HYPERBOLIC EQUATIONS WITH MULTIPLE CHARACTERISTICS(1)

BY

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1. **Introduction.** The definition of a linear hyperbolic partial differential equation with constant coefficients was given by Gårding [2]. In the case of "distinct characteristics" (referred to here as the strictly hyperbolic case) the extension of the definition to variable coefficients is immediate and the Cauchy problem has been discussed for linear strictly hyperbolic operators with variable coefficients by Gårding and others (see [3] for further references). It is harder to define a linear hyperbolic equation with variable coefficients if multiple characteristics are allowed. A. Lax [4] has given such a definition for two independent variables and has solved the Cauchy problem using an iteration scheme. This paper deals with the equations of A. Lax and uses the methods of Gårding and Leray to solve the Cauchy problem for a wide variety of Cauchy data including distributions and with a minimum of differentiability requirements on the coefficients.

Let the independent variables be $x=(x^1, x^2)$ and let D_i denote differentiation with respect to x^i. We consider operators of the form

$$\Lambda = D_1 + \lambda(x) D_2. \tag{1}$$

If $\Lambda' = D_1 + \lambda'(x) D_2$ we assume that either $\lambda(x) \equiv \lambda'(x)$ or $\lambda(x) \neq \lambda'(x)$ for all x. Let

$$a = \Lambda_0 \cdots \Lambda_m \tag{2}$$

(the factors Λ_i are not necessarily distinct). Associated with a are two left modules $\mathfrak{S}(a)$ and $\overline{\mathfrak{S}}(a)$ over a ring of functions. $\mathfrak{S}(a)$ consists of all linear combinations of differential operators obtained from a by deleting one or more factors from the product a and writing the remaining factors in the same order. $\overline{\mathfrak{S}}(a)$ is spanned by $\mathfrak{S}(a)$ and a itself. The ring of functions may be all functions or all functions satisfying certain differentiability conditions. The operators of A. Lax are operators of the form $b=a+M$, $M \in \mathfrak{S}(a)$. a will be called the principal part of b, $a=Pb$, even though when (2) is multiplied out a will contain more than the highest order terms.

In addition certain other operators (first introduced in [3]), the partial adjoints, are treated. Let M and N be products of operators of the form (1). We shall consider the tensor product $\overline{\mathfrak{S}}(M) \otimes \overline{\mathfrak{S}}(N)$ but if $Q \otimes R$ lies in this

Received by the editors February 25, 1957.

(1) The results of this paper were submitted in a thesis to the University of Chicago in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

product we shall use the notation $QR^* = Q \otimes R$. Suppose that the factors of M and N, taken together, are precisely the factors of a. Then a partial adjoint E of a (or b) is an element of $\overline{\mathfrak{S}}(M) \otimes \overline{\mathfrak{S}}(N)$ one of whose terms is MN^*. MN^* is called the principal part of E, $MN^* = PE$. $E = A$ is a partial adjoint of the first kind if $A - PA$ lies in $\mathfrak{S}(M) \otimes \overline{\mathfrak{S}}(N)$ and if degree $M > 0$, and $E = B$ is a partial adjoint of the second kind if $B - PB$ is in $\overline{\mathfrak{S}}(M) \otimes \mathfrak{S}(N)$ and if degree $N > 0$. For example $b = b \otimes 1$ is an operator of the first kind.

We wish to acknowledge our gratitude to Professor Lars Gårding who suggested this problem and whose encouragement and advice made this paper possible.

2. **The Friedrichs-Lewy inequality.** Some preliminary notations and properties of the module $\mathfrak{S}(a)$ are introduced. Then a number of norms and Banach spaces are defined which include the ones used by Gårding in [3]. Finally an inequality of the Friedrichs-Lewy type is proved.

Let $\alpha = (\alpha_1, \alpha_2)$ be a pair of non-negative integers and write $D_\alpha = D_1^{\alpha_1} D_2^{\alpha_2}$. Let $|\alpha| = \alpha_1 + \alpha_2$ and let $[\alpha] \leqq p, q$ mean $|\alpha| \leqq p + q$ and $\alpha_1 \leqq p$.

We shall work in a strip $V = V_t : 0 \leqq x^1 \leqq t \leqq 1$. Lip$(p, q)$ will denote the class of all functions f bounded in V and such that $D_\alpha f$ exists a.e. and is bounded in V whenever $[\alpha] \leqq p, q$. A differential operator $b \in$ Lip(p, q) if all the coefficients of b are in this class. If a is given by (1.2) and $\Lambda_i \in$ Lip$(p+i, q)$, a will be said to be of class (p, q). (This depends on the order of the product.) If $\Lambda_i \in$ Lip$(p+m, q)$, a will be said to be of class $\{p, q\}$. In either event the modules $\mathfrak{S}(a)$ and $\overline{\mathfrak{S}}(a)$ will be taken over the ring of functions in Lip(p, q) and b and the various partial adjoints will be said to be of the appropriate class. The number (p, q, b) is then defined to be the ess sup of all the various derivatives that are asserted to be a.e. bounded by asserting that b is of class (p, q). A similar definition is given for $\{p, q, b\}$. A partial adjoint E, $PE = MN^*$, is of class $\{p, q\}$ if M and N are of class $\{p, q\}$ and if the modules $\overline{\mathfrak{S}}(M)$, $\overline{\mathfrak{S}}(N)$ are taken over Lip(p, q). The number $\{p, q, E\}$ is defined similarly.

Let a' be a rearrangement of the product a and suppose that a and a' are both of class (p, q) when considered in their respective arrangements of their factors. It is proved in [4] that

$$a' - a \in \mathfrak{S}(a). \tag{1}$$

This shows that the hyperbolic operator $b = a + M$ remains hyperbolic when the factors of the principal part are rearranged.

Let S_τ denote the line $x^1 = \tau$, and let c^* denote the complex conjugate of a complex number c. By (f, g, S_τ) and (f, g) are meant respectively the integrals $\int f(x)g(x)^* dx^2$ and $\int f(x)g(x)^* dx$ taken over the line S_τ in the first case and the strip V in the second. If $E = MN^*$, (Ef, g) means (Mf, Ng) and (Ef, g) is defined for any partial adjoint by linearity. (Ef, g, S_τ) is defined similarly. $\mathfrak{C}$ will denote the class of indefinitely differentiable functions with compact support and $\mathfrak{C}_\tau$ will refer to those functions in $\mathfrak{C}$ which vanish in a neighborhood of the line S_τ.

