# ON THE DIMENSIONS OF THE IRREDUCIBLE MODULES OF LIE ALGEBRAS OF CLASSICAL TYPE(1)

BY

CHARLES W. CURTIS

**Introduction.** In this and another paper [4], partial answers are given to two of the questions raised in §II.5 of [3]. Because the results in [4] hold for a more extensive class of Lie algebras of classical type than those considered in this paper, the two questions have been treated separately. Let $\mathfrak{L}$ be a Lie algebra over an algebraically closed field $\Omega$ of characteristic $p>7$ whose Killing form is nondegenerate, and let $M$ be an irreducible restricted right $\mathfrak{L}$-module. We shall give a computable sufficient condition on $M$ in order for Weyl's formula [9, p. 359], suitably interpreted, to give the dimension of $M$ over $\Omega$. The problem of calculating the dimension of $M$ in all cases remains unsolved. In the last section we obtain an upper bound for the dimension of any irreducible restricted right $\mathfrak{L}$-module, and from this result it follows that Weyl's formula as we have interpreted it does not always give the dimension of $M$.

1. **Definitions and preliminary results.** Familiarity with the papers [1; 2] and [3] is assumed. First we list some notations.

$\Omega$ algebraically closed field of characteristic $p>7$;

$\mathfrak{L}$ Lie algebra over $\Omega$ whose Killing form is nondegenerate;

$M$ irreducible restricted right $\mathfrak{L}$-module;

$\lambda^*$ maximal weight of $M$;

$\Delta=\{\alpha_1, \cdots, \alpha_l\}$ a maximal simple system of roots of $\mathfrak{L}$ with respect to a fixed Cartan subalgebra $\mathfrak{H}$ (see [2, p. 97]);

$h_i$ the unique element in $[\mathfrak{L}_{-\alpha_i}\mathfrak{L}_{\alpha_i}]$ such that $\alpha_i(h_i)=2$.

By Corollary 1, p. 107, of [2], $\Delta$ is a linearly independent set, and it follows that $h_1, \cdots, h_l$ is a basis of $\mathfrak{H}$, so that $\lambda^*$ is uniquely determined by the vector $(\lambda^*(h_1), \cdots, \lambda^*(h_l))$ with coefficients in the prime field $\Omega_0$ of $\Omega$. By the main theorem of [2, p. 104], there exists a semi-simple Lie algebra $\mathfrak{L}'$ over the complex field with the following properties. Let $\mathfrak{H}'$ be a Cartan subalgebra of $\mathfrak{L}'$. Then there exists a one-to-one mapping $\alpha\rightarrow\alpha'$ of the set of roots of $\mathfrak{L}$ with respect to $\mathfrak{H}$ onto the set of roots of $\mathfrak{L}'$ with respect to $\mathfrak{H}'$ which is compatible with the operations of addition and subtraction defined on the system of roots. If $\Delta'=\{\alpha_1', \cdots, \alpha_l'\}$ then $\Delta'$ is a maximal simple system of roots of $\mathfrak{L}'$. For each root $\alpha'$ of $\mathfrak{L}'$, let $H_{\alpha'}$ be the unique element in $[\mathfrak{L}'_{-\alpha'}, \mathfrak{L}'_{\alpha'}]$ such that $\alpha'(H_{\alpha'})=2$, and let $H_i=H_{\alpha'_i}$, $1\leqq i\leqq l$. We can choose

Received by the editors August 26, 1959.

(1) This research was supported in part by the Office of Naval Research.

$E_{\alpha'} \in \mathfrak{L}'_{\alpha'}$ such that $[E_{-\alpha'}, E_{\alpha'}] = H_{\alpha'}$ for all $\alpha'$, and the elements $E_{\alpha'}$ together with the $H_{i, 1 \leqq i \leqq l}$, form a basis $(X_i)$ of $\mathfrak{L}'$ such that all the constants of structure of this basis belong to the ring $Z_p$ of $p$-adic integers (see [2, pp. 102–104]). Let $\mathfrak{L}_0'$ be the Lie ring $\sum Z_p X_i$; then $\mathfrak{L}_0'/p\mathfrak{L}_0'$ is a Lie algebra over $\Omega_0$ which becomes isomorphic to $\mathfrak{L}$ upon extension of the base field from $\Omega_0$ to $\Omega$.

