SOME REMARKS ON COMMUTATIVE ALGEBRAS OF OPERATORS ON BANACH SPACES(1)

BY

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Introduction. In this paper a series of propositions are given concerning commutative algebras of operators on a Banach space and more especially commutative algebras of scalar operators. A number of the results are known and due to W. G. Bade [1; 2] but different proofs are given here. The inspiration for this paper, both in the choice of the subject matter and of method, has largely been derived from [2; 7] and [10].

The material presented here is divided into four paragraphs. The first, which is introductory, contains various results on spectral families of measures. The principal propositions are contained in paragraphs 2 and 3. Theorem 1, proved in paragraph 2, is a generalisation of a theorem due to W. G. Bade [2], and almost all the other results of this same paragraph are more or less consequences of it. In paragraph 3 it is shown that, under certain conditions, an algebra of scalar operators can be identified, in a sense made precise below, with a von Neumann algebra. This fact makes it possible to reduce many results concerning algebras of scalar operators or σ-complete boolean algebras of projections, in Banach spaces, to corresponding results in Hilbert spaces. Various remarks on spectral families of measures are made in paragraph 4.

1. **Spectral families.** Let Z be a compact space, $C(Z)$ the set of complex-valued continuous functions defined on Z, $B_0(Z)$ the set of complex-valued Baire measurable functions defined on Z and $B(Z)$ the set of complex-valued Borel measurable functions defined on Z(2).

Let X be a Banach space, X' the dual of X and $\mathfrak{F}=(\mu_{x,x'})_{x\in X, x'\in X'}$ a family of Radon measures defined on Z. We shall say that $\mathfrak{F}$ is *semi-spectral* if:

(1) $(x, x')\rightarrow\mu_{x,x'}$ is a bilinear mapping;

(2) there is a constant $M(\mathfrak{F})\geqq 1$ which satisfies the inequalities $\|\mu_{x,x'}\| \leqq M(\mathfrak{F})\|x\| \|x'\|$ for every $x\in X$, $x'\in X'$.

We shall say that a function $f\in B(Z)$ is $\mathfrak{F}$-negligible if $|\mu_{x,x'}|^*(|f|)=0$ for every $x\in X$, $x'\in X'$. A Borel measurable set $A\subset Z$ is $\mathfrak{F}$-negligible if the characteristic function of A, ϕ_A, is $\mathfrak{F}$-negligible. For every function $f\in B(Z)$ we shall put $N_\infty(f, \mathfrak{F})=\inf E(f)$ where $E(f)$ is the set of numbers $\alpha>0$ for which

Presented to the Society, January 22, 1959; received by the editors October 13, 1958.

(1) This paper was sponsored by the Office of Scientific Research, United States Air Force, under contract SAR AF 49 (638)-153.

(2) For the definition of Baire sets and Borel sets see [12]. For other definitions and results of integration theory see [6].

the set $\{z \mid |f(z)| > \alpha\}$ is $\mathfrak{F}$-negligible. If f is continuous then $N_\infty(f, \mathfrak{F}) = \sup_{z \in S} |f(z)|$ where S is the closure of $\bigcup_{x \in X, x' \in X'} S(\mu_{x,x'})$; for a Radon measure μ on Z, $S(\mu)$ will denote its support.

We shall write $B_0^\infty(Z) = \{f \in B_0(Z) \mid N_\infty(f, \mathfrak{F}) < \infty\}$ and $B^\infty(Z) = \{f \in B(Z) \mid N_\infty(f, \mathfrak{F}) < \infty\}$; $B_0^\infty(Z)$ and $B^\infty(Z)$ are algebras and $f \to N_\infty(f, \mathfrak{F})$ is a semi-norm on $B_0^\infty(Z)$ and also on $B^\infty(Z)$. It is easy to see that $B_0^\infty(Z)$ and $B^\infty(Z)$ are complete for the semi-norm $f \to N_\infty(f, \mathfrak{F})$ and that, for $f, g, h \in B^\infty(Z)$, we have $N_\infty(f\bar{f}, \mathfrak{F}) = N_\infty(f, \mathfrak{F})^2$ and $N_\infty(gh, \mathfrak{F}) \leqq N_\infty(g, \mathfrak{F}) N_\infty(h, \mathfrak{F})$.

For each $f \in B^\infty(Z)$ we denote by $U_{\mathfrak{F},f}$ the operator in $\mathcal{L}(X, X'')$(3) which satisfies the equations $\langle U_{\mathfrak{F},f}x, x' \rangle = \int f d\mu_{x,x'}$ for all $x \in X$, $x' \in X'$; $\int$ means always $\int_Z$. When there will be no ambiguity we shall write only U_f instead of $U_{\mathfrak{F},f}$. In what follows we shall always suppose that all the semi-spectral families considered are such that $U_f \in \mathcal{L}(X, X)$ for $f \in C(Z)$ and $U_1 = I$. It is easy to see that if $\mathfrak{F}$ is a semi-spectral family, $f \in B^\infty(Z)$ and $U_f \in \mathcal{L}(X, X)$, then U_f belongs to the strong closure (= weak closure) of $\mathfrak{A}(\mathfrak{F}) = \{U_f \mid f \in C(Z)\}$. A family $\mathfrak{F} = (\mu_{x,x'})_{x \in X, x' \in X'}$ of Radon measures defined on Z is a *spectral family* if it is semi-spectral and if:

$$f \cdot \mu_{x,x'} = \mu_{U_f x, x'} \qquad \text{for all } f \in C(Z),\ x \in X,\ x' \in X'. \tag{3}$$

If $\mathfrak{F}$ is spectral then $f \to U_f$ is a continuous representation of the algebra $C(Z)$, endowed with the semi-norm $f \to N_\infty(f, \mathfrak{F})$ into $\mathcal{L}(X, X)$; it follows that $\mathfrak{A}(\mathfrak{F})$ is an algebra. Conversely if $\mathfrak{F}$ is semi-spectral and $f \to U_f$ is a representation of the algebra $C(Z)$ into $\mathcal{L}(X, X)$, then $\mathfrak{F}$ is spectral. Let us also remark that (3) implies $f \cdot \mu_{x,x'} = \mu_{x, {}^tU_f x'}$ for every $f \in C(Z)$, $x \in X$, $x' \in X'$; ${}^tU_f$ denotes the Banach adjoint of U_f.