Following Gårding [3, p. 53] a variety of norms is defined on $\mathfrak{C}$. First let $|D^{p,q}f, S_\tau|^2=\sum(D_\alpha f, D_\alpha f, S_\tau)$, the sum being over all $[\alpha]\leqq p, q$. The closure of $\mathfrak{C}$ under $|D^{p,q}f, S_\tau|$ is called $H^{p,q}(S_\tau)$. Now let

$$|D^{p,q}f|_1 = \int_0^t |D^{p,q}f, S_\tau|\,d\tau,$$

$$|D^{p,q}f|_2 = |D^{p,q}f| = \left(\int_0^t |D^{p,q}f, S_\tau|^2 d\tau\right)^{1/2},$$

$$|D^{p,q}f|_\infty = \operatorname{ess\,sup} |D^{p,q}f, S_\tau|, \qquad 0\leqq \tau \leqq t.$$

The closures of $\mathfrak{C}$ under these norms are respectively called $L^{p,q}$, $H^{p,q}$, and $C^{p,q}$. The functions $f\in L^{p,q}$ with $|D^{p,q}f|_\infty<\infty$ form a Banach space under this norm, called $B^{p,q}$. The inclusions $C^{p,q}\subset B^{p,q}\subset L^{p,q}$ are evident. If in the above construction $\mathfrak{C}_\tau$ is used instead of $\mathfrak{C}$, the resultant spaces are denoted with a subscript τ.

Various properties of these spaces are given in [3]. In particular it is immediate that if $f\in L^{p,q}$, $[\alpha]\leqq p, q$, a meaning may be given to $D_\alpha f$ (it is a derivative in the sense of L. Schwartz) and $D_\alpha f\in L^{0,0}$. Similar considerations hold for the other spaces. In the case of $H^{p,q}(S_\tau)$ however, the functions $D_\alpha f$ are not all determined by one another and in fact a function $f\in H^{p,q}(S_\tau)$ if and only if there is a collection of $p+1$ functions f_i, $0\leqq i\leqq p$, $f_i\in H^{0,p+q-i}(S_\tau)$, such that $D_1^i f=f_i$. The functions f_i are entirely independent of one another and may also be characterized by the fact that f_i is square integrable on S_τ and for each j, $1\leqq j\leqq p+q-i$, the Schwartz derivative $D_2^j f_i$ is a function square integrable on S_τ.

In the case of multiple characteristics it is necessary to define more norms. Roughly speaking the idea is to include in the norms of f not only all derivatives of f up to a certain order but also certain directional derivatives of f taken in the characteristic directions of a. More precisely we define $|D^{a+p,q}f, S_\tau|^2$ and $|\overline{D}^{a+p,q}f, S_\tau|^2$ by the sum $\sum|D^{p,q}Mf, S_\tau|^2$, the sum being taken over all monomials M in $\mathfrak{S}(a)$ in the first case and $\overline{\mathfrak{S}}(a)$ in the second. The 1, 2 and ∞ norms are defined, for $\mathfrak{S}(a)$ and $\overline{\mathfrak{S}}(a)$, the same as above. The spaces obtained are denoted by $H^{a+p,q}(S_\tau)$, $\overline{H}^{a+p,q}(S_\tau)$, etc. If a is strictly hyperbolic these norms for the module $\mathfrak{S}(a)$ are equivalent to the usual norms with indices $m+p$, q. If a is not strictly hyperbolic but has some distinct factors more complicated equivalent norms are given below. If $b=a+M$ the norms and spaces with respect to b are defined to be those with respect to a.

If $f\in\overline{L}^{a+p,q}$ and $M\in\overline{\mathfrak{S}}(a)$, a meaning may be given to Mf, and $Mf\in L^{p,q}$. Similar considerations hold for the other spaces and for the module $\mathfrak{S}(a)$ and it will be shown below (Lemma 3.1) that the collection of functions Mf, taken with certain equations relating them to one another, serve to characterize an $f\in\bar{L}^{a+p,q}$. As in the case $a=1$, when $f\in\bar{H}^{a+p,q}(S_\tau)$, some of the Mf may be independent of one another. A collection (f_M), $f_M\in H^{p,q}(S)$, $M\in\bar{\mathfrak{S}}(a)$ will be called *consistent* if the map $M\to f_M$ is a linear map where $\bar{\mathfrak{S}}(a)$ and $H^{p,q}(S_\tau)$ are thought of as modules over the ring of functions in Lip(p,q). If $f\in\bar{H}^{a+p,q}(S_\tau)$, $(f_M)=(Mf)$ is a consistent collection and Lemma 3.1 gives the converse of this statement.

We will frequently use a simple inequality [3, p. 18] which says that if $\phi(t)$, $\psi(t)$ are non-negative nondecreasing functions and if $\psi(t)\leqq c\int_0^t\psi(\tau)d\tau+\phi(t)$, $c\geqq 0$, then

$$\psi(t)\leqq e^{ct}\phi(t). \tag{2}$$

THEOREM 2.1 (FRIEDRICHS-LEWY INEQUALITY). *If b is of class (p, q) and $f\in\mathfrak{C}$,*

$$\left| D^{b+p,q}f\right|_\infty \leqq c\left| D^{b+p,q}f, S_0\right| + c\left| D^{p,q}bf\right|_1 \tag{3}$$

where $c>0$ depends only on (p, q, b).

Proof. It suffices to prove (3) for $b=a$, for, assuming (3) for a one obtains as a bound for the left side of (3),

$$c\left| D^{b+p,q}f, S_0\right| + c\left| D^{p,q}bf\right|_1 + c\left| D^{p,q}Mf\right|_1$$
$$\leqq c\left| D^{b+p,q}f, S_0\right| + c\left| D^{p,q}bf\right|_1 + c\left| D^{b+p,q}f\right|_1.$$

(3) then follows from (2).

If $a=\Lambda$ and $f\in\mathfrak{C}$, $f(x)=f(0,\phi(0))+\int_0^{x^1}\Lambda f(\tau,\phi(\tau))d\tau$ where $\phi(\tau)$ is a solution of the differential equation $\phi'=\lambda(\tau,\phi)$, $\phi(x^1)=x^2$. Squaring this and integrating, one obtains after some manipulation $|f, S_{x^1}|\leqq c|D^0f, S_0|+c\int_0^{x^1}|\Lambda f, S_\tau|d\tau$ which yields (3) for Λ with $p=q=0$. Applying this to $D_\alpha f$, $[\alpha]\leqq p$, q and adding, (3) is obtained for $a=\Lambda$.

(3) is proved for any a by induction on the degree of a. If $a=\Lambda a'$, we have $|D^{a'+p,q}f|_\infty\leqq c|D^{a'+p,q}f, S_0|+c|D^{p,q}a'f|_1\leqq c|D^{a'+p,q}f, S_0|+c|D^{p,q}a'f|_\infty\leqq c|D^{a+p,q}f, S_0|+c|D^{p,q}af|_\infty$. Adding to this the inequality (3) for Λ and using (2), the inequality (3) for a is obtained, proving the theorem.

The result may be written in several different but equivalent ways. First we observe that if $p>0$,

$$\left| D^{b+p,q}f, S_0\right| \backsim \left| D^{p-1,q}bf, S_0\right| + \left| D^{b,p+q}f, S_0\right|. \tag{4}$$

It is obvious that the right side of (4) is majorized by a constant times the left side. Conversely since D_1 and all the Λ_i differ by derivatives in the x^2 direction any $M\in\mathfrak{S}(a)$ may be written as a sum of operators of the form $D_\alpha b$, $[\alpha]\leqq p-1$, q, and D_2^jN, $j\leqq p+q$, $N\in\mathfrak{S}(a)$. (4) follows. Hence if $p>0$, (3) may be written

$$\left| D^{b+p,q}f\right|_\infty \leqq c\left| D^{p-1,q}bf, S_0\right| + c\left| D^{b,p+q}f, S_0\right| + c\left| D^{p,q}bf\right|_1. \tag{5}$$

(5) has the advantage that the S_0 terms only involve the initial data and derivatives of the initial data of a Cauchy problem.