Let $\mathfrak{F}$ be the set of *integral linear functions* on $\mathfrak{H}'$ i.e. $\mathfrak{F}$ consists of those linear functions $\lambda$ such that $\lambda(H_i)$ is a rational integer for $1 \leqq i \leqq l$. The dominant integral functions are those such that $\lambda(H_i) \geqq 0$, $1 \leqq i \leqq l$. The elements of $\mathfrak{F}$ span a vector space over the field of rational numbers, and this vector space carries a scalar product whose value at the pair $\lambda$, $\mu$ in $\mathfrak{F}$ is given as follows. Let $B(X, Y)$ be the Killing form on $\mathfrak{L}'$, and for each $\lambda \in \mathfrak{F}$ let $J_\lambda$ be the unique element in $\mathfrak{H}'$ such that $\lambda(H) = B(J_\lambda, H)$ for all $H$ in $\mathfrak{H}'$. The value of the scalar product of $\lambda$ and $\mu$ is given by $\langle \lambda, \mu \rangle = B(J_\lambda, J_\mu)$, and it is known that $\langle \lambda, \mu \rangle$ is a rational number for $\lambda$, $\mu$ in $\mathfrak{F}$. Because in this case the Killing form is nondegenerate modulo $p$ we shall prove that we have

$$\langle \lambda, \mu \rangle \in Z_p, \qquad \lambda, \mu \in \mathfrak{F}. \tag{1}$$

Let $\lambda \in \mathfrak{F}$; then $\lambda = \sum_{i=1}^{l} q_i \alpha_i'$ where the $q_i$ are rational numbers. Consider the system of equations

$$\lambda(H_j) = \sum_{i=1}^{l} q_i \alpha_i'(H_j), \qquad j = 1, 2, \cdots, l.$$

The numbers $\lambda(H_j)$ belong to $Z_p$, and the $\alpha_i'(H_j)$ are elements of $Z_p$ such that $\det(\alpha_i'(H_j)) \not\equiv 0 \pmod{p}$ (see [1, p. 164]). It follows that the $q_i \in Z_p$ for all $i$, and since $\langle \alpha_i', \alpha_j' \rangle \in Z_p$[2], $1 \leqq i, j \leqq l$, we have proved (1).

Now let $\{H_i\}$ and $\{K_i\}$ be dual bases of $\mathfrak{H}'$ with respect to the Killing form, and choose elements $F_{\alpha'} \in \mathfrak{L}'$ such that $[F_{-\alpha'} F_{\alpha'}] = J_{\alpha'}$ for each root $\alpha' \neq 0$. We then form the *Casimir element*

$$\Gamma = \sum_{\alpha' \neq 0} F_{-\alpha'} F_{\alpha'} + \sum_{i=1}^{l} H_i K_i,$$

which is known to belong to the center of the universal associative algebra $U(\mathfrak{L}')$ of $\mathfrak{L}'$. For each irreducible finite dimensional right $\mathfrak{L}'$-module $V$ with maximal weight $\Lambda$, we have

$$v\Gamma = \gamma(\Lambda) v \qquad v \in V,$$

where $\gamma(\Lambda)$ is a fixed nonnegative rational number which is given explicitly by

$$\gamma(\Lambda) = \langle \Lambda, \Lambda \rangle + \sum_{\alpha' > 0} \langle \Lambda, \alpha' \rangle = \langle \Lambda, \Lambda + 2\rho \rangle \tag{2}$$

---

[2] To prove this assertion, we begin with the known result that $2\langle \alpha_i', \alpha_j' \rangle \langle \alpha_j', \alpha_j' \rangle^{-1}$ is a rational integer for all $i$ and $j$. Then by the argument in the proof of Proposition 6, p. 104 of [2], we know that $\langle \alpha_j', \alpha_j' \rangle = B(J_{\alpha'_j}, J_{\alpha'_j})$ is a unit in $Z_p$, and hence $\langle \alpha_i', \alpha_j' \rangle \in Z_p$ as required.