Let A be a nonempty bounded subset of X and suppose that $\mathfrak{F}$ is spectral. For every $f \in B_0(Z)$ define $p(f, A) = \sup_{x \in A, \|x'\| \leq 1} |\mu_{x,x'}|^*(|f|)$; let $B_0(Z, A) = \{f \in B_0(Z) \mid p(f, A) < \infty\}$. If $T \in \mathcal{L}(X, X)$ and if $TU_f x = U_f T x$ for all $f \in C(Z)$ and $x \in A$ then it is easy to see that $p(f, T(A)) \leqq \|T\| p(f, A)$ for every $f \in B_0(Z)$ and hence that $B_0(Z, A) \subset B_0(Z, T(A))$. If $A_1 \subset A_2$ then $B_0(Z, A_1) \supset B_0(Z, A_2)$.

Proposition 1. (i) *$B_0(Z, A)$ is a linear space and $B_0(Z, A) \supset B_0^\infty(Z)$;* (ii) *$f \to p(f, A)$ is a semi-norm on $B_0(Z, A)$;* (iii) *$B_0(Z, A)$ is complete with respect to the semi-norm $f \to p(f, A)$.*

Assertions (i) and (ii) are obvious. To prove (iii) let $(f_n)_{1 \leq n < \infty}$ be a sequence of functions belonging to $B_0(Z, A)$ for which $\sum_{n=1}^\infty p(f_{n+1} - f_n, A) < \infty$. Then for every $x \in A$ and $\|x'\| \leqq 1$ we have $\sum_{n=1}^\infty |\mu_{x,x'}|^*(|f_{n+1} - f_n|) < \infty$. Hence $\lim_{n \to \infty} f_n(z)$ exists for $z \notin N_A$ where N_A is a Baire set which is $|\mu_{x,x'}|$-negligible for every $x \in A$ and $x' \in X'$. If we put $f(z) = 0$ for $z \in N_A$ and $f(z) = \lim_{n \to \infty} f_n(z)$ for $z \notin N_A$ then $f \in B_0(Z)$, and for $n \geqq 1$, $x \in A$ and $\|x'\| \leqq 1$ we

(3) If X, Y are Banach spaces we shall denote by $\mathcal{L}(X, Y)$ the space of linear continuous mappings of X into Y, endowed with the usual norm.

have $|\mu_{x,x'}|^*(|f-f_n|) \leqq \sum_{j=n}^{\infty} |\mu_{x,x'}|^*(|f_{j+1}-f_j|) \leqq \sum_{j=n}^{\infty} p(f_{j+1}-f_j, A)$ and hence $p(f-f_n, A) \leqq \sum_{j=n}^{\infty} p(f_{j+1}-f_j, A)$. This implies that $f \in B_0(Z, A)$ and that $\lim_{n\to\infty} p(f-f_n, A)=0$.

For each bounded nonempty subset $A \subset X$ we denote by $B_0^1(Z, A)$ the closure of $B_0^\infty(Z)$ in $B_0(Z, A)$ and by $L^1(Z, A)$ the closure of $C(Z)$ in $B_0(Z, A)$. It is obvious that $fg \in B_0^1(Z, A)$ if $f \in B_0^1(Z, A)$ and $g \in B_0^\infty(Z)$ and that $fg \in L^1(Z, A)$ if $f \in L^1(Z, A)$ and $g \in L^1(Z, A) \cap B_0^\infty(Z)$. Also it is easy to prove that if $f \in L^1(Z, A) \cap B_0^\infty(Z)$ then there is a sequence of functions $(f_n)_{1 \leq n < \infty}$ belonging to $C(Z)$ such that we have $\lim_{n\to\infty} p(f_n - f, A) = 0$ and $\|f_n\| = \sup_{z\in Z} |f_n(z)| \leqq N_\infty(f, \mathfrak{F})$ for $n=1, 2, \cdots$. If $T \in \mathfrak{L}(X, X)$ is such that $TU_fx = U_fTx$ for all $f \in C(Z)$ and $x \in A$ then we have $B_0^1(Z, A) \subset B_0^1(Z, T(A))$ and $L^1(Z, A) \subset L^1(Z, T(A))$. If $A_1 \subset A_2$ then $B_0^1(Z, A_1) \supset B_0^1(Z, A_2)$ and $L_0^1(Z, A_1) \supset L_0^1(Z, A_2)$. In connection with the spaces $B_0(Z, A)$, $B_0^1(Z, A)$, $L^1(Z, A)$ and Proposition 1 see also [15].

PROPOSITION 2. *For every bounded nonempty subset $A \subset X$ and $f \in B_0^\infty(Z)$*

$$(1/M(\mathfrak{F}))p(f, A) \leqq \sup_{x\in A} \|U_fx\| \leqq p(f, A). \tag{4}$$

The first inequality, $(1/M(\mathfrak{F}))p(f, A) \leqq \sup_{x\in A} \|U_fx\|$, follows from the relations

$$\begin{aligned} p(f, A) &= \sup_{x\in A, \|x'\|\leqq 1} \left(\sup_{\|g\|\leqq 1} \left| \int fg d\mu_{x,x'} \right| \right) \\ &\leqq M(\mathfrak{F}) \sup_{x\in A, \|x'\|\leqq 1} |\langle U_fx, x'\rangle| \leqq M(\mathfrak{F}) \sup_{x\in A} \|U_fx\|. \end{aligned}$$

The second inequality, $\sup_{x\in A} \|U_fx\| \leqq p(f, A)$, is obvious. Hence the proof is complete.