In (3) and (5) either the left side or the last term on the right side or both may be replaced by $|\overline{D}^{b+p,q}f|_1$.

The inequality in the above theorem is not as sharp as the Friedrichs-Lewy inequality given by Leray [3, p. 50]. This is because the possibility of some distinct factors in (1.2) has not been used. (3) may be sharpened by giving another, but equivalent, definition of the norms with respect to the modules $\mathfrak{S}(a)$ and $\overline{\mathfrak{S}}(a)$. This equivalent form will not be proved as it is not difficult. The principal fact used in the proof is that the mth degree divisors of a polynomial of degree $m+1$ with distinct roots form a basis for the vector space of all polynomials of degree m. This fact must be used in one form or another in establishing the Leray form of the Friedrichs-Lewy inequality.

Suppose there are s distinct factors in the product (1.2) and order them in some fashion. With each monomial $M\in\mathfrak{S}(a)$ associate a sequence of non-negative integers $(k)=(k_1, \cdots, k_s)$ by letting k_i be the number of times the ith factor must be deleted from a to obtain M. Not all of the k_i can be zero since $\sum k_i=\text{degree } a-\text{degree } M>0$. For each such sequence let $\sigma(k)=\sigma=\sum k_i-\max k_i$. Then if a is of class $\{p, q\}$

$$|D^{a+p,q}f, S_\tau| \backsim \sum |D^{\sigma+p,q}Mf, S_\tau| \tag{6}$$

the sum being taken over all $M\in\mathfrak{S}(a)$. Similar expressions can be given for the other norms.

A consequence of (6) is that if $M\in\mathfrak{S}(a)$ is a monomial obtained from a by deleting one of each of the distinct factors in (1.2), then

$$|D^{a+p,q}f, S_\tau| \backsim |\overline{D}^{M+p+s-1,q}f, S_\tau|. \tag{7}$$

Thus every norm with respect to $\mathfrak{S}(a)$ is also a norm with respect to $\mathfrak{S}(M)$ for some $M\in\mathfrak{S}(a)$. The results obtained in the next section for spaces of the latter kind will apply immediately to spaces of the former kind.

Let $d(a)=\sup 1/|\lambda_i(x)-\lambda_j(x)|$, $x\in V$, where Λ_i, Λ_j range over all pairs of distinct factors of (1.2). The constants involved in the equivalences (6) and (7) depend only on $\{p, q, a\}+d(a)$.

3. Linear spaces. The object of this section is to establish for the spaces defined in the preceding section the various properties given in [3] for the strictly hyperbolic case. These properties are all needed in the solution of the Cauchy problem for distributions.

The first property is, for b of class (p, q), the inclusion

$$\overline{L}^{b+p,q} \subset C^{b+p,q} \tag{1}$$

which is an immediate consequence of the inequality

$$|D^{b+p,q}f|_\infty \leqq c|\overline{D}^{b+p,q}f|_1, \qquad f\in\mathfrak{C}. \tag{2}$$

One may obtain (2) from (2.3) applied to the strip between S_τ and S_t with the left side of (2.3) replaced by $|D^{b+p,q}f,\ S_{t_1}|$, $0\leqq\tau\leqq t_1\leqq t$, if, first, the integral of the resultant inequality is taken with respect to τ from 0 to t_1 and then the sup is taken with respect to t_1 from 0 to t.

The next result is a characterization in terms of weak derivatives of the elements of the function spaces introduced in §2. It amounts to a theorem of Friedrichs [1] that the weak and strong extensions of a hyperbolic operator are the same. The proof uses Theorem 5.3 but as will be seen this theorem depends only on the results of §4 and some results of this section in the case $a=1$. The results of §4 are self-contained.

LEMMA 3.1. *If a is of class $\{p,\ q\}$ a set of elements (f_M), $f_M\in H^{p,q}(S_0)$, $M\in\overline{\mathfrak{S}}(a)$, determines an $f\in\overline{H}^{a+p,q}(S_0)$ if and only if it is a consistent set. A function $g\in H^{a+p,q}(S_0)$ and a set of functions $f_M\in L^{p,q}$ (resp. $B^{p,q}$), $M\in\overline{\mathfrak{S}}(a)$, determine an $f\in\overline{L}^{a+p,q}$ (resp. $\overline{B}^{a+p,q}$) if and only if*

$$(3)\qquad (f_{\Lambda M},\ h)+(f_M,\ \Lambda h)=-(Mg,\ h,\ S_0)-((D_2\lambda)f_M,\ h)$$

for all $h\in\mathfrak{C}_t$ and all $\Lambda M\in\overline{\mathfrak{S}}(a)$.

Proof. The necessity of both conditions is immediate. The sufficiency of the first condition is proved by induction on $p+$degree a. Let $a_i=\Lambda_i\cdots\Lambda_m$. We may choose $m+p+2$ independent functions $f\in H^{0,p+q}(S_0)$, $a_if\in H^{0,p+q}(S_0)$, $0\leqq i\leqq m$, $D_1^kaf\in H^{0,p+q-k}(S_0)$, $1\leqq k\leqq p$. Pick a sequence $g_j\in\mathfrak{C}$ approximating f in the norm of $H^{0,p+q}(S_0)$. By induction and (2.7) the $m+p+1$ functions a_if, D_1^kaf specify a function in $H^{a+p,q}(S_0)$. So does the collection, for fixed j, a_ig_j, $D_1^kag_j$. A sequence of functions of the form $f_j=g_j+x^1h_j\in\mathfrak{C}$ will be found which tend to f in $\overline{H}^{a+p,q}(S_0)$. If the a_if_j and $D_1^kf_j$ are expanded on S_0 they are seen to involve a set of differential operators on the h_j which constitute a basic set of derivatives for a function in $H^{a+p,q}(S_0)$. Define $\bar{h}_j\in H^{a+p,q}(S_0)$ by setting the appropriate differential operator applied to $\bar{h}_j$ equal to $a_if-a_ig_j$ or $D_1^kaf-D_1^kag_j$, as the case may be. Pick $h_j\in\mathfrak{C}$ such that $|D^{a+p,q}h_j-\bar{h}_j,\ S_0|\leqq 1/j$. With this choice of h_j, the f_j converge to f.

For the second condition, from Theorem 5.3 there is an $\bar{f}\in\overline{L}^{a+p,q}$, $\bar{f}|S_0=g$, $a\bar{f}=f_a$. Applying (3) successively with $A=a_i$, $B=\Lambda_ia_{i+1}$, it is seen that $a_i\bar{f}=f_{a_i}$ until finally $\bar{f}=f$. If all the $f_M\in B^{p,q}$, the above gives $f\in\overline{L}^{a+p,q}$ with $|\overline{D}^{a+p,q}f|_\infty$ finite. This completes the proof.

