A PROPERTY OF THE DIFFERENTIAL IDEAL $[y^p]$

BY

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Introduction. Let y be a *differential indeterminate* over the rational number field R, that is, we consider the polynomial ring $R[y_0, y_1, y_2, \cdots]$ in a sequence of (algebraic) indeterminates $y_0 = y, y_1, y_2, \cdots$ together with the mapping $a \rightarrow a'$ of $R[y_0, y_1, y_2, \cdots]$ into itself which has the properties: (1) $(a+b)' = a'+b'$, (2) $(ab)' = a'b+ab'$, (3) $y_i' = y_{i+1}$; there is one and only one such mapping, and the operation of passing from a to a' is called *differentiation*. By a *differential ideal* in $R[y_0, y_1, y_2, \cdots]$ we mean an ideal in the ring-theoretic sense which has the property that if a is in the ideal, then also a' is in the ideal. Notationally, $[y^p]$ stands for the differential ideal generated by y^p, that is, for the ideal generated in the usual ring-theoretic sense by $y^p, (y^p)', ((y^p)')', \cdots$.

A study of the structure of differential ideals yields many unsolved problems even for the relatively simple ideal $[y^p]$. It is shown from a simple calculation that $y_1^{2p-1} \equiv O[y^p]$, whence it follows that some power of each y_i is in $[y^p]$. The following question was singled out for investigation by J. F. Ritt [3]: what is the smallest q such that $y_i^q = O[y^p]$? For $i=1$, $q=2p-1$ is stated by him without proof to be the answer. In Part I we give a proof of this result, and in Part II we solve the problem for $i=2$, $p \geqq 2$. For arbitrary i we conjecture the answer to be $q=(i+1)(p-1)+1$.

The following notation and results of H. Levi we use extensively. Let $P = y_0^{\alpha_0} y_1^{\alpha_1} \cdots y_n^{\alpha_n}$ be a power product (pp.) of degree

$$d = \sum_{i=0}^{n} \alpha_i$$

and weight

$$w = \sum_{i=1}^{n} i\alpha_i.$$

Write $d = a(p-1)+b$ where a and b are integers such that $0 \leqq a$, $0 < b \leqq p-1$. We let $f(p, d) = a(a-1)(p-1)+2ab$. It is helpful to note, as Levi has done, that $f(p, d)$ is the weight of the first d factors of the formal infinite product $y_0^{p-1} y_2^{p-1} y_4^{p-1} \cdots$. A sufficient condition for $P \equiv O[y^p]$ is that the weight be small with respect to the degree. More precisely, we have the following theorem of H. Levi:

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THEOREM 0.1. *Let $p>1$ and P be a pp. in the y_i of degree d and weight w. Then if $w<f(p, d)$, $P\equiv O[y^p]$.*

Levi's theorem may be restated in terms of the notion of *weight sequence* introduced by D. G. Mead in [2].

DEFINITION 0.2. *If $P=y_{i_1}y_{i_2}\cdots y_{i_n}$, where the i_j are monotonically non-decreasing, then $(a_1, \cdots, a_n)$ is called the weight sequence of P and a_n the excess weight of P, where*

$$a_j = \sum_{k=1}^{j} i_k - f(p, j).$$

There is a one-to-one correspondence between weight sequences and pp. in the y_i, therefore a pp. and its associated weight sequence may be used interchangeably. Theorem 0.1 says that if one of the entries in the weight sequence of P is negative, then $P\equiv O[y^p]$. That this condition is not necessary was shown by Mead in [2].

Another basic result of H. Levi concerns the so-called weak pp.

DEFINITION 0.3. *$P=y_0^{\alpha_0}y_1^{\alpha_1}\cdots y_n^{\alpha_n}$ is called a weak pp. if, for $i=0, \cdots, n-1$, one has $\alpha_i+\alpha_{i+1}<p$.*

THEOREM 0.4. *No linear combination of weak pp. is in $[y^p]$, unless all the coefficients are zero.*

This theorem furnishes us with a starting point for our work. By a sequence of congruence relations we reduce a pp. to a linear combination of weak pp., and then we have only to inspect the coefficients involved to ascertain whether or not $P\equiv O[y^p]$. In the reduction process used by Levi, if P is not weak, so that it contains a factor $y_i^a y_{i+1}^{p-a}$, this factor is replaced modulo $[y^p]$ by the other terms in the $[ia+(i+1)(p-a)]$th derivative of y^p, that is by

$$-\sum c_{(\beta)} \prod y_j^{\beta_j}$$

where

$$\sum \beta_j = p,$$
$$\sum j\beta_j = ia + (i+1)(p-a)$$

and

$$c_{(\beta)} = -\frac{i!^a(i+1)!^{p-a}}{\prod (j!)^{\beta_j}}\cdot\frac{a!(p-a)!}{\prod (\beta_j!)}.$$

This gives rise to a congruence relation $P\equiv -\sum c_i Q_i$ where the Q_i are pp. of the same weight and degree as P, but less than P if the pp. in the y_i are ordered lexicographically. The Q_i may in turn be replaced by linear combinations of pp. R_{i_j}, each R_{i_j} being less than Q_i, until P is written congruent to a linear combination of weak pp. This process usually requires elaborate computations which sometimes may be simplified by the notation of an M-congruence [2].

