<0$. It is now clear that to H there corresponds an ϵ_2 and a γ_1 such that

$$\mathrm{Re}\,[d_u(z^{q-1}-s^{q-1})] \leqq -\gamma_1 r^{q-1}, \qquad |z|=r,$$

if $z\in\bigcup_{j\neq k} A_j(1, \epsilon_2)$ and $qu+d\in H$.

From (1.3),

$$\log|f_u(z)/f_u(s)| \leqq \mathrm{Re}\,[d_u(z^{q-1}-s^{q-1})]+o(r^{q-1}).$$

The last inequality alone makes clear the existence of the number A. The last two inequalities show the existence of the number γ.

Now let $f\in\mathfrak{G}$. Set $d_u=2u+d$. In this case the convenient way is to consider $f_u(z)/f_u(\zeta(r))$. The set $E(1, \epsilon)$ consists of the sectors $A_0(1, \epsilon)$ and $A_1(1, \epsilon)$. Put $z=-\zeta^*(r)e^{i\theta}$, θ real, $|\theta|\leqq\epsilon$. Then $z\in A_l(1, 2\epsilon)$, $l=1-k$, for large r. Put $w=-\zeta^*(r)$. From Lemma 2 one gets

$$\log[f_u(z)/f_u(w)] = ia_{f_u}(w)\theta-\theta^2 b_{f_u}(w)/2+\eta(w, \theta).$$

There is an ϵ_2 for which $|\eta(w, \theta)|\leqq\mathrm{Re}\, b_{f_u}(w)\theta^2/4$ if $|\theta|\leqq\epsilon_2$. Therefore the real part of the sum of the last two terms is negative. On the other hand, in view of the definition of w and Lemma 5, $\mathrm{Re}[ia_{f_u}(w)\theta]=o(r)$. Thus we assert that

$$\log|f_u(z)/f_u(\zeta(r))| \leqq \log|f_u(-\zeta^*(r))/f_u(\zeta(r))|+o(r),$$

uniformly for u in $D(0, R)$. But

$$|f_u(w)/f_u(-w^*)| = |e^{d_u(w+w^*)}\phi_u(w)/\phi_u(-w^*)|.$$

Using Lemma 4 we find that (3.13) and (3.14) hold for $f\in\mathfrak{G}$.

Lemma 7 follows from (2.5) for $|\theta|\leqq\epsilon_1$, (3.13), and (3.12) with $\epsilon=\min(\epsilon_1, \epsilon_2)$ if we choose r so that $\zeta(r)=\psi_{nk}(u)$.

To establish (2.5) on $\epsilon_1\leqq|\theta|\leqq\pi$ it is enough to use (3.14) and (3.12) with $\epsilon=\min(\epsilon_1, \epsilon_2)$.

The remaining conditions (2.6) and (2.7) are easy to check.

Taking that r for which $\zeta(r)$ becomes $\psi_{nk}(u)$, we get (3.10) from (2.8) since

$$[2\pi b_{f_u}(\psi_{nk}(u))]^{1/2} = (2\pi qn)^{1/2}(1+o(1)).$$

4. The principal theorems.

4.1. *An auxiliary theorem.* The tool to be used in determining $\mathfrak{L}_f$ is the following generalization of Jentzsch's theorem due to Ganelius [1, p. 33].

THEOREM 3. *Let* $\{F_n\}_{n=1}^{\infty}$ *be a sequence of functions holomorphic in a region containing the closure of the bounded region* G. *Let* z_0 *be a point of* G. *Let* $\lambda_n = \operatorname{Sup}_{z\in G} \log |F_n(z)|$. *Suppose that* $\lambda_n \to \infty$ *and that there is a positive number* δ *such that* $|F_n(z_0)| \geqq \delta$, $n \geqq 1$.

Set $\log^+ x = \max(\log x, 0)$.