In solving the Cauchy problem one is confronted with certain bounded sequences from which a limit must be extracted. A sequence $f_j\in\overline{B}^{a+p,q}$ is said to converge weakly to an $f\in\overline{B}^{a+p,q}$ if for each $M\in\mathfrak{S}(a)$, each $[\alpha]\leqq p,\ q$ and each $h\in L^0$, $(D_\alpha Mf_j,\ h)\rightarrow(D_\alpha Mf,\ h)$.

LEMMA 3.2. *Any bounded sequence $f_j\in\overline{B}^{a+p,q}$ contains a subsequence g_j converging weakly to an $f\in\overline{B}^{a+p,q}$ and*

$$(4)\qquad |\overline{D}^{a+p,q}f|_\infty\leqq\liminf|\overline{D}^{a+p,q}g_j|_\infty.$$

Proof. When $a=$constant the result is [3, Lemma 8.1]. Since for each monomial $M\in\mathfrak{S}(a)$, Mf_j is a bounded sequence in $B^{p,q}$ the usual diagonalization argument enables a subsequence g_j to be chosen such that Mg_j converges weakly to some $f_M\in B^{p,q}$. If $M\in\mathfrak{S}(a)$ and $\Lambda M\in\overline{\mathfrak{S}}(a)$, for each $h\in\mathfrak{C}_t$, $(Mg_j, h, S_0)=-(\Lambda Mg_j, h)-(Mg_j, \Lambda h)-((D_2\lambda)Mg_j, h)$ converges so $Mg_j\to g_M \in H^0(S_0)$. Applying this argument to $D_\alpha g_j$, $[\alpha]\leqq p$, q, we find $g_M\in H^{p,q}(S_0)$ so by Lemma 3.1 $g\in H^{a+p,q}(S_0)$ and $(Mg, h, S_0)=-(f_{\Lambda M}, h)-(f_M, \Lambda h) +((D_2\lambda)Mf, h)$. Again applying Lemma 3.1, $f\in\overline{B}^{a+p,q}$. (4) follows in the same way as in [3].

The spaces discussed so far may be thought of as consisting of functions with generalized derivatives in the sense of Sobolev. Following [3] certain dual spaces are now defined which will be spaces of distributions. We define

$$|\overline{D}^{-a-p,-q}f|_\infty = \sup |(f, h)| / |\overline{D}^{a+p,q}h|_1, \tag{5}$$

$$|\overline{D}^{-a-p,-q}f|_1 = \sup |(f, h)| / |D^{a+p,q}h|_\infty, \tag{6}$$

the suprema being taken over all $h\in\mathfrak{C}_t$. The corresponding norms with respect to $\mathfrak{S}(a)$ are defined in the same way, removing the bar from both sides of (5), (6). The closures of $\mathfrak{C}$ with respect to (5), (6) are respectively called $\overline{C}^{-a-p,-q}$ and $\overline{L}^{-a-p,-q}$.

The elements of these spaces are linear functionals on $\mathfrak{C}_t$. As an example, let $[\alpha]\leqq p$, q, $M\in\mathfrak{S}(a)$, $g\in L^0$. Then the linear functional

$$F(h) = (D_\alpha Mh, g), \qquad h\in\mathfrak{C}_t \tag{7}$$

is in $\overline{L}^{-a-p,-q}$. To see this it is necessary to use one of the mollifiers

$$J^{\pm}f(x) = \int f(y)j(\pm(x-y))dy \tag{8}$$

where $j(x)=\epsilon^{-2}j_0(x\epsilon^{-1})$, $j_0\in\mathfrak{C}_0\cap\mathfrak{C}_t$, $j_0\geqq 0$, $\int j_0(x)dx=1$. The properties of these smoothing operators are fairly well known [3, p. 65] and will be assumed here. Let $F_\epsilon(h)=(D_\alpha Mh, J^+g)$. Since $J^+g\in\mathfrak{C}_0$, $F_\epsilon(h)=(h, ND_\alpha J^+g)$ is in $\overline{L}^{-a-p,-q}$. Also $|F_\epsilon(h)-F(h)|\leqq c|\overline{D}^{a+p,q}h|_\infty|J^+g-g|_1$ so $F_\epsilon(h)\to F(h)$ as $\epsilon\to 0$. The same argument shows that the F_ϵ form a Cauchy sequence in $\overline{L}^{-a-p,-q}$.

Also $\overline{C}_t^{-a-p,-q}$ and $\overline{B}^{-a-p,-q}$ will denote respectively the dual spaces of $\overline{C}_t^{a+p,q}$ and $\overline{L}_t^{a+p,q}$ with respect to the duality (f, g). The norms in these spaces are respectively (6) and (5) where (f, h) is interpreted to mean the value of the linear functional f on h. We have the inclusions

$$\overline{C}^{-a-p,-q}\subset\overline{B}^{-a-p,-q}\subset\overline{L}^{-a-p,-q}\subset\overline{C}_t^{-a-p,-q}.$$

The next lemma will state that $\overline{B}_t^{a+p,q}$ is the dual of $\overline{L}^{-a-p,-q}$. The proof will use a certain formal identity involving the commutator $[A, B]=AB-BA$, of two operators A and B. Let $E_0, \cdots, E_m$ be operators. Let $\beta_i=(j_\mu)$ $1\leqq\mu\leqq i$, be an i-tuple of integers between 0 and m and arranged in increasing order, and let γ_{m+1-i} be the complementary set to β_i in $(0, \cdots, m)$, also arranged in increasing order. Let $L[\beta_i]=E_{j_1}\cdots E_{j_i}$ and, for any operator A, let $A(\beta_i)=[E_{j_1}, [\cdots [E_{j_i}, A]\cdots]]$. For example, $A(\beta_0)=A$, $L[\beta_0]=1$, $L[\beta_{m+1}]=L=E_0\cdots E_m$. The identity in question asserts that

$$[L, A] = \sum A(\beta_i)L[\gamma_{m+1-i}], \tag{9}$$

the sum being over all β_i with $1\leqq i\leqq m+1$. The proof can be given by induction and will be omitted.

LEMMA 3.3. *If a is of class $\{p, q\}$, $\overline{B}_t^{a+p,q}$ is the dual of $\overline{L}^{-a-p,-q}$.*

Proof. Let L be a continuous linear functional on $\overline{L}^{-a-p,-q}$. Then L is continuous on $L^{-p,-q}$ so by [3], Lemma 8.1, $L(f)=(f, g)$ for some $g\in B_t^{p,q}$. Similarly if $M\in\overline{\mathfrak{S}}(a)$, $L(Mf)$ is a continuous functional of $f\in L^{-p,-q}$ so $L(Mf)=(f, g_M)$ for some $g_M\in B_t^{p,q}$. It is seen from Lemma 3.1 with the strip V reversed that $g\in\overline{B}_t^{a+p,q}$.

