AREA AS THE INTEGRAL OF LENGTHS OF CONTOURS⁽¹⁾

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Introduction. Let Q be the unit square in E^2 and $T:Q \rightarrow E^k$ a continuous mapping of Q into Euclidean k -space. Let $T = 1 \circ m$ be a monotone light factorization of T with monotone factor m , light factor 1, and middle space M . For $a, b \in M$ let $\Gamma(a, b, T) = \{f: [0, 1] \rightarrow M: f \text{ is continuous and } f(0) = a, f(1) = b\}$. The geodesic distance, $G(a, b, T)$, between 1(a) and 1(b) on the surface T is defined by $G(a, b, T) = \inf_{f \in \Gamma(a, b, T)} \text{length } 1 \circ f$.

For a fixed point p of Q , let $D(t, G, T) = \{q \in Q: G(m(p), m(q), T) > t\}$. Let $L(D(t, G, T))$ denote the length of T restricted to the boundary of $D(t, G, T)$ as defined by Cesari in [2]⁽²⁾ and $A(T)$ denote the Lebesgue area of T . This paper investigates the relationship between $A(T)$ and $\int_0^\infty L(D(t, G, T))dt$.

For nondegenerate mappings T of finite area, Theorem III asserts that if p is a point of Q such that $G(m(p), b, T)$ is not infinite for every $b \in M$ different from $m(p)$ then equality holds. If T is a continuous mapping of finite area and $F = \{b \in M: G(m(p), b, T) < \infty\}$ then there is a monotone retraction $r_F: M \rightarrow \bar{F}$ of M onto the closure of F . If T_F is defined by $T_F = 1 \circ r_F \circ m$, Theorems IV and V assert that $A(T_F) = \int_0^\infty L(D(t, G, T_F))dt = \int_0^\infty L(D(t, G, T))dt$.

The paper consists of four parts. In Part I geodesic and μ_j -geodesic distance are defined and the relationship between them is discussed. These concepts have been used previously by Silverman in [10] and [11]. Some properties of the length of a mapping restricted to the boundary of an open set as defined by Cesari in [2] are stated and for a connected open set an inequality between this definition and the definition given by Federer in [6] is given.

In Part II the inequality $A(T) \geq \int_0^\infty L(D(t, G, T))dt$ for nondegenerate mappings T is proved. The use of μ_j -geodesic distances makes it possible to give a proof of this inequality which is very similar to the proof of the Cavalieri inequality given by Cesari in [2].

Part III is concerned with proving Theorem III. Two examples are given to show the hypothesis of Theorem III are necessary. Morrey's representation theorem and a recent result found independently by Federer [7] and L. C. Young [13] are used in the proof of Theorem III.

In Part IV the cyclic additivity of $L(D(t, G, T))$ is discussed. Theorems

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⁽²⁾ The numbers in brackets refer to the bibliography at the end of the paper.

IV and V are proved using Theorem III, and the cyclic additivity of Lebesgue area and $L(D(t, G, T))$.

Part I. Let $f: [0, 1] \rightarrow Y$ be a continuous mapping of the unit interval into a metric space Y with metric d . Let $t_0 \leq t_1 \leq t_2 \leq \dots \leq t_j$ be a collection of $j+1$ points of $[0, 1]$. Let $I_k = \{t: t_{k-1} \leq t \leq t_k\}$ for $k=1, 2, 3, \dots, j$. The following notion of length is a slight modification of a length used in [10].

DEFINITION I. The μ_j -length of f is defined:

$$\mu_j\text{-length } f = \sup_{\substack{\text{all subdivisions} \\ \text{as above by } j+1 \text{ points}}} \sum_{k=1}^j \text{diameter } f(I_k).$$

The following properties of μ_j -length follow from the definition:

- (a) $\mu_j\text{-length } f \leq \mu_{j+1}\text{-length } f$.
- (b) $|\mu_j\text{-length } f - \mu_j\text{-length } g| \leq 2j \sup_{t \in [0,1]} d(f(t), g(t))$.
- (c) $\mu_j\text{-length } f \leq j \text{ diameter } f([0, 1])$.
- (d) Let f, g be functions from $[0, 1]$ into Y such that $f(1) = g(0)$. Let $f \# g$ be defined by

$$f \# g(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Then $\mu_j\text{-length } f \# g \leq \mu_j\text{-length } f + \mu_j\text{-length } g$.

Let $T: Q \rightarrow E^k$ be a continuous mapping of the unit square Q in E^2 into E^k . Let $T = 1 \circ m$, $m: Q \rightarrow M$, $1: M \rightarrow E^k$ be a monotone light factorization of T with monotone factor m , light factor 1 and middle space M . The middle space M will be understood to be metrized by the metric $d(a, b)$ defined as follows: let a and b be points of M , and let C denote the class of continua K in Q that meet $m^{-1}(a)$ and $m^{-1}(b)$, then $d(a, b) = \inf_{K \in C} \text{diameter } T(K)$. For $a, b \in M$ let $\Gamma(a, b, T) = \{f: [0, 1] \rightarrow M: f(0) = a, f(1) = b\}$.

