ON THE INVERSE LIMIT OF EUCLIDEAN N-SPHERES(1)

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In [1] Bing constructed a 1-dimensional hereditarily indecomposable continuum which is the inverse limit of a sequence of circles C_i and such that the maps $f_i\colon C_i \to C_{i-1}$ were of degree 1. In this paper we show that the dimension cannot be raised, i.e., if $S = \text{Lim}(S_i^N, f_i)$ where S_i^N is an N-sphere and f_i is essential, then S is *not* hereditarily indecomposable. In doing so we further generalize Bing's definition of "ϵ-crooked." Lemma 2 shows how to construct hereditarily indecomposable continua by making the bonding maps sufficiently crooked. In Lemma 3 we get a necessary crookedness condition on the bonding maps if the limit space is to be hereditarily indecomposable. The remainder of the paper is devoted to proving that in the case of N-spheres ($N > 1$) this condition cannot be satisfied.

DEFINITIONS AND NOTATION. Let X_i be a sequence of compact metric spaces, and for $i \geqq 2$ let f_i be a map of X_i into X_{i-1}. Then the subspace(2) (3)

$$S = \left\{ z \in \prod_1^\infty X_i \,\middle|\, f_{ij}(z_j) = z_i \right\}$$

of $\prod_1^\infty X_i$ is the *inverse limit space* of the *inverse system* (X_i, f_i).

If d_i is the diameter of X_i then a metric for $\prod_1^\infty X_i$ (and hence for S) is given by $|z - z'| = \sum_1^\infty 2^{-i} d_i^{-1} |z_i - z_i'|$.

Let $f\colon X \to Y$ where X, Y are metric spaces. Then for $\epsilon > 0$(4),

$$L(\epsilon, f) = \sup \left\{ \delta \,\middle|\, \begin{array}{l} x, y \in X \text{ and } |x - y| < \delta \\ \text{implies } |f(x) - f(y)| < \epsilon \end{array} \right\}.$$

If X is compact then $L(\epsilon, f) > 0$ for all $\epsilon > 0$.

Let $f\colon X \to Y$ where X is a topological space and Y is a metric space. Let $\epsilon > 0$. Then f is *ϵ-crooked* if for each path $g\colon I \to X$(5) there exist real numbers t_1, t_2 such that $0 \leqq t_1 \leqq t_2 \leqq 1$ and:

Presented to the Society April 25, 1959 under the title *A note on higher dimensional indecomposable continua*; received by the editors September 25, 1959.

(1) This paper was written while the author was an Office of Naval Research Fellow at the University of Michigan.

(2) $f_{ij} = f_{i+1} f_{i+2} \cdots f_j$, $f_{ii} = 1$.

(3) If $z \in \prod_1^\infty X_i$, then z_i denotes the ith coordinate of z. Hence $z = (z_i)$.

(4) $|x - y|$ denotes the distance from x to y.

(5) I denotes the unit interval [01].

$$|fg(0) - fg(t_2)| < \epsilon,$$
$$|fg(t_1) - fg(1)| < \epsilon.$$

A metric space Y is ϵ-crooked if the identity map $1: Y \to Y$ is ϵ-crooked(6).

A continuum M is *indecomposable* if it is not the union of two proper subcontinua. If every subcontinuum of M is indecomposable then M is *hereditarily indecomposable*.

LEMMA 1. *Let K be a metric space. Suppose $K_1, K_2, \cdots$ is a sequence of Peanian continua in K such that $K_1 \supset K_2 \supset \cdots$. Then $\bigcap_1^\infty K_n$ is hereditarily indecomposable if and only if there is a null sequence (ϵ_n) of positive real numbers such that K_n is ϵ_n-crooked.*

Proof of sufficiency. Suppose M is a subcontinuum of $\bigcap_1^\infty K_n$ and $M = A \cup B$, where A, B are proper subcontinua of M. Let $a \in A - B$, $b \in B - A$. Then there is an n such that $\epsilon_n < \min(|a - B|, |b - A|)$(7). Let O_A, O_B be connected open subsets of K_n such that $A \subset O_A$, $B \subset O_B$, $|a - \overline{O}_B| > \epsilon_n$, and $|b - \overline{O}_A| > \epsilon_n$. Let $x \in O_A \cap O_B$. Then there are paths $\alpha, \beta: I \to K_n$ such that $\alpha(0) = a$, $\alpha(1) = x$, $\beta(0) = x$, $\beta(1) = b$, $\alpha(I) \subset O_A$, and $\beta(I) \subset O_B$. Let $g: I \to K$ where

$$g(t) = \begin{cases} \alpha(2t), & 0 \leqq t \leqq 1/2, \\ \beta(2t - 1), & 1/2 \leqq t \leqq 1. \end{cases}$$

Since K_n is ϵ_n-crooked there exist t_1, t_2 such that $t_1 \leqq t_2$, $|a - g(t_2)| < \epsilon_n$ and $|g(t_1) - b| < \epsilon_n$. Hence $t_2 < 1/2$ and $t_1 > 1/2$. But this contradicts the fact that $t_1 \leqq t_2$.

Proof of necessity. Suppose $\bigcap_1^\infty K_n$ is hereditarily indecomposable and for some $\epsilon > 0$ there is a sequence n_i such that K_{n_i} is not ϵ-crooked. Then for each i there is a path $\alpha_i: I \to K_{n_i}$, and real numbers t_i such that $|\alpha_i([0 t_i]) - \alpha_i(1)| \geqq \epsilon$ and $|\alpha_i([t_i 1]) - \alpha_i(0)| \geqq \epsilon$(8). Let $A_{n_i} = \alpha_i([0 t_i])$, $B_{n_i} = \alpha_i([t_i, 1])$, $M_{n_i} = \alpha_i(I)$. Then lim sup A_{n_i}, lim sup B_{n_i} are proper subcontinua of the continuum lim sup M_{n_i}, and lim sup $M_{n_i} =$ lim sup $A_{n_i} \cup$ lim sup B_{n_i}.

LEMMA 2. *Let $S = \lim(X_i, f_i)$ where the X_i are Peanian continua with diameters d_i. Suppose for all n, f_n is ϵ_n-crooked where $\epsilon_n < \min_{i<n-1} L(2^{-n} d_i, f_{i\, n-1})$. Then S is hereditarily indecomposable.*

Proof. Let $K_n = \{z \in \prod_1^\infty X_i \mid f_{ij}(z_j) = z_i \text{ for } i < j \leqq n\}$. Then K_n is homeomorphic to X_n and $S = \bigcap_{n=1}^\infty K_n$. Now K_n is 2^{3-n}-crooked. For if $g: I \to K_n$ is a path in K_n, then(9)

(6) It may be noted that both definitions of ϵ-crooked (for maps and for spaces) depend on the given metric.

(7) $|x - Y|$ denotes the distance from x to Y.

(8) We may assume without loss of generality that $\alpha_i(t_i)$ converges.

(9) π_n denotes the map collapsing each point of $\prod_1^\infty X_i$ onto its nth coordinate.

$$I \xrightarrow{g} K_n \xrightarrow{\pi_n} X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{i\,n-1}} X_i.$$

Since f_n is ϵ_n-crooked, there exist real numbers t_1, t_2 such that $0 \leqq t_1 \leqq t_2 \leqq 1$, $|f_n\pi_n g(0) - f_n\pi_n g(t_2)| < \epsilon_n$, and $|f_n\pi_n g(t_1) - f_n\pi_n g(1)| < \epsilon_n$. Let $\alpha = g(0), \beta = g(t_2)$ $\cdot(\alpha, \beta \in K_n)$. Then $|f_n\pi_n(\alpha) - f_n\pi_n(\beta)| < \epsilon_n$, i.e. $|\alpha_{n-1} - \beta_{n-1}| < \epsilon_n$. Since $\epsilon_n < L(2^{-n}d_i, f_{i\,n-1})$, $|\alpha_i - \beta_i| < 2^{-n}d_i$ for $i = 1, 2, \cdots, (n-2)$. Hence

$$\begin{aligned} |\alpha - \beta| = \sum_1^{\infty} 2^{-i}d_i^{-1}|\alpha_i - \beta_i| &\leqq \sum_1^{n-2} 2^{-i}d_i^{-1}|\alpha_i - \beta_i| + 2^{2-n} \\ &\leqq \sum_1^{n-2} 2^{-i}d_i^{-1}2^{-n}d_i + 2^{2-n} \\ &\leqq 2^{-n}\sum_1^{n-2} 2^{-i} + 2^{2-n} \\ &< 2^{-n} + 2^{2-n} \\ &< 2^{3-n}. \end{aligned}$$

Hence $|g(0) - g(t_2)| < 2^{3-n}$. Similarly, $|g(t_1) - g(1)| < 2^{3-n}$. Hence K_n is 2^{3-n}-crooked. But $S = \bigcap_{n=1}^{\infty} K_n$, so by Lemma 1, S is hereditarily indecomposable.

LEMMA 3. *Let $S = \lim(X_i, f_i)$ where the X_i are Peanian continua and S is hereditarily indecomposable. Then for any $\epsilon > 0$ there is an n such that f_{1n} is ϵ-crooked.*

Proof. By Lemma 1 there is an n such that K_n is $2^{-1}\epsilon d_1^{-1}$-crooked. Let $g: I \to X_n$ be a path in X_n. Let $x_i \in X_i$ for $i = n+1, n+2, \cdots$. Let $\bar{g}: I \to K_n$ by

$$\bar{g}(t) = (f_{1n}g(t), f_{2n}g(t), \cdots, g(t), x_{n+1}, x_{n+2}, \cdots).$$

Since K_n is $2^{-1}\epsilon d_1^{-1}$-crooked there exist $t_1 \leqq t_2$ such that $|\bar{g}(0) - \bar{g}(t_2)| < 2^{-1}d_1^{-1}\epsilon$, $|\bar{g}(t_1) - \bar{g}(1)|\, 2^{-1}d_1^{-1}\epsilon$. Now $|\bar{g}(0) - \bar{g}(t_2)| = \sum_{i=1}^{n} 2^{-i}d_i^{-1}|f_{in}g(0) - f_{in}g(t_2)|$. Hence $\sum_{i=1}^{n} 2^{-i}d_i^{-1}|f_{in}g(0) - f_{in}g(t_2)| < 2^{-1}d_1^{-1}\epsilon$. In particular

$$2^{-1}d_1^{-1}|f_{1n}g(0) - f_{1n}g(t_2)| < 2^{-1}d_1^{-1}\epsilon, \text{ or } |f_{1n}g(0) - f_{1n}g(t_2)| < \epsilon.$$

Similarly $|f_{in}g(t_1) - f_{in}g(1)| < \epsilon$. Hence f_{1n} is ϵ-crooked.

LEMMA 4. *Let Y be a metric space, X a topological space, and f, g two maps of X into Y such that $\|f - g\| < \delta$. Then if f is ϵ-crooked, g is $(\epsilon + 2\delta)$-crooked.*

Proof. Suppose $\alpha: I \to X$ is a path in X. Since f is ϵ-crooked there exist $t_1 \leqq t_2$ such that $|f\alpha(0) - f\alpha(t_2)| < \epsilon$, $|f\alpha(t_1) - f\alpha(1)| < \epsilon$. Hence

$$\begin{aligned} |g\alpha(0) - g\alpha(t_2)| &\leqq |g\alpha(0) - f\alpha(0)| + |f\alpha(0) - f\alpha(t_2)| + |f\alpha(t_2) - g\alpha(t_2)| \\ &< \|g - f\| + \epsilon + \|g - f\| \\ &< \epsilon + 2\delta. \end{aligned}$$

Similarly $|g\alpha(t_1)-g\alpha(1)|<\epsilon+2\delta$.

Let T be a triangulation of the N-sphere S^N $(N>1)$ and suppose $\bar{f}$: $T^n \to T^m$ is a chain map of the nth barycentric subdivision of T onto the mth barycentric subdivision of T. Let f: $|T^n| \to |T^m|$ be the induced map of S^N upon itself and suppose f is essential. Finally let σ be a N-simplex of T. An open subset o of S^N is said to be *inessential* mod $|\sigma|$ if there is a homotopy F: $S^N \times I \to S^N$ such that

(a) $F_0=f$,

(b) $F_t|(S^N-o)=f$,

(c) $F_1(o)\subset|\sigma|$.

Otherwise o is *essential* mod $|\sigma|$.

LEMMA 5. *Let o be essential* mod $|\sigma|$. *Then if F is any homotopy of $S^N\times I$ into S^N satisfying the above conditions* (a) *and* (b), $F_1(o)\supset S^N-|\sigma|$.

Proof. Suppose $x_0\in S^N-[|\sigma|\cup F_1(o)]$. Since $|\sigma|$ is a strong deformation retract of S^N-x_0, there is a homotopy G: $(S^N-x_0)\times I\to S^N$ such that $G_0=1$, $G_t||\sigma|=1$, and $G_1(S^N-x_0)\subset|\sigma|$. Let H: $S^N\times I\to S^N$ by:

$$H(x,t)=\begin{cases} F(x,2t), & 0\leqq t\leqq 1/2,\\ f(x), & 1/2\leqq t\leqq 1,\ x\notin o,\\ G(F_1(x),2t-1), & 1/2\leqq t\leqq 1,\ x\in o.\end{cases}$$

Then $H_0=f$, $H_t|(S^N-o)=f$, and $H_1(o)=G_1(F_1(o))\subset G_1(S^N)\subset|\sigma|$. But this contradicts the assumption that o is essential mod $|\sigma|$.

LEMMA 6. *Suppose o_1, o_2 are disjoint open subsets of S^N. Then if each is inessential* mod $|\sigma|$ *so is their union.*

Proof. Let F, G be the homotopies corresponding to o_1, o_2 respectively. Let H: $S^N\times I\to S^N$ by:

$$H(x,t)=\begin{cases} F(x,2t), & 0\leqq t\leqq 1/2,\\ G(x,2t-1), & 1/2\leqq t\leqq 1,\ x\notin o_1,\\ F(x,1), & 1/2\leqq t\leqq 1,\ x\in o_1.\end{cases}$$

LEMMA 7. *If X is a component of $f^{-1}(|\sigma|)$, then at least one complementary domain of X is essential* mod $|\sigma|$.

Proof. Let $o_1, o_2, \cdots, o_r$ be the complementary domains of X. If each o_i is inessential then by Lemma 6 so is their union S^N-X. Hence there is a homotopy F: $S^N\times I\to S^N$ such that $F_0=f$, $F_t|X=f$, and $F_1(S^N-X)\subset|\sigma|$. But then $F_1(S^N)\subset|\sigma|$. Hence F_1 is inessential. But F_1 is homotopic to f and f is essential.

LEMMA 8. *Suppose σ_1, σ_2, σ_3 are three N-simplexes of T whose geometric realizations are pairwise disjoint. Then there is an integer i_0 ($i_0=1, 2, 3$), a component X of $f^{-1}(|\sigma_{i_0}|)$, and a complementary domain o of X such that:*

(a) *o is essential* mod $|\sigma_{i_0}|$.

(b) *If Y is any component of $f^{-1}(|\sigma_i|)$ $(i=1, 2, 3)$ in o and o' is a complementary domain of Y lying in o, then o' is inessential* mod $|\sigma_i|$.

Proof. Let X_1 be a component of $f^{-1}(|\sigma_1|)$. By Lemma 7, X_1 has a complementary domain o_1 which is essential mod $|\sigma_1|$. If the pair (X_1, o_1) does not satisfy condition (b) there is an integer i_2 $(i_2=1, 2, 3)$, a component X_2 of $f^{-1}(|\sigma_{i_2}|)$, and a complementary domain o_2 of X_2 such that $o_2 \subset o_1$ and o_2 is essential mod $|\sigma_2|$. Continuing in this fashion we must arrive at the required pair (X, o) after a finite number of steps.

LEMMA 9. *Let $\sigma_1, \sigma_2, \sigma_3$ be three N-simplexes of T whose geometric realizations are pairwise disjoint. Let p_1, p_2, p_3 be interior points of $|\sigma_1|, |\sigma_2|, |\sigma_3|$ respectively. Then for some permutation (i, j, k) of $(1, 2, 3)$ there is a component X of $f^{-1}(|\sigma_i|)$, a complementary domain o of X, and points p_j', p_k' of o such that*:

(a) $f(p_j')=p_j$,

(b) $f(p_k')=p_k$,

(c) $f^{-1}(|\sigma_j|)$ *does not separate p_k' from X*,

(d) $f^{-1}(|\sigma_k|)$ *does not separate p_j' from X.*

Proof. Let i, o, X be the i_0, o, X of Lemma 8. Let j, k be any permutation of the remaining integers. Let $Q_1, Q_2, \cdots, Q_w$ be those components of $f^{-1}(|\sigma_j|)$ in o which are not separated from X by any other component of $f^{-1}(|\sigma_j|)$. For $1 \leqq t \leqq w$ let W_t be the union of the complementary domains of Q_t not containing X. Then the W_t are pairwise disjoint. From condition (b) of Lemma 8, each component of W_t is inessential mod $|\sigma_j|$. Hence by Lemma 6, each W_t is inessential mod $|\sigma_j|$. Again, by Lemma 6, $\bigcup_1^w W_t$ is inessential mod $|\sigma_j|$. Hence there is a homotopy $F: S^N \times I \rightarrow S^N$ such that $F_0=f$, $F_t|(S^N-\bigcup_1^w W_t)=f$, and $F_1(\bigcup_1^w W_t) \subset |\sigma_j|$. By Lemma 5, since o is essential mod $|\sigma_i|$, there is a point $p_k' \in o$ such that $F_1(p_k')=p_k$. Since $F_1(\bigcup_1^w W_t) \subset |\sigma_j|$, $p_k' \notin \bigcup_1^w W_t$. Hence no Q_t (and therefore no component of $f^{-1}(|\sigma_j|)$) separates p_k' from X. It follows from the unicoherence[10] of S^N that $f^{-1}(|\sigma_j|)$ does not separate p_k' from X. In the same manner we can find a $p_j' \in o \cap f^{-1}(p_j)$ such that $f^{-1}(|\sigma_k|)$ does not separate p_j' from X.

LEMMA 10. *Let T be a triangulation of S^N $(N>1)$ and $\bar{f}: T^n \rightarrow T^m$ a chain map such that the induced map $f: |T^n| \rightarrow |T^m|$ is essential. Suppose $\sigma_1, \sigma_2, \sigma_3$ are N-simplexes of T whose geometric realizations are pairwise disjoint. Let p_1, p_2, p_3 be points interior to $|\sigma_1|, |\sigma_2|, |\sigma_3|$ respectively. Finally, suppose $\epsilon < \min_i D(p_i, S^N-|\sigma_i|)$. Then f is not ϵ-crooked.*

Proof. The hypotheses of this lemma include those of Lemma 9. Rather than restating the conclusions of Lemma 9 let us bodily incorporate its con-

[10] It is here that the proof breaks down for $N=1$.

clusions and notation. It follows from conditions (c) and (d) and from the unicoherence of S^N that there exist arcs $[p_j' x_1]$ and $[p_k' x_2]$ in S^N such that $x_1 \cup x_2 \subset X$, $[p_j' x_1] \cap f^{-1}(|\sigma_k|) = [p_k' x_2] \cap f^{-1}(|\sigma_j|) = 0$, $[p_k' x_2] \cap X = x_2$, and $[p_j' x_1] \cap X = x_1$. Since X is a connected polyhedron there is an arc $[x_1 x_2]$ in X. Let h_1, h_2, h_3 be homeomorphisms of I into $[p_j' x_1]$, $[x_1 x_2]$, and $[x_2 p_k']$ respectively, such that $h_1(0) = p_j'$, $h_2(0) = x_1$, $h_3(0) = x_2$. Let $g: I \to S^N$ be defined by:

$$g(t) = \begin{cases} h_1(3t), & 0 \leqq t \leqq 1/3, \\ h_2(3t - 1), & 1/3 \leqq t \leqq 2/3, \\ h_3(3t - 2), & 2/3 \leqq t \leqq 1. \end{cases}$$

If f were ϵ-crooked there would be real numbers t_1, t_2 such that $t_1 \leqq t_2$, $|fg(0) - fg(t_2)| < \epsilon$, and $|fg(t_1) - fg(1)| < \epsilon$. Now $fg(0) = p_j$ and $fg(1) = p_k$.

Since $([p_j' x_1] \cup [x_1 x_2]) \cap f^{-1}(|\sigma_k|) = 0$, $|fg(t) - p_k| > \epsilon$ for $0 \leqq t \leqq 2/3$. Hence $t_1 > 2/3$. Similarly $|fg(t) - p_j| > \epsilon$ for $1/3 \leqq t \leqq 1$. Hence $t_2 < 1/3$. But this contradicts the assumption that $t_1 \leqq t_2$. Hence f is not ϵ-crooked.

THEOREM 1. *Let $N > 1$. Then there is an $\epsilon > 0$ such that no essential map of S^N upon itself is ϵ-crooked.*

Proof. Let T be a triangulation of S^N fine enough to insure the existence of three N-simplexes σ_1, σ_2, σ_3 whose geometric realizations are pairwise disjoint. Let p_i be an interior point of $|\sigma_i|$ $(i = 1, 2, 3)$ and let

$$2\epsilon < \min_i D(p_i, S^N - |\sigma_i|).$$

Suppose now that g is any essential map of S^N upon itself. By the Simplicial Approximation Theorem there are barycentric subdivisions T^n, T^m of T and a chain map $\bar{f}: T^n \to T^m$ such that if f is the map of $|T^n| \to |T^m|$ induced by $\bar{f}$, then f is homotopic to g and $\|f - g\| < \epsilon/2$. Suppose g is ϵ-crooked. Then by Lemma 4 f is 2ϵ-crooked. But by Lemma 10 f cannot be 2ϵ-crooked. Hence g cannot be ϵ-crooked.

THEOREM 2. *Let $X = \lim(X_i, f_i)$ where for some $N > 1$ each X_i is an N-sphere and f_i is essential. Then X is not hereditarily indecomposable.*

Proof. This theorem is a direct consequence of Theorem 1 and Lemma 3.

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