Conversely if $g\in\overline{B}_t^{a+p,q}$ it must be shown that (f, g) gives rise to a continuous functional on $\overline{L}^{-a-p,-q}$. Setting $J=J^-$ this follows from

$$\left| D^{p,q}[a, J]g\right|_\infty \leqq c\left| D^{a+p,q}g\right|_\infty, \qquad g\in\overline{B}_t^{a+p,q}. \tag{10}$$

For (10) implies that $\left|\overline{D}^{a+p,q}Jg\right|_\infty\leqq c\left|\overline{D}^{a+p,q}g\right|_\infty$. Now $Jg\in\mathfrak{C}_t$ and, if $f\in\mathfrak{C}$, $(f, Jg)\rightarrow(f, g)$. Hence $\left|(f, g)\right|/\left|\overline{D}^{a+p,q}g\right|_\infty\leqq c\left|\overline{D}^{-a-p,-q}f\right|_1$. Since $\mathfrak{C}$ is dense in $\overline{L}^{-a-p,-q}$, (f, g) can be extended to a continuous functional on this space. It therefore remains to prove (10). This will be done in a number of steps.

First the effect of commuting J with D_1 or D_2 must be determined. It is evident that J commutes with D_2 and its powers. A computation yields $[D_1^p, J]g=0$ when g and its first $p-1$ derivatives in the x^1 direction vanish on S_t. This condition is satisfied when $g\in B_t^{p,q}$.

Define operators $J\{n\}$ and $J(n)$ inductively by $J\{0\}=J(0)=J$, $J\{n\}=u[v, J\{n-1\}]$, $J(n)=u[\Lambda, J(n-1)]$, where u, v are functions and Λ is an operator of the form (1.1).

One can show by induction the following facts. First, when applied to functions g which vanish on S_t, $[D_i, J\{n\}]$ is a sum of operators of the form $J\{n\}$. Again if $g=0$ on S_t, $J(n)$ is a sum of operators of the form $J\{i\}D_2^j$, $0\leqq j\leqq i$, $1\leqq i\leqq n$. Finally if $[\alpha]\leqq p, q$, $[D_\alpha, J\{n\}]$ when applied to a $g\in B_t^{p,q}$ is a sum of operators of the form $J\{i\}M$ where M is a differential operator of degree $\leqq p, q-1-n+i$. From these facts we will derive the inequality

$$\left| D^{p,q}J(n)g\right|_\infty \leqq c\left| D^{p,q}g\right|_\infty, \qquad g\in B_t^{p,q}. \tag{11}$$

It suffices to show that, if $[\alpha]\leqq p, q, j\leqq i\leqq n$, and M is an operator of degree $\leqq p,q-1-i+k$, the right side of (11) majorizes $\left|J\{i\}D_\alpha D_2^jg\right|_\infty$ and $\left|J\{k\}MD_2^jg\right|_\infty$. The second quantity is of the same type as the first and we shall show that $J\{i\}D_\alpha D_2^jg$ is bounded by a sum of terms of the form

$$\int \eta(y-x)\left| h(y)\right| dy \tag{12}$$

where h is a derivative of g of order $\leqq p$, q and η has bounded integral for all $\epsilon>0$. Observe that if $J\{n\}=[u_1, [\cdots [u_n, J]\cdots]]$, then

$$J\{n\}g=\int j(y-x)g(y)\prod [u_i(x)-u_i(y)]dy.$$

Hence after an integration by parts, $J\{i\}D_\alpha D_2^j g(x)$ is bounded by

$$\int \left| D_{2y}^i \left\{ j(y-x)\prod_1^i [u_r(x)-u_r(y)]\right\}\right| \left| D_\alpha g(y)\right| dy$$

where the subscript y indicates derivatives with respect to the y variable. Every derivative of $j(y-x)$ brings out a factor $1/\epsilon$ and since $j\leqq i$ each ϵ may be placed under one of the factors $u_r(x)-u_r(y)$. $j(y-x)$ vanishes outside a set $|y-x|\leqq c\epsilon$ so using a Lipschitz condition on the u_r the integrand remains bounded save for the ϵ^{-2} that occurs in the definition of $j(x)$. Hence the integral remains bounded so the integral is of the form (12). Squaring (12) and integrating over S_τ it is seen that (12) is bounded by $c|h|_\infty$, which completes the proof of (11).

The proof of (10) can now be given. By (9) the left side of (10) is majorized by $|D^{p,q}J(\beta_i)L[\gamma_{m+1-i}]g|_\infty$ and, since $i\geqq 1$, $L[\gamma_{m+1-i}]\in\mathfrak{S}(a)$. Hence using (11), (10) follows completing the proof of the lemma.

To obtain the full generality of results available in the strictly hyperbolic case it is necessary to allow p and q to have different signs. Following [3], let $\mathfrak{C}'$ denote the infinitely differentiable functions f in V such that $P(x)Q(D)f$ is bounded for any polynomials P, Q. ($\mathfrak{C}'$ is the space of "rapidly decreasing" functions of Schwartz.) $\mathfrak{C}'\supset\mathfrak{C}$ and $\mathfrak{C}'$ is dense in all the spaces introduced above in which $\mathfrak{C}$ is dense. Define $T^{0,-2q}=\sum(-1)^j D_2^{2j}$, $0\leqq j\leqq q$. Evidently $T^{0,-2q}\mathfrak{C}'\subset\mathfrak{C}'$ and it follows from a consideration of the Fourier transform that $T^{0,-2q}$ is a (1-1) map of $\mathfrak{C}'$ onto $\mathfrak{C}'$. Hence it has an inverse, which is denoted $T^{0,2q}$. We define $|\overline{D}^{a+p,-q}f, S_\tau|=|\overline{D}^{a+p,q}T^{0,2q}f, S_\tau|$, $f\in\mathfrak{C}'$, and $\overline{H}^{a+p,-q}(S_\tau)$ as the closure of $\mathfrak{C}'$ (or $\mathfrak{C}$) under this norm. The other norms are defined in terms of this in the same way as in the case $+q$. It is clear that the maps $T^{0,\pm 2q}$ extend to homeomorphisms between the spaces $\overline{H}^{a+p,\pm q}(S_\tau)$, $\overline{L}^{a+p,\pm q}$, etc.

More generally it is possible, using the methods of the preceding lemma, to show that $\sum|D^{p,\pm q+2r}T^{0,2r}Mf, S_\tau|^2$, M a monomial in $\overline{\mathfrak{S}}(a)$, is a norm equivalent to $|\overline{D}^{a+p,\pm q}f, S_\tau|^2$. Using this it is seen that the maps $T^{0,\pm 2r}$ extend to homeomorphisms between $\overline{H}^{a+p,\pm q}(S_\tau)$ and $\overline{H}^{a+p,\pm q+2r}(S_\tau)$ and similarly for the other spaces.

$\overline{H}^{a+p,\pm q}(S_\tau)$ are evidently Hilbert spaces. One can also show that they are dual to each other under an extension of the duality

$$\langle f, g, S_\tau\rangle=\sum(D_\alpha Mf, D_\alpha Mg, S_\tau), \qquad |\alpha|\leqq p, M\in\overline{\mathfrak{S}}(a).$$

The proofs of these facts are not included as they are not needed in the sequel.