For every $f \in B_0(Z)$ let $D(f) = \{x \in X \mid f \in B_0^1(Z, \{x\})\}$; using the inequalities (4) (for $A = \{x\}$, $x \in X$) we can easily prove that there is a mapping U_f of $D(f)$ into X'' such that $\langle U_fx, x'\rangle = \int f d\mu_{x,x'}$ for every $x \in D(f)$ and $x' \in X'$. If $f \in L_0^1(Z, \{x\}) (\subset B_0^1(Z, \{x\}))$ then $U_fx \in X$. Again using the inequalities (4) and the definition of the spaces $B_0^1(Z, A)$ we see that:

PROPOSITION 3. *For every nonempty bounded subset $A \subset D(f)$ and $f \in B_0^1(Z, A)$*

$$(1/M(\mathfrak{F}))p(f, A) \leqq \sup_{x\in A} \|U_fx\| \leqq p(f, A). \tag{5}$$

For every nonempty bounded subset $A \subset X$ let $X(A)$ be the linear space of all bounded families $y = (y_x)_{x\in A}$ of elements belonging to X, endowed with the norm: $\|y\| = \sup_{x\in A} \|y_x\|$; $X(A)$ is a Banach space. If $\mathfrak{F} = (\mu_{x,x'})_{x\in X, x'\in X'}$ is a spectral family of measures we shall denote by $\mathfrak{A}(\mathfrak{F}, A)$ the closure of the subset $\{(U_fx)_{x\in A} \mid f \in C(Z)\} \subset X(A)$.

PROPOSITION 4. *For every nonempty bounded subset $A \subset X$ $\mathfrak{A}(\mathfrak{F}, A) = \{(U_fx)_{x\in A} \mid f \in L^1(Z, A)\}$.*

Let $f\in L^1(Z, A)$ and let $(f_n)_{1\leq n<\infty}$ be a sequence of functions belonging to $C(Z)$ such that $\lim_{n\to\infty} p(f-f_n, A)=0$. Then $\lim_{n\to\infty} (\sup_{x\in A} \|U_{f_n}x-U_fx\|)=0$ and hence $(U_fx)_{x\in A}\in \mathfrak{A}(\mathfrak{F}, A)$. Conversely let $y=(y_x)_{x\in A}\in\mathfrak{A}(\mathfrak{F}, A)$. Then there is a sequence $(f_n)_{1\leq n<\infty}$ of functions belonging to $C(Z)$ such that (in $X(A)$) $y=\lim_{n\to\infty} y^n$ where for each $n=1, 2, \cdots, y^n=(U_{f_n}x)_{x\in A}$. From (4) we deduce that $\lim_{n,m\to\infty} p(f_n-f_m, A)=0$ and hence that there is a function $f\in L^1(Z, A)$ for which $\lim_{n\to\infty} p(f_n-f, A)=0$. Using (5) we now deduce that $y=(U_fx)_{x\in A}$.

Let $\mathfrak{F}$ be a spectral family of measures defined on Z. If $U_g=U_h$, where $g, h\in B^\infty(Z)$, then using (3) we deduce that $g-h$ is $\mathfrak{F}$-negligible. This remark implies that we can introduce an involution in $\mathfrak{A}(\mathfrak{F})$ by writing $U_f^*=U_{\bar{f}}$. The inequalities (4) show that the mapping $T\to T^*$ of $\mathfrak{A}(\mathfrak{F})$ onto $\mathfrak{A}(\mathfrak{F})$, just defined, is strongly continuous and it can accordingly be extended to the strong closure, $s(\mathfrak{A}(\mathfrak{F}))$, of $\mathfrak{A}(\mathfrak{F})$; it is clear that the extension is an involution on $s(\mathfrak{A}(\mathfrak{F}))$.

In what follows we shall often be concerned with spectral families $\mathfrak{F}$ which have the property:

(E) $U_f\in\mathfrak{L}(X, X)$ for every $f\in B_0^\infty(Z)$.

Let $S_0(Z)$ be the class of all Baire sets $D\subset Z$. For every $D\in S_0(Z)$ define $E_{\mathfrak{F}}(D)=U_{\phi_D}$; then $E_{\mathfrak{F}}$ is a strongly countably additive (s.c.a.) spectral measure on $S_0(Z)$ such that $E_{\mathfrak{F}}(Z)=I$. Conversely, if E is a s.c.a. spectral measure on $S_0(Z)$ such that $E(Z)=I$ and if we put $\mu_{x,x'}(f)=\int f(z)d\langle E(z)x, x'\rangle$ for all $f\in C(Z)$, $x\in X$, $x'\in X'$, then $\mathfrak{F}=(\mu_{x,x'})_{x\in X,x'\in X'}$ is a spectral family having property (E) and $E_{\mathfrak{F}}=E$.

If $\mathfrak{F}$ is a spectral family having property (E) then it is easy to see that $U_f\in\mathfrak{L}(X, X)$ for every $f\in B^\infty(Z)$. This can be deduced, for instance, using the following result: (*) Let $\mathfrak{F}=(\mu_{x,x'})_{x\in X,x'\in X'}$ be a spectral family of measures on a compact space Z having property (E) and let $x\in X$; then there is a positive Radon measure ν_x on Z such that (for $N\in S_0(Z)$) $\nu_x(N)=0$ if and only if $|\mu_{x,x'}|(N)=0$ for all $x'\in X'$. Let us remark that every U_f, $f\in B^\infty(Z)$, is a scalar operator whose resolution of the identity is a s.c.a. spectral measure on $S_0(\sigma(U_f))$ [10, pp. 341–342, Lemma 6] and that $U_{gh}=U_gU_h$ for every $g, h\in B^\infty(Z)$.

For a spectral family $\mathfrak{F}$ having property (E) we have, for each $f\in B^\infty(Z)$, the following inequality which may be established by the method used in the proof of Proposition 2, [7, pp. 177–178]:

$$N_\infty(f, \mathfrak{F}) \leqq \|U_f\| \leqq M(\mathfrak{F})N_\infty(f, \mathfrak{F}). \tag{6}$$

Let $\mathfrak{F}$ be a spectral family on Z, having property (E), and let $\mathfrak{B}_0(\mathfrak{F})=\{U_f|f\in B_0^\infty(Z)\}$ and $\mathfrak{B}(\mathfrak{F})=\{U_f|f\in B^\infty(Z)\}$. Using (6) we see that $\mathfrak{B}_0(\mathfrak{F})$ and $\mathfrak{B}(\mathfrak{F})$ are uniformly closed and that each is algebraically and topologically isomorphic with an algebra $C(\hat{Z})$ where $\hat{Z}$ is a compact space. It is easy to see that $\mathfrak{B}_0(\mathfrak{F})$ is the smallest uniformly closed algebra containing $\{E_{\mathfrak{F}}(A)|A\in S_0(Z)\}$. Using the fact that, if $f\in C(Z)$, $N_\infty(f, \mathfrak{F})=\sup_{z\in S}|f(z)|$, where S is the closure of $\bigcup_{x\in X, x'\in X'} S(\mu_{x,x'})$, we see that $\mathfrak{A}(\mathfrak{F})$ is algebraically and topologically isomorphic with $C(S)$ (this last assertion remains true also for spectral families which do not satisfy condition (E)).