DEFINITION II. The μ_j -geodesic distance between $T(p)$ and $T(q)$ on T , $\mu(j, p, q, T)$, is defined:

$$\mu(j, p, q, T) = \inf_{f \in \Gamma(m(p), m(q), T)} \mu_j\text{-length } 1 \circ f.$$

The function $\mu(j, p, q, T)$ has the following properties:

- (a) $\mu(j, p, q, T) \leq \mu(j+1, p, q, T)$.
- (b) $\mu(j, p, q, T) \leq j \inf_{f \in \Gamma(m(p), m(q), T)} \text{diameter } 1 \circ f([0, 1])$.
- (c) $\mu(j, p, \cdot, T)$ is continuous.
- (d) Let T and T' be continuous mappings. Let $T = 1 \circ m$ and $T' = 1' \circ m'$ be monotone light factorizations of T and T' with middle spaces

M and M' . If there is a homeomorphism $H: M \rightarrow M'$ such that $m' = H \circ m$ then

$$|\mu(j, p, q, T) - \mu(j, p, q, T')| \leq 2j|T - T'|.$$

(e) If T is quasi linear and q and q' belong to the same triangle on which T is linear then

$$|\mu(j, p, q, T) - \mu(j, p, q', T)| \leq |T(q) - T(q')|.$$

(f) If T is Lipschitzian with Lipschitz constant K then

$$|\mu(j, p, q, T) - \mu(j, p, q', T)| \leq K|q - q'|.$$

For each mapping T two geodesic distance functions will be used, one defined on the middle space M , and the other defined on the parameter domain Q . The geodesic function on the middle space, $G(a, b, T)$, is defined by $G(a, b, T) = \inf_{f \in \Gamma(a, b, T)} \text{length } 1 \circ f$. The geodesic distance function defined on Q , $G(p, q, T)$, is defined by

$$G(p, q, T) = \inf_{f \in \Gamma(m(p), m(q), T)} \text{length } 1 \circ f = G(m(p), m(q), T).$$

Let $\mu(p, q, T) = \lim_{j \rightarrow \infty} \mu(j, p, q, T)$. (Note that the limit exists since $\mu(j+1, p, q, T) \geq \mu(j, p, q, T)$.)

LEMMA I.1. $G(p, q, T) = \mu(p, q, T)$.

Notice that for a collection of curves it is not true that $\inf_{\alpha} \text{length } f_{\alpha} = \lim_{j \rightarrow \infty} \inf_{\alpha} \mu_j\text{-length } f_{\alpha}$. Let S_k be the circle in E^2 of diameter $1/k$ and center $(0, 1/2k)$. Let f_k be the function which traverses S_k k times starting from $(0, 0)$. Then $\mu_j\text{-length } f_k \leq j/k$, so $\inf_k \mu_j\text{-length } f_k = 0$ for every j but $\inf_k \text{length } f_k = \pi$.

Proof of Lemma I.1. $G(p, q, T) \geq \mu(j, p, q, T)$ for every j hence $G(p, q, T) \geq \mu(p, q, T)$. To show the other inequality it may be assumed that $\mu(p, q, T) < \infty$. Let $f_j \in \Gamma(m(p), m(q), T)$ be such that $\mu_j\text{-length } 1 \circ f_j - \mu(j, p, q, T) < 1/j$. Since $\mu_k\text{-length } 1 \circ f_k \geq \mu_j\text{-length } 1 \circ f_k$ for $k \geq j$, $\mu_j\text{-length } 1 \circ f_k < \mu(p, q, T) + 1$ for $k \geq j$.

Let $\{f_{\alpha}\}$ be a collection of functions $f_{\alpha}: [0, 1] \rightarrow Y$ where Y is a compact metric space. According to [8] a necessary and sufficient condition that there exist a collection $\{f'_{\alpha}\}$ which is relatively compact in the topology of uniform convergence and such that f_{α} and f'_{α} are Frechet equivalent is that the $\{f_{\alpha}\}$ be equally divisible i.e. for each $\epsilon > 0$ there is an integer n such that for each function f_{α} the interval $[0, 1]$ can be subdivided into n subintervals on which the oscillation of f_{α} is less than ϵ . (The same n must work for all the f_{α} 's.)

Let ϵ be a positive number and N an integer greater than $(\mu(p, q, T) + 1)\epsilon^{-1}$. For each integer $k > N$ the interval $[0, 1]$ can be divided into fewer than N subintervals such that oscillation $1 \circ f_k$ is $\leq \epsilon$ on each interval in the following manner:

Let $t_1 = \sup \{t: \text{diameter } 1 \circ f_k([s: 0 \leq s \leq t]) \leq \epsilon\}$.

Let $t_2 = \sup \{t: \text{diameter } 1 \circ f_k([s: t_1 \leq s \leq t]) \leq \epsilon\}$, etc.

There are at most $N-1$ of these t_i 's which are less than one. Suppose not. Then

$$\text{diameter } 1 \circ f_k([t_i \leq s \leq t_{i+1}]) = \epsilon \text{ if } t_{i+1} < 1$$

and

$$\mu_N\text{-length } 1 \circ f_k \geq N\epsilon > \mu(p, q, T) + 1 > \mu_N\text{-length } 1 \circ f_k,$$

a contradiction.