The same considerations apply when the module $\mathfrak{S}(a)$ is used instead of $\overline{\mathfrak{S}}(a)$.

As a consequence the various versions of the Friedrichs-Lewy inequality given in §2 remain valid if q is replaced by $-q$. Also (3.1), (3.2) and Lemmas 3.2 and 3.3 remain true if q is replaced by $-q$. (3.7) may be modified to make $F\in\overline{L}^{-a-p,q}$ by requiring $|\alpha|\leqq p$ and $g\in L^{0,q}$.

4. The density theorem. This section is devoted to proving the density theorem and deriving some of its consequences. The density theorem, which asserts that $a\mathfrak{C}_t$ is dense in $L_t^{p,q}$ is the key result needed in solving the various Cauchy problems considered in §5. As a consequence of the density theorem there is obtained a variety of inequalities for partial adjoints which include as special cases the Friedrichs-Lewy inequality and Gårding's "basic inequality" for hyperbolic operators [3, p. 70].

This proof of the density theorem was suggested by Gårding.

THEOREM 4.1. *If a is of class (p, q) and $\Lambda_0\in\mathrm{Lip}(1)$ then $a\mathfrak{C}_t$ is dense in $L_t^{p,q}$.*

Proof. Note that if $p=0$ the subscript t must be dropped. The proof is by induction on m. Suppose $a=\Lambda$, and let $\phi(x)$ satisfy the differential equation $\partial\phi/\partial x^1=\lambda(x^1,\phi)$, $\phi(0,x^2)=x^2$. The map $F: x\rightarrow(x^1,\phi)$ is a (1-1) map of class $\mathrm{Lip}(p, q)$ of V onto itself (as may be seen by an easy modification of [5, p. 40]) and $F\Lambda=D_1F$. Let $f\in\mathfrak{C}_t$, and let $D_1v=Ff$; then $u=F^{-1}v$ is a solution of class $\mathrm{Lip}(p, q)$ of $\Lambda u=f$, $u=0$ on S_t. To show $\Lambda\mathfrak{C}_t$ dense in $L_t^{p,q}$ it suffices to show $\Lambda\mathfrak{C}_t$ dense in $H_t^{p,q}$ and to show this it suffices to show $\Lambda\mathfrak{C}_t$ dense in a dense subset of $H_t^{p,q}$, namely $\mathfrak{C}_t$. Let $f\in\mathfrak{C}_t$ and let $\Lambda u=f$ as above. Let $v=J^-u\in\mathfrak{C}_t$. We show that $\Lambda v\rightarrow f$ in $H_t^{p,q}$ by showing that $|D^{p,q}\Lambda v|_2$ is bounded and that $G(\Lambda v)\rightarrow G(f)$ where G is one of a dense set of linear functionals on $H_t^{p,q}$. $|D^{p,q}\Lambda v|_2\leqq|D^{p,q}u|_2+|D^{p,q}[\Lambda, J]u|_2$ is bounded by (3.10). Also if $h\in\mathfrak{C}_0\cap\mathfrak{C}_t$, $(\Lambda v, h)=(v,\Lambda'h)\rightarrow(u,\Lambda'h)=(f,h)$ where $\Lambda'h=-D_1h-D_2(\lambda h)$.

Now let a be as in (1.2) and let $b=\Lambda_1\cdots\Lambda_m$. Let $f\in L_t^{p,q}$ and by induction pick $g\in\mathfrak{C}_t$ such that $|D^{p,q}\Lambda_0g-f|_1$ is small and $v\in\mathfrak{C}_t$ such that $|D^{p+1,q}bv-g|_1$ is small. Then $|D^{p,q}av-f|_1\leqq|D^{p,q}av-\Lambda_0g|_1+|D^{p,q}\Lambda_0g-f|_1\leqq c|D^{p+1,q}bv-g|_1+|D^{p,q}\Lambda_0g-f|_1$ is small, completing the induction.

Because of the multiplicity of characteristics there are a larger variety of inequalities for partial adjoints than in the strictly hyperbolic case. The next theorem gives some of these inequalities. (Recall that degree $M>0$ (degree $N>0$) for a partial adjoint of the first (second) kind.)

THEOREM 4.2. *Let A and B be partial adjoints of the first and second kinds respectively with $PA=PB=MN^*$. Let A and B be of class $\{p, q\}$ and let N satisfy the hypotheses of Theorem 4.1. Then if $f\in\overline{L}^{M,\pm q}$*

$$(1)\qquad |\overline{D}^{M,\pm q}f|_1\leqq c|D^{M,\pm q}f, S_0|+c\sup|(Af,h)|/|\overline{D}^{N,\pm q}h|_\infty,$$

and if $f\in\overline{B}^{M,\pm q}$

$$(2)\qquad |\overline{D}^{M,\pm q}f|_\infty\leqq c|D^{M,\pm q}f, S_0|+c\sup|(Bf,h)|/|D^{N,\mp q}h|_\infty$$

where the suprema are taken over all $h \in \mathfrak{C}_t$ *and the* S_0 *term in* (2) *is to be dropped if degree* $M=0$.

Proof. We first prove (2) with the lower signs. By the density theorem and (2.3) with the strip V reversed, $|D^{0,-q}Mf|_\infty \leqq c \sup |(Mf, Nh)|/|D^{N,q}h|_\infty$. The left side of this inequality majorizes $c^{-1}|D^{M,-q}f|_\infty - |D^{M,-q}f, S_0|$. Since $|((PB-B)f, h)| \leqq c|\overline{D}^{M,-q}f|_1 |D^{N,q}h|_\infty$ an application of (2.2) gives (2). For the upper signs apply (2) to $T^{0,-2q}f$.

In the case $B=N^*$, $M=1$, (2) implies that if $f \in L^{0,\mp q}$ and $(f, Nh)=0$ for all $h \in \mathfrak{C}_t$, then $|D^{0,\mp q}f|_1 \leqq |D^{0,\mp q}f|_\infty = 0$ so $f=0$. Hence $N\mathfrak{C}_t$ is weakly dense in $B^{0,\pm q}$, the dual space of $L^{0,\mp q}$. (This is a weak analogue of Theorem 4.1.) With this weak density, (1) is proved in the same way as (2).

To give more inequalities an inequality is needed in the strictly hyperbolic case which is the same as but not quite a consequence of Gårding's inequalities for strictly hyperbolic partial adjoints [3, p. 83]. Namely, if $B=MN^*$ is a strictly hyperbolic partial adjoint (the factors of M and N are distinct) and if $|b|$ denotes the order of any differential operator b,

$$|D^{|M|,\pm q}f|_\infty \leqq c|D^{|M|-1,1\pm q}f, S_0| + c \sup |(Bf, h)|/|D^{|N|-1,\mp q}h|_\infty, \quad h \in \mathfrak{C}_t. \tag{3}$$

Let $N=\Lambda L$, let Λ_i be one of the $|M|$ distinct factors of M and let M_i be the product M with the factor Λ_i deleted. Applying (2) to the partial adjoint $\Lambda M_i \Lambda_i^* L^*$ and summing on i, (3) is obtained.