2. **Commutative algebras of operators.** Let $\mathfrak{F}=(\mu_{x,x'})_{x\in X, x'\in X'}$ be a spectral family of Radon measures defined on a compact space.

THEOREM 1. *Let A be a nonempty bounded subset of X and suppose that $T\in\mathfrak{L}(X, X)$ is an operator such that*: (i) *$TU_f x=U_f Tx$ for all $f\in C(Z)$ and $x\in A$*; (ii) *$(Tx)_{x\in A}\in\mathfrak{A}(\mathfrak{F}, A)$. Then there is a function $g\in L^1(Z, A)$ such that*: (j) $\|g\|\leqq M(\mathfrak{F})\|T\|$; (jj) *$U_g x=Tx$ for every x belonging to the closed linear space $\mathfrak{M}$ spanned by $\bigcup_{x\in A}\mathfrak{A}(\mathfrak{F}, \{x\})$.*

By Proposition 4 there is an $h\in L^1(Z, A)$ such that $U_h x=Tx$ for every $x\in A$. Let $B=\{z \mid |h(z)|\geqq(1+\lambda)M(\mathfrak{F})\|T\|\}$ where $\lambda>0$ (we can suppose $T\neq 0$). It is obvious that $h\phi_B\in B_0^1(Z, A)$. Choose a sequence $(h_n)_{1\leqq n<\infty}$ of functions belonging to $C(Z)$ such that $\lim_{n\to\infty} p(h_n-h, A)=0$. By (5) this gives $\lim_{n\to\infty} U_{h_n}x=Tx$ for each $x\in A$, and hence (since the condition (ii) implies $\mu_{Tx,x'}=\mu_{x,{}^tTx'}$ for $x\in A$ and $x'\in X'$)

$$\langle U_{\phi_B}x, {}^tTx'\rangle = \int \phi_B d\mu_{x,{}^tTx'} = \int \phi_B d\mu_{Tx,x'}$$
$$= \lim_{n\to\infty}\int \phi_B d\mu_{U_{h_n}x,x'} = \int h\phi_B d\mu_{x,x'} = \langle U_{h\phi_B}x, x'\rangle;$$

the above equations imply $\|U_{\phi_B}x\|\,\|T\|\geqq\|U_{h\phi_B}x\|$ for each $x\in A$. Using again (5) we deduce that $p(\phi_B, A)\|T\|\geqq\sup_{x\in A}\|U_{\phi_B}x\|\,\|T\|\geqq\sup_{x\in A}\|U_{h\phi_B}x\|$ $\geqq(1/M(\mathfrak{F}))p(h\phi_B, A)\geqq(1+\lambda)p(\phi_B, A)\|T\|$. It follows that $p(\phi_B, A)=0$ and hence that $|h(z)|\leqq M(\mathfrak{F})\|T\|$ except when $z\in N_A$, where N_A is a Baire set such that $|\mu_{x,x'}|(N_A)=0$ for every $x\in A$ and $x'\in X'$. If we define g as follows: $g(z)=h(z)$ if $z\notin N_A$ and $g(z)=0$ if $z\in N_A$, then $g\in L^1(Z, A)$, $\|g\|\leqq M(\mathfrak{F})\|T\|$ and obviously $U_g x=Tx$ for each $x\in A$ (in fact we have $p(g-h, A)=0$). Take a uniformly bounded sequence $(g_n)_{1\leqq n<\infty}$ of functions belonging to $C(Z)$ such that $\lim_{n\to\infty} p(g_n-g, A)=0$. For each $f\in C(Z)$ we have then: $g\in L^1(Z, U_f(A))$, $\lim_{n\to\infty} p(g_n-g, U_f(A))=0$, $gf\in L^1(Z, A)$ and $\lim_{n\to\infty} p(g_nf-gf, A)=0$. We deduce that, when $x\in A$, $U_gU_fx=\lim_{n\to\infty} U_{g_n}U_fx=\lim_{n\to\infty} U_fU_{g_n}x=U_fU_gx$ $=U_fTx=TU_fx$. Since $\mathfrak{M}$ is the closure of the set of all finite sums $\sum U_{f_i}x_i(f_i\in C(Z), x_i\in A)$ we deduce (jj) and hence the proof of the theorem is complete.

REMARK. The assertion (jj) can be strengthened. We have $U_{g^p}=T^px$ for every $p=1, 2, \cdots$ and $x\in\mathfrak{M}$. In fact suppose the equations valid for $p-1$ $(p>1)$ and remark that $T(\mathfrak{M})\subset\mathfrak{M}$. Then for all $x\in A$, $x'\in X'$

$$\langle U_{g^p}x, x'\rangle = \lim_{n\to\infty}\int g_n^{p-1} g d\mu_{x,x'} = \lim_{n\to\infty}\int g d\mu_{x,{}^tU_{g(n,p-1)}x'}$$
$$= \lim_{n\to\infty}\langle Tx, {}^tU_{g(n,p-1)}x'\rangle = \lim_{n\to\infty}\langle U_{g(n,p-1)}Tx, x'\rangle = \langle T^px, x'\rangle$$

(we denoted g_n^{p-1} by $g(n, p-1)$); hence $U_{g^p} = T^p x$ for $x \in A$ and this implies (as in the proof of Theorem 1) $U_{g^p} x = T^p x$ for every $x \in \mathfrak{M}$.

If $\mathfrak{D} \subset \mathfrak{L}(X, X)$ we denote by $s(\mathfrak{D})$ the strong closure of $\mathfrak{D}$ and by $\mathfrak{D}_r$ $(r > 0)$ the set $\{T \in \mathfrak{D} \mid \|T\| \leqq r\}$. From Theorem 1 we can deduce the following (see [14]):

COROLLARY 1. *If* $\mathfrak{B} \subset s(\mathfrak{A}(\mathfrak{F}))$ *then* $\mathfrak{B}_1 \subset s(\mathfrak{A}(\mathfrak{F})_{M(\mathfrak{F})^2})$.