Since it is possible to find a number N' so that for each of the finite number of functions $1 \circ f_1, \dots, 1 \circ f_N$ the interval $[0, 1]$ may be subdivided into N' intervals on which the corresponding functions have oscillation less than ϵ , it's seen that the collection $\{1 \circ f_k\}_{k=1}^\infty$ is equally divisible.

From the definition of the metric in M , $\text{diameter } 1 \circ f_k([t_i \leq s \leq t_{i+1}]) \geq \text{diameter } f_k([t_i \leq s \leq t_{i+1}])$. Hence the $\{f_k\}$ are equally divisible. Let $\{f_k^*\}$ be a collection of functions such that f_k and f_k^* are Frechet equivalent and the collection $\{f_k^*\}$ is relatively compact. By selecting a subsequence and relabeling, it may be assumed that the f_k^* converge to a function f . It's seen that f_k^* and $f \in \Gamma(m(p), m(q), T)$ and

$$\mu_j\text{-length } 1 \circ f_k = \mu_j\text{-length } 1 \circ f_k^*.$$

If $k \geq j$,

$$\mu_j\text{-length } 1 \circ f_k^* \leq \mu_k\text{-length } 1 \circ f_k^* \leq \mu(p, q, T) + 1/k.$$

By property (b) of μ_j -lengths

$$\lim_{k \rightarrow \infty} \mu_j\text{-length } 1 \circ f_k^* = \mu_j\text{-length } 1 \circ f \leq \mu(p, q, T),$$

$$\lim_{j \rightarrow \infty} \mu_j\text{-length } 1 \circ f = \text{length } 1 \circ f \leq \mu(p, q, T).$$

Hence $G(p, q, T) \leq \mu(p, q, T)$.

COROLLARY I (OF THE PROOF). *For $a, b, \in M$ there is an $f \in \Gamma(a, b, T)$ such that $\text{length } 1 \circ f = G(a, b, T)$.*

COROLLARY II. *$G(p, \cdot, T)$ and $G(m(p), \cdot, T)$ are lower semicontinuous.*

The functions $\mu(j, p, \cdot, T)$ are an increasing sequence of continuous functions such that $\lim_{j \rightarrow \infty} \mu(j, p, q, T) = G(p, q, T)$ so $G(p, \cdot, T)$ is lower semicontinuous. $\{b \in M: G(m(p), b, T) \leq t\} = m(\{q \in Q: G(p, q, T) \leq t\})$ and hence is a closed set for each t . Therefore $G(m(p), \cdot, T)$ is lower semicontinuous.

Let Q be the unit square with the relative topology as a subset of E^2 . For subsets of Q open, closed, boundary, etc., will be understood to be with respect to this topology. For a set A , A^* will generally be used to denote its

boundary. For the square Q it will be necessary to speak of its boundary both using the convention mentioned above and as a subset of E^2 , Q^* will be used for the relative boundary and $Q^\#$ for the boundary as a subset of E^2 .

For a continuous mapping $T: Q \rightarrow E^k$ and an open subset D of Q , let $L(T, D^*)$ denote the length of T restricted to the boundary of D as defined by Cesari in [2].

The following statements about $L(T, D^*)$ are proved (although not stated in quite this manner) in [1; 2] or [3] or follow directly from the definition of length.

LEMMA I.2. *Let $T: Q \rightarrow E^k$ be a continuous mapping and D an open subset of Q .*

Let $(D)_\rho = \{q \in Q: \text{dist}(q, Q-D) \geq \rho\}$. If $\epsilon > 0$ then there is a $\rho > 0$ such that if T' and D' are a continuous mapping and an open subset of Q such that $|T - T'| < \rho$ and $(D)_\rho \subset D' \subset D$ then $L(T', D'^) \geq L(T, D^*) - \epsilon$.*

LEMMA I.3. *If D is an open subset of Q and T and T' are continuous mappings of Q into E^k such that $T|D^* = T'|D^*$ then $L(T, D^*) = L(T', D^*)$.*

LEMMA I.4. *Let T and T' be Frechet equivalent mappings of Q into E^k . By [9] or [5] there exist monotone light factorizations $T = 1 \circ m$ and $T' = 1 \circ m'$ of T and T' with a common light factor 1 and a common middle space M . Let D be an open subset of M and let $D(T)$ and $D(T')$ be $m^{-1}(D)$ and $(m')^{-1}(D)$ respectively. Then $L(T, D(T)^*) = L(T', D(T')^*)$.*

LEMMA I.5. *Let α be a connected open subset of Q and $T: Q \rightarrow E^k$ a continuous mapping. Let γ be a component of α^* and $A(\gamma)$ be the associated open set with γ as its boundary defined in [2]. If $L(T, A(\gamma)^*) < \infty$ there is a set B which can be taken to be a circle or a closed interval and a function $f: B \rightarrow \gamma$ such that:*

- (a) $T \circ f: B \rightarrow E^k$ is continuous.
- (b) The continua of constancy of T which contain a point of $f(B)$ cover γ .
- (c) $L(T, A(\gamma)^*) = \text{length } T \circ f$.

In Part III essential use will be made of a length defined by Federer in [6]. The following lemma gives a comparison of $L(T, D^*)$ and this length.