Write $MN^* = YQZ^*R^*$ where YZ^* is a strictly hyperbolic part of MN^*. By permuting the factors of M and N if necessary such a strictly hyperbolic part may in general be separated out in a variety of ways.

THEOREM 4.3. *Let* a, M, N *be of class* $\{p, q\}$. *Then if* $f \in \overline{L}^{|Y|+Q,\pm q}$,

$$|\overline{D}^{|Y|+Q,\pm q}f|_1 \leqq c|D^{Q+|Y|-1,\pm q}f, S_0| + c \sup |(Af, h)|/|\overline{D}^{|Z|+R,\mp q}h|_\infty, \quad h \in \mathfrak{C}_t. \tag{4}$$

If $f \in \overline{B}^{|Y|+Q,\pm q}$,

$$|\overline{D}^{|Y|+Q,\pm q}f|_\infty \leqq c|D^{Q+|Y|-1,1\pm q}f, S_0| + c \sup |(Bf, h)|/|D^{R+|Z|-1,\mp q}h|_\infty \quad h \in \mathfrak{C}_t. \tag{5}$$

Proof. By (3),

$$|D^{|Y|,\pm q}Qf|_\infty \leqq c|D^{|Y|-1,1\pm q}Qf, S_0| + c \sup |(YQf, Zh)|/|D^{|Z|-1,\mp q}h|_\infty$$

and by the density theorem the last term is majorized by the last term of (5) so (5) follows. (4) is proved in the same way.

5. **The Cauchy problem.** The equation $bu=f$ will be solved for a wide class of f and $u|S_0$ including the case when these are distributions. In order to make this meaningful we write $(bu, h)=(f, h)$ where h is any function in

$\mathfrak{C}_t$ and integrate the left side by parts, throwing the derivatives on h and obtaining the adjoint of b plus some integrals on S_0. It may not be necessary to throw all the derivatives on h and this is why the partial adjoints of b are considered. Corresponding to each of the inequalities in the preceding chapter is a Cauchy problem. First the Cauchy problems arising from Theorem 4.2 are given and then the Cauchy problems coming from Theorem 4.3 are stated (the proofs are the same). Finally the extreme cases when no integrations by parts are made and when all of them are made are treated.

THEOREM 5.1. *Let A be a partial adjoint of the first kind satisfying the hypotheses of Theorem* 4.2 *and let $PA = MN^*$. Then the equations*

$$Au = f \in \overline{L}^{-N,\pm q}, \tag{1}$$

$$u \mid S_0 = g \in H^{M,\pm q}(S_0) \tag{2}$$

have a unique solution $u \in \overline{L}^{M,\pm q}$ satisfying

$$\left| \overline{D}^{M,\pm q} u \right|_1 \leqq c \left| D^{M,\pm q} g, S_0 \right| + c \left| \overline{D}^{-N,\pm q} f \right|_1. \tag{3}$$

Proof. Equations (1), (2) are meaningful for any $u \in \overline{L}^{M,\pm q}$. Let $\mathfrak{A} = \overline{L}^{-N,\pm q} \oplus H^{M,\pm q}(S_0)$. By Lemma 3.3 and the fact that the latter summand is a Hilbert space, the dual of $\mathfrak{A}$ is $\mathfrak{A}' = \overline{B}_t^{N,\mp q} \oplus H^{M,\pm q}(S_0)$. Define a map G: $\mathfrak{C} \to \mathfrak{A}$ by $Gv = Av \oplus v \mid S_0$. ($Av \in \overline{L}^{-N,\pm q}$ by (3.7).) $G\mathfrak{C}$ is dense in $\mathfrak{A}$, for let $h \oplus g \in \mathfrak{A}'$ and suppose that $(Av, h) = \{v, g, S_0\} = 0$ for all $v \in \mathfrak{C}$ where $\{\ \}$ is the inner product on $H^{M,\pm q}(S_0)$. Picking $v \in \mathfrak{C}_0$ and applying (4.2) to $(Bh, v)^* = (Av, h)$ with the strip V reversed, $\left| \overline{D}^{N,\mp q} h \right|_\infty = 0$ so $h = 0$. Therefore $g = 0$ too so $G\mathfrak{C}$ is dense in $\mathfrak{A}$. Approximating the data $f \oplus g$ of (1), (2) by a sequence Gv_j and using (4.1), the v_j converge to an $u \in \overline{L}^{M,\pm q}$ which solves the Cauchy problem. (3) and the uniqueness follow from (4.1).

Observe that the parameter q determines the regularity of the solution. By choosing $-q$ small enough the Cauchy data f, g may be taken as distributions and the solution u will be a distribution.

THEOREM 5.2. *Let B be a partial adjoint of the second kind satisfying the hypotheses of Theorem* 4.2, *and let $PB = MN^*$. Then the equations*

$$Bu = f \in C_t^{-N,\pm q}, \tag{4}$$

$$u \mid S_0 = g \in H^{M,\pm q}(S_0) \tag{5}$$

have a unique solution $u \in \overline{B}^{M,\pm q}$ satisfying

$$\left| \overline{D}^{M,\pm q} u \right|_\infty \leqq c \left| D^{M,\pm q} g, S_0 \right| + c \left| D^{-N,\pm q} f \right|_1. \tag{6}$$

If $|M| = 0$, the S_0 term should be dropped.

Proof. Equations (4), (5) are meaningful for any $u \in \overline{B}^{M,\pm q}$ if (4) is interpreted to mean $(Bu, h) = (f, h)$, $h \in C_t^{N,\mp q}$. Let $\mathfrak{A} = C_t^{N,\mp q} \oplus H^{M,\pm q}(S_0)$ with dual space $\mathfrak{A}' = C_t^{-N,\pm q} \oplus H^{M,\pm q}(S_0)$. A map G: $\mathfrak{C} \to \mathfrak{A}'$ may be defined as follows.

For $h\in C_t^{N,\mp q}$, $v\in\mathfrak{C}$, write $(Bv, h)=(B_1h, v)^*+(B_2h, v, S_0)^*$ where the partial adjoint B_1 is obtained from B by integration by parts, throwing one factor of N onto M. Then $|(Bv, h)|\leqq c|D^{N,\mp q}h|_\infty$ so Bv is a linear functional on $C_t^{N,\mp q}$. Define $Gv=Bv\oplus v|S_0$. $G\mathfrak{C}$ is weakly dense in $\mathfrak{A}'$, for let $h\oplus g\in\mathfrak{A}$ and suppose $(Bv, h)+\{v, g, S_0\}=0$, for all $v\in\mathfrak{C}$. Then $(B_1h, v)=0$ for all $v\in\mathfrak{C}_0$. If $PB_1=M_1N_1^*$, two distinct factors Λ_1 and Λ_2 are picked from M_1, N_1, and Theorem 4.3 is applied with $YZ^*=\Lambda_1\Lambda_2^*$ and with the strip V reversed, one obtains $h=0$. Hence $g=0$ so $G\mathfrak{C}$ is weakly dense in $\mathfrak{A}'$. Hence if $f\oplus g$ are the data of (4), (5), there is a sequence $v_j\in\mathfrak{C}$ such that for any $h\in\mathfrak{C}_t$, $(Bv_j, h)\to(f, h)$ and $\{v_j, h, S_0\}\to\{g, h, S_0\}$ and

$$|D^{-M,\pm q}Bv_j|_1+|D^{M,\pm q}v_j, S_0|\leqq|D^{-N,\pm q}f|_1+|D^{M,\pm q}g, S_0|.$$

By (4.2), v_j is a bounded sequence in $\overline{B}^{M,\pm q}$ so by Lemma 3.2 there is a subsequence converging weakly to an $u\in\overline{B}^{M,\pm q}$ which solves the Cauchy problem. Uniqueness and (6) follow from (4.2).