If $T \in \mathfrak{B}_1$ and A is a finite part of X then conditions (i) and (ii) are satisfied. Hence there is a $g \in L^1(Z, A)$ such that $\|g\| \leqq M(\mathfrak{F})$ and $U_g x = Tx$ for $x \in A$. Now let $(g_n)_{1 \leqq n < \infty}$ be a sequence of functions belonging to $C(Z)$ for which $\lim_{n \to \infty} p(g_n - g, A) = 0$ (we can suppose that $\|g_n\| \leqq M(\mathfrak{F})$ for $n = 1, 2, \cdots$, whence $\|U_{g_n}\| \leqq M(\mathfrak{F})^2$). Then $\lim_{n \to \infty} \|Tx - U_{g_n} x\| = 0$ for every $x \in A$ and this implies that $T \in s(\mathfrak{A}(\mathfrak{F})_{M(\mathfrak{F})^2})$.

REMARK. From Corollary 1 it follows that if $\mathfrak{A}(\mathfrak{F})_1$ is strongly closed then $\mathfrak{A}(\mathfrak{F})$ is strongly closed.

COROLLARY 2. *Suppose that* $\mathfrak{F}$ *has property* (E) *and that there exists a denumerable set* $A \subset X$ *such that the closed linear space spanned by* $\bigcup_{x \in A} \mathfrak{A}(\mathfrak{F}, \{x\})$ *is* X. *Then* $s(\mathfrak{A}(\mathfrak{F})) = \{U_f \in \mathfrak{L}(X, X) \mid f \in B_0^\infty(Z)\} = \{U_f \in \mathfrak{L}(X, X) \mid f \in B^\infty(Z)\}$.

We can suppose that $\sum_{x \in A} \|x\| < \infty$. If $T \in s(\mathfrak{A}(\mathfrak{F}))$ we deduce, using Corollary 2, that there is a directed family $(T_j)_{j \in I}$ of operators belonging to $\mathfrak{A}(\mathfrak{F})$ which converges uniformly to T, on A. Then by Theorem 1 there is a $g \in B_0^\infty(Z)$ such that $T = U_g$; hence $s(\mathfrak{A}(\mathfrak{F})) \subset \{U_f \in \mathfrak{L}(X, X) \mid f \in B_0^\infty(Z)\}$. Since $\{U_f \in \mathfrak{L}(X, X) \mid f \in B_0^\infty(Z)\} \subset \{U_f \in \mathfrak{L}(X, X) \mid f \in B^\infty(Z)\} \subset s(\mathfrak{A}(\mathfrak{F}))$ the corollary is completely proved.

Let X be a Banach space. We shall say that an algebra $\mathfrak{A} \subset \mathfrak{L}(X, X)$ has *property* (P_1) if there exists a compact space Z and a spectral family $\mathfrak{F} = (\mu_{x,x'})_{x \in X, x' \in X'}$ of measures defined on Z such that $\mathfrak{A} = \mathfrak{A}(\mathfrak{F})$. Evidently an algebra $\mathfrak{A} \subset \mathfrak{L}(X, X)$ has property (P_1) if and only if there is a compact space Z and a continuous representation of the algebra $C(Z)$ onto $\mathfrak{A}$ (we can show that an algebra $\mathfrak{A}$ has property (P_1) if and only if it is algebraically and topologically isomorphic to an algebra $C(Z)$, where Z is a compact space). An algebra $\mathfrak{A} \subset \mathfrak{L}(X, X)$ has *property* (P_2) if there is a compact space Z and a spectral family $\mathfrak{F}$ on Z having property (E) and such that $\mathfrak{A} = \mathfrak{A}(\mathfrak{F})$. If X is sequentially weakly complete then every algebra having property (P_1) clearly has property (P_2).

If an algebra $\mathfrak{A}$ has property (P_2) and $\mathfrak{F}$ is a spectral family such that $\mathfrak{A} = \mathfrak{A}(\mathfrak{F})$, then F has property (E). This follows for instance from:

PROPOSITION 5. *Let* $\mathfrak{F}$ *be a spectral family. Then* $\mathfrak{F}$ *has property* (E) *if and only if* $\mathfrak{A}(\mathfrak{F})_1$ *is relatively weakly compact.*

It is obvious that $\mathfrak{F}$ has property (E) if $\mathfrak{A}(\mathfrak{F})_1$ is relatively weakly compact. The fact that $\mathfrak{A}(\mathfrak{F})_1$ is relatively weakly compact if $\mathfrak{F}$ has property (E) follows from Theorem 6, [11, p. 160] ($f \to U_f$ is weakly compact; we take on $\mathcal{L}(X, X)$ the strong topology and we remark that every bounded closed part is complete). Theorem 3.2, [4, pp. 300–301] can also be used for the same purpose.

If $\mathfrak{D} \subset \mathcal{L}(X, X)$ we denote by $\mathfrak{D}^{(p)}$ the set of all projections belonging to $\mathfrak{D}$.

COROLLARY. *If* $\mathfrak{A} \subset \mathcal{L}(X, X)$ *is a strongly closed algebra having property* (P_2) *then* $\mathfrak{A}^{(p)}$ *is a (bounded) complete boolean algebra.*

Let Z be the spectrum of $\mathfrak{A}$ and $\mathfrak{F}$ a spectral family on Z such that $\mathfrak{A} = \mathfrak{A}(\mathfrak{F})$. Then $\mathfrak{A}^{(p)} = \{E_{\mathfrak{F}}(\omega) \mid \omega \in S_0(Z)\}$ and hence $\mathfrak{A}^{(p)}$ is bounded (by $M(\mathfrak{F})$). The completeness of $\mathfrak{A}^{(p)}$ follows from the fact that $\mathfrak{A}(\mathfrak{F})_{M(\mathfrak{F})}$ is weakly compact and from Theorem 1, [5, pp. 313–314] (see also [13, pp. 162–163]).

THEOREM 2. (i) *If* $\mathfrak{A}$ *has property* (P_1) *then* $s(\mathfrak{A})$ *has property* (P_1); (ii) *if* $\mathfrak{A}$ *has property* (P_2) *then* $s(\mathfrak{A})$ *has property* (P_2).