Suppose that there is a region G_1 *containing* z_0 *such that*

$$\lim_{n\to\infty} \lambda_n^{-1} \log^+ |F_n(z)| = 0, \qquad z \in G_1. \tag{4.1}$$

Let E *be the largest such region.*

Let v *be a boundary point of* E *that belongs to* G. *Then, to each neighborhood* V *of* v *there corresponds a subsequence* $\{F_{n_k}\}$ *and a positive number* K *such that the number of zeros of* F_{n_k} *in* V *is not less than* $K\lambda_{n_k}$.

4.2. *The set* $\mathfrak{L}_f$. Recalling from §1.3 the definition of $\mathfrak{R}_f$, let us note that $\mathfrak{R}_f$ is simply the complement of $\bigcup_{k=0}^{q-1} S_k$, where S_k is the sector defined in (3.11).

THEOREM 4. *Let* $f \in \mathfrak{F} \cup \mathfrak{G}$. *Let* $v \in \mathfrak{R}_f$ *and let* V *be a neighborhood of* v. *There exist numbers* N *and* K *such that the number of zeros of* $f^{(n)}$ *in* V *is not less than* $Kn^{1-1/q}$ *if* $n > N$.

Proof. We may suppose that v lies on the ray between S_k and S_{k+1}. Let G be the disk with center at v and unit radius. (We may suppose V to be a smaller, concentric disk.) Fix z_0 in $G \cap S_k$. Using the notation of Theorem 2, set

$$F_n(z) = \frac{f^{(n)}(z)[\psi_{nk}(z)]^n[2\pi qn]^{1/2}}{n!f(z+\psi_{nk}(z))}.$$

The functions F_n are certainly holomorphic in a region containing the closure of G for all n sufficiently large, and the zeros of F_n in V are exactly those of $f^{(n)}$.

Out of (3.8) and (3.10) we shall establish that

$$F_n(z) = 1 + o(1), \qquad z \in S_k \cap G,$$

$$\log |F_n(z)| \geqq Mn^{1-1/q}, \qquad z \in S_{k+1} \cap G.$$

In fact, the first of these is quite clear from the definition of F_n and (3.10). To prove the second, one must treat two expressions: $|\psi_{nk}(z)/\psi_{n,k+1}(z)|$ and $|f(z+\psi_{n,k+1}(z))/f(z+\psi_{nk}(z))|$. In each instance it is convenient to distinguish the cases $f \in \mathfrak{F}$ and $f \in \mathfrak{G}$.

Set $\omega = e^{-2\pi i/q}$ and $\Psi = |\psi_{nk}(z)/\psi_{n,k+1}(z)|$.

If $f \in \mathfrak{F}$,

$$\Psi = \left| \frac{t_{nk} - m + o(1)}{t_{nk} - m\omega + o(1)} \right|,$$

where $m = (qz + d/c)(q-1)/q^2$ and $t_{nk} = \beta_k(c)(n/q)^{1/q}$. Consequently $\log \Psi = \operatorname{Re}\,[m(\omega-1)/t_{nk}] + o(n^{-1/q})$. With z in S_{k+1}, $\arg m = \arg t_{nk} + 2\pi/q + \tau$, $|\tau| < \pi/q$. It follows that $\log \Psi \geqq Kn^{-1/q}$.

Let $f \in \mathfrak{G}$ and suppose, without loss of generality, that $c > 0$. Then from parts C and D of Lemma 6

$$\Psi = \left| \frac{t_{nk} - m + o(1)}{t_{nk} + m^* + o(1)} \right|$$

and $\log \Psi = \operatorname{Re}\,[-(m + m^*)/t_{nk}] + o(n^{-1/2})$. The conclusion $\log \Psi \geqq Kn^{-1/2}$ may be drawn from the fact that $\operatorname{Re} m \operatorname{Re} t_{nk} < 0$.

Now set $\Phi = |f(z + \psi_{n,k+1}(z))/f(z + \psi_{nk}(z))|$. Using (1.3) and (3.8) it is not difficult to show that

$$\log \Phi = \operatorname{Re} \left\{ \frac{nm}{q-1} \left[\frac{1}{t_{n,k+1}} - \frac{1}{t_{nk}} \right] \right\} + o(n^{1-1/q})$$

if $f \in \mathfrak{F}$. Calling on Lemma 4, one shows as readily that

$$\log \Phi = -4c \operatorname{Re} m \operatorname{Re} t_{nk} + o(n^{1/2})$$

if $f \in \mathfrak{G}$. From each of these it follows that $\log \Phi \geqq Kn^{1-1/q}$.