Let $P \equiv \sum cQ$ be the congruence obtained after one step in Levi's reduction process. Let $P = \prod y_k^{\alpha_k}$ and let $Q = \prod y_k^{\beta_k}$ be the monomial on the right side of the congruence. Placing $m(P, Q) = (\prod (k!)^{\alpha_k}]/[\prod (k!)^{\beta_k}]$, $m(P, Q)$ is called the first factor of the step from P to Q, $M(P, Q) = c/m(P, Q)$ the second. Note that $m(P, Q)$ depends only on P, Q, not on the step. After several steps in the reduction, suppose we come to a congruence $P \equiv \sum dR$. Calling a sequence of successive steps a *path*, there may be several paths in the given reduction leading from P to R. The first factor of a path from P to R is defined to be the product of the first factors of the steps; clearly this is $m(P, R)$ and hence the first factor is the same for any path and can be designated without confusion as $m(P, R)$. If, quite generally, $P \equiv \sum dR$, we also write $P \equiv^M \sum (d/m(P, R))R$ and call this an M-congruence. Because of the independence of the first factor from the path joining P and R, the M-congruence is what is obtained in the Levi reduction provided the first factors are suppressed at each step of the reduction. As remarked by Mead, in testing $P \equiv O[y^p]$, we may restrict the computations to M-congruences.

Part I. In this section we show that the smallest q such that $y_1^q \equiv O[y^p]$ is $q = 2p-1$. This result was known to Ritt and a proof that $y_1^{2p-1} \equiv O[y^p]$ is given in [3].

Lemma 1.0. $y_1^{2p-1} \equiv O[y^p]$.

Proof. $P = y_1^{2p-1}$ is of degree $2p-1$ and weight $2p-1$. We find that $f(p, 2p-1) = 2p+2$. Hence, by Theorem 0.1, $P \equiv O[y^p]$.

Lemma 1.1. *If $P = y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2}$ is of degree $2p-2$ and has excess weight zero, then*

$$Q = y_0^{\alpha_0+1} y_1^{\alpha_1-1} y_2^{\alpha_2} \equiv O[y^p].$$

Proof. Since P has excess weight zero, $w(P) = f(p, 2p-2) = \alpha_1 + 2\alpha_2$. $w(Q) = \alpha_1 - 1 + 2\alpha_2$. Therefore $w(Q) - f(p, 2p-2) = -1$, and hence, by Theorem 0.1, $Q \equiv O[y^p]$.

Lemma 1.2. *Let $P = y_0^{\alpha_0} y_1^{\alpha_1} \cdots y_s^{\alpha_s}$, $\alpha_s > 0$, be of degree $2p-2$ and have excess weight zero. Then if $s > 2$ we have $P \equiv O[y^p]$.*

Proof. We examine the weight of $Q = y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_s^{\alpha_s - 1}$. Since P has excess weight zero, $w(P) = f(p, 2p-2) = 2p-2$. Hence $w(Q) = w(P) - s = 2p-2-s < 2p-4 = f(p, 2p-3)$. By Theorem 0.1, $Q \equiv O[y^p]$, and therefore also $P \equiv O[y^p]$.

LEMMA 1.3. $y_1^{2p-2} \not\equiv O[y^p]$.

Proof. We show by induction on $p-k$ that $y_0^k y_1^{2p-2-2k} y_2^k \not\equiv O[y^p]$, $0 \leqq k \leqq p-1$. For the case $p-k=1$, $y_0^{p-1} y_2^{p-1} \not\equiv O[y^p]$ by Theorem 0.1. We assume the lemma true for values less than $p-k$. By Lemma 1.1, we have $y_0^{k+1} y_1^{2p-3-2} y_2^k \equiv O[y^p]$, whence $(y_0^{k+1} y_1^{2p-3-2k} y_2^k)' \equiv O[y^p]$. Expanding and applying Lemma 1.2, we have

$$y_0^k y_1^{2p-2-2k} y_2^k \equiv -\frac{2p-2k-3}{k+1} y_0^{k+1} y_1^{2p-4-2k} y_2^{k+1} [y^p].$$

By our induction hypothesis, $y_0^{k+1} y_1^{2p-4-2k} y_2^{k+1} \not\equiv O[y^p]$, and $(2p-2k-3)/(k+1) \neq 0$ since $2p-2k-3$ is an odd number. Therefore the induction holds and our lemma follows by taking $k=0$.

COROLLARY 1.4. $y_1^{2p-2} \equiv d y_0^{p-1} y_2^{p-1} [y^p]$, $d \neq 0$.

This follows from the proof of the lemma, or from it directly upon observing that there is only one weak pp. of the same degree and weight as $y_0^{p-1} y_2^{p-1}$. (More generally, it is known, and easily proved, that there is only one weak pp. of the same degree and weight as $y_0^{p-1} y_2^{p-1} \cdots y_{2s}^{p-1}$).

THEOREM 1.5. *The smallest q such that $y_1^q \equiv O[y^p]$ is $q=2p-1$.*

Proof. Lemmas 1.0 and 1.3.

Part II. In this section we find that the smallest q such that $y_2^q \equiv O[y^p]$ is $q=3p-2$.

LEMMA 2.0. $y_2^{3p-2} \equiv O[y^p]$.

Proof. $P = y_2^{3p-2}$ has degree $3p-2$ and weight $6p-4$. We find that $f(p, 3p-2) = 6p$. Hence $P \equiv O[y^p]$ by Theorem 0.1.

LEMMA 2.1. *If $P = y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3} y_4^{\alpha_4}$ is of degree $3p-3$ and excess weight zero, then*

(a) $Q_0 = y_0^{\alpha_0+1} y_1^{\alpha_1-1} y_2^{\alpha_2} y_3^{\alpha_3} y_4^{\alpha_4} \equiv O[y^p]$,
(b) $Q_1 = y_0^{\alpha_0} y_1^{\alpha_1+1} y_2^{\alpha_2-1} y_3^{\alpha_3} y_4^{\alpha_4} \equiv O[y^p]$,
(c) $Q_2 = y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2+1} y_3^{\alpha_3-1} y_4^{\alpha_4} \equiv O[y^p]$,
(d) $Q_3 = y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3+1} y_4^{\alpha_4-1} \equiv O[y^p]$.