Let us prove first (i). Take a compact space Z and a spectral family of measures on Z such that $\mathfrak{A} = \mathfrak{A}(\mathfrak{F})$. Let us introduce an involution on $\mathfrak{A}$ as follows: $U_f^* = U_{\bar{f}}$ for each $f \in C(Z)$; let us extend this involution by continuity to $s(\mathfrak{A})$. Let $T \in s(\mathfrak{A})$; then by Corollary 1 of Theorem 1, there is a uniformly bounded directed family of operators belonging to $\mathfrak{A}$, $(U_{f(j)})_{j \in I}$, which converges strongly to T. A simple use of the inequalities (4) and of the Cauchy inequality gives, for each $x \in X$ and $j \in I$: $\|U_{f(j)}x\|^2 \leqq p(f(j), \{x\})^2 \leqq M(\mathfrak{F})\|x\| p(f(j)\bar{f}(j), \{x\}) \leqq M(\mathfrak{F})^2 \|x\| \, \|U_{f(j)} U_{f(j)}^* x\|$. We deduce $\|Tx\|^2 \leqq M(\mathfrak{F}) \|x\| \, \|TT^*x\|$ for every $x \in X$ and hence $\|T\|^2 \leqq M(\mathfrak{F})\|TT^*\|$. Since this last inequality implies that $s(\mathfrak{A})$ is isomorphic algebraically and topologically with an algebra $C(\tilde{Z})$ the proof of (i) is complete. The second part of the theorem follows from the first, from Proposition 5 and from Corollary 1 of Theorem 1.

3. **Strongly closed algebras of operators.** For every s.c.a. spectral measure E, defined on a tribe $S_0(Z)$, we shall write $\mathfrak{R}(E) = \{E(\omega) \mid \omega \in S_0(Z)\}$. It is obvious that $\mathfrak{R}(E)$ is a σ-complete boolean algebra of projections.

Let E be a s.c.a. spectral measure defined on a tribe $S_0(Z)$ and let $\mathfrak{F}$ be the corresponding spectral family of measures. Let Z' be the spectrum of $\mathfrak{B}_0(\mathfrak{F})$ and $\mathfrak{F}' = (\mu'_{x,x'})_{x \in X, x' \in X'}$ a spectral family on Z' such that $\mathfrak{B}_0(\mathfrak{F}) = \mathfrak{A}(\mathfrak{F}')$; it is easy to see that the closure of $\bigcup_{x \in X, x' \in X'} S(\mu'_{x,x'})$ is Z'.

Suppose $\mathfrak{R}(E)$ complete. Then Z' is stonean and every measure $\mu'_{x,x'}$ is normal. For the sake of completeness we give here a direct proof of these assertions. Let $U \subset Z'$ be an open subset and let

$$P' = \sup\{P \in \mathfrak{R}(E) \mid P \leqq U_{\mathfrak{F}', \phi_U}\}.$$

There is then an open and closed subset $U' \subset Z'$ such that $P' = U_{\mathfrak{F}', \phi_{U'}}$. If $U - U' \neq \varnothing$ we can find a function $g' \in C(Z')$, $g' \neq 0$, whose support is contained in $U - U'$. Let $g \in \mathcal{B}_0(\mathfrak{F})$ be such that $U_{\mathfrak{F},g} = U_{\mathfrak{F}',g'}$ and let $E' = E(\{z | g(z) \neq 0\})$. By direct computation we obtain $E'P' = 0$ and if K is the support of g', $E' \leqq U_{\mathfrak{F}',\phi_K} \leqq U_{\mathfrak{F}',\phi_U}$; hence $P' < E' + P' \leqq U_{\mathfrak{F}',\phi_U}$. It follows necessarily that $U \subset U'$ and $P' = U_{\mathfrak{F}',\phi_U}$. Therefore $U' - U$ is $\mathfrak{F}'$-negligible and since $U' - \overline{U}$ is open, $U' = \overline{U}$. We deduce that Z' is stonean and that every measure $\mu'_{x,x'}$ is normal.

THEOREM 3. *Let E be a s.c.a. spectral measure defined on a tribe $S_0(Z)$ and let $\mathfrak{F}$ be the corresponding spectral family of measures. Suppose that $\mathcal{R}(E)$ is complete. Then there is a Hilbert space H, a von Neumann algebra of operators $\mathcal{B}$ on H and an algebraic isomorphism ϕ of $\mathcal{B}_0(\mathfrak{F})$ onto $\mathcal{B}$ such that*: (i) *ϕ is bicontinuous when $\mathcal{B}_0(\mathfrak{F})$ and $\mathcal{B}$ are endowed with their uniform topologies*; (ii) *the restriction of ϕ to bounded sets is weakly and strongly bicontinuous*; (iii) *$\phi(h(T)) = h(\phi(T))$ for every $T \in \mathcal{B}_0(\mathfrak{F})$, $h \in S_0(\sigma(T))$*; (iv) *$\phi(T^*) = \phi(T)^*$ for every $T \in \mathcal{B}_0(\mathfrak{F})$.*