LEMMA I.6. *Let $T: Q \rightarrow E^k$ be a continuous mapping. Let α be a connected open subset of Q . Let $T = 1 \circ m$ be a monotone light factorization of T with middle space M . Let W denote the set of points at which $m(\alpha^*)$ is of positive dimension. Let $N(x, 1, W)$ equal the number, possibly infinite, of points a of W such that $1(a) = x$. Let H^1_x denote 1-dimensional Hausdorff measure in E^k . Then $L(T, \alpha^*) \geq \int_{E^k} N(x, 1, W) dH^1_x$.*

It is no loss of generality to assume $L(T, \alpha^*) < \infty$. Let γ be a component of α^* . Since $L(T, \alpha^*) < \infty$, $L(T, A(\gamma)^*) < \infty$ and Lemma I.5 may be applied. Let f be the function described in Lemma I.5. By the theorem relating lengths and Hausdorff measure for curves

$$L(T, A(\gamma)^*) = \text{length } T \circ f = \int_{E^k} N(x, T \circ f, B) dH_k^1.$$

By (b) of Lemma I.5,

$$N(x, 1, m(\gamma)) = N(x, 1, m \circ f(B)) \leq N(x, T \circ f, B),$$

hence

$$L(T, A(\gamma)^*) \geq \int_{E^k} N(x, 1, m(\gamma)) dH_k^1.$$

Let $E = \{\gamma \in \alpha^*: L(T, A(\gamma)^*) \neq 0\}$. By 20.2(i) of [2] $L(T, A(\gamma)^*) = 0$ if and only if T is constant on γ . Since each point of E belongs to a γ on which T is not constant, $m(E) \subset W$. Let S be the family of all countable collections $\{C_i\}$ of disjoint nondegenerate continua contained in $m(\alpha^*)$. By Theorem 3.3 of [6]

$$\sup_{\{C_i\} \in S} \sum_{i=1}^{\infty} \text{diameter } 1(C_i) = \int_{E^k} N(x, 1, W) dH_k^1.$$

Let $\{C_i\}$ be a countable collection of disjoint nondegenerate continua of $m(\alpha^*)$. Since m is monotone, each C_i must be contained in some $m(\gamma)$ where $\gamma \in E$ hence

$$\sum_{i=1}^{\infty} \text{diameter } 1(C_i) \leq \sum_{i=1}^{\infty} \int_{E^k} N(x, 1, C_i) dH_k^1 \leq \int_{E^k} N(x, 1, m(E)) dH_k^1$$

so $\int_{E^k} N(x, 1, m(E)) dH_k^1 = \int_{E^k} N(x, 1, W) dH_k^1$. Now

$$\begin{aligned} L(T, \alpha^*) &= \sum_{\gamma \ni L(T, A(\gamma)^*) \neq 0} L(T, A(\gamma)^*) \\ &\geq \sum_{\gamma \ni L(T, A(\gamma)^*) \neq 0} \int_{E^k} N(x, 1, m(\gamma)) dH_k^1 \\ &\geq \int_{E^k} N(x, 1, m(E)) dH_k^1 = \int_{E^k} N(x, 1, W) dH_k^1. \end{aligned}$$

Part II. Let $D(t, \mu_j, T) = \{q \in Q : \mu(j, p, q, T) > t\}$. Since $\mu(j, p, \cdot, T)$ is continuous $D(t, \mu_j, T)$ is an open set. Denote $L(T, D(t, \mu_j, T)^*)$ more briefly by $L(D(t, \mu_j, T))$.

LEMMA II.1. *Let T_n, T be continuous mappings such that T_n converges uniformly to T and all the T_n 's and T are light or all the T_n 's and T have $Q^\#$ as their only nondegenerate continuum of constancy. Then*

$$\liminf_{\tau \rightarrow t^+} \liminf_{n \rightarrow \infty} L(D(\tau, \mu_j, T_n)) \geq L(D(t, \mu_j, T)).$$

Let $T_n = 1_n \circ m_n$, $T = 1 \circ m$ be monotone light factorizations of T and T_n with middle spaces M and M_n . If T and T_n are light m and m_n are homeomorphisms. In the other case m and m_n define homeomorphisms of the quotient space $Q/Q^\#$ with M and M_n . Hence property (d) of the function $\mu(j, p, q, T)$ may be used to show $\mu(j, p, \cdot, T_n)$ converges uniformly to $\mu(j, p, \cdot, T)$.

Let ϵ be a positive number and ρ be the corresponding number for T , $D(t, \mu_j, T)$ described in Lemma I.2. Let $t' = \min_{q \in (D(t, \mu_j, T))_\rho} \mu(j, p, q, T)$. Then $t' > t$. Let τ be any number $t < \tau < t'$ and let $\sigma = \min(\tau - t, t' - \tau)$. Let N be an integer so that if $n > N$, $|T_n - T| < \rho$ and $|\mu(j, p, \cdot, T_n) - \mu(j, p, \cdot, T)| < \sigma$.

If $q \in (D(t, \mu_j, T))_\rho$ and $n > N$

$$\mu(j, p, q, T_n) > \mu(j, p, q, T) - \sigma \geq t' - \sigma \geq \tau$$

so $D(\tau, \mu_j, T_n) \supset (D(t, \mu_j, T))_\rho$.