Finally, $\log |F_n(z)| = n \log \Psi + \log \Phi$ if $z \in S_{k+1} \cap G$; consequently $\log |F_n(z)| \geqq Kn^{1-1/q}$.

The inequality $K_1 n^{1-1/q} \leqq \lambda_n \leqq K_2 n^{1-1/q}$ is a consequence of the estimate just obtained and Lemma 7.

The point v is a boundary point of the maximal region E for the sequence $\{F_n\}$ and for each of its subsequences as well. The assertion of Theorem 4 follows now from Theorem 3.

THEOREM 5. *If $f \in \mathfrak{F} \cup \mathfrak{G}$, then $\mathfrak{L}_f = \mathfrak{R}_f$.*

Proof. Theorem 4 implies $\mathfrak{R}_f \subseteq \mathfrak{L}_f$. The reverse inclusion follows from the fact that (3.10) holds uniformly in compact subsets of S_k.

5. **Some subclasses of $\mathfrak{F}$ and $\mathfrak{G}$.** 5.1. *A subclass of $\mathfrak{F}$.* Let

$$f(z) = z^m P(z) e^{Q(z)} \tag{5.1}$$

where $Q(z) = \sum_{k=0}^{q} b_k z^{q-k}$ and P is a canonical product of genus p, $0 \leqq p \leqq q-2$. In order that $f \in \mathfrak{F}$, with $c = b_0$ and $d = b_1$, it suffices that f have at most a finite set of zeros in some set $E(b_0, \rho)$. The lemma of [9, p. 53] serves well as the basis of a proof.

5.2. *A subclass of* $\mathcal{G}$. Let f have the form (5.1) with Q of degree 2 and P of genus 1. Suppose that $b_0>0$, that f has but a finite number of zeros in some $E(1, \rho)$, and that $\sum_{k=1}^{\infty} |\mathrm{Re}\ (1/a_k)| < \infty$, where $\{a_k\}$ is the sequence of zeros of f. Then $f \in \mathcal{G}$ with $c=b_0$ and $d=b_1+\sum_{k=1}^{\infty} \mathrm{Re}\ (1/a_k)$. The lemma mentioned above again serves well in the proof.

6. **A class outside** $\mathfrak{F} \cup \mathcal{G}$. In the description of the subclass of $\mathfrak{F}$ given above, let the requirement that in some $E(b_0, \rho)$ f has a finite set of zeros be replaced by the requirement that f has in each disk in $E(b_0, \rho)$ at most one zero, counted according to multiplicity. (A finite set of zeros may be excepted.) It is possible to find $\mathfrak{L}_f$ in this instance by a modification of the methods used for $\mathfrak{F}$ and $\mathcal{G}$. In fact, $\mathfrak{R}_f \subseteq \mathfrak{L}_f$, but the inclusion may be proper. $\mathfrak{L}_f$ contains the half-line $S_k \cap \{z: \mathrm{Im}\ [z/\alpha_k(b_0)] = \eta\}$ if and only if for every $\delta>0$ the half-strip

$$S_k \cap \{z: \eta - \delta \leqq \mathrm{Im}\ [(qb_0z - (q-1)b_1)(q^2b_0\alpha_k(b_0))^{-1}] \leqq \eta + \delta\}$$

contains an infinite set of zeros of f. These additional points in $\mathfrak{L}_f$ arise from the presence of zeros of f near the path on which

$$\max_{u \in D(z,r)} \left| \exp\left[b_0u^q + b_1u^{q-1}\right] \right|, \qquad z \in S_k,$$

occurs.

It seems reasonable to conjecture that for each function f of the form (5.1), without restriction on the location of the zeros but with $0 \leqq p \leqq q-2$, the set $\mathfrak{L}_f$ will have the description just given.

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