Let Z' be the spectrum of $\mathcal{B}_0(\mathfrak{F})$ and $\mathfrak{F}' = (\mu'_{x,x'})_{x \in X, x' \in X'}$ a spectral family on Z' such that $\mathcal{B}_0(\mathfrak{F}) = \mathcal{A}(\mathfrak{F}')$. Then Z' is hyperstonean and every measure $\mu'_{x,x'}$ is normal. Then there is a Hilbert space H, a von Neumann algebra $\mathcal{B}$ on H and a $*$-isometry $f \rightarrow T_f$ of $C(Z')$ onto $\mathcal{B}$ (see for instance [8]). If we write $\phi(U_{\mathfrak{F}',f}) = T_f$, then ϕ is an algebraic isomorphism of $\mathcal{B}_0(\mathfrak{F})$ onto $\mathcal{B}$ which has properties (i) and (iv). The weak continuity of ψ, the mapping inverse to ϕ, is a consequence of the fact that every measure $\mu'_{x,x'}$ is normal and hence that there are complex numbers c_1, c_2, c_3, c_4 and a_1, a_2, a_3, $a_4 \in H$ such that $\langle U_{\mathfrak{F}',f}x, x' \rangle = \sum_{i=1}^{4} c_i (T_f a_i | a_i)$. The weak continuity of ϕ, on bounded sets, follows from the fact that $\mathcal{B}_1$ is weakly compact. Therefore the first part of (ii) is proved. Now let $\mathcal{P}$ be the set of linear forms $T \rightarrow \langle Tx, x' \rangle$, defined on $\mathcal{B}_0(\mathfrak{F})$, which are positive on the elements TT^*; it is obvious that the topology on $\mathcal{B}_0(\mathfrak{F})$ defined by the set of semi-norms $\{ |\rho| \,|\, \rho \in \mathcal{P} \}$ coincide with the weak topology (see the remark at the end of paragraph 4). Denote by τ the topology on $\mathcal{B}_0(\mathfrak{F})$ defined by the set of semi-norms $\{ T \rightarrow \rho(TT^*)^{1/2} | \rho \in \mathcal{P} \}$. Since T converges to T_0 in the topology τ if and only if $(T - T_0)(T - T_0)^*$ converges weakly to zero it follows that the restriction of ϕ to bounded sets is bicontinuous when $\mathcal{B}_0(\mathfrak{F})$ is endowed with the topology τ and $\mathcal{B}$ with the strong topology. Therefore, to complete the proof of (ii), it is enough to show that on each bounded subset of $\mathcal{B}_0(\mathfrak{F})$ the strong topology coincides with τ. It is obvious that τ is weaker than the strong topology. Conversely suppose that the $U_{\mathfrak{F}',f}$ are uniformly bounded and that they converge to $U_{\mathfrak{F}',f(0)}$ in the topology τ. Then, for every $x \in X$, $x' \in X'$, $\lim \int |f - f(0)|^2 d|\mu'_{x,x'}| = 0$ and hence f converges in $|\mu'_{x,x'}|$-measure to $f(0)$. Since, for fixed $x \in X$, $\{ \mu'_{x,x'} | \|x'\| \leqq 1 \}$ is relatively weakly compact, we have $\lim \int f d\mu'_{x,x'} = \int f(0) d\mu'_{x,x'}$, uniformly with respect to $\|x'\| \leqq 1$; $x \in X$ being arbitrary it follows that $U_{\mathfrak{F}',f}$ converges strongly to $U_{\mathfrak{F}',f(0)}$. Hence the proof of (ii) is complete. The assertion (iii) is an immediate consequence of (ii).

REMARKS. (1) The involution in $\mathcal{B}_0(\mathfrak{F})$ is introduced by $U_{\mathfrak{F}',f} = U_{\mathfrak{F}',\bar{f}}$ or, equivalently, by $U_{\mathfrak{F},f} = U_{\mathfrak{F},\bar{f}}$. If $T \in \mathcal{B}_0(\mathfrak{F})$ and E^T is the resolution of the identity of T, defined on $S_0(\sigma(T))$, it is easily seen that $T^* = \int_{\sigma(T)} \bar{\lambda} dE^T(\lambda)$. (2) Every $P \in \mathcal{B}^{(p)}$ is a self-adjoint projection and ϕ is an order isomorphism of $\mathcal{R}(E) = \mathcal{B}_0(\mathfrak{F})^{(p)}$ onto $\mathcal{B}^{(p)}$. (3) In connection with Theorem 3 see also [3, p. 37, Theorem 9.2]. (4) Some of the arguments involved in the proof of Theorem 3 can be avoided if we remark that (for an equivalent norm) $\mathcal{B}_0(\mathfrak{F})$ is an AW*-algebra and use various known results concerning such algebras (see for instance [18]).

COROLLARY 1. *Let E be a s.c.a. spectral measure, defined on a tribe $S_0(Z)$, and $\mathfrak{F}$ the corresponding spectral family. Then the following statements are equivalent*: (j) *$\mathcal{R}(E)$ is complete*; (jj) $\mathcal{B}_0(\mathfrak{F}) = s(\mathcal{B}_0(\mathfrak{F}))$; (jjj) $\mathcal{R}(E) = s(\mathcal{R}(E))$.

If $\mathcal{R}(E)$ is a complete boolean algebra, (ii) implies that $\mathcal{B}_0(\mathfrak{F})_1$ is strongly complete and hence that $\mathcal{B}_0(\mathfrak{F})_1 = s(\mathcal{B}_0(\mathfrak{F})_1)$; therefore we have $\mathcal{B}_0(\mathfrak{F}) = s(\mathcal{B}_0(\mathfrak{F}))$. If $\mathcal{B}_0(\mathfrak{F}) = s(\mathcal{B}_0(\mathfrak{F}))$ then $\mathcal{B}_0(\mathfrak{F})^{(p)} = \mathcal{R}(E)$ is obviously strongly closed. If $\mathcal{R}(E) = s(\mathcal{R}(E))$ it follows from Theorem 1, [5, pp. 313–314], that $\mathcal{R}(E)$ is complete.

REMARKS. (1) The results stated in Corollary 1 are due to W. G. Bade [2, p. 358, Theorem 4.5]; the proof of the implications (jj)→(jjj)→(j) is essentially the same as the one given in [2]. These assertions are justified by the proposition [2, p. 349]: If $\mathcal{G}$ is a σ-complete boolean algebra of projections in a Banach space then there exists a compact space Z and a s.c.a. spectral measure defined on $S_0(Z)$ such that $\mathcal{G} = \mathcal{R}(E)$. (2) From Corollary 1 follows that if $\mathfrak{F}$ is a spectral family on Z having property (P_2) and such that $\mathcal{B}_0(\mathfrak{F})$ is σ-finite (= every orthogonal set of projections belonging to the considered algebra is denumerable) then $\mathcal{B}_0(\mathfrak{F})$ is strongly closed; this result can also be reduced to the corresponding one in Hilbert spaces if we use Theorem 3.

Let H be a Hilbert space and E^H a s.c.a. spectral measure, defined on a tribe $S_0(Z)$, such that $\mathcal{R}(E) \subset \mathfrak{L}(H, H)$. We shall say that E^H is self-adjoint if $E^H(\omega)$ is self-adjoint for each $\omega \in S_0(Z)$.