Proof. (a) $w(Q_0) = (\alpha_1 - 1) + 2\alpha_2 + 3\alpha_3 + 4\alpha_4$. Since P has excess weight zero, $w(P) = f(p, 3p-3) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4$. Therefore $w(Q_0) - f(p, 3p-3) = -1$ and $Q_0 \equiv O[y^p]$ by Theorem 0.1. The proofs for Parts b, c, and d are similar.

LEMMA 2.2. *Let $P = y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3} y_4^{\alpha_4} \cdots y_s^{\alpha_s}$, $\alpha_s > 0$ be of degree $3p-3$ and have excess weight zero, then if $s > 4$, $P \equiv O[y^p]$.*

Proof. Assuming that $\alpha_s > 0$, we compute the weight of $Q = y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3} y_4^{\alpha_4}$

$\cdots y_s^{\alpha_s-1}$. Since P has excess weight zero, $w(P)=f(p,\ 3p-3)=6p-6$. $w(Q)=w(P)-s=6p-6-s<6p-10=f(p,\ 3p-4)$. By Theorem 0.1, $Q\equiv O[y^p]$, and therefore also $P\equiv O[y^p]$.

THEOREM 2.3. *$y_2^{3p-3}\equiv\sum_k c_kP_k[y^p]$, where $P_k=(y_1^2y_4)^{2k}(y_1y_3)^{(3p-6k-3)/2}$, p odd, $0\leqq 2k\leqq p-1$, or $P_k=(y_1^2y_4)^{2k+1}(y_1y_3)^{(3p-6k-6)/2}$, p even, $0\leqq 2k\leqq p-2$. Also $c_{(p-1)/2}\neq 0$ for p odd and $c_{(p-2)/2}\neq 0$ for p even.*

Proof. By Lemma 2.1b, we may write $y_2^{3p-3}\equiv -(3p-4)y_1y_2^{3p-5}y_3[y^p]$. Lemma 2.1b is applied again to obtain

$$y_1y_2^{3p-5}y_3 \equiv -\frac{1}{2}[(3p-6)y_1^2y_2^{3p-7}y_3^2+y_1^2y_2^{3p-6}y_4][y^p].$$

Now Lemmas 2.1b, 2.2 are applied repeatedly to eliminate the y_2 factor. Applying these lemmas to $y_1^ay_2^by_3^cy_4^d$ (for appropriate a, b, c, d), we can write this pp. congruent to a linear combination (with negative coefficients) of $y_1^{a+1}y_2^{b-2}y_3^{c+1}y_4^d$ and $y_0^{a+1}y_2^{b-1}y_3^{c-1}y_4^{d+1}$. Thus repeated application of Lemmas 2.1b, 2.2 yields y_2^{3p-3} to be congruent to a linear combination of monomials $y_1^xy_2^{(3p-3)-x-y}y_3^{y-z}y_4^z$, $x=y+z$. By setting $3p-3-x-y=0$, we may write y_2^{3p-3} congruent to a linear combination of monomials of the form $y_1^xy_3^{y-z}y_4^z$ with (1) $3p-3=x+y$; (2) $x=y+z$. We are interested only in those monomials with non-negative weight sequences; hence (3) $z\leqq p-1$; also (4) $x+3y+z=6p-6$; and (5) $y\geqq z$ so $x\geqq 2z$. From (3) and (4) we have

$$y_1^xy_3^{y-z}y_4^z=(y_1^2y_4)^z(y_1y_3)^{x-2z}=(y_1^2y_4)^z(y_1y_3)^{3(p-z-1)/2}.$$

In order to guarantee that the exponent $3(p-z-1)/2$ be an integer, we set $z=2k$, $0\leqq 2k\leqq p-1$, if p is odd, and $z=2k+1$, $0\leqq 2k\leqq p-2$, if p is even. Thus $y_2^{3p-3}\equiv\sum_k c_kP_k[y^p]$ as stated.

The transition from y_2^{3p-3} to $y_1y_2^{3p-5}y_3$ is a step, or a path of length one, and the transition from y_2^{3p-3} to $y_1^xy_2^{3p-3-x-y}y_3^{y-z}y_4^z$ is by means of paths of length x. The coefficient of $y_1^xy_2^{3p-3-x-y}y_3^{y-z}y_4^z$ at the end of the path is positive or negative according as x is even or odd; the total coefficient is the sum of the coefficients for the separate paths, and since all the paths are the same length, the total coefficient is positive or negative according as x is even or odd. In particular the coefficient of $(y_1^2y_4)^{p-1}$ is positive and this concludes the proof of Theorem 2.3.

The following lemma is a generalization of a statement of Mead [2, p. 430].

LEMMA 2.4. *If $P=y_0^{\alpha_0}y_1^{\alpha_1}y_2^{\alpha_2}y_3^{\alpha_3}y_4^{\alpha_4}\equiv O[y^p]$ is of degree $3p-3$ and excess weight zero, then $Q=y_0^{\alpha_4}y_1^{\alpha_3}y_2^{\alpha_2}y_3^{\alpha_1}y_4^{\alpha_0}\equiv O[y^p]$.*

Proof. In the reduction of P, by Lemma 2.2 we may simply neglect modulo $[y^p]$ the terms having a factor y_i, $i>4$. Also if P is of degree $3p-3$

and excess weight zero, then Q is of degree $3p-3$ and excess weight zero, so the same remark holds for Q. With this understanding, the mapping $y_i \to y_{4-i}$, $0 \leqq i \leqq 4$, not only maps P into Q, but also maps a valid M-congruence used in the reduction of P into a valid M-congruence for the reduction of Q.