It is easy to see that the functions in $C^{0,1}$ are all continuous, so $f=\delta\in C_t^{-0,-1}$ where δ is the Dirac distribution at any x_0 in V. Hence if, in the above theorem, the regularity parameter is $-q=-1$, the equation $Bu=\delta$ may be solved for $u\in\overline{B}^{M,-1}$. u is then a Green's "function" for the operator b.

The other inequalities of §4 also give rise to Cauchy problems but as the proofs are the same we merely state the results. Let A satisfy the hypotheses of Theorem 4.3, $PA=YQZ^*R^*$ with YZ^* strictly hyperbolic. Then $Au=f\in L^{-|Z|-R,\pm q}$, $u|S_0=g\in H^{Q+|Y|-1,\pm q}(S_0)$ has a unique solution $u\in\overline{L}^{|Y|+Q,\pm q}$. This is a stronger result than Theorem 5.1 but the same is not the case for the following result about partial adjoints of the second kind. Let B be a partial adjoint of the second kind satisfying the hypotheses of Theorem 4.3. Then $Bu=f\in C_t^{-|Z|+1-R,\pm q}$, $u|S_0=g\in H^{|Y|-1+Q,1\pm q}(S_0)$ has a unique solution $u\in\overline{B}^{|Y|+Q,\pm q}$. In both cases there are inequalities analogous to (3) and (6).

Finally two theorems are stated concerning the extreme cases for partial adjoints of the first and second kinds. The proofs are the same as for Theorems 5.1 and 5.2.

THEOREM 5.3. *If b is of class $\{p, q\}$ the equations*

$$bu=f\in L^{p,\pm q}, \tag{7}$$

$$u|S_0=g\in H^{a,p\pm q}(S_0) \tag{8}$$

have a unique solution $u\in\overline{L}^{p+a,\pm q}$ satisfying

$$|\overline{D}^{a+p,\pm q}u|_1\leqq c|D^{a,p\pm q}g, S_0|+c|D^{p-1,\pm q}f, S_0|+c|D^{p,\pm q}f|_1. \tag{9}$$

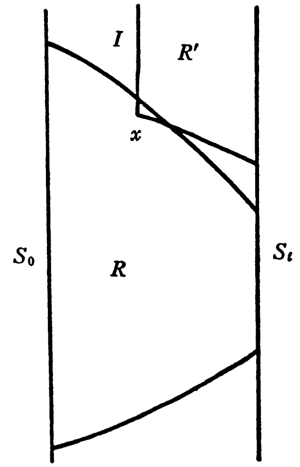
THEOREM 5.4. *If b is of class $\{p, q\}$ the equation*

$$b^*u=f\in C_t^{-b-p,\pm q} \tag{10}$$

has a unique solution $u\in\overline{B}^{-p,\pm q}$ satisfying

$$|D^{-p,\pm q}u|_{\infty} \leqq c|D^{-b-p,\pm q}f|_1. \tag{11}$$

The proof of Theorem 5.1 involves the results of §3 for spaces defined with respect to N. In Theorem 5.3, $N=1$ and the results of §3 are used only in the simplest case so Theorem 5.3 may be used in the proof of Lemma 3.1.



In solving a hyperbolic equation it is expected that the values of the solution on a bounded interval of the line S_t depend only on a bounded set of values of the data, f and g. This expectation is fulfilled with the Cauchy problem considered here. With any factor Λ one associates characteristic curves $x^2=\phi(x^1)$ where ϕ is a solution of the differential equation $d\phi/dx^1=\lambda(x^1, \phi)$. A characteristic curve of a hyperbolic operator b is a characteristic curve of one of the factors of its principal part. If I is a connected interval of S_t, a dependence domain $R\subset V$ of b is the set of segments lying in V of all the characteristic curves which pass through I. If E is any measurable set in V, an E placed in any of the norms with positive indices that have been defined will indicate that the integrations used in defining the norm are to extend only over E.

The proof of Theorem 2.1 actually proves the more precise inequality

$$|D^{a+p,q}f, R|_{\infty} \leqq c|D^{a+p,q}f, R\cap S_0| + c|D^{p,q}f, R|_1 \tag{12}$$

and similarly for the other versions of this inequality given in §2.

The density theorem may also be modified to include the dependence domain. If $E\subset V$, let $\mathfrak{C}(E)$ be those functions in $\mathfrak{C}$ whose supports are in E. Then $a\mathfrak{C}_t(R)$ is dense in $L_t^{p,q}(R)$. This is proved by induction in the same way as Theorem 4.1. To start the induction it is necessary to show that if R is the dependence domain of a and if Λ is a factor of a, $\Lambda\mathfrak{C}_t(R)$ is dense in $L_t^{p,q}(R)$. As in Theorem 4.1, let $f\in\mathfrak{C}_t(R)$ and let $\Lambda u=f$, $v=J^{-}u$, so $\Lambda v\rightarrow f$. If

$x \in R$, I, and R' are as in the figure, and if x is so close to the boundary of R that $f=0$ in R', then an application of (12) with the strip reversed gives $u=0$ in R'. This argument may also be used on the lower boundary of R and it follows that the support of u is contained in R and u vanishes in a neighborhood of the boundary of R. Hence for small ϵ, $v \in \mathfrak{C}_t(R)$ so $\Lambda\mathfrak{C}_t(R)$ is dense in a dense subspace of $L_t^{p,q}(R)$.

These remarks show that R is indeed the dependence domain for the various Cauchy problems considered. (12) shows that, in Theorem 5.3, if $f=g=0$ in R, then $u=0$ in R. When the regularity parameter q is negative or zero the proof of Theorem 4.2 uses only the density theorem and the Friedrichs-Lewy inequality and so may be carried through with the dependence domain. This shows that in Theorems 5.1 and 5.2, if the regularity parameter is nonpositive and if $f=g=0$ in R, the solution $u=0$ in R. Finally a solution for positive regularity parameter is equally a solution for zero regularity parameter so in all cases $f=g=0$ in R implies $u=0$ in R.

The proof, or validity, of Theorem 4.2 with positive q and a dependence domain R seems to be more difficult.

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