COROLLARY 2. *Let E be a s.c.a. spectral measure, defined on a tribe $S_0(Z)$. Then*: (j) *there is a Hilbert space H, a s.c.a. self-adjoint spectral measure $E^H(\mathcal{R}(E^H) \subset \mathfrak{L}(H, H))$, defined on $S_0(Z)$ and an order isomorphism ϕ of $\mathcal{R}(E)$ onto $\mathcal{R}(E^H)$ which is uniformly, strongly and weakly bicontinuous*; (jj) *ϕ can be extended (in a unique way) to an algebraic isomorphism, of the strongly closed algebra $\mathcal{A}(E)$ spanned by $\mathcal{R}(E)$ onto the strongly closed algebra $\mathcal{A}(E^H)$ spanned by $\mathcal{R}(E^H)$, having the properties* (i)–(iv) *(we replace here $\mathcal{B}_0(\mathfrak{F})$ by $\mathcal{A}(E)$ and $\mathcal{B}$ by $\mathcal{A}(E^H)$) formulated in Theorem* 3; (jjj) *$\mathcal{R}(E)$ is complete if and only if $\mathcal{R}(E^H)$ is.*

Let $\mathfrak{F}$ be the spectral family corresponding to E; obviously $\mathcal{R}(E) \subset s(\mathcal{A}(\mathfrak{F}))$. By Theorem 3 there is a Hilbert space H, a von Neumann algebra $\mathcal{B} \subset \mathfrak{L}(H, H)$

and an algebraic isomorphism ϕ of $s(\mathfrak{A}(\mathfrak{F}))$ onto $\mathfrak{B}$, which has the properties (i)–(iv). If we take $E^H(\omega)=\phi(E(\omega))$, for $\omega\in S_0(Z)$, then E^H is a s.c.a. self-adjoint spectral measure on $S_0(Z)(\mathfrak{R}(E^H)\subset\mathfrak{L}(H, H))$ and ϕ is an order isomorphism of $\mathfrak{R}(E)$ onto $\mathfrak{R}(E^H)$ which is uniformly, strongly and weakly bicontinuous. Hence the proof of (j) is complete. The results (jj) (we use for its proof Corollary 1 of Theorem 1) and (jjj) are now obvious.

REMARKS. (1) An immediate consequence of Corollary 2 (see the remarks which follow Theorem 3) is the proposition: On a σ-complete boolean algebra of projections the weak and the strong topology coincide. (2) Theorem 4.7, [2, p. 359] can be deduced from Corollary 5.3, [17, p. 38] if we use Theorem 3. (3) Using Theorem 3 we can also prove the following proposition: If $\mathfrak{A}$ is a strongly closed algebra containing I, σ-finite, generated (in the strong topology) by a denumerable set and having property (P_2) then there is an operator $T\in\mathfrak{A}$ with real spectrum such that every $U\in\mathfrak{A}$ is of the form $h(T)$ where $h\in B_0^\infty(\sigma(T))$.

Using Theorem 3 and, for instance, Corollary 1, [9, p. 57] we deduce the following:

COROLLARY 3. *Let* $\mathfrak{A}(1)\subset\mathfrak{L}(X_1, X_1)$, $\mathfrak{A}(2)\subset\mathfrak{L}(X_2, X_2)$ *be two strongly closed algebras having property* (P_2) *and* ϕ *an algebraic isomorphism of* $\mathfrak{A}(1)$ *onto* $\mathfrak{A}(2)$ *such that* $\phi(T^*)=\phi(T)^*$. *Then* ϕ *is bicontinuous when* $\mathfrak{A}(1)$ *and* $\mathfrak{A}(2)$ *are endowed with their uniform topologies. The restriction of* ϕ *to bounded sets is strongly and weakly bicontinuous.*

4. **Remarks on spectral families having property** (E). Proposition (*) stated at the end of paragraph 1 follows from Theorem 1.4 and Lemma 2.3 [4] or from Theorems 1.3 and 1.4 [4]. Instead of Lemma 2.3 or Theorem 1.3 [4] we can use, for the proof of (*), Theorem 2 [11].

From Theorem 1.4 [4], taking into account the form of the measure ν (constructed in the proof of this theorem), we can deduce the following result: (**) Let $\mathfrak{C}$ be a weakly relatively compact set of Radon measures on a compact space Z. Then there is a Radon measure $\nu\geqq 0$ such that: (1) $\nu(f)\leqq\sup_{\mu\in\mathfrak{C}}|\mu|(f)$ for $f\geqq 0$; $\nu(N)=0$ (for $N\in S_0(Z)$) if and only if $|\mu|(N)=0$ for every $\mu\in\mathfrak{C}$. Using (**) we can immediately prove the following proposition due to W. G. Bade (Theorem 3.1, [2, pp. 351–353]): Let Z be a compact space and $\mathfrak{F}=(\mu_{x,x'})_{x\in X,x'\in X'}$ a spectral family of measures given on Z, having property (E). Then for every $y\in X$ there is a $\nu_y=\mu_{y,y'}\geqq 0$ such that every $\mu_{y,x'}$ is absolutely continuous with respect to ν_y. In fact take $\mathfrak{C}=\{\mu_{y,x'}\mid \|x'\|\leqq 1\}$ and $\nu_y=\nu$. We then have $\nu_y(|f|)\leqq p(f, \{y\})$ for every $f\in C(Z)$. Define β on $X_y=\{U_fy\mid f\in C(Z)\}$ by the equation: $\beta(U_fy)=\int f d\nu_y$. Using the inequalities $\nu_y(|f|)\leqq p(f, \{y\})\leqq M(\mathfrak{F})\|U_fy\|$ we deduce that β is defined on X_y without ambiguity and that it is continuous; β can therefore be extended to X. If we denote the extension by y', then $\nu_y=\mu_{y,y'}$.

By the same method, but without using proposition (**) or (*), we prove that every measure $|\mu_{x,x'}|$ is a measure $\mu_{y,y'}$.

Let $\mathfrak{A}$ be a strongly closed algebra having property (P_2) and $\mathfrak{F}$ a spectral family of measures defined on the spectrum of $\mathfrak{A}$ such that $\mathfrak{A}=\mathfrak{A}(\mathfrak{F})$. We have then the following proposition (which we shall state without proof): Every measure $\mu_{x,x'}$ is normal and every normal measure on Z is absolutely continuous with respect to a measure $\mu_{x,x'}\geqq 0$. In the case of Hilbert spaces more precise results are valid (see [8] and [16]).

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