ON MONOSPLINES OF LEAST DEVIATION(1)

BY

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Introduction and statement of results. Let

(1) $$\xi_1 < \xi_2 < \cdots < \xi_k$$

be k given real numbers, and n a non-negative integer. Let the function S be defined in each of the intervals $(-\infty, \xi_1)$, $[\xi_1, \xi_2)$, $\cdots$, $[\xi_k, \infty)$ by a separate polynomial of degree not exceeding n, so that the composite function S be continuous together with its first $n-1$ derivatives (no continuity requirement for $n=0$). $S^{(n)}$ is a step-function with (possible) discontinuities at the points (1). A function S of this kind is called a *spline function* [7] of class (n, k), the points (1) being referred to as its *knots*.

It is convenient to introduce the notation

(2) $$x_+ = \begin{cases} x, & x \geqq 0 \\ 0, & x < 0 \end{cases},$$

in terms of which the most general spline function of class (n, k) with knots (1) may be written as

(3) $$S(x) = P_n(x) + \sum_{\nu=1}^{k} \rho_\nu (x - \xi_\nu)_+^n,$$

where P_n is a polynomial of degree at most n and the ρ_ν are arbitrary numbers. Thus S depends on $n+2k+1$ arbitrary parameters.

Conventions. Throughout the remainder of this paper it is to be understood that $n \geqq 1$, $k \geqq 0$, whenever these symbols are used, unless the contrary is stated. We agree to take $(0)_+^0 = 1$, $(x)_+^0 = 0$ for $x < 0$.

A *monospline* of class (n, k) with knots (1) is a function M of the form

(4) $$M(x) = x^n + S(x),$$

where S is a spline function of class $(n-1, k)$ with knots (1). Thus a monospline of class (n, k) depends on $n+2k$ parameters.

Among all polynomials of degree n with leading coefficient one, the Tchebycheff polynomial T_n [5, p. 36] defined by

(5) $$T_n(x) = 2^{-(n-1)} \cos(n \arccos x), \qquad -1 \leqq x \leqq 1,$$

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deviates least from zero on the interval $[-1, 1]$. The maximum absolute deviation is clearly $2^{-(n-1)}$, and it is achieved at the $n+1$ points $x=\cos(\pi j/n)$, $0 \leqq j \leqq n$, with alternating signs. We investigate here the existence, uniqueness, and properties of monosplines of class (n, k) which deviate least from zero on $[-1, 1]$. Letting $M_{n,k}^*$ denote any such monospline of class (n, k) we have

$$M_{n,0}^* = T_n, \tag{6}$$

since a monospline of class $(n, 0)$ is a polynomial of degree n with leading coefficient one.

It is shown in §2 that a monospline of class (n, k) can have at most $n+2k$ zeros. Thus a monospline of class (n, k), $n \geqq 3$, being proportional to the integral of a continuous monospline of class $(n-1, k)$, can have at most $n+2k-1$ relative extrema. This limitation holds also for $n=1$, $n=2$. The main results obtained in this paper are contained in the following two theorems.

THEOREM 1. *Given any set of numbers* $\{e_1, \cdots, e_{n+2k-1}\}$, *such that*

$$\begin{matrix} e_{n+2k-1} < e_{n+2k-2}, & e_{n+2k-3} < e_{n+2k-4}, \cdots, \\ e_{n+2k-2} > e_{n+2k-3}, & e_{n+2k-4} > e_{n+2k-5}, \cdots, \end{matrix} \qquad n + 2k \geqq 2,$$

there exists a monospline $M_{n,k}$ *of class* (n, k) *which has the* e_ν *as its relative extrema, in the given order. That is, for* $n \geqq 2$ *there is a sequence* $x_1 < x_2 < \cdots < x_{n+2k-1}$ *such that* $M_{n,k}(x_\nu) = e_\nu$, $1 \leqq \nu \leqq n+2k-1$, *and that the* e_ν *are the relative extrema of* $M_{n,k}$. *For* $n=1$, *there is a sequence* $x_1 < x_2 < \cdots < x_k$ *such that* $M_{1,k}(x_\nu -) = e_{2\nu-1}$, $M_{1,k}(x_\nu +) = e_{2\nu}$, $1 \leqq \nu \leqq k$, *and that the* e_ν *are the relative extrema of* $M_{1,k}$.

THEOREM 2. *For each* (n, k) *there exists a unique monospline* $M_{n,k}^*$ *of class* (n, k) *which deviates least from zero on* $[-1, 1]$. *For* $n \geqq 2$, $M_{n,k}^*$ *achieves its maximum absolute deviation, with alternating signs, at precisely* $n+2k+1$ *points of* $[-1, 1]$, *including both end-points, and this condition determines* $M_{n,k}^*$ *uniquely.*

It is to be noted that the family $\mathcal{S}_{n-1,k}$ of $(n-1, k)$-splines is not solvent (for $k>0$) in the sense of Motzkin [4]. For, since $\mathcal{S}_{n-1,k}$ is an $(n+2k)$-parametric family of continuous functions (for $n>1$), any member S of this family such that $S(-1+2\nu/(n+2k-1)) = (-1)^\nu$, $\nu = 0, 1, \cdots, n+2k-1$, would necessarily have at least $n+2k-1$ zeros, whereas such a function S can have at most $n+k-1$ zeros by Theorem 4. Furthermore, the subfamilies $\mathcal{S}_{n-1,k}^N$ of $(n-1, k)$-splines which are bounded in absolute value by N in some fixed interval are not closed, as is easily seen by considering the sequence

$$S_{n-1,2}^{(m)}(x) = \frac{m}{2(n-1)}\left[\left(x + \frac{1}{m}\right)_+^{n-1} - \left(x - \frac{1}{m}\right)_+^{n-1}\right], \qquad m = 2, 3, \cdots.$$

for any fixed integer $n>1$. Each $S_{n-1,2}^{(m)}$ is an element of $\mathcal{S}_{n-1,2}$, and $|S_{n-1,2}^{(m)}(x)| \leqq (3/2)^{n-2}$, $-1 \leqq x \leqq 1$, but

$$S(x) = \lim_{m\to\infty} S_{n-1,2}^{(m)}(x) = (x)_+^{n-2}$$

is not an element of $\mathcal{S}_{n-1,2}$ (but rather of $\mathcal{S}_{n-2,2}$). Hence we may not appeal to the results of [4] in proving the above theorems. We are indebted to the referee for calling our attention to this reference.

The above theorems are proved in §3. For polynomials, $k=0$, Theorem 1 is a special case of a theorem of MacLane [3, pp. 100, 101]. A qualitative result of a similar nature is obtained by Kempner [2], where it is shown that the relative magnitudes of the maxima and the minima of a polynomial with a maximal number of extrema may be prescribed arbitrarily, subject to the necessary alternation of maxima and minima.

In §1 we construct the functions $M_{n,k}^*$ for $n=1, 2, 3$, and 4, using Theorem 2 for $n=3$ and 4, and offer a short proof of the existence of a monospline of class $(n, 1)$ having $n+3$ absolutely equal extrema in $[-1, 1]$. In §2 we discuss the zeros of spline functions and monosplines. In §4 we investigate the magnitude of the absolute deviation from zero of $M_{n,k}^*$ in $[-1, 1]$.

1. **Explicit formulas.** We consider the problem of constructing the monospline $M_{n,k}^*$ of class (n, k) which deviates least from zero in $[-1, 1]$, for arbitrary k and small values of n. When we speak of the scaled (Tchebycheff) polynomial τ_n appropriate to an interval $[a, b]$,we refer to the polynomial

$$T_n(x) = \left(\frac{b-a}{2}\right)^n T_n\left[\frac{2x-(b+a)}{b-a}\right], \qquad a \leqq x \leqq b. \tag{7}$$

The cases $n=1$ and $n=2$ are particularly simple, by the following reasoning. The interval $[-1, 1]$ is subdivided into $r+1 \leqq k+1$ subintervals by the $r \leqq k$ knots of $M_{n,k}^*$ which lie in $[-1, 1]$. In each of these subintervals $M_{n,k}^*$ cannot deviate less from zero than does the scaled polynomial τ_n appropriate to that interval. Hence it is clear that, if the k knots be so chosen as to subdivide $[-1, 1]$ into $k+1$ equal subintervals, and if the scaled polynomials τ_n appropriate to these intervals join smoothly enough at the knots that the composite function be a monospline, then this monospline is the function $M_{n,k}^*$, whose existence and uniqueness are thereby proved.

(a) For $n=1$ there are no continuity requirements, so the above conditions can surely be satisfied. The function $M_{1,k}^*$ is easily seen to be given by

$$\begin{aligned} M_{1,k}^*(x) &= \frac{1}{k+1} T_1[(k+1)x-k], && \frac{k-1}{k+1} \leqq x, \\ M_{1,k}^*\left(x-\frac{2j}{k+1}\right) &= M_{1,k}^*(x), && \frac{k-1}{k+1} \leqq x < 1, \qquad 1 \leqq j \leqq k-1, \\ M_{1,k}^*(-x) &= -M_{1,k}^*(x), && \frac{k-1}{k+1} < x. \end{aligned} \tag{8}$$

(b) For $n=2$, the separate parabolic arcs of $M_{2,k}^*$ must join continuously at the end-points of the subintervals determined by the knots ξ_j. Since $T_2(-1)=T_2(1)$, this requirement is fulfilled by the appropriately scaled polynomials τ_2. Thus the function $M_{2,k}^*$ is given by

$$
\begin{aligned}
&M_{2,k}^*(x) = \frac{1}{(k+1)^2} T_2[(k+1)x - k], \qquad \frac{k-1}{k+1} \leqq x,\\
(9)\quad &M_{2,k}^*\left(x - \frac{2j}{k+1}\right) = M_{2,k}^*(x), \qquad \frac{k-1}{k+1} \leqq x < 1, \qquad 1 \leqq j \leqq k-1,\\
&M_{2,k}^*(-x) = M_{2,k}^*(x), \qquad \frac{k-1}{k+1} < x.
\end{aligned}
$$

(c) The above device will not work for $n=3$, since $T_3(-1)=-T_3(1)$, and a direct construction seems difficult. However, Theorem 2 provides enough additional information about $M_{3,k}^*$ to permit us to determine it rather easily. Instead, we shall verify that

$$
\begin{aligned}
&M_{3,k}^*(x) = \frac{1}{((3^{1/2}/2)k+1)^3} T_3\left\{\frac{3^{1/2}}{2}\left[\left(k + \frac{2}{3}3^{1/2}\right)x - k\right]\right\},\\
&\qquad\qquad\qquad \frac{k-1}{k+(2/3)3^{1/2}} \leqq x,\\
(10)\quad &M_{3,k}^*\left(x - \frac{2j}{k+(2/3)3^{1/2}}\right) = M_{3,k}^*(x),\\
&\qquad \frac{k-1}{k+(2/3)3^{1/2}} \leqq x < \frac{k+1}{k+(2/3)3^{1/2}}, \qquad 1 \leqq j \leqq k-1,\\
&M_{3,k}^*(-x) = -M_{3,k}^*(x), \qquad \frac{k-1}{k+(2/3)3^{1/2}} < x.
\end{aligned}
$$

First of all, noting that the absolutely equal extrema of $T_3(y)$ are located at $y=\pm 1$ and $y=\pm 1/2$, we see that $M_{3,k}^*(x)$ has absolutely equal extrema at the points

$$x = \frac{k \pm (1/3)3^{1/2}}{k+(2/3)3^{1/2}}.$$

Hence, $M_{3,k}^*$ has two absolutely equal extrema in each of the intervals

$$\frac{k-2j-1}{k+(2/3)3^{1/2}} < x \leqq \frac{k-2j+1}{k+(2/3)3^{1/2}}, \qquad 0 \leqq j \leqq k,$$

and one more at each of the points ± 1, making $2k+4$ in all. According to Theorem 2, if we can show that

$$M_{3,k}^{*(i)}\left(\frac{k-2j-1}{k+(2/3)3^{1/2}}-\right)=M_{3,k}^{*(i)}\left(\frac{k-2j-1}{k+(2/3)3^{1/2}}+\right),\quad 0\leqq j\leqq k-1, i=0,1,$$

it will follow that $M_{3,k}^*$ is correctly defined by (10). Hence it is enough to show that

$$T_3^{(i)}\left(-\frac{1}{2}3^{1/2}\right)=T_3^{(i)}\left(\frac{1}{2}3^{1/2}\right),\qquad i=0,1,$$

which is clearly the case since T_3' is an even function and $T_3(\pm(1/2)3^{1/2})=0$.

(d) For $n=4$ we again appeal to Theorem 2 in verifying that $M_{4,k}^*$ is correctly defined by

$$M_{4,k}^*(x)=\frac{1}{((1/2)2^{1/2}k+1)^4}T_4\left\{\frac{2^{1/2}}{2}[(k+2^{1/2})x-k]\right\},\quad \frac{k-1}{k+2^{1/2}}\leqq x,$$

(11) $$M_{4,k}^*\left(x-\frac{2j}{k+2^{1/2}}\right)=M_{4,k}^*(x),\quad \frac{k-1}{k+2^{1/2}}\leqq x<\frac{k+1}{k+2^{1/2}},\quad 1\leqq j\leqq k-1,$$

$$M_{4,k}^*(-x)=M_{4,k}^*(x),\qquad \frac{k-1}{k+2^{1/2}}<x.$$

Since the absolutely equal extrema of $T_4(y)$ are located at $y=0$, $y=\pm 2^{1/2}/2$, and $y=\pm 1$, we see that $M_{4,k}^*$ has absolutely equal extrema at the points

$$x=\frac{k}{k+2^{1/2}},\qquad x=\frac{k\pm 1}{k+2^{1/2}}.$$

Hence, $M_{4,k}^*$ has two absolutely equal extrema in each of the intervals

$$\frac{k-2j-1}{k+2^{1/2}}<x\leqq\frac{k-2j+1}{k+2^{1/2}},\qquad 0\leqq j\leqq k,$$

and one more at each of the points $x=\pm 1$, $x=-(k+1)/(k+2^{1/2})$, making $2k+5$ in all. By Theorem 2, if we can show that

$$M_{4,k}^{*(i)}\left(\frac{k-2j-1}{k+2^{1/2}}-\right)=M_{4,k}^{*(i)}\left(\frac{k-2j-1}{k+2^{1/2}}+\right),\quad 0\leqq j\leqq k-1, i=0,1,2,$$

it will follow that $M_{4,k}^*$ is correctly defined by (11). Hence it is enough to show that

$$T_4^{(i)}\left(-\frac{1}{2}2^{1/2}\right)=T_4^{(i)}\left(\frac{1}{2}2^{1/2}\right),\qquad i=0,1,2;$$

these equations are satisfied, since T_4 and $T_4^{(2)}$ are even functions, while $T_4'(\pm(1/2)2^{1/2})=0$.

It might be inferred from the foregoing that in all cases $M^*_{n,k}$ was made up of pieces of the polynomial T_n. In order for this to be so, it would have to be true that

$$T_n^{(j)}(x-b) = T_n^{(j)}(x), \qquad 0 \leqq j \leqq n-2,$$

for some pair (x, b) such that $-1 \leqq x-b \leqq 1$, $-1 \leqq x \leqq 1$, $b \neq 0$, and such a pair does not exist for $n \geqq 5$. To see this, we may use the well known result

$$T_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k}\binom{n-k}{k} 4^{-k}x^{n-2k} \tag{12}$$

to show that

$$\begin{aligned} T_n^{(n-5)}(x) &= \frac{n!}{5!}x^5 - \frac{n(n-2)!}{4!}x^3 + \frac{n(n-3)!}{32}x, && n \geqq 5,\\ T_n^{(n-3)}(x) &= \frac{n!}{3!}x^3 - \frac{n(n-2)!}{4}x, && n \geqq 3,\\ T_n^{(n-2)}(x) &= \frac{n!}{2!}x^2 - \frac{n(n-2)!}{4}, && n \geqq 2, \end{aligned} \tag{13}$$

By the third of these equations, in order that $T_n^{(n-2)}(x-b) = T_n^{(n-2)}(x)$ we must have $b(b-2x)=0$. If $b=2x$, the equations to be satisfied are $T_n^{(j)}(-x) = T_n^{(j)}(x)$, $0 \leqq j \leqq n-2$. Since $T_n^{(n-3)}$ and $T_n^{(n-5)}$ are odd functions, we must have $T_n^{(n-3)}(x) = T_n^{(n-5)}(x) = 0$. Now the nonzero roots of $T_n^{(n-3)}(x)=0$ are

$$x = \pm\left(\frac{3}{2(n-1)}\right)^{1/2},$$

while those of $T_n^{(n-5)}(x)=0$ are

$$x = \pm\left[\frac{5 \pm \left(\frac{2n-7}{n-2}\right)^{1/2}}{2(n-1)}\right]^{1/2},$$

and there is no common solution. Hence $b=0$, and the desired condition cannot be satisfied.

It is of course possible to write down the equations which must be satisfied by $M^*_{n,k}$ as a consequence of Theorem 2, and these equations suffice in principle to determine the solution completely. They are very cumbersome, however, and no explicit formulas have been obtained for $n \geqq 5$.

It may be of interest to note that the existence of a monospline of class $(n, 1)$ having $n+3$ absolutely equal extrema in $[-1, 1]$ can be shown more directly than via the argument of §3. Such a monospline must be the function $M^*_{n,1}$, by Theorem 2.

In view of the uniqueness guaranteed by Theorem 2, such a function must have its single knot ξ at the origin, and must be even or odd according as n is even or odd. We consider first the case n odd and $\geqq 3$, and let

$$f_n(x) = \begin{cases} x^{n-1}, & x \geqq 0, \\ -x^{n-1}, & x < 0. \end{cases} \tag{14}$$

Let Q_n be the Tchebycheffian approximation to f_n on $[-1, 1]$ by a polynomial of degree at most $n+1$. Q_n is known to exist, be unique, and to be such that $f_n - Q_n$ has $n+3$ absolutely equal extrema on $[-1, 1]$ (see [5, pp. 25–32]). Since f_n is odd and Q_n is unique, Q_n must itself be odd, and so

$$f_n(x) - Q_n(x) = \sum_{j=1}^{(n+1)/2} \alpha_{2j-1} x^{2j-1} + x^{n-1} - 2(-x)_+^{n-1}. \tag{15}$$

We assert that $\alpha_n \neq 0$. For, if $\alpha_n = 0$, then $f_n - Q_n$ is a spline function of class $(n-1, 1)$, and as such can have at most n zeros (see Theorem 4), while it is known to change sign at least $n+2$ times. Hence we may divide by α_n, obtaining the function $M_{n,1}$ defined by

$$M_{n,1}(x) = x^n + \frac{x^{n-1} - 2(-x)_+^{n-1}}{\alpha_n} + \sum_{j=1}^{(n-1)/2} \frac{\alpha_{2j-1}}{\alpha_n} x^{2j-1} \tag{16}$$

having $n+3$ absolutely equal extrema in $[-1, 1]$. It is easy to see that extrema must occur at both points $x = \pm 1$, and thus $M_{n,1} = M_{n,1}^*$. When n is even, we approximate instead of f_n the function g_n defined by $g_n(x) = |x|^{n-1}$, and proceed exactly as above.

We note also that, with the aid of equations (8)–(11) and the above discussion, we can determine the Tchebycheffian approximation C_n to the function h_n defined by

$$h_n(x) = x^{n-1} + (-1)^n 2(-x)_+^{n-1}, \qquad n = 2, 3, 4,$$

(called f_n or g_n above according as n was odd or even) on $[-1, 1]$ by a polynomial of degree at most n. The explicit formulas are:

$$h_2(x) = |x|, \qquad C_2(x) = x^2 + \frac{1}{8}; \tag{17}$$

$$h_3(x) = \begin{cases} x^2, & x \geqq 0 \\ -x^2, & x < 0 \end{cases}, \quad C_3(x) = \left(\frac{3 + 2(3^{1/2})}{9}\right) x^3 + \left(\frac{4(3^{1/2}) - 6}{3}\right) x; \tag{18}$$

$$h_4(x) = |x|^3, \quad C_4(x) = \left(\frac{1 + 2^{1/2}}{4}\right) x^4 + (2^{1/2} - 1)x^2 - \frac{1}{8}(2^{1/2} - 1)^3. \tag{19}$$

2. **The zeros of spline functions and monosplines.** Let S be a spline function of class (n, k), $n \geqq 0$, with knots (1), and assume for the moment that

there is no interior interval in which S vanishes identically, i.e., if $\xi_1 < x < \xi_k$ there is a deleted neighborhood $\hat{N}_x$ of x such that $S(t) \neq 0$ for all t in $\hat{N}_x$. Then, if $x \neq \xi_\nu$, $1 \leqq \nu \leqq k$, S has a zero of order $r \leqq n$ at x if and only if

$$S(x) = S'(x) = \cdots = S^{(r-1)}(x) = 0, \quad S^{(r)}(x) \neq 0. \tag{20}$$

The same definition will also be used when $x = \xi_\nu$ for some $1 \leqq \nu \leqq k$, provided that $r \leqq n-1$. However, the case $x = \xi_\nu$, $r = n$, requires special treatment. In this event clearly

$$S(x) = \begin{cases} A(x - \xi_\nu)^n, & x < \xi_\nu, \\ B(x - \xi_\nu)^n, & x \geqq \xi_\nu, \end{cases} \tag{21}$$

in some neighborhood of the point ξ_ν. We make the convention that

$$\begin{aligned} &\text{if } AB > 0, && \xi_\nu \text{ is a zero of order } n; \\ &\text{if } AB \leqq 0, && \xi_\nu \text{ is a zero of order } n+1. \end{aligned} \tag{22}$$

If there is a (maximal) interior interval $[a, b]$ in which S vanishes identically, we assign to that interval a zero of whatever order the above definition ascribes to the "composite point" $\{a, b\}$ if the interval is considered as being shrunk to a point. Let $S_{n,k}$ denote an arbitrary spline function of class (n, k), and let $Z(S_{n,k})$ denote the number of zeros of $S_{n,k}$, counting multiplicities as above. Then

THEOREM 3. *$Z(S'_{n,k}) \geqq Z(S_{n,k}) - 1$, where $S'_{n,k}(x) \equiv (d/dx)S_{n,k}(x)$.*

Proof. In view of the above convention about maximal intervals of identical vanishing, it is enough to consider the case in which $S_{n,k}$ has only isolated zeros. If $S_{n,k}$ has a zero of order $r_\nu \geqq 1$ at each of the distinct points x_ν, $1 \leqq \nu \leqq m$, then $S'_{n,k}$ has by definition a zero of order $r_\nu - 1$ at each x_ν. If $n \geqq 2$, $S'_{n,k}$ is continuous, and so has at least one zero in each of the $m-1$ intervals $(x_\nu, x_{\nu+1})$, $1 \leqq \nu \leqq m-1$, by Rolle's theorem. If $n=1$, $|S_{1,k}|$ has a maximum in each interval $(x_\nu, x_{\nu+1})$, say at $x = t_\nu$. (If there is a maximal subinterval throughout which $|S_{1,k}|$ assumes its constant maximum value, let t_ν be the left-hand endpoint of that subinterval.) Now $S'_{1,k}$ is a spline function of class $(0, k)$, and $S'_{1,k}(t_\nu -) \times S'_{1,k}(t_\nu +) \leqq 0$, using again the convention about intervals of identical vanishing if necessary. Hence $S'_{1,k}$ has a zero of order one at t_ν, by definition. Thus in any case

$$Z(S'_{n,k}) \geqq \sum_{\nu=1}^{m} (r_\nu - 1) + (m - 1) = \sum_{\nu=1}^{m} r_\nu - 1 = Z(S_{n,k}) - 1.$$

THEOREM 4. *$Z(S_{n,k}) \leqq n + k$, $n \geqq 0$.*

Proof. By n-fold application of Theorem 3 we find that $Z(S_{n,k}) \leqq Z(S^{(n)}_{n,k}) + n$. But $S^{(n)}_{n,k}$ is a spline function of class $(0, k)$, and as such can clearly have no more zeros than knots. Hence $Z(S^{(n)}_{n,k}) \leqq k$, $Z(S_{n,k}) \leqq n + k$.

Thus we have $Z(S_{n,k}) \leqq Z(S'_{n,k})+1 \leqq \cdots \leqq Z(S^{(n)}_{n,k})+n \leqq n+k$. Hence, if $Z(S_{n,k})=n+k$, we have $Z(S^{(j)}_{n,k})=n+k-j$, $0\leqq j\leqq n$.

Slightly different definitions will be employed in the case of monosplines $M_{n,k}$ of class (n, k) with knots (1). Here there is of course no possibility of an interval of identical vanishing. The definition (20) of a zero of order r at a point x will still be used if x is not a knot, or if x is a knot ξ_ν and $r\leqq n-2$. If $x=\xi_\nu$ and

$$M_{n,k}(\xi_\nu) = M'_{n,k}(\xi_\nu) = \cdots = M^{(n-2)}_{n,k}(\xi_\nu) = 0,$$

then clearly

$$M_{n,k}(x) = \begin{cases}(x-\xi_\nu)^n + A(x-\xi_\nu)^{n-1}, & x < \xi_\nu \\ (x-\xi_\nu)^n + B(x-\xi_\nu)^{n-1}, & x \geqq \xi_\nu\end{cases}, \qquad A \neq B, \tag{23}$$

in some neighborhood of the point ξ_ν. We make the convention that

$$\begin{aligned}&\text{if } AB > 0,\ \xi_\nu \text{ is a zero of order } n-1;\\ &\text{if } AB < 0,\ \xi_\nu \text{ is a zero of order } n;\\ &\text{if } AB = 0 \text{ and } B-A > 0,\ \xi_\nu \text{ is a zero of order } n;\\ &\text{if } AB = 0 \text{ and } B-A < 0,\ \xi_\nu \text{ is a zero of order } n+1.\end{aligned} \tag{24}$$

Then as above we have

THEOREM 5. $Z(M'_{n,k})\geqq Z(M_{n,k})-1$, $n\geqq 2$, $M'_{n,k}(x)\equiv (d/dx)M_{n,k}(x)$.

Proof. If $M_{n,k}$ has a zero of order $r_\nu \geqq 1$ at each of the distinct points x_ν, $1\leqq \nu\leqq m$, then $M'_{n,k}$, being proportional to a monospline of class $(n-1, k)$, has by definition a zero of order $r_\nu-1$ at each x_ν. If $n\geqq 3$, $M'_{n,k}$ is continuous, and so has at least one zero in each of the $m-1$ intervals $(x_\nu, x_{\nu+1})$, $1\leqq\nu\leqq m-1$, by Rolle's theorem. If $n=2$, $|M_{2,k}|$ has a maximum in each interval $(x_\nu, x_{\nu+1})$, say at $x=t_\nu$. Now either t_ν is a continuity point of $M'_{2,k}$, in which case $M'_{2,k}(t_\nu)=0$, or else t_ν is a knot of $M_{2,k}$ and

$$M'_{2,k}(t_\nu-) \times M'_{2,k}(t_\nu+) \leqq 0.$$

But $(1/2)M'_{2,k}$ is a monospline of class $(1, k)$, and so this inequality implies that $M'_{2,k}(t_\nu)=0$, by (23) and (24). Hence in any case $Z(M'_{n,k})\geqq \sum_{\nu=1}^{m}(r_\nu-1)+(m-1)=Z(M_{n,k})-1$.

THEOREM 6. $Z(M_{n,k})\leqq n+2k$.

Proof. By $(n-1)$-fold application of Theorem 5 we find that $Z(M_{n,k})\leqq Z(M^{(n-1)}_{n,k})+(n-1)$. But $M^{(n-1)}_{n,k}$ is proportional to a monospline of class $(1, k)$, and thus can clearly have at most $2k+1$ zeros. Hence $Z(M_{n,k})\leqq n+2k$.

Thus we have

$$Z(M_{n,k}) \leqq Z(M'_{n,k}) + 1 \leqq \cdots \leqq Z(M^{(n-1)}_{n,k}) + n - 1 \leqq n + 2k.$$

Hence, if $Z(M_{n,k})=n+2k$, we have $Z(M^{(j)}_{n,k})=n+2k-j$, $0\leqq j\leqq n-1$.

Let $\mathfrak{M}_{n,k}$ be the set of all monosplines of class (n, k) having $n+2k$ zeros in $[0, 1]$, and let

$$\sigma_{n,k} = \sup\{ | M_{n,k}(x) | : M_{n,k} \in \mathfrak{M}_{n,k}, 0 \leqq x \leqq 1\}. \tag{25}$$

We conjecture that $\sigma_{n,k}=\sigma_{n,k+1}$, in which case $\sigma_{n,k}=1$, since clearly $\sigma_{n,0}=1$. At any rate we can prove

LEMMA 1. $\sigma_{n,k} \leqq n!$.

Proof. Let $M_{n,k} \in \mathfrak{M}_{n,k}$; then $(1/n!)M_{n,k}^{(n-1)} \in \mathfrak{M}_{1,k}$, by the remark following Theorem 6. Since an element of $\mathfrak{M}_{1,k}$ must be of the form $x-c$, $0 \leqq c \leqq 1$, at every point x of $[0, 1]$, we see that $\sigma_{1,k}=1$. Let a_j be any zero of $M_{n,k}^{(j)}$, $0 \leqq j \leqq n-2$. Now

$$M_{n,k}(x) = n! \int_{a_0}^{x} \int_{a_1}^{t_{n-1}} \cdots \int_{a_{n-2}}^{t_2} \frac{1}{n!} M_{n,k}^{(n-1)}(t_1) dt_1 dt_2 \cdots dt_{n-1}, \tag{26}$$

and so

$$\begin{aligned} | M_{n,k}(x) | &\leqq n! \left| \int_{a_0}^{x} \int_{a_1}^{t_{n-1}} \cdots \int_{a_{n-2}}^{t_2} \left| \frac{1}{n!} M_{n,k}^{(n-1)}(t_1) \right| dt_1 dt_2 \cdots dt_{n-1} \right| \\ &\leqq n! \int_0^1 \int_0^1 \cdots \int_0^1 \sigma_{1,k} dt_1 \cdots dt_{n-1} = n!. \end{aligned} \tag{27}$$

Hence $\sup\{ | M_{n,k}(x) | : M_{n,k} \in \mathfrak{M}_{n,k}, 0 \leqq x \leqq 1\} = \sigma_{n,k} \leqq n!$.

We shall also require the following

LEMMA 2. *For any monospline $M_{n,k}$ of class (n, k),*

$$\int_0^1 | M_{n,k}(t) | \, dt \geqq 4(4k+4)^{-(n+1)}. \tag{28}$$

Proof. $M_{n,k}$ has at most k knots in $[0, 1]$, and hence there is at least one subinterval $[a, b] \subset [0, 1]$ of length not less than $(k+1)^{-1}$ in which $M_{n,k}$ is a polynomial P_n of degree n with leading coefficient one. Thus

$$\begin{aligned} &\int_0^1 | M_{n,k}(t) | \, dt \\ &\geqq \int_a^b | P_n(t) | \, dt = \frac{1}{2}(b-a) \int_a^b \left| P_n\left[\frac{1}{2}(b-a)x + \frac{1}{2}(b+a) \right] \right| dx \\ &= \left[\frac{1}{2}(b-a) \right]^{n+1} \int_{-1}^{1} \left| \frac{P_n[(2^{-1})(b-a)x + (2^{-1})(b+a)]}{[(2^{-1})(b-a)]^n} \right| dx \\ &\geqq (2k+2)^{-(n+1)} \inf_{P \in \Pi_n} \int_{-1}^{1} | P(x) | \, dx = (2k+2)^{-(n+1)} \lambda_n, \end{aligned} \tag{29}$$

where Π_n denotes the class of all polynomials of degree n with leading coefficient one. But it is known that $\lambda_n > 0$, and in fact

$$\lambda_n = \int_{-1}^{1} | U_n(x) | \, dx = 2^{-(n-1)}, \tag{30}$$

where

$$U_n(x) = \frac{1}{2^n} \frac{\sin [(n+1) \arccos x]}{(1-x^2)^{1/2}}, \qquad -1 \leqq x \leqq 1, \tag{31}$$

the modified Tchebycheff polynomial of degree n [5, pp. 302–309].

3. **Proofs of Theorems 1 and** 2. In our proof of Theorem 1 we shall make use of a theorem of Schoenberg [8] which we state as

THEOREM 7. *Given any sequence of numbers* $x_1 < x_2 < \cdots < x_n + 2k$, *there exists a unique monospline* $M_{n,k}$ *of class* (n, k) *such that* $M_{n,k}(x_\nu) = 0$, $1 \leqq \nu \leqq n+2k$. $M_{n,k}$ *depends continuously on the sequence* $\{x_\nu\}$.

We remark that the knots $\{\xi_\nu\}$ of this monospline $M_{n,k}$ are related to its zeros $\{x_\nu\}$ by

$$\begin{array}{lll} x_{2\nu} < \xi_\nu < x_{n+2\nu-1}, & n > 1 & \\ x_{2\nu} = \xi_\nu, & n = 1 & \end{array} \qquad 1 \leqq \nu \leqq k. \tag{32}$$

The result for $n=1$ may be verified directly; in fact, the monospline $M_{1,k}$ is given by

$$M_{1,k}(t) = (t - x_1) - \sum_{\nu=1}^{k} (x_{2\nu+1} - x_{2\nu-1})(t - x_{2\nu})_+^0, \tag{33}$$

(see (23) and (24)). For $n > 1$, we write

$$M_{n,k}(t) = t^n + P_{n-1}(t) + \sum_{j=1}^{k} \rho_j (t - \xi_j)_+^{n-1}.$$

Then if $\xi_\nu \geqq x_{n+2\nu-1}$ the function M defined by

$$M(t) = t^n + P_{n-1}(t) + \sum_{j=1}^{\nu-1} \rho_j (t - \xi_j)_+^{n-1}$$

has at least $n+2\nu-1$ zeros, and is evidently a monospline of class $(n, \nu-1)$, contradicting Theorem 6. Similarly, if $\xi_\nu \leqq x_{2\nu}$ the function M defined by

$$M(t) = t^n + P_{n-1}(t) + \sum_{j=1}^{\nu} \rho_j (t - \xi_j)^{n-1} + \sum_{j=\nu+1}^{k} \rho_j (t - \xi_j)_+^{n-1}$$

has at least $n+2(k-\nu)+1$ zeros, and is evidently a monospline of class $(n, k-\nu)$, again contradicting Theorem 6.

We also notice that, if $k=1$, (32) represents the necessary and sufficient condition that arbitrary values can be interpolated at any $n+1$ of the $n+2$ points x_ν by a spline function S of class $(n, 1)$ with knot ξ_1 (see [7, pp. 256–257]).

Before turning to the proof of Theorem 1, we dispose of the special case $n=1$ of that theorem. Here the situation is somewhat different than for larger values of n, in that the extrema $e_{2\nu-1}$, $1\leqq\nu\leqq k$, are not actually assumed, i.e. they are suprema rather than maxima. We observe that the function $M_{1,k}$ defined by

$$(34)\qquad M_{1,k}(t) = t - \sum_{j=1}^{k} (e_{2j-1} - e_{2j})\left[t - e_{2j-1} - \sum_{i=1}^{j-1} (e_{2i-1} - e_{2i})\right]_+^0$$

is a monospline of class $(1, k)$ with the properties:

(i) $M_{1,k}$ is monotone increasing in each of the intervals

$$\left[e_{2\nu-1} + \sum_{j=1}^{\nu-1} (e_{2j-1} - e_{2j}),\ e_{2\nu+1} + \sum_{j=1}^{\nu} (e_{2j-1} - e_{2j})\right),\qquad 1 \leqq \nu \leqq k-1$$

$$(-\infty, e_1),\ \left[e_{2k-1} + \sum_{j=1}^{k-1} (e_{2j-1} - e_{2j}),\ \infty\right);$$

$$\text{(ii)}\quad M_{1,k}\left\{\left[e_{2\nu-1} + \sum_{j=1}^{\nu-1} (e_{2j-1} - e_{2j})\right]-\right\} = e_{2\nu-1},\qquad 1 \leqq \nu \leqq k;$$

$$\text{(iii)}\quad M_{1,k}\left\{\left[e_{2\nu-1} + \sum_{j=1}^{\nu-1} (e_{2j-1} - e_{2j})\right]+\right\} = e_{2\nu},\qquad 1 \leqq \nu \leqq k.$$

Hence $M_{1,k}$ is a function of the type whose existence is asserted by Theorem 1 for $n=1$; in the sequel we consider only $n\geqq 2$, and prove the following

THEOREM 8. *Suppose* $N\geqq 3$, *let* $x=(x_2, \cdots, x_{N-1})$ *be an* $(N-2)$*-tuple of real numbers with* $x_1=0<x_2<\cdots<x_{N-1}<1=x_N$, *let* t *be a real number, and let* f *be a real-valued function of* t *and* x *satisfying the following conditions*:

(i) $f(t, x)$ *is continuous in* t *for* $0\leqq t\leqq 1$, *except possibly at some or all of the points* $t=x_j$;

(ii) $|f(t, x)|\leqq h$ *for* x *as above*, $0\leqq t\leqq 1$;

(iii) $G(t, x)=\int_0^t|f(s, x)|\,ds$ *is continuous in* x *for* x *as above*;

(iv) $G(1, x)\geqq g>0$ *for all* x *as above*;

(v) $f(x_j, x)=0$ *for each* $1\leqq j\leqq N$ *such that* $f(t, x)$ *is continuous at* $t=x_j$, *while* $f(t, x)\neq 0$ *for all other* $0\leqq t\leqq 1$;

(vi) $f(t, x)$ *changes sign at each point* $t=x_j$, $1\leqq j\leqq N$; *for definiteness we suppose that* $f(t, x)<0$ *for* $x_{N-1}<t<1$, *so that* $\operatorname{sgn}[f(t, x)]=(-1)^{N-j-1}$, $x_{j-1}<t<x_j$, $2\leqq j\leqq N$.

Let $F(t, x)=\int_0^t f(s, x)ds$, *and let* $\beta_1, \beta_2, \cdots, \beta_N$ *be a sequence of real numbers such that* $\beta_1=0$,

$$\beta_N < \beta_{N-1}, \qquad \beta_{N-2} < \beta_{N-3}, \cdots,$$

$$\beta_{N-1} > \beta_{N-2}, \qquad \beta_{N-3} > \beta_{N-4}, \cdots,$$

so that $\operatorname{sgn}[\beta_j-\beta_{j-1}]=(-1)^{N-j-1}$. *Then there exists a vector* x *and a number* $\lambda>0$ *such that*

$$F(x_j, x) = \lambda\beta_j, \qquad 1 \leqq j \leqq N. \tag{35}$$

Before proving Theorem 8, we show that it implies Theorem 1, $n\geqq 2$. In the above notation, let $N=n+2k-1$ and let f/n be the monospline of class $(n-1, k)$ which has a simple zero at each of the points x_j. Such a monospline exists uniquely, and depends continuously on the vector x, by Theorem 7. Conditions (i), (v), and (vi) are clearly satisfied: f is indeed negative in $(x_{N-1}, 1)$, since otherwise, being positive for t sufficiently large, it would have a zero of even order at $t=1$. Conditions (ii) and (iv) are also fulfilled, according to Lemmas 1 and 2. Condition (iii) is surely met if $n\geqq 3$, since then f is itself continuous in t and x. If $n=2$, $f/2$ is as given by (33) with $x_1=0$, $x_{2k+1}=1$, and we find that

$$\begin{aligned} G(t, x) = \sum_{\nu=1}^{k} \{ &[(t \cap x_{2j}) - x_{2j-1}]^2 - [(t \cap x_{2j-1}) - x_{2j-1}]^2 \\ &+ [x_{2j+1} - (t \cap x_{2j})]^2 - [x_{2j+1} - (t \cap x_{2j+1})]^2 \}, \end{aligned} \tag{36}$$

where $a\cap b=\min\{a, b\}$. Hence G is continuous in x, and the conditions of Theorem 8 are all satisfied. Let $\beta_j=e_j-e_1$, $1\leqq j\leqq n+2k-1$. Then by Theorem 8 there exists a vector x and a $\lambda>0$ such that (35) holds. F is surely a monospline of class (n, k), and hence is of the form

$$F(t, x) = t^n + S_{n-1,k}(t)$$

for some spline function $S_{n-1,k}$ of class $(n-1, k)$. But then the function $M_{n,k}$ defined by

$$M_{n,k}(t) = t^n + \frac{1}{\lambda} S_{n-1,k}(\lambda^{1/n}t) + e_1$$

is a monospline of the type whose existence is asserted by Theorem 1.

The cases $n\geqq 2$, $n+2k-1=1$ or 2, i.e. $k=0$, $n=2$ or 3, are not covered by Theorem 8. However, we may take for F the functions defined by

$$F(t, x) = \begin{cases} t^2, & n = 2,\ k = 0, \\ 2^{-1}(2t^3 - 3t^2), & n = 3,\ k = 0, \end{cases}$$

and the above argument still applies.

We turn now to the proof of Theorem 8, for which we shall require two additional lemmas. Let Φ denote the family of all real, continuous, strictly increasing functions ϕ defined on $[0, 1]$, such that $\phi(0)=0$. For ϕ, ψ in Φ, define $A(\phi, \psi)=\max\{|\phi(z)-\psi(z)| : 0\leqq z\leqq 1\}$; then clearly (Φ, A) is a metric space. Let $\gamma_2, \gamma_3, \cdots, \gamma_N$ be a sequence of positive numbers.

LEMMA 3. *For every* $\phi\in\Phi$ *there is a unique subdivision* $0<z_2<\cdots<z_{N-1}<1$ *of the interval* $[0, 1]$ *such that*

$$\frac{\phi(z_2)}{\gamma_2}=\frac{\phi(z_3)-\phi(z_2)}{\gamma_3}=\cdots=\frac{\phi(1)-\phi(z_{N-1})}{\gamma_N}. \tag{37}$$

Proof. The existence of such a subdivision, but not its uniqueness, is a consequence of the n-lattice theorem [6]. We give a direct proof in this simple case. For any vector $z=(z_2, \cdots, z_{N-1})$ which satisfies (37) it is true that

$$\frac{\phi(z_2)}{\gamma_2}=\frac{\phi(z_3)}{\gamma_2+\gamma_3}=\cdots=\frac{\phi(1)}{\sum_{j=1}^{N}\gamma_j}. \tag{38}$$

Hence, $t=z_\nu$ is the unique solution of the equation

$$t=\phi^{-1}\left\{\frac{\sum_{i=2}^{\nu}\gamma_i}{\sum_{i=2}^{N}\gamma_i}\phi(1)\right\}, \qquad 2\leqq\nu\leqq N-1. \tag{39}$$

For fixed γ_i and ϕ as above, let $z^\phi=(z_2^\phi, \cdots, z_{N-1}^\phi)$ be the corresponding subdivision of $[0, 1]$, and let $z_1^\phi=0$, $z_N^\phi=1$.

LEMMA 4. z^ϕ *is a continuous function of* ϕ.

Proof. By (37) and (38) we have

$$\frac{\phi(z_j^\phi)}{\phi(1)}=\frac{\sum_{i=2}^{j}\gamma_i}{\sum_{i=2}^{N}\gamma_i}=\lambda_j, \qquad 1\leqq j\leqq N. \tag{40}$$

Suppose $A(\phi_\nu, \phi)\to 0$, and that for each positive integer ν and each j we have also

$$\frac{\phi_\nu(z_j^{\phi_\nu})}{\phi_\nu(1)}=\lambda_j, \qquad 1\leqq j\leqq N. \tag{41}$$

Given $0<\epsilon<1/2$, suppose that $A(\phi_\nu, \phi)<\epsilon$ for $\nu>\nu_\epsilon$. Then for such values of ν we have

$$(42)\qquad (1-\epsilon)\phi(\overset{\phi}{z_j}) < \phi_\nu(\overset{\phi^\nu}{z_j}) < (1+\epsilon)\phi(\overset{\phi}{z_j}).$$

Suppose it has been established that $|z_j^\phi - z_j^{\phi_\nu}| < \epsilon$ for $j<\mu$ and $\nu>\nu_\mu \geqq \nu_\epsilon$. If $\mu = N$ the proof is complete by the definition of z_N^ϕ. If $2\leqq \mu < N$ there are two cases to consider.

(i) $z_\mu^\phi \leqq z_\mu^{\phi_\nu}$. Then

$$\phi_\nu(\overset{\phi_\nu}{z_\mu}) < (1+\epsilon)\{\phi(\overset{\phi_\nu}{z_\mu}) - [\phi(\overset{\phi_\nu}{z_\mu}) - \phi(\overset{\phi}{z_\mu})]\},$$

$$0 \leqq \phi(\overset{\phi_\nu}{z_\mu}) - \phi(\overset{\phi}{z_\mu}) < \phi(\overset{\phi_\nu}{z_\mu}) - \phi_\nu(\overset{\phi_\nu}{z_\mu}) + \frac{\epsilon}{1+\epsilon}\phi_\nu(\overset{\phi_\nu}{z_\mu}) < \epsilon[1+\epsilon+\phi(1)].$$

(ii) $z_\mu^{\phi_\nu} \leqq z_\mu^\phi$. Then

$$(1-\epsilon)\{\phi(\overset{\phi_\nu}{z_\mu}) + [\phi(\overset{\phi}{z_\mu}) - \phi(\overset{\phi_\nu}{z_\mu})]\} < \phi_\nu(\overset{\phi_\nu}{z_\mu}),$$

$$0 \leqq \phi(\overset{\phi}{z_\mu}) - \phi(\overset{\phi_\nu}{z_\mu}) < \phi_\nu(\overset{\phi_\nu}{z_\mu}) - \phi(\overset{\phi_\nu}{z_\mu}) + \frac{\epsilon}{1-\epsilon}\phi(\overset{\phi_\nu}{z_\mu}) < \epsilon[1+2\phi(1)].$$

Hence in either case $\phi(z_\mu^{\phi_\nu}) \to \phi(z_\mu^\phi)$ as $\nu \to \infty$, and so $z_\mu^{\phi_\nu} \to z_\mu^\phi$ since ϕ is continuous and strictly increasing. Thus there is a $\nu_{\mu+1} \geqq \nu_\mu$ such that $|z_j^\phi - z_j^{\phi_\nu}| < \epsilon$ for $j < \mu+1$ and $\nu > \nu_{\mu+1} \geqq \nu_\epsilon$, and the truth of the lemma follows by induction.

Let x and f be as in the statement of Theorem 8, and let $(\gamma_2, \cdots, \gamma_N)$ be a fixed sequence of positive numbers. Surely the function G defined in condition (iii) is an element of (Φ, A); hence by Lemma 3 there exists a vector $z^x = (z_2^x, \cdots, z_{N-1}^x)$ such that $0 < z_2^x < \cdots < z_{N-1}^x < 1$, and such that

$$\frac{G(\overset{x}{z_2}, x)}{\gamma_2} = \frac{G(\overset{x}{z_3}, x) - G(\overset{x}{z_2}, x)}{\gamma_3} = \cdots = \frac{G(1, x) - G(\overset{x}{z}_{N-1}, x)}{\gamma_N}.$$

By Lemma 4 the mapping T: $(x_2, \cdots, x_{N-1}) \to (z_2^x, \cdots, z_{N-1}^x)$ is continuous on the open set $\{x: 0 < x_2 < \cdots < x_{N-1} < 1\}$, since z^x is a continuous function of G and G is continuous in x. The above equations may be rewritten as

$$(43)\qquad \int_{\overset{x}{z}_{j-1}}^{\overset{x}{z_j}} |f(t,x)|\,dt = \frac{\gamma_j \int_0^1 |f(t,x)|\,dt}{\sum_{i=2}^N \gamma_i}, \qquad 2 \leqq j \leqq N,$$

so that

$$(\overset{x}{z_j} - \overset{x}{z}_{j-1}) \sup\{|f(t,x)| : \overset{x}{z}_{j-1} \leqq t \leqq \overset{x}{z_j}\} \geqq \frac{\gamma_j \int_0^1 |f(t,x)|\,dt}{\sum_{i=2}^N \gamma_i},$$

by the first mean-value theorem, and

$$(44)\qquad (\overset{x}{z}_j - \overset{x}{z}_{j-1}) \geqq \frac{g\gamma_j}{h\sum_{i=2}^{N}\gamma_i}$$

by (ii) and (iv) of Theorem 8. Choose any positive ϵ less than

$$\min\left\{\frac{1}{N}, \frac{g\gamma_2}{h\sum_{i=2}^{N}\gamma_i}, \cdots, \frac{g\gamma_N}{h\sum_{i=2}^{N}}\right\}.$$

Then we have shown that

$$(45)\qquad \overset{x}{z}_j - \overset{x}{z}_{j-1} \geqq \epsilon > 0, \qquad 2 \leqq j \leqq N.$$

But this implies that the function T above maps the closed $(N-2)$-simplex

$$\{x: \epsilon \leqq x_2 \leqq x_3 - \epsilon \leqq \cdots \leqq x_{N-1} - (N-3)\epsilon \leqq 1 - (N-2)\epsilon\}$$

continuously into itself, and so the Brouwer Fixed Point Theorem (see [9, pp. 242–245]) guarantees the existence of at least one vector x for which $z^x = x$.

We can now complete the proof of Theorem 8. For, since $|\beta_j - \beta_{j+1}| > 0$, $1 \leqq j \leqq N-1$, there exists by the above discussion a vector x for which

$$(46)\qquad \frac{G(x_2, x)}{|\beta_1 - \beta_2|} = \frac{G(x_3, x) - G(x_2, x)}{|\beta_2 - \beta_3|} = \cdots = \frac{G(1, x) - G(x_{N-1}, x)}{|\beta_{N-1} - \beta_N|}.$$

But since $f(t, x)$ is of constant sign in each interval (x_{j-1}, x_j) it follows from (vi) that

$$(47)\qquad \begin{aligned} G(x_j, x) - G(x_{j-1}, x) &= \int_{x_{j-1}}^{x_j} |f(t, x)|\, dt = (-1)^{N-j-1}\int_{x_{j-1}}^{x_j} f(t, x)dt \\ &= (-1)^{N-j-1}[F(x_j, x) - F(x_{j-1}, x)], \end{aligned}$$

$2 \leqq j \leqq N$, and hence (46) may be rewritten as

$$(48)\qquad \begin{aligned} \frac{F(x_2, x)}{\beta_2} &= \frac{F(x_3, x) - F(x_2, x)}{\beta_3 - \beta_2} = \cdots = \frac{F(1, x) - F(x_{N-1}, x)}{\beta_N - \beta_{N-1}} \\ &= \lambda > 0, \end{aligned}$$

so that (35) holds and Theorem 8 is proved.

We base our proof of Theorem 2 on Theorem 1 and the following

THEOREM 9. *If $n \geqq 2$, and if there exists a monospline $M_{n,k}$ of class (n, k) and a sequence of points $-1 = x_0 < x_1 < \cdots < x_{n+2k} = 1$ such that $M_{n,k}$ has a* *relative extremum at each point* x_ν *with* $M_{n,k}(x_\nu)=(-1)^\nu M_{n,k}(x_0)$, $0\le\nu\le n+2k$, *then* $M_{n,k}=M^*_{n,k}$, *so that the latter exists uniquely.*

Proof. The unique $M_{2,k}$ with extrema in $[x_0, x_{2+2k}]$ at $\{x_\nu\}_0^{2+2k}$ is given by

$$(49)\quad \begin{aligned} M_{2,k}(x) &= M_{2,k}(x_0) - (x_1-x_0)^2 + (x-x_1)^2 \\ &\quad - 2\sum_{\nu=1}^{k}(x_{2\nu+1}-x_{2\nu-1})(x-x_{2\nu})_+, \end{aligned}$$

as may easily be verified. The requirements $x_0=-1$, $M_{2,k}(x_1)=-M_{2,k}(x_0)$ imply that $2M_{2,k}(-1)=(x_1+1)^2$, and by induction we find $(x_j-x_{j-1})^2=(x_1+1)^2$, $1\leqq j\leqq 2k+2$. Hence the x_ν's are equally spaced, and $M_{2,k}=M^*_{2,k}$ as given in (9) of §1. Thus we need consider only $n\geqq 3$.

Let $M_{n,k}$ be a monospline of class (n, k), $n\geqq 3$, let $-1=x_0<x_1<\cdots<x_{n+2k}=1$ be a sequence of points such that $M_{n,k}(x_\nu)=(-1)^{n-\nu}e_0$ for some $e_0>0$, with $|M_{n,k}(x)|<e_0$ for $x\neq x_\nu$, and let $N_{n,k}$ be any other monospline of class (n, k) such that $|N_{n,k}(x)|\leqq e_0$, $-1\leqq x\leqq 1$. Assume that $M_{n,k}\neq N_{n,k}$; then $\Delta=M_{n,k}-N_{n,k}$ is a spline function of class $(n-1, r)$ for some $0\leqq r\leqq 2k$. Denote the interval $[-1, 1]$ by J, and let I be any maximal open subinterval of J in which Δ vanishes identically. It is clear that an end-point of I must be either a knot of $M_{n,k}$ or a knot of $M_{n,k}$ or one of the points $x=\pm 1$. Suppose that I contains ν points which are knots of either $M_{n,k}$ or $N_{n,k}$, and note that they must in fact be common to both functions $M_{n,k}$ and $N_{n,k}$. Now $M_{n,k}$ is a monospline of class (n, ν) in $\bar I$, and $M'_{n,k}$ is continuous. Hence $M'_{n,k}$ must vanish at each extremum of $M_{n,k}$ in $\bar I$, unless one of the points $x=\pm 1$ is an end-point of I. Thus $M_{n,k}$ has at most $n+2\nu-1$ extrema in $\bar I$, by Theorem 6, unless one of the points $x=\pm 1$ is an end-point of I, in which case $M_{n,k}$ has at most $n+2\nu$ extrema in $\bar I$.

Suppose there are exactly p such intervals I_j, with ν_j knots common to both $M_{n,k}$ and $N_{n,k}$ in I_j. Then Δ has exactly $np-p'$ zeros in $\bigcup_1^p \bar I_j$, where $p'=(p-2, p-1, p)$ according as (both, one, neither) of the points $x=\pm 1$ are found among the end-points of the intervals I_j. We wish to count the number of zeros of Δ in $K=J-\bigcup_1^p \bar I_j$. In doing this, according to the prescription given in §2, we must collapse each interval I_j to a point. But then Δ is actually a spline function of class $(n-1, 2k-p'-2\sum_1^p \nu_j)$, since $2\nu_j+1$ of the original $2k$ knots are lost when each interval I_j is collapsed to a point, unless one of the points $x=\pm 1$ is an end-point of I_j, when only $2\nu_j$ knots are lost.

For each j, let I'_j be a relatively open subinterval of J which contains $\bar I_j$ but which contains no zeros or knots of Δ or extrema of $M_{n,k}$ which are not already contained in $\bar I_j$. Then $M_{n,k}$ has at least $n+2k+1-np+p'-2\sum_1^p \nu_j$ extrema in $K'=J-\bigcup_1^p I'_j$, according to the above discussion, and Δ has only isolated zeros in K', and the number of zeros of Δ is the sum of those to be found in K' and the $np-p'$ already found in $\bigcup_1^p \bar I_j$. We remark that, by Theorem 4, $np-p'\leqq Z(\Delta)\leqq n+2k-1-p'-2\sum_1^p \nu_j\leqq n+2k-1-p'$, and hence $p \leqq [(n+2k-1)/n]$. Note also that K' is the disjoint union of $p'+1$ closed subintervals of J.

To complete the proof of Theorem 2 we require two lemmas.

LEMMA 5. *If the closed interval $K_j \subset K'$ contains m of the points x_ν, then Δ has at least $m-1$ zeros in K_j.*

Proof. Since we consider only $n \geqq 3$, $M_{n,k}$, $N_{n,k}$, and Δ all have continuous first derivatives. Consider the subintervals $[x_{\nu_1}, x_{\nu_2})$, $[x_{\nu_2}, x_{\nu_3})$, $\cdots$, $[x_{\nu_{m-1}}, x_{\nu_m}]$ of K_j, noting that x_{ν_j} is a maximum (minimum) of $M_{n,k}$ according as $n-\nu_j$ is even (odd). Suppose it has been established that, for some $2 \leqq j \leqq m$, either

(i) Δ has at least $j-1$ zeros on $[x_{\nu_1}, x_{\nu_{j-1}}]$, or

(ii) Δ has at least $j-2$ zeros on $[x_{\nu_1}, x_{\nu_{j-1}})$,

where (i) and (ii) are not necessarily exclusive. One of these conditions is surely met if $j=2$. Then, in case (i), Δ has at least $(j+1)-2$ zeros on $[x_{\nu_1}, x_{\nu_j})$. In case (ii), either

(a) $\Delta(x_{\nu_{j-1}})=0$, and $\Delta'(x_{\nu_{j-1}})=0$ if $j \geqq 3$, or

(b) $\Delta(x_{\nu_j})=0$, and $\Delta'(x_{\nu_j})=0$ if $j \leqq m-1$, or

(c) $N_{n,k}$ is above (below) $M_{n,k}$ at $x_{\nu_{j-1}}$, and below (above) $M_{n,k}$ at x_{ν_j}, according as $n-\nu_j$ is even (odd), and hence $\Delta(x)=0$ for some $x_{\nu_{j-1}} < x < x_{\nu_j}$.

In any case we recover one of the conditions (i) or (ii), with j replaced by $j+1$. If we are in case (i) for $j=m$, the proof is complete. If we are in case (ii) for $j=m$, we may apply the induction argument once more, and again the proof is complete.

LEMMA 6. *If each of the q disjoint closed intervals $K_j \subset K'$, and if $\bigcup_1^q K_j$ contains m of the points x_ν, then Δ has at least $m-q$ zeros in $\bigcup_1^q K_j$.*

Proof. This is clearly so for $q=2$, using Lemma 5, and the result for arbitrary $q \leqq m$ follows by induction.

Returning now to the theorem proper, since K' contains at least $n+2k+1-np+p'-2\sum_1^p \nu_j$ of the points x_ν, and K' is the disjoint union of $p'+1$ closed intervals, Δ has at least $n+2k-np-2\sum_1^p \nu_j$ zeros in K' by Lemmas 5 and 6, and hence at least $n+2k-p'-2\sum_1^p \nu_j$ zeros in all. But we have already seen that Δ is a spline function of class $(n-1,\ 2k-1-p'-2\sum_1^p \nu_j)$, and thus it can have at most $n+2k-1-p'-2\sum_1^p \nu_j$ zeros, by Theorem 4. Thus $\Delta=0$, and the theorem is proved.

Finally, we must show that Theorem 2, $n \geqq 2$, follows from Theorems 1 and 9, the case $n=1$ of Theorem 2 having been settled in §1. Let $M_{n,k}$ be any monospline of class (n, k) which has as its relative extrema the numbers $e_\nu=(-1)^{n-\nu}$, $1 \leqq \nu \leqq n+2k-1$, with $M_{n,k}(x_\nu)=e_\nu$, $x_j < x_{j+1}$. Such a monospline exists, by Theorem 1. Let x_0 be the unique point to the left of x_1 such that $M_{n,k}(x_0)=(-1)^n$, and similarly let x_{n+2k} be the unique point to the right of x_{n+2k-1} such that $M_{n,k}(x_{n+2k})=1$. Such points surely exist, since $M_{n,k}$ is continuous and monotone outside the interval $[x_1, x_{n+2k-1}]$, by the remark following Theorem 6, and $M_{n,k}(x_{n+2k-1}) = -1$, $M_{n,k}(x_1) = (-1)^{n-1}$. Let $2\lambda = x_{n+2k} - x_0$, and $2\mu = x_{n+2k} + x_0$, and define the monospline N of class (n, k) by

$$N(x) = \lambda^{-n} M_{n,k}(\lambda x + \mu). \tag{49}$$

Since N has absolutely equal extrema at $n+2k+1$ points of the interval $[-1, 1]$, including both end-points, with alternating signs, $N = M^*_{n,k}$ by Theorem 9, and the proof of Theorem 2 is complete.

4. **The order of the approximation.** Let $\delta_{n,k}$ denote the maximum value of $M^*_{n,k}(x)$ on the interval $[-1, 1]$. In the light of equations (8)–(11) it is not unreasonable to conjecture that $\delta_{n,k} = 2^{1-n}(1 + \lambda_n k)$, with $\lambda_n \to 0$ as $n \to \infty$. At any rate, it is of interest to bound $\delta_{n,k}$ above and below.

A presumably very crude lower bound may be obtained as follows. The k knots of $M^*_{n,k}$ divide the interval $[-1, 1]$ into $k+1$ subintervals, at least one of which must be of length $2(k+1)^{-1}$ or more. In this subinterval $M^*_{n,k}$ is a polynomial of degree n with leading coefficient one, and hence cannot depart less from zero than does the polynomial $(1+k)^{-n} T_n[(k+1)x]$ in the interval $[-(1+k)^{-1}, (1+k)^{-1}]$, where T_n is as in (5). Hence

$$\delta_{n,k} \geqq 2^{1-n}(1 + k)^{-n}. \tag{50}$$

As to an upper bound, the Bernoulli polynomials (see [1, chapter 5]) B_n defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n, \qquad |z| < 2\pi, \qquad 0 \leqq x \leqq 1, \tag{51}$$

are known to satisfy the equation

$$B_n(x + 1) = B_n(x) + nx^{n-1}. \tag{52}$$

From a consideration of the periodic extension of period unity of B_n it is clear that the polynomial B_n, its leading coefficient being one, may be considered as a component of a monospline of class (n, k). For n odd, the functions B_n are properly centered, i.e. $\max B_n(x) = -\min B_n(x)$. For n even this is not the case, but it may be brought about by subtracting the quantity $2^{-n}B_n(0)$. Thus we consider instead of B the function $B^\#$ defined by

$$B^\#_n(x) = B_n(x) - 2^{-n}B_n(0), \tag{53}$$

$B_n(0)$ being equal to zero for n odd. Finally we change the variable so that the interval $[0, 1]$ is mapped onto the interval $[0, 2(1+k)^{-1}]$, defining

$$B^*_{n,k}(x) = \frac{2^n}{(k + 1)^n} B^\#_n\left[\frac{1}{2}(k + 1)x\right], \qquad 0 \leqq x \leqq 2(k + 1)^{-1}. \tag{54}$$

Since $B^*_{n,k}$ is a component of a properly centered monospline of class (n, k),

$\delta_{n,k}$ is surely not greater than the maximum value of $B^*_{n,k}$ on the interval $[0, 2(k+1)^{-1}]$. For n even, this remark permits the estimate

$$\delta_{n,k} \leqq \frac{|B_n(0)|}{(k+1)^n}(2^n - 1) \leqq \frac{n!}{3\pi^{n-2}(k+1)^n}, \tag{55}$$

while for n odd we find similarly

$$\delta_{n,k} \leqq \frac{2^{n-1}n|B_{n-1}(0)|}{\pi(k+1)^n} \leqq \frac{n!}{3\pi^{n-2}(k+1)^n}, \tag{56}$$

so that the estimate

$$\delta_{n,k} \leqq \frac{n!}{3\pi^{n-2}(k+1)^n} \tag{57}$$

is available in all cases. A comparison of (50) with (57) is disappointing in so far as the behavior of $\delta_{n,k}$ for fixed k is concerned, as the coefficient of $(k+1)^{-n}$ tends strongly to zero in the first case and strongly to infinity in the second, as n tends to infinity.

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