THEOREM 2.5.

$$(y_1^2 y_4)^t (y_1^3 y_3^3)^{(p-1)/2} \equiv O[y^{p+t}], \qquad t \geqq 0, \quad p \text{ odd}, \quad p > 1.$$

$$(y_1^2 y_4)^{t+1} (y_1^3 y_3^3)^{(p-2)/2} \equiv O[y^{p+t}], \qquad t \geqq 0, \quad p \text{ even}, \quad p > 2.$$

Proof. For $p=2$, nothing is being asserted. For p even, $p>2$, write $p+t=(p-1)+(t+1)$; the theorem for $p=p$ then follows from the theorem for $p=p-1$. So we may assume p odd, $p\geqq 3$. We prove the theorem first for $p=3$. For arbitrary p, replacing $y_3^3 y_4^{p-3}$ by the other terms in the $(4p-3)$rd derivative of y^p, we find

$$(1) \quad y_1^{2p-3} y_3^3 y_4^{p-3} \equiv \left\{ \frac{-9(p-3)!}{4(p-1)!} y_1^{2p-2} y_4^{p-1} - \frac{9(p-3)!}{2(p-2)!} y_1^{2p-3} y_2 y_3 y_4^{p-2} \right\} [y^p].$$

By Lemmas 2.1b, 2.2, we have

$$(2) \qquad y_1^{2p-3} y_2 y_3 y_4^{p-2} \equiv \frac{-1}{2p-2} y_1^{2p-2} y_4^{p-1} [y^p].$$

Therefore, from (1) and (2) $y_1^{2p-3} y_3^3 y_4^{p-3} \equiv O[y^p]$, $p \geqq 3$. Placing $p-3=t$, we have $y_1^3 y_3^3 (y_1^2 y_4)^t \equiv O[y^{3+t}]$, which is the theorem for $p=3$.

Letting T denote an odd integer such that $3 \leqq T < p$, we have as our induction hypothesis

$$\text{I.H.2.5.} \quad (y_1^2 y_4)^t (y_1^3 y_3^3)^{(T-1)/2} \equiv O[y^{T+t}], \qquad t \geqq 0.$$

Before proceeding with the proof of Theorem 2.5, we insert a number of lemmas.

LEMMA 2.6. *If $y_1^a y_3^{3k} y_4^b$ is of degree $3p+3t-3$ and weight $6p+6t-6$, then $a=2p+2t-k-2$ and $b=p+t-2k-1$, so that $y_1^a y_3^{3k} y_4^b = (y_1^2 y_4)^{p+t-2k-1} (y_1^3 y_2^3)^k$.*

Proof. We obtain a and b by solving simultaneously $a+3k+b=3p+3t-3$ and $a+9k+4b=6p+6t-6$.

LEMMA 2.7. *If $y_1^a y_3^{3k} y_4^b$ is of degree $3p+3t-3$ and weight $6p+6t-6$, and $0<k<(p-1)/2$, then $y_1^a y_3^{3k} y_4^b \equiv O[y^{p+t}]$.*

Proof. By Lemma 2.6 and I.H.2.5

$$y_1^a y_3^{3k} y_4^b \equiv O[y^{(2k+1)+(p+t-2k-1)}].$$

LEMMA 2.8.

$$P = (y_0 y_3^2)^k (y_1^2 y_4)^l (y_1^3 y_3^3)^{(T-1)/2} \equiv O[y^{T+k+l}]$$

if

$$(T-1)/2 + \min(k, l) < (p-1)/2.$$

Proof. By Lemma 2.4, interchanging y_0 and y_4, y_1 and y_3, we see that the lemma is symmetric in k, l; hence we may, and will, assume that $k \geqq l$ and $(T-1)/2+l<(p-1)/2$.

To $P=y_0^k y_1^{2l+3m} y_3^{2k+3m} y_4^l$, where $m=(T-1)/2$, we apply Lemmas 2.1d, 2.2 to eliminate the y_4 factor. Applying these lemmas to $y_0^a y_1^b y_2^c y_3^d y_4^e$, (for appropriate a, b, c, d, e,) we can write this pp. congruent to a linear combination of $y_0^{a-1} y_1^{b+1} y_2^c y_3^{d+1} y_4^{e-1}$, $y_0^a y_1^{b-1} y_2^{c+1} y_3^{d+1} y_4^{e-1}$, $y_0^a y_1^b y_2^{c-1} y_3^{d+2} y_4^{e-1}$; and after repeated applications, the pp. will be congruent to a linear combination of monomials of the form:

$$y_0^{a-u} y_1^{b+u-v} y_2^{c+v-w} y_3^{d+w+x} y_4^{e-x}, \qquad u+v+w=x.$$

Taking $x=e$, P can be written as congruent to a linear combination of monomials of the form:

$$y_0^{k-u} y_1^{2l+3m+u-v} y_2^{v-w} y_3^{2k+3m+w+l}, \qquad u+v+w=l.$$

We propose to show that each of these is congruent to zero, modulo $[y^{T+k+l}]$. Applying Lemma 2.4, it is sufficient to see that

$$y_1^{2k+3m+l+w} y_2^{v-w} y_3^{2l+3m+u-v} y_4^{k-u} \equiv O[y^{T+k+l}].$$

We apply Lemmas 2.1b, 2.2 to eliminate the y_2 factor. One application of these lemmas to $y_1^a y_2^b y_3^c y_4^d$ replaces the pp. by a linear combination of $y_1^{a+1} y_2^{b-2} y_3^{c+1} y_4^d$ and $y_1^{a+1} y_2^{b-1} y_3^{c-1} y_4^{d+1}$; after repeated applications, by a linear combination of pp. of the form

$$y_1^{a+x} y_2^{b-x-y} y_3^{c+y-z} y_4^{d+z}, \qquad x=y+z.$$

Taking $v-w-x-y=0$, it will be sufficient to show that

$$y_1^{2k+l+3m+v-y} y_3^{2l+3m+u-v+y-z} y_4^{k-u+v-w-2y} \equiv O[y^{T+k+l}].$$

Recalling that $l=u+v+w$, $x=y+z$, and $v-w-x-y=0$, we see that $2l+3m+u-v+y-z=3(m+u+w+y)$, and since $y=v-w-x\leqq v$, we have $3(m+u+w+y)\leqq 3(m+l)<(3/2)(p-1)$. By Lemma 2.6, then, it remains to see that

$$(y_1^2 y_4)^{k-u+z}(y_1^3 y_3^3)^{m+u+w+y} \equiv O[y^{T+k+l}],$$

and this follows from Lemma 2.7. This completes the proof of Lemma 2.8.