If $q \in D(\tau, \mu_j, T_n)$ and $n > N$

$$\mu(j, p, q, T) > \mu(j, p, q, T_n) - \sigma > \tau - \sigma \geq t$$

so $D(\tau, \mu_j, T_n) \subset D(t, \mu_j, T)$.

Hence by Lemma I.2 for $n > N$, $L(D(\tau, \mu_j, T_n)) \geq L(D(t, \mu_j, T)) - \epsilon$. Taking limits

$$\liminf_{\tau \rightarrow t^+} \liminf_{n \rightarrow \infty} L(D(\tau, \mu_j, T_n)) \geq L(D(t, \mu_j, T)) - \epsilon.$$

Since ϵ was arbitrary the conclusion of the lemma follows.

LEMMA II.2. Let $T: Q \rightarrow E^k$ be a continuous mapping. Let $D(t, G, T) = \{q \in Q: G(p, q, T) > t\}$. Then $\liminf_{j \rightarrow \infty} L(D(t, \mu_j, T)) \geq L(D(t, G, T))$.

Let ϵ be a positive number and ρ be the corresponding number for T , $(D(t, G, T))$ described in Lemma I.2. Since $\mu(j, p, q, T) \leq G(p, q, T)$, $D(t, \mu_j, T) \subset D(t, G, T)$. Since $\lim_{j \rightarrow \infty} \mu(j, p, q, T) = G(p, q, T)$ the sets $D(t, \mu_j, T)$ cover $D(t, G, T)$. Since $(D(t, G, T))_\rho$ is compact there are a finite number of sets $D(t, \mu_j, T)$ which cover $(D(t, G, T))_\rho$. Let J be the largest of the integers j corresponding to these sets $D(t, \mu_j, T)$. Since $\mu(j+1, p, q, T) \geq \mu(j, p, q, T)$, $(D(t, G, T))_\rho \subset D(t, \mu_j, T)$ for $j > J$.

By Lemma I.2, $L(D(t, \mu_j, T)) \geq L(D(t, G, T)) - \epsilon$ for $j > J$. Hence $\liminf_{j \rightarrow \infty} L(D(t, \mu_j, T)) \geq L(D(t, G, T)) - \epsilon$ and since ϵ was arbitrary the conclusion of the lemma follows.

LEMMA II.3. Let $T: Q \rightarrow E^k$ be a quasi linear mapping then $A(T) \geq \int_0^\infty L(D(t, \mu_j, T)) dt$.

Let $\Omega = \{\tilde{\Delta}_i\}$ be a subdivision of Q into triangles on which T is linear. Let M be a Lipschitz constant for T . Let $\tilde{\Delta}_i$ be a triangle so that $T(\tilde{\Delta}_i)$ is a non-degenerate triangle in E^k . Then $T|_{\tilde{\Delta}_i}$ has an inverse $(T|_{\tilde{\Delta}_i})^{-1}: T(\tilde{\Delta}_i) \rightarrow \tilde{\Delta}_i$ which is linear and hence Lipschitzian on $T(\tilde{\Delta}_i)$. Let N be the maximum of the Lipschitz constants of the $(T|_{\tilde{\Delta}_i})^{-1}$.

Let S_n denote a strip of width $2(2/n)^{1/2}$ about the edges of the triangles of Ω . Let C_p denote the square of center q and sides $1/n$ about q . Let $\mu^n(q) = 1/n^2 \int_{C_q} \mu(j, p, r, T) dL_2(r)$. The functions $\mu^n(q)$ are continuously differentiable, converge uniformly to $\mu(j, p, \cdot, T)$ and using property (f) of the function $\mu(j, p, q, T)$ it is seen that

$$|\mu^n(q') - \mu^n(q)| \leq M |q' - q|.$$

By 20.4(ii) of [2] there is a quasi linear function $\phi^n: Q \rightarrow E^1$ such that $0 < \mu(j, p, q, T) - \phi^n(q) < \epsilon_n$, where ϵ_n is a sequence of numbers tending to 0, and $|\text{grad } \phi^n(q) - \text{grad } \mu^n(q)| < 1/Nn$ at each interior point of a triangle of linearity of ϕ^n .

Let $\Omega^n = \{\Delta_i\}$ be a triangulation of Q so that ϕ^n and T are linear on each triangle of Ω^n , each Δ_i is contained in a $\tilde{\Delta}_i$ of Ω , and Δ_i is contained either in S_n or $Q - S_n$.