where $\rho$ is one half the sum of the positive roots of $\mathfrak{L}'$ with respect to $\mathfrak{H}'$. The number $\gamma(\lambda)$ is defined by (2) for every dominant integral function $\lambda$, and is by (1) an element of $Z_p$. In particular if $\lambda$ is a dominant integral function which is a weight of $V$, then it is known [5, p. 373] that

$$\gamma(\lambda) < \gamma(\Lambda), \qquad \lambda \neq \Lambda. \tag{3}$$

Any $\mathfrak{L}'$-module $V$ is also a completely reducible module for the three dimensional simple subalgebra $\mathfrak{U}_{\alpha'}$ of $\mathfrak{L}'$ with basis $E_{\alpha'}$, $E_{-\alpha'}$, $H_{\alpha'}$, where $\alpha'$ is any nonzero root. This fact together with the explicit determination of the irreducible $\mathfrak{U}_{\alpha'}$-modules will be used frequently throughout the paper to obtain information about the linear transformations on $V$ induced by $E_{\alpha'}$ and $E_{\alpha'}E_{-\alpha'}$. For any $\alpha' \neq 0$, the irreducible modules of $\mathfrak{U}_{\alpha'}$ are described as follows. There exists one and only one module of dimension $k$ for each $k=1, 2, \cdots$, and if $[v_0, v_1, \cdots, v_{k-1}]$ is a basis for this module, the action of $\mathfrak{U}_{\alpha'}$ is given by

$$\begin{aligned} v_iE_{-\alpha'} &= v_{i+1},\ 0 \leqq i \leqq k-2; \qquad v_{k-1}E_{-\alpha'} = 0; \\ v_iE_{\alpha'} &= i(k-i)v_{i-1}, \qquad 0 \leqq i \leqq k-1; \\ v_iH_{\alpha'} &= (k-1-2i)v_i, \qquad 0 \leqq i \leqq k-1. \end{aligned} \tag{4}$$

From these formulas it is immediate that $VE^m_{\alpha'}=0$ for all roots $\alpha' \neq 0$ if and only if

$$l(V) = \max_{\alpha',\lambda} \left| \lambda(H_{\alpha'}) \right| < m,$$

where the maximum is taken over all roots $\alpha' \neq 0$ and all weight $\lambda$ of $V$ (see [6, p. 306]).

All these preparations are brought to bear on the irreducible $\mathfrak{L}$-module $M$ by means of the following definition.

DEFINITION. Let $\Lambda$ be the dominant integral function such that $\Lambda(H_i)$ is the integer, $0 \leqq \Lambda(H_i) < p$, whose residue modulo $p$ is equal to $\lambda^*(h_i)$, $1 \leqq i \leqq l$. Let $V$ be the irreducible $\mathfrak{L}'$-module whose maximal weight is $\Lambda$. Then $V$ is called the *associated module* of $M$.

Now we can state the principal result of the paper.

THEOREM 1. *Let $M$ be an irreducible restricted $\mathfrak{L}$-module whose associated module $V$ satisfies the following conditions*: (i) $l(V) < p$; *and* (ii) *for any dominant weight $\lambda$ different from the maximal weight $\Lambda$ of $V$*, $\gamma(\lambda) \not\equiv \gamma(\Lambda)$ (mod $p$). *Then we have* $\dim M = \dim V$, *and this dimension is given by Weyl's formula*

$$\dim M = \dim V = \prod_{\alpha'>0} \frac{(\Lambda+\rho)(H_{\alpha'})}{\rho(H_{\alpha'})}. \tag{5}$$

2. **Proof of Theorem** 1. We remark first that in view of (3) the hypothesis (ii) amounts to the assumption that the inequalities (3) cannot be replaced

by congruence modulo $p$. We point out also that it is sufficient to prove that $\dim M = \dim V$, since the formula (5) is known to give the dimension of $V$.

We begin with an observation about modules over $Z_p$. Let $Q$ be a free $Z_p$-module with basis $x_1, \cdots, x_n$. A set of vectors $y_1, \cdots, y_m$ is called a *p-independent set* provided that $\sum a_i y_i \in pQ$ implies $a_i \in pZ_p$ for any set of $a_i$ from $Z_p$. We shall prove that *a set $y_1, \cdots, y_n$ of vectors in $Q$ is a $Z_p$-basis if the $y_i$ are p-independent.* Let $y_i = \sum a_{ij} x_j$, $a_{ij} \in Z_p$. Because the $y_i$'s are $p$-independent, we have $\det (a_{ij}) \not\equiv 0 \pmod p$. Since an element of $Z_p$ not divisible by $p$ is a unit, the matrix $(a_{ij})$ is invertible in the set of $n$ by $n$ matrices with coefficients in $Z_p$. From this it follows that $y_1, \cdots, y_n$ is a $Z_p$-basis of $Q$.

Now let $V$ be the associated module of $M$, and let $v$ be a maximal vector in $V$. Let $V_0$ be the smallest $Z_p$-submodule of $V$ containing $v$ and invariant relative to $\mathfrak{L}_0'$. Then $V_0$ is a finitely generated free $Z_p$-module which spans $V$ over the complex field, and by the argument on p. 170 of [1], $V_0$ is generated by vectors of the form $vE_{\gamma'_1} \cdots E_{\gamma'_s}$, $s \geqq 0$, where the $\gamma_i'$ are negative roots. Because these vectors are all weight vectors it follows that

$$V_0 = \sum_\lambda \oplus (V_0 \cap V_\lambda) \tag{6}$$

where the sum is taken over the weights $\lambda$ of $V$. From (6) it is clear that $V_0$ has a $Z_p$-basis consisting of weight vectors.

LEMMA 1. *Suppose that $l(V) < p$, and let $\alpha_i \in \Delta$. Then $V_0$ has a $Z_p$-basis $(v_j^{(i)}) 1 \leqq j \leqq \dim V$ of weight vectors $v_j^{(i)}$ which are proper vectors of $E_{\alpha'_i} E_{-\alpha'_i}$ with proper values which are either zero or units in $Z_p$.*

**Proof.** We begin with some known facts about representation of the three dimensional simple Lie algebra $\mathfrak{U}_{\alpha'_i}$ which are more or less immediate consequences of the formulas (4). Let $v \in V_\lambda$ be a weight vector such that $vE_{\alpha'_i} = 0$. Then the vectors $\{vE^k_{-\alpha'_i}\}$, $k \geqq 0$, generate an irreducible $\mathfrak{U}_{\alpha'_i}$-submodule of $V$. Because $0 \leqq \lambda(H_i) \leqq l(V) < p$, $vE^p_{-\alpha'_i} = 0$ and the nonzero vectors $vE^k_{-\alpha'_i}$ are weight vectors, and are proper vectors of $E_{\alpha'_i} E_{-\alpha'_i}$ whose proper values are either zero or units in $Z_p$. Now let $N^i$ be the null space of $E_{\alpha'_i}$ in $V$, and let $N_0^i = N^i \cap V_0$. We prove first that

$$N_0^i = \sum_\lambda \oplus (N^i \cap V_\lambda \cap V_0). \tag{7}$$

It is sufficient to prove that $N_0^i$ is contained in the sum on the right hand side. Let $v \in N_0^i$; then by (6), $v = \sum v_\lambda$ where the $v_\lambda \in V_\lambda \cap V_0$, and the $\lambda$'s are distinct weights. Applying $E_{\alpha'_i}$ we have

$$0 = vE_{\alpha'_i} = \sum_\lambda v_\lambda E_{\alpha'_i},$$

and since the $v_\lambda E_{\alpha'_i}$ belong to the distinct weights $\lambda + \alpha_i'$, we have $v_\lambda E_{\alpha'_i} = 0$ for all $\lambda$, and the $v_\lambda \in N^i \cap V_\lambda \cap V_0$ as required.

Because of (7) we can find a $Z_p$-basis of $N_0^i$ consisting of weight vectors $w_j$; then the $w_j$ span $N^i$ over the complex field. From the formulas (4) and the fact that $V$ is a completely reducible $\mathfrak{U}_{\alpha'_i}$-module, it follows that the nonzero vectors $w_jE_{-\alpha'_i}^k$ are linearly independent and span $V$ over the complex field. These vectors satisfy all the requirements of the lemma if we can prove that they are $p$-independent, for then by the remark preceding Lemma 1, they will form a $Z_p$-basis of $V_0$. Let

$$w = \sum a_{jk}w_jE_{-\alpha'_i}^k \in pV_0, \qquad a_{jk} \in Z_p.$$

Write $w$ in the form

$$w = \sum a_{jN}w_jE_{-\alpha'_i}^N + \sum a_{j,N-1}w_jE_{-\alpha'_i}^{N-1} + \cdots;$$

then from the formulas (4) we have

$$wE_{\alpha'_i}^N = \sum a_{jN}\zeta_{jN}w_j \in pV_0,$$

where the $\zeta_{jN}$ are units in $Z_p$. Because the $w_j$ are $p$-independent we have $a_{jN}\in pZ_p$, and the terms $\sum a_{jN}w_jE_{-\alpha_i}^N$ can be dropped from the expression for $w$. Continuing in this way we see that the $w_jE_{-\alpha'_i}^k$ are $p$-independent, and Lemma 1 is proved.

Lemma 2. *Let $M$ be an irreducible restricted $\mathfrak{L}$-module with maximal weight $\lambda^*$, and let $V$ be the associated module $M$. Then $\overline{V}=(V_0/pV_0)^\Omega$ is a (not necessarily restricted) $\mathfrak{L}$-module, and $\overline{V}$ has a restricted homomorphic image $R$ which is $\mathfrak{L}$-isomorphic to $M$*(3).

**Proof.** We introduce first some notation which will be used here and later in the proof of the theorem. Let $X\rightarrow X^*$ be the natural mapping of $\mathfrak{L}_0'\rightarrow\mathfrak{L}_0'/p\mathfrak{L}_0'$. Because $(\mathfrak{L}_0'/p\mathfrak{L}_0')^\Omega=\mathfrak{L}$, we may regard the $X^*$ as elements of $\mathfrak{L}$ which span $\mathfrak{L}$ over $\Omega$. For any $\eta\in Z_p$ we let $\eta^*$ be the image of $\eta$ under the natural map of $Z_p\rightarrow\Omega_0$. The elements $H_i^*$, $1\leqq i\leqq l$ form a basis of $\mathfrak{H}$, and we may assume that $\mathfrak{L}_\alpha=\Omega E_{\alpha'}^*$ for all roots $\alpha\neq 0$. We shall denote the image of $v$ in $V_0$ under the natural map of $V_0\rightarrow V_0/pV_0$ by $[v]$. $\overline{V}$ becomes a right $\mathfrak{L}$-module if we define

$$a^*[v] = [av], \qquad a \in Z_p, \qquad v \in V_0,$$

and

$$[v]X^* = [vX], \qquad v \in V_0, \qquad X \in \mathfrak{L}_0'.$$

If $\Lambda$ is the maximal weight of $V$ and $v_0$ the maximal vector in $V_0$ which generates $V_0$, then $[v_0]$ is a maximal vector in $\overline{V}$ of weight $\lambda^*$. Let $S$ be a maximal $\mathfrak{L}$-submodule of $\overline{V}$ which does not contain $[v_0]$; then $R=\overline{V}/S$ is an irreducible

(3) This lemma is an improvement of Theorem 6 of [1].

$\mathfrak{L}$-module with maximal weight $\lambda^*$. If we can show that $R$ is a restricted $\mathfrak{L}$-module, then $R \cong M$ by Theorem 1 of [3]. It is sufficient to prove that $R(E^*_{\alpha'})^p = 0$ for all roots $\alpha'$ and $R[(H^*_i)^p - H^*_i] = 0$. In any case $(E^*_{\alpha'})^p$ belongs to the centralizer of $R$, so that by Schur's Lemma $(E^*_{\alpha'})^p = \xi \cdot 1$ for some $\xi \in \Omega$. On the other hand, $E_{\alpha'}$ is nilpotent on $V$, hence $E^*_{\alpha'}$ is nilpotent on $R$, and we have $\xi = 0$. In order to discuss $(H^*_i)^p - H^*_i$, we recall that $\overline{V}$ is spanned by the elements $\bar{v} = [v_0]E^*_{\gamma'_1}, \cdots, E^*_{\gamma'_s}$, and for such an element we have

$$\begin{aligned} \bar{v}H_i^* &= [v_0]E^*_{\gamma'_1}, \cdots, E^*_{\gamma'_s}H^*_i = [v_0 E_{\gamma'_1}, \cdots, E_{\gamma'_s}H_i] \\ &= [\mu(H_i)v_0] = \mu(H_i)^* v_0 \end{aligned}$$

where $\mu = \Lambda + \gamma'_1 + \cdots + \gamma'_s$ and $\Lambda$ is the maximal weight of $V$. Since $\mu(H_i)^* \in \Omega_0$, $\bar{v}(H^*_i)^p - \bar{v}H^*_i = 0$, and we have proved that $R$ is restricted. This completes the proof of Lemma 2.

Now consider the Casimir element $\Gamma$. Since we can express $F_{\alpha'} = 2^{-1}\langle \alpha', \alpha' \rangle E_{\alpha'}, \alpha' > 0, F_{-\alpha'} = E_{-\alpha'}, \alpha' > 0$, and solve the equations $K_i = \sum b_{ij}H_j$ for $b_{ij} \in Z_p$ because $B(K_i, H_j) = \delta_{ij}, \cdots$, we can assume that the elements $F_{\alpha'}$, $H_i$, and $K_i$ all belong to $\mathfrak{L}'_0$. Therefore $\Gamma^* = \sum F^*_{-\alpha'}F^*_{\alpha'} + \sum H^*_i K^*_i$ is an element of the center of the universal associative algebra $U(\mathfrak{L})$ of $\mathfrak{L}$.

LEMMA 3. *Let $Q$ be a restricted right $\mathfrak{L}$-module and let $N$ be an irreducible composition factor of $Q$. Suppose that $\Gamma^* = \delta \cdot 1$ on $Q$ for some $\delta \in \Omega$. Then $\delta = \gamma(\mu)^*$, where $\mu$ is the maximal weight of the associated module $W$ of $N$.*

**Proof.** On $W$ we have $\Gamma = \gamma(\mu) \cdot 1$. Therefore on $\overline{W} = (W_0/pW_0)^\Omega$ we have $\Gamma^* = \gamma(\mu)^* \cdot 1$. By Lemma 2, $\overline{W}$ has a homomorphic image isomorphic to $N$, and it follows at once from this that $\delta = \gamma(\mu)^*$.

Finally we come to the proof of the theorem. Because $l(V) < p$, we have $V_0(E^*_{\alpha'})^p = 0$ for all roots $\alpha'$, and it follows that $\overline{V}$ is a restricted $\mathfrak{L}$-module. We suppose the theorem is false; then $\overline{V}$ contains a proper submodule $T$. The enveloping algebra of the transformations $E^*_{\alpha'}$, $\alpha' > 0$, on $T$ is a nilpotent algebra, so that $T$ contains a maximal vector $[w] \neq 0$. We may assume that $[w]$ is a weight vector. Then we can express $w = A + B + \cdots$, where $A \in V_0 \cap V_{\lambda_A}$, $B \in V_0 \cap V_{\lambda_B}, \cdots$, and where $\lambda_A(H_i) \equiv \lambda_B(H_i) \pmod p$ for $i = 1, \cdots, l$. We may assume also that none of the components $A, B, \cdots$ of $w$ lie in $pV_0$. For $\alpha_i \in \Delta$, let $\{v_j^{(i)}\}$ be the $Z_p$-basis of $V_0$ chosen according to Lemma 1, and let $A = \sum a_j v_j^{(i)}$, for example, where the $v_j^{(i)}$ all have weight $\lambda_A$. Because $[w]$ is a maximal vector in $\overline{V}$, we have $[w]E^*_{\alpha'_i} = [wE_{\alpha'_i}] = 0$, and hence $AE_{\alpha'_i}E_{-\alpha'_i} = \sum a_j \mu_j v_j^{(i)} \in pV_0$, where the $\mu_j$ are either zero or units in $Z_p$. Therefore $a_j\mu_j \in pZ_p$, and if $\mu_j \neq 0$, then $a_j \in pZ_p$. Since $A \notin pV_0$, at least one $\mu_j = 0$. By Lemma 1 and the formulas (4), the basis vectors $v_j^{(i)}$ for which $v_j^{(i)}E_{\alpha'_i}E_{-\alpha'_i} = 0$ are among those for which $v_j^{(i)}H_i = \lambda v_j^{(i)}$ for a non-negative $\lambda$, and it follows that $\lambda_A(H_i) \geqq 0$. Similarly $\lambda_B(H_i) \geqq 0$, etc. and the weights $\lambda_A$, $\lambda_B$ are dominant weights of $V$. Because $l(V) < p$, no two dominant weights are congruent modulo $p$, and it follows that we may assume that $w = A \in V_{\lambda_A}$, where $\lambda_A$ is a dominant weight different from $\Lambda$. If $\lambda'$ is the weight of $[w]$ then $T$ has a restricted irreducible composition factor $N$ whose maximal weight is $\lambda'$. Then $\lambda_A$ is the maximal weight of the associated module of $N$. On $\overline{V}$ we have $\Gamma^* = \gamma(\Lambda)^* \cdot 1$, and applying Lemma 3 we obtain $\gamma(\Lambda)^* = \gamma(\lambda_A)^*$. This contradicts the hypothesis (ii) of the theorem. Therefore $\overline{V}$ is irreducible, restricted, and is isomorphic to $M$ by Theorem 1 of [3]. Thus dim $M$ = dim $V$ and Theorem 1 is proved.

3. **Remarks and examples.** We begin with the following estimate of dim $M$ for any irreducible module $M$.

THEOREM 2. *Let $M$ be an irreducible restricted $\mathfrak{L}$-module and let $x_0$ be a maximal vector of $M$. For each positive root $\alpha$, let $m_\alpha$ be the integer such that $x_0 e_{-\alpha}^{m_\alpha} \neq 0$, $x_0 e_{-\alpha}^{m_\alpha + 1} = 0$. Then*

$$\dim M \leqq \prod_{\alpha > 0} (m_\alpha + 1). \tag{8}$$

**Proof.** Let $A$ be the universal associative algebra of $\mathfrak{L}$ regarded as an ordinary Lie algebra (i.e. not taking account of the $p$-power operation). Then $M$ is an irreducible $A$-module, and the argument given by Harish-Chandra [7, p. 52] can be applied almost verbatim to prove (8).

Now let $V$ be the associated module of $M$, and let $\Lambda$ be the maximal weight of $V$. For each $\alpha' > 0$, let $m_\alpha'$ be the non-negative integer $< p$ such that $\Lambda(H_{\alpha'}) \equiv m_\alpha'$ (mod $p$). Then it is easily proved that $m_\alpha = m_\alpha'$.

Now let $\mathfrak{L}'$ be the simple Lie algebra of type $A_2$ defined over the complex field, and let $H_{\alpha'_1}$, $H_{\alpha'_2}$ be the basis of the Cartan subalgebra of $\mathfrak{L}'$ corresponding to the roots $\alpha_1'$, $\alpha_2'$ in a maximal simple system of roots, where $\alpha_1'(H_{\alpha'_1}) = \alpha_2'(H_{\alpha'_2}) = 2$. Then $H_{\alpha'_1 + \alpha'_2} = H_{\alpha'_1} + H_{\alpha'_2}$, and it follows that if $\Lambda$ is a dominant integral function such that $\Lambda(H_{\alpha'_i}) = a_i$, $i = 1, 2$, then the dimension of the irreducible $\mathfrak{L}'$-module $V$ whose maximal weight is $\Lambda$ is given by

$$\dim V = \prod_{\alpha' > 0} \frac{(\Lambda + \rho)(H_{\alpha'})}{\rho(H_{\alpha'})} = \frac{1}{2}(a_1 + 1)(a_2 + 1)(a_1 + a_2 + 2), \tag{9}$$

where $\rho = (\sum_{\alpha' > 0} \alpha')/2$ (see also [9, p. 289]).

If $p \neq 3$, then the Killing form of $\mathfrak{L}'$ is nondegenerate modulo $p$, and it is easily shown by means of (8) and (9) that for any $p > 7$ (so that the general theory is applicable) there exist irreducible restricted $\mathfrak{L}$-modules $M$ with associated modules $V$ such that dim $M \neq$ dim $V$. In fact, it is enough to choose $\Lambda$ such that $a_1 + a_2 = p$, $a_i > 0$, $i = 1, 2$. Let $V$ be the irreducible module whose maximal weight is $\Lambda$, and $M$ the irreducible restricted $(\mathfrak{L}_0'/p\mathfrak{L}_0')^\Omega$ module whose maximal weight is $\lambda^*$, where $\lambda^*(H_i^*) = a_i^*$, $i = 1, 2$. Then since $H^*_{\alpha'_1 + \alpha'_2} = H^*_{\alpha'_1} + H^*_{\alpha'_2}$, we have $\lambda^*(H^*_{\alpha'_1 + \alpha'_2}) = a_1^* + a_2^* = 0$, and by (8)

$$\dim M \leqq (a_1 + 1)(a_2 + 1) < \dim V.$$

It is also worthwhile pointing out where our argument in the proof of Theorem 1 fails to apply to arbitrary Lie algebras of classical type in the sense of [8]. For the simple Lie algebra $\mathfrak{L}$ of classical type of class $A_n$ with $p|n+1$, the Killing form of $\mathfrak{L}$ is degenerate. In this case $\mathfrak{L}$ is not a modular Lie algebra coming from a complex simple Lie algebra of the same type. Therefore the apparatus of the associated modules and the Casimir element $\Gamma^*$ as we have used it does not seem to be available.

## References

1. C. W. Curtis, *Modular Lie algebra* I, Trans. Amer. Math. Soc. vol. 82 (1956) pp. 160–179.

2. ———, *Modular Lie algebra* II, Trans. Amer. Math. Soc. vol. 86 (1957) pp. 91–108.

3. ———, *Representations of Lie algebras of classical type*, J. Math. Mech. vol. 9 (1960) pp. 307–326.

4. ———, *On projective representations of certain finite groups*, Proc. Amer. Math. Soc., to appear.

5. H. Freudenthal, *Zur Berechnung der Charaktere der halbeinfachen Lieschen Gruppen* I, Nederl. Akad. Wetensch. Proc. Ser. A. vol. 57 (1954) pp. 369–376.

6. Harish-Chandra, *Lie algebras and the Tannaka duality theorem*, Ann. of Math. vol. 51 (1950) pp. 299–330.

7. ———, *On some applications of the universal enveloping algebra of a semi-simple Lie algebra*, Trans. Amer. Math. Soc. vol. 70 (1951) pp. 28–96.

8. W. H. Mills and G. B. Seligman, *Lie algebras of classical type*, J. Math. Mech. vol. 6 (1957) pp. 519–548.

9. H. Weyl, *Selecta*, Basel, 1956.

University of Wisconsin,
Madison, Wisconsin
Cornell University,
Ithaca, New York