Returning to the proof of Theorem 2.5, we distinguish the cases $p=4m-1$ and $p=4m+1$. The following lemmas are proved for $p=4m-1$. The corresponding results are stated for $p=4m+1$ and are proved in a similar fashion.

LEMMA 2.9a. *Let* $p=4m-1$, *then*

$$P = y_1^{6m-3+2t} y_3^{6m-3} y_4^{t} \equiv \sum_{k=0}^{m-1} c_k Q_k [y^{4m-1+t}]$$

where

$$Q_k = (y_0 y_3^2)^{m-k-1}(y_1^2 y_4)^{3m-1-k+t}(y_1 y_3)^{2k} y_2^{2k}$$

LEMMA 2.9b. *Let* $p=4m+1$, *then*

$$P = y_1^{6m+3t} y_3^{6m} y_4^{t} \equiv \sum_{k=0}^{m} d_k S_k [y^{4m+1+t}]$$

where

$$S_0 = (y_0 y_3^2)^{m}(y_1^2 y_4)^{3m+t}$$

and

$$S_k = (y_0 y_3^2)^{m-k}(y_1^2 y_4)^{3m+1-k+t}(y_1 y_3)^{2k-1} y_2^{2k-1}$$

for $k>0$.

The exact values of the c_k and d_k will be given below.

Proof a. Replacing $y_3^{4m-1}y_4^t$ by the other terms in the $(12m+4t-3)$rd derivative of y^{4m-1+t}, we write P congruent to a linear combination of pp. of the form:

$$Q = y_1^{6m-3+2t} y_3^{2m-2} y_0^{i_0} y_1^{i_1} y_2^{i_2} y_3^{i_3} y_4^{i_4}$$

with (1) $i_0+i_1+i_2+i_3+i_4=4m-1+t$,

(2) $i_1+2i_2+3i_3+4i_4=12m-3+4t$.

We have $i_1+2i_2+3i_3+4i_4=4(3m-1+t)+1$, and moreover,

(3) $i_0<m$, as $i_0 \geqq m$ implies $i_1+i_2+i_3+i_4 \leqq 3m-1+t$, whence $3i_1+2i_2+i_3 \leqq -1$, a contradiction. We also note that

(4) $i_4>t$.

Using the relations (1) and (2), one verifies that Q can be written in the form:

$$Q = (y_0y_3^2)^{i_0}(y_1y_3)^{2m-2-2i_0+i_3}(y_1^2y_4)^{i_4}y_2^{i_2}.$$

We divide the Q into three types: type iii, those for which $2m-2-2i_0+i_3=i_2$, types i and ii take up the rest, type i having $i_2=0$ and type ii having $i_2\neq 0$.

For terms of type i we claim

(5) $i_0+(2m-2-2i_0+i_3)/3\leqq 2m-2$.

For if not, then $i_0+i_3+3>4m-1$, which with (1) gives $3-i_1-i_2-(i_4-t)>0$, and with (2) gives $9+3i_0-i_1-2i_2-4(i_4-t)>0$, the first of which yields $i_4-t=1$ or 2. If $i_4-t=2$, then $i_1=i_2=0$; and the last inequality makes $i_0=0$. Thus $i_3+(i_4-t)=4m-1$ and $3i_3+4i_4-4t=12m-3$, so that $i_4-t=0$ contradicting (4). Thus $i_4-t=1$. Recalling that $i_2=0$, we have $i_1=0$ or 1. But there are no terms of the form $y_0^{i_0}y_3^{i_3}y_4^{t+1}$ or $y_0^{i_0}y_1y_3^{i_3}y_4^{i_4}$ of the desired degree and weight. Hence (5) is established, and with it, from Lemma 2.8, we conclude that pp. of type i are zero modulo $[y^{4m-1+t}]$.

We now assert, for all Q,

(6) $2m-2-2i_0+i_3\geqq i_2$.

For if not, then $4i_0-2i_3+2i_2+3>4m-1$. From (1) and (2) we get

(7) $4i_0+3i_1+2i_2+i_3=4m-1$,

and with the last inequality, $3-3i_1-3i_3>0$, whence $i_1=i_3=0$. Hence $4i_0+2i_2=4m-1$, which is impossible, as an even number cannot be odd.

To take care of the other monomials we insert a lemma:

LEMMA 2.10a. *Let $p=4m-1$. If a pp. $Q=(y_0y_3^2)^a(y_1y_3)^b(y_1^2y_4)^cy_2^d$ has the same degree and weight as P, and if $a\leqq m-1$, $b>d$, and $b+2c\geqq 6m-3+2t$, then $Q\equiv O[y^{4m-1+t}]$.*

LEMMA 2.10b. *Let $p=4m+1$. If a pp. $S=(y_0y_3^2)^a(y_1y_3)^b(y_1^2y_4)^cy_2^d$ has the same degree and weight as P, and if $a\leqq m-1$, $b>d$, and $b+2c\geqq 6m+2t$, then $S\equiv O[y^{4m+1+t}]$.*

Proof a. Applying Lemmas 2.1b, 2.2 to Q, we can write Q congruent to a linear combination of pp. of the same form and with smaller d, namely,

$$(y_0y_3^2)^{a-1}(y_1y_3)^{b+2}(y_1^2y_4)^cy_2^{d-1}$$

$$(y_0y_3^2)^{a}(y_1y_3)^{b+1}(y_1^2y_4)^cy_2^{d-2}$$

$$(y_0y_3^2)^{a}(y_1y_3)^{b-1}(y_1^2y_4)^{c+1}y_2^{d-1}.$$

Thus we may assume $d=0$. With $d=0$ and $b\geqq 2m-2$, Q is of type i and hence $Q\equiv O[y^{4m-1+t}]$. With $d=0$, if $b<2m-2$, then $a+b/3<m-1+(2m-2)/3=(5/3)(m-1)<(p-1)/2$, so that by Lemma 2.8, $Q\equiv O[y^{4m-1+t}]$. This completes the proof of Lemma 2.10.