Let Δ_i be a triangle of Ω^n such that $T(\Delta_i)$ is a nondegenerate triangle. Let $D(T|_{\Delta_i})^{-1}(x)$ be the differential of $(T|_{\Delta_i})^{-1}$ at a point x of $T(\Delta_i)$.

$$\begin{aligned} & \|\text{grad } \phi^n \circ (T|_{\Delta_i})^{-1}(x) - \text{grad } \mu^n \circ (T|_{\Delta_i})^{-1}(x)\| \\ &= \|\text{grad } \phi^n[D(T|_{\Delta_i})^{-1}(x)] - \text{grad } \mu^n[D(T|_{\Delta_i})^{-1}(x)]\| \\ &\leq \|\text{grad } \phi^n[D(T|_{\Delta_i})^{-1}(x)] - \text{grad } \mu^n[D(T|_{\Delta_i})^{-1}(x)]\| \\ &\leq \|\text{grad } \phi^n((T|_{\Delta_i})^{-1}(x)) - \text{grad } \mu^n((T|_{\Delta_i})^{-1}(x))\| D(T|_{\Delta_i})^{-1}(x) \\ &\leq (1/Nn)N = 1/n. \end{aligned}$$

Hence

$$|\text{grad } \phi^n \circ (T|_{\Delta_i})^{-1}(x)| \leq |\text{grad } \mu^n \circ (T|_{\Delta_i})^{-1}(x)| + 1/n.$$

If q, q' belong to a triangle of Ω^n contained in $Q - S_n$ then

$$\begin{aligned} & |\mu^n(q) - \mu^n(q')| \\ &\leq 1/n^2 \int_{C_q} |\mu(j, p, r, T) - \mu(j, p, r + q - q', T)| dL_2(r) \\ &\leq 1/n^2 \int_{C_q} \mu(j, r, r + q - q', T) dL_2(r) \\ &= 1/n^2 \int_{C_q} |T(r) - T(r + q - q')| dL_2(r) \\ &= 1/n^2 \int_{C_q} |T(q - q')| dL_2(r) = |T(q) - T(q')| 1/n^2 \int_{C_q} dL_2(r) \\ &= |T(q) - T(q')|. \end{aligned}$$

Hence if Δ_i is a triangle of Ω^n contained in $Q - S_n$ so that $T(\Delta_i)$ is a nondegenerate triangle, $|\text{grad } \mu^n \circ (T|_{\Delta_i})^{-1}| \leq 1$. This implies that

$$|\operatorname{grad} \phi^n \circ (T| \Delta_i)^{-1}(x)| \leq 1 + 1/n.$$

If Δ_i is any triangle of Ω^n for which $T(\Delta_i)$ is nondegenerate and $x, y \in T(\Delta_i)$

$$\begin{aligned} |\mu^n \circ (T| \Delta_i)^{-1}(x) - \mu^n \circ (T| \Delta_i)^{-1}(y)| \\ \leq M |(T| \Delta_i)^{-1}(x) - (T| \Delta_i)^{-1}(y)| \leq MN |x - y|. \end{aligned}$$

Hence $|\operatorname{grad} \mu^n \circ (T| \Delta_i)^{-1}(x)| \leq MN$ which implies $|\operatorname{grad} \phi^n \circ (T| \Delta_i)^{-1}(x)| \leq MN + 1/n$.

Let $D(t, \phi^n) = \{q \in Q: \phi^n(q) > t\}$. By Lemma 20.4(i) of [2]

$$(1 + 1/n)A(T| \Delta_i) \geq \int_0^\infty L(T| \Delta_i, [D(t, \phi^n) \cap \Delta_i]^*) dt$$

if Δ_i is a triangle contained in $Q - S_n$ so that $T(\Delta_i)$ is a nondegenerate triangle and

$$(MN + 1/n)A(T| \Delta_i) \geq \int_0^\infty L(T| \Delta_i, [D(t, \phi^n) \cap \Delta_i]^*) dt$$

if Δ_i is a triangle contained in S_n so that $T(\Delta_i)$ is a nondegenerate.

Since

$$\begin{aligned} L(T, D(t, \phi^n)^*) &= \sum_{\{\Delta_i: T(\Delta_i) \text{ is nondegenerate}\}} L(T| \Delta_i, [D(t, \phi^n) \cap \Delta_i]^*), \\ (1 + 1/n) \sum_{\Delta_i \subset Q - S_n} A(T| \Delta_i) &+ (MN + 1/n) \sum_{\Delta_i \subset S_n} A(T| \Delta_i) \\ &\geq \int_0^\infty L(T, D(t, \phi^n)) dt. \end{aligned}$$

Since ϕ^n converges uniformly to $\mu(j, p, q, T)$ and $\phi^n \leq \mu(j, p, q, T)$, Lemma I.2 implies $\liminf_{n \rightarrow \infty} L(T, D(t, \phi^n)) \geq L(D(t, \mu_j, T))$. By taking limits, applying Fatou's lemma, and noting that $\liminf_{n \rightarrow \infty} \sum_{\Delta_i \subset S_n} A(T| \Delta_i) = 0$, it's concluded that $A(T) \geq \int_0^\infty L(D(t, \mu_j, T)) dt$.

LEMMA II.4. *Let T be a light mapping or a mapping whose only nondegenerate continuum of constancy is $Q^\#$; then $A(T) \geq \int_0^\infty L(D(t, \mu_j, T)) dt$.*

By [9] if T is light T may be approximated by a sequence T_n of light quasi linear mappings such that $A(T_n)$ converges to $A(T)$ or if $Q^\#$ is the only continuum of constancy of T , T may be approximated by a sequence T_n of quasi linear mappings with the same property such that $A(T_n)$ converges to $A(T)$.