Continuing the proof of Lemma 2.9, Lemma 2.10 shows that all Q of type ii are zero. Thus we are left with terms of type iii. Terms of type iii are of the form:

$$(y_0y_3^2)^{m-k-1}(y_1y_4^2)^{3m-1-k-l+t}(y_1y_3)^{2k+l}y_2^{2k+l}$$
$$= y_1^{6m-3+2t}y_2^{2m-2}y_0^{m-k-1}y_1^{1-l}y_2^{2k+l}y_3^{l}y_4^{3m-1-k-l+t},$$

$0 \leqq k \leqq m-1$, $l \geqq 0$. One observes that $l \leqq 1$ since the term is divisible by $y_1^{6m-3+2t}$. By the formula given in the introduction we note that the coefficient for $l=0$ is

$$-\frac{3!^{4m-1}4!^t(4m-1)!t!}{2^{2k}4!^{3m-1-k+t}(2k)!(3m-1-k+t)!(m-k-1)!},$$

and the coefficient for $l=1$ is

$$-\frac{3!^{4m-1}4!^t(4m-1)!t!}{2^{2k+1}3!4!^{3m-2-k+t}(2k+1)!(3m-2-k+t)!(m-k-1)!}.$$

Suppose $l=1$. An application of Lemmas 2.1b, 2.2 to such a term yields it to be congruent to a linear combination of three pp.:

$$(y_0y_3^2)^{m-k-2}(y_1y_4^2)^{3m-2-k+t}(y_1y_3)^{2k+1}y_2^{2k} \qquad .$$
$$(y_0y_3^2)^{m-k-1}(y_1y_4^2)^{3m-2-k+t}(y_1y_3)^{2k+2}y_2^{2k-1}$$
$$(y_0y_3^2)^{m-k-1}(y_1y_4^2)^{3m-1-k+t}(y_1y_3)^{2k}y_2^{2k},$$

where the coefficient of the last term, Q_k, is

$$-\frac{2m-1}{6m-2+2t}.$$

The first two pp. are zero by Lemma 2.10. Thus we have $P \equiv \sum_{k=0}^{m-1} c_kQ_k$ as stated.

REMARK. By the last paragraph we see that

$$c_k = -\frac{3!^{4m-1}4!^t(4m-1)!t!}{2^{2k}4!^{3m-1-k+t}(2k)!(3m-1-k+t)!(m-k-1)!},$$
$$+\frac{(2m-1)}{(6m-2+2t)}\frac{3!^{4m-1}4!^t(4m-1)!t!}{2^{2k+1}3!4^{3m-2-k+t}(2k+1)!(3m-2-k+t)!(m-k-1)!}.$$

Similarly one finds

$$d_0 = -\frac{3!^{4m+1}4!^t(4m+1)!t!}{3!4!^{3m+t}(3m+t)!m!},$$

$$d_k = -\frac{3!^{4m+1}4!^t(4m+1)!(t)!}{2^{2k-1}4!^{3m-k+1+t}(2k-1)!(3m-k+1+t)!(m-k)!}$$
$$+\frac{(2m)}{(6m+1+2t)}\frac{3!^{4m+1}4!^t(4m+1)!t!}{2^{2k}3!4!^{3m-k+t}(2k)!(3m-k+t)!(m-k)!},$$

when

$$k > 0.$$

LEMMA 2.11a. *Let $p=4m-1$, then for $0\leqq k\leqq m-1$,*

$$Q_k \equiv \frac{(2m-2)!(6m-2+2t)!}{(2m-2k-2)!(6m-2+2k+2t)!}$$
$$\times y_0^{m-k-1}y_1^{6m-2+2t+2k}y_2^{2m-2-2k}y_4^{3m-1+k+t}\left[y^{4m-1+t}\right].$$

LEMMA 2.11b. *Let $p=4m+1$, then for $1\leqq k\leqq m$,*

$$S_k \equiv (-1)^{2k-1}\frac{(2m-1)!(6m+1+2t)!}{(2m-2k)!(6m+2k+2t)!}$$
$$\times y_0^{m-k}y_1^{6m+2k+2t}y_3^{2m-2k}y_4^{3m+k+t}\left[y^{4m+1+t}\right].$$

Proof a. Let

$$Q_{k,j} = y_0^{m-k-1}y_1^{6m-2+2t+j}y_2^{2k-j}y_3^{2m-2j}y_4^{3m-1-k+t+j}$$
$$= (y_0y_3^2)^{m-k-1}(y_1y_4^2)^{3m-k+t+j-1}(y_1y_3)^{2k-j}y_2^{2k-j}.$$

We will prove that

$$Q_{k,j} \equiv \frac{-(2m-2-j)}{6m-1+2t+j}Q_{k,j+1}\left[y^{4m-1+t}\right],$$

$0\leqq j<2k$. By Lemmas 2.1b, 2.2 we may write $Q_{k,j}$ congruent to a linear combination of three monomials:

$$(y_0y_3^2)^{m-k-2}(y_1y_4^2)^{3m-1-k+t+j}(y_1y_3)^{2k+2-j}y_2^{2k-j-1},$$
$$(y_0y_3^2)^{m-k-1}(y_1y_4^2)^{3m-1-k+t+j}(y_1y_3)^{2k+1-j}y_2^{2k-j-2},$$
$$(y_0y_3^2)^{m-k-1}(y_1y_4^2)^{3m-k+t+j}(y_1y_3)^{2k-1-j}y_2^{2k-j-1}.$$

The first two pp. are zero modulo $[y^{4m-1+t}]$ by Lemma 2.10, and the third pp. is $Q_{k,j+1}$. Taking into account the coefficients, we have

$$Q_{k,j} \equiv \frac{-(2m-2-j)}{6m-1+2t+j}\, Q_{k,j+1}[y^{4m-1+t}],$$

and Lemma 2.11a follows.

LEMMA 2.12a. *Let $p=4m-1$, then for $1 \leqq k \leqq m$, $k \leqq j \leqq m$,*

$$y_0^{m-k} y_1^{6m-4+2k+2t} y_2^{m-j} y_3^{2j-2k} y_4^{4m-2+k+t-j}$$
$$\equiv \left(-\frac{3}{2}\right)^{j-k} \frac{(2j-2k)!(4m+k-2+t-j)!}{2^{j-k}(j-k)!(4m-2+t)!}$$
$$\times\, y_0^{m-k} y_1^{6m-4+2k+2t} y_2^{m-k} y_4^{4m-2+t}[y^{4m-1+t}].$$

LEMMA 2.12b. *Let $p=4m+1$, then for $0 \leqq k \leqq m$, $k \leqq j \leqq m$,*

$$y_0^{m-k} y_1^{6m+2t+2k} y_2^{m-j} y_3^{2j-2k} y_4^{4m+t+k-j}$$
$$\equiv \left(-\frac{3}{2}\right)^{j-k} \frac{(2j-2k)!(4m+t+k-j)!}{2^{j-k}(j-k)!(4m+t)!}$$
$$\times\, y_0^{m-k} y_1^{6m+2t+2k} y_2^{m-k} y_4^{4m+t}[y^{4m+1+t}].$$

Proof. Let

$$Q_j = y_0^{m-k} y_1^{6m-4+2k+2t} y_2^{m-j} y_3^{2j-2k} y_4^{4m-2+k+t-j}.$$

We will show

$$Q_j \equiv \left(-\frac{3}{2}\right)\frac{2j-2k-1}{4m+k+t-1-j}\, Q_{j-1}[y^{4m-1+t}].$$

Let $Q_j \leftrightarrow \overline{Q}_j$ under the mapping $y_i \leftrightarrow y_{4-i}$, $0 \leqq i \leqq 4$. By an application of Lemmas 2.1a, 2.2 to

$$\overline{Q}_j = y_0^{4m-2+k+t-j} y_1^{2j-2k} y_2^{m-j} y_3^{6m-4+2k+2t} y_4^{m-k},$$

we see that $\overline{Q}_j$ is congruent to a linear combination of the following three pp.:

$$y_0^{4m-1+k+t-j} y_1^{2j-2k-2} y_2^{m-j+1} y_3^{6m-4+2k+2t} y_4^{m-k},$$

$$y_0^{4m-1+k+t-j} y_1^{2j-2k-1} y_2^{m-j-1} y_3^{6m-3+2k+2t} y_3^{m-k},$$

$$y_0^{4m-1+k+t-j} y_1^{2j-2k-1} y_2^{m-j} y_3^{6m-5+2k+2t} y_4^{m-k+1}.$$

Of these three terms, the first is $\overline{Q}_{j-1}$. By Lemma 2.4, the second two are in $[y^p]$ if the following are:

$$y_0^{m-k} y_1^{6m-3+2k+2t} y_2^{m-j-1} y_3^{2j-2k-1} y_4^{4m-1+k+t-j},$$

$$y_0^{m-k+1} y_1^{6m-5+2k+2t} y_2^{m-j} y_3^{2j-2k-1} y_4^{4m-1+k+t-j},$$

and this is so since

$$w(y_0^{m-k} y_1^{6m-3+2k+2t}) < f(4m-1+t, 7m-3+k+2t) = 6m-2+2k+2t,$$

and

$$w(y_0^{m-k+1} y_1^{6m-5+2k+2t}) < f(4m-1+t, 7m-4+k+2t) = 6m-4+2k+2t.$$

Taking into account the coefficients, we have

$$\overline{Q}_j \equiv -\frac{2j-2k-1}{4m-1+k+t-j}\overline{Q}_{j-1}[y^{4m-1+t}].$$

Hence we have:

$$\overline{Q}_j \equiv^M -\frac{2j-2k-1}{4m-1+k+t-j}\,\frac{1}{m(\overline{Q}_j, \overline{Q}_{j-1})}\overline{Q}_{j-1}[y^{4m-1+t}],$$

$$Q_j \equiv^M -\frac{2j-2k-1}{4m-1+k+t-j}\,\frac{1}{m(\overline{Q}_j, \overline{Q}_{j-1})}Q_{j-1}[y^{4m-1+t}],$$

(by Lemma 2.4), and

$$Q_j \equiv -\frac{2j-2j-1}{4m-1+k+t-j}\,\frac{m(Q_j, Q_{j-1})}{m(\overline{Q}_j, \overline{Q}_{j-1})}Q_{j-1}[y^{4m-1+t}].$$

We also have

$$\frac{m(Q_j, Q_{j-1})}{m(\overline{Q}_j, \overline{Q}_{j-1})} = \frac{3}{2}$$

and our lemma follows.

LEMMA 2.13a. *Let* $p=4m-1$, *then for* $1 \leqq k \leqq m$,

$$y_0^{m-k} y_1^{6m-4+2t+2k} y_2^{m-k} y_4^{4m-2+t}$$
$$\equiv (-1)^{m-k}\frac{(m-k)!}{(8m-5+2t)(8m-7+2t)\cdots(6m-3+2k+2t)}$$
$$\times\, y_1^{8m-4+2t} y_4^{4m-2+t}[y^{4m-1+t}].$$

LEMMA 2.13b. *Let* $p=4m+1$, *then for* $0 \leqq k \leqq m$,

$$y_0^{m-k}\,y_1^{6m+2k+2t}\,y_2^{m-k}\,y_4^{4m+t}$$

$$\equiv (-1)^{m-k}\frac{(m-k)!}{(8m-1+2t)(8m-3+2t)\cdots(6m+1+2k+2t)}$$

$$\times\, y_1^{8m+2t}\,y_4^{4m+t}\left[y^{4m+1+t}\right].$$

Proof a. By Lemmas 1.1, 1.2 we have

$$y_1^{8m-4+2t} \equiv -(8m-5+2t)\,y_0 y_1^{8m-6+2t}\,y_2\left[y^{4m-1+t}\right].$$

A second application of Lemmas 1.1, 1.2 gives

$$y_1^{8m-4+2t} \equiv \frac{(8m-5+2t)}{1}\,\frac{(8m-7+2t)}{2}\,y_0^2 y_1^{8m-8+2t}\,y_2^2\left[y^{4m-1+t}\right].$$

Repeated applications of Lemmas 1.1, 1.2 gives

$$y_1^{8m-4+2t} \equiv (-1)^j\,\frac{(8m-5+2t)(8m-7+2t)\cdots(8m-3+2t-2j)}{j!}$$

$$\times\, y_0^j y_1^{8m+2t-(2j+4)}\,y_2^j\left[y^{4m-1+t}\right].$$

Taking $j=m-k$ and multiplying by y_4^{4m-2+t} we have our lemma.

LEMMA 2.14a. *Let $p=4m-1$, then for $0\leqq k\leqq m-1$,*

$$Q_k \equiv \left(\frac{3}{2}\right)^{m-k-1}\frac{(2m-2)!(6m-2+2t)!}{(8m-4+2t)!}\,y_1^{8m-4+2t}\,y_4^{4m-2+t}\left[y^{4m-1+t}\right].$$

LEMMA 2.14b. *Let $p=4m+1$, then*

$$S_0 \equiv \left(\frac{3}{2}\right)^{m}\frac{(2m)!(6m+2t)!}{(8m+2t)!}\,y_1^{8m+2t}\,y_4^{4m+t}\left[y^{4m+1+t}\right],$$

$$S_k \equiv -\left(\frac{3}{2}\right)^{m-k}\frac{(2m-1)!(6m+2t+1)!}{(8m+2t)!}\,y_1^{8m+2t}\,y_4^{4m+t}\left[y^{4m+1+t}\right],\quad 1\leqq k\leqq m.$$

Proof a. Lemmas 2.11, 2.12 with $j=m$, and 2.13.

Continuing with the proof of Theorem 2.5, we have by Lemma 2.14, if $p=4m-1$,

$$P = y_1^{6m-3+2t}\,y_3^{6m-3}\,y_4^{t}$$

$$\equiv \text{Constant}\times\sum_{j=1}^{m}\frac{2^{2j-2}\{6m^2-8mj+3j-1+(2m-2j)t\}}{(2j-1)!(m-j)!(3m-j+t)!}$$

$$\times\, y_1^{8m-4+2t}\,y_4^{4m-2+t}\left[y^{4m-1+t}\right];$$

or if $p=4m+1$,

$$P = y_1^{6m+2t} y_3^{6m} y_4^{t}$$
$$\equiv c \sum_{j=0}^{m} \frac{2^{2j+2}(6m^2 - 8mj - 2jt + 2mt + 2m - j)}{(m-j)!(2j)!(3m-j+t+1)!} y_1^{8m+2t} y_4^{4m+t} [y^{4m+1+t}].$$

Theorem 2.5 will follow from Lemma 2.15.

LEMMA 2.15a. *For $p=4m-1$,*

$$\sum_{j=1}^{m} \frac{2^{2j-2}(6m^2 - 8mj + 3j - 1 + 2mt - 2jt)}{(2j-1)!(m-j)!(3m-j+t)!} = 0.$$

LEMMA 2.15b. *For $p=4m+1$,*

$$\sum_{j=0}^{m} \frac{2^{2j+2}(6m^2 - 8mj - 2jt + 2mt + 2m - j)}{(m-j)!(2j)!(3m-j+t+1)!} = 0.$$

Proof a.

$$\sum_{j=1}^{m} \frac{2^{2j-2}(6m^2 - 8mj + 3j - 1 + (2m-2j)t)}{(2j-1)!(m-j)!(3m-j+t)!}$$
$$= \sum_{j=1}^{m} \frac{2^{2j-1}(3m-j+t)(m-j)}{(2j-1)!(m-j)!(3m-j+t)!}$$
$$- \sum_{j=0}^{m-1} \frac{2^{2j}(2j+1)(j)}{(2j+1)!(m-j-1)!(3m-j-1+t)!}$$
$$= \sum_{j=0}^{m} \frac{2^{2j}j(2j+1)(3m-j+t)(m-j)}{(2j+1)!(m-j)!(3m-j+t)!} - \sum_{j=0}^{m-1} \frac{2^{2j}j(2j+1)(3m-j+t)(m-j)}{(2j+1)!(m-j)!(3m-j+t)!}$$
$$= 0.$$

THEOREM 2.16. *The smallest q such that $y_2^q \equiv O[y^p]$ is $q=3p-2$.*

Proof. By Lemma 2.0, $y_2^{3p-2} \equiv O[y^p]$. By Theorems 2.3, 2.5 $y_2^{3p-3} \equiv c y_1^{2p-2} y_4^{p-1}[y^p]$, $c \neq 0$. By Corollary 1.4, $y_1^{2p-2} \equiv d y_0^{p-1} y_2^{p-1}[y^p]$, $d \neq 0$. Hence $y_2^{3p-3} \equiv cd y_0^{p-1} y_2^{p-1} y_4^{p-1}[y^p]$, $cd \neq 0$ whence $y_2^{3p-3} \not\equiv O[y^p]$ by Theorem 0.4.

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