Let $\phi(\tau) = \liminf_{n \rightarrow \infty} L(D(\tau, \mu_j, T_n))$. By Lemma II.3 and Fatou's lemma

$$A(T) = \lim_{n \rightarrow \infty} A(T_n) \geq \liminf_{n \rightarrow \infty} \int_0^\infty L(D(\tau, \mu_j, T_n)) d\tau \geq \int_0^\infty \phi(\tau) d\tau.$$

Let $h = \tau - t$ and define $\phi(\tau) = 0$ for $\tau < 0$. Then $\int_0^\infty \phi(\tau) d\tau = \int_0^\infty \phi(t+h) dt$. By Lemma II.1 and Fatou's lemma

$$\liminf_{h \rightarrow 0^+} \int_0^\infty \phi(t+h) dt \geq \int_0^\infty \liminf_{h \rightarrow 0^+} \phi(t+h) dt \geq \int_0^\infty L(D, T, \mu_j, T) dt.$$

Hence

$$A(T) \geq \int_0^\infty L(D(t, \mu_j, T)) dt.$$

LEMMA II.5. *Let T be as in Lemma II.4. Then $A(T) \geq \int_0^\infty L(D(t, G, T)) dt$.*

By Lemma II.2, Lemma II.4, and Fatou's lemma

$$\begin{aligned} A(T) &\geq \liminf_{j \rightarrow \infty} \int_0^\infty L(D(t, \mu_j, T)) \\ &\geq \int_0^\infty \liminf_{j \rightarrow \infty} L(D(t, \mu_j, T)) dt \geq \int_0^\infty L(D(t, G, T)) dt. \end{aligned}$$

THEOREM I. *Let T be an open or closed nondegenerate mapping, then $A(T) \geq \int_0^\infty L(D(t, G, T)) dt$.*

Since T is open or closed nondegenerate T is Frechet equivalent to a mapping T' of the type described in Lemma II.4. Let p' be a point of Q such that $m(p) = m'(p')$ and let $D(t, G, T') = \{q \in Q : G(m'(p'), m'(q), T') > t\}$. Then $m(D(t, G, T)) = m'(D(t, G, T'))$ and Lemma II.5 can be applied to give,

$$A(T) = A(T') \geq \int_0^\infty L(D(t, G, T')) dt = \int_0^\infty L(D(t, G, T)) dt.$$

Part III. The following example shows that $\int_0^\infty L(D(t, G, T)) dt$ need not equal $A(T)$ if $A(T)$ is infinite. Let (u, v) be coordinate variables in Q . Let $\phi: [0, 1] \rightarrow E^1$ be a nowhere differentiable real valued function defined on the unit interval. Let $T: Q \rightarrow E^2$ be defined by $T(u, v) = (\phi(u), \phi(v))$.

ϕ is a light mapping, for if ϕ were constant on some nondegenerate connected set, the set would contain an interval on which ϕ was constant and ϕ would be differentiable on the interior of the interval. This implies T is a light mapping. If $I: Q \rightarrow Q$ is the identity mapping of Q onto Q , $T = T \circ I$ is a monotone light factorization of T with middle space Q .

Let p, q be distinct points of Q and $f \in \Gamma(p, q, T)$. Let (x_1, x_2) be coordinates in E_2 and $f_i: [0, 1] \rightarrow [0, 1]$ be the component mappings of f . For $i = 1, 2$ let $\pi_i: E^2 \rightarrow E^1$ be defined by $\pi_i(x_1, x_2) = x_i$. π_i is Lipschitzian with Lipschitz constant 1. Since p and q are distinct at least one of $f_i[0, 1]$ covers a nondegenerate interval $[c, d]$. Now $\pi_i \circ T \circ f(t) = \phi(f_i(t))$ and

$$\begin{aligned}
\text{length } T \circ f &= \int_{E^2} N(x, T \circ f, [0, 1]) dH_2^1 \\
&\geq \int_{E^1} N(x_i, \pi_i \circ T \circ f, [0, 1]) dH_1^1 \\
&\geq \int_{E^1} N(x_i, \phi, [c, d]) dH_1^1 = \infty
\end{aligned}$$

since if the last expression was finite ϕ would be of bounded variation on $[c, d]$ and hence differentiable almost everywhere on $[c, d]$.

This implies $G(p, q, T) = \infty$ if $p \neq q$ so $D(t, G, T)$ is the complement of the point p for every $t > 0$ and hence $L(D(t, G, T)) = 0$ for $t > 0$. Therefore $\int_0^\infty L(D(t, G, T)) dt = 0$.

There are many ways of showing $A(T) = \infty$. One is that if this were not so Theorem II proved below would be contradicted.

The following is an example of a mapping of finite area, such that there is a single point at an infinite geodesic distance from every other point. Let C be the closed unit disc in E^2 . Let (ρ, θ) be polar coordinates in E^2 and (r, ω, z) be cylindrical coordinates in E^3 . Define $F: C \rightarrow E^3$ by $r = \rho, \omega = \theta, z = \rho \sin 1/\rho$. Let H be any homeomorphism of Q onto C and let $T = F \circ H$. Then point which maps into the center of C is at an infinite geodesic distance from every other point of Q .

Most of the remainder of this part will be concerned with proving the following equality corresponding to the inequality of Theorem I.

THEOREM III. *Let T be an open or closed nondegenerate mapping of Q into E^k for which $A(T) < \infty$. Let $1 \circ m$ be a monotone light factorization of T with middle space M . Let p be a point of Q such that $G(m(p), b, T)$ is finite for some $b \in M$ different from $m(p)$. Then $\int_0^\infty L(D(t, G, T)) dt = A(T)$.*

If T is a closed nondegenerate mapping T is Frechet equivalent to a mapping T' which has $Q^\#$ as its only nondegenerate continua of constancy [9]. By [9] or [5] there is a monotone mapping $m': Q \rightarrow M$ so that $1 \circ m'$ is a monotone light factorization of T' with the same light factor 1 and middle space M as in the monotone light factorization of T and such that $m(Q^\#)$ and $m'(Q^\#)$ are the same single point of M .

Let Q' be a square contained in $Q - Q^\#$. Then $T'|_{Q'}$ is an open nondegenerate mapping, $m'|_{Q'}$ is a homeomorphism and $1 \circ (m'|_{Q'})$ is a monotone light factorization of $T'|_{Q'}$ with middle space $m'(Q') \subset M$.

In order to handle the cases of open and closed nondegenerate mappings simultaneously let $T' = T$ and $Q' = Q$ if T is open nondegenerate. Then if T is either an open or closed nondegenerate mapping of finite area $T'|_{Q'}$ is an open nondegenerate mapping of finite area. Morrey's theorem may be applied to obtain an almost conformal mapping $T'': Q \rightarrow E^k$ such that T'' and $T'|_{Q'}$

are Frechet equivalent. There will be a monotone mapping $m'': Q \rightarrow m'(Q') \subset M$ such that $1 \circ m''$ is a monotone light factorization of T'' with the same light factor as T and middle space $m'(Q')$ contained in the middle space M of T .

Let (u, v) be coordinate variables in Q and $x_1, \dots, x_k$ be the component mappings of T'' . Since T'' is almost conformal it has the following properties: (a) T'' is BVT and ACT (of bounded variation and absolutely continuous in the sense of Tonelli). Hence T'' has partial derivatives almost everywhere in Q .

$$(b) \quad \begin{aligned} \text{If } E(q) &= \sum_{i=1}^k x_{iu}^2 = \left| \frac{\partial T''}{\partial u} \right|^2, & G(q) &= \sum_{i=1}^k x_{iv}^2 = \left| \frac{\partial T''}{\partial v} \right|^2, \\ F(q) &= \sum_{i=1}^k x_{iu} x_{iv} = \frac{\partial T''}{\partial u} \cdot \frac{\partial T''}{\partial v}, \end{aligned}$$

and $J(q)$ is the Jacobian matrix of the partial derivatives then $E(q) = G(q)$, and $F(q) = 0$ almost everywhere in Q and

$$A(T'') = \int_Q |J(q)| dL_2 = \int_Q (E(q)G(q) - F(q)^2)^{1/2} dL_2 = \int_Q E(q) dL_2.$$

Since T'' is ACT, for almost every parallel to the u -axis and for almost every parallel to the v -axis T'' restricted to this parallel is absolutely continuous as a function of one variable.

LEMMA III.1. *Let $d \in M$, $e \in m'(Q')$. Let $f \in \Gamma(d, e, T)$ be such that length $1 \circ f = G(d, e, T)$. Then there is a continuum $K \subset (m'')^{-1}(f([0, 1]))$ which joins $(m'')^{-1}(d)$ and $(m'')^{-1}(e)$ or $(m'')^{-1}(e)$ and $Q^\#$. If T is open nondegenerate K joins $(m'')^{-1}(d)$ and $(m'')^{-1}(e)$.*

If $f[0, 1] \subset m'(Q')$ let $K = (m'')^{-1}(f([0, 1]))$. Note that if T is open nondegenerate $f([0, 1])$ is always contained in $m'(Q')$ since $m'(Q') = M$ in this case. If $f([0, 1])$ is not contained in $m'(Q')$ let $\tau = \sup \{t: f(t) \in M - m'(Q')\}$. Let $m'(Q')^\#$ denote the boundary of $m'(Q')$. Then $f([\tau, 1])$ meets $m(Q')^\#$. Since m'' is monotone and $m'(Q')$ is a two cell $m''(Q^\#) \subset m'(Q')^\#$ [9]. Therefore $K = (m'')^{-1}f([\tau, 1])$ joins $(m'')^{-1}(e)$ and $Q^\#$.

THEOREM II. *Let $T: Q \rightarrow E^k$ be an open or closed nondegenerate mapping of finite area. Let $I = \{a \in M: G(a, b, T) = \infty \text{ for every } b \in M \ni b \neq a\}$. Then $G(a, b, T) < \infty$ if $a, b \in M - I$ and $(M - I)^\# = M$.^(*)*

If a, b are points of $M - I$ there are points c, d of M such that $a \neq c$, $b \neq d$, $G(a, c, T) < \infty$, and $G(b, d, T) < \infty$. If T is closed nondegenerate at least one point in each of the pairs a, c and b, d is not equal to $m(Q^\#)$ say for

(*) The notation $(M - I)^\#$ is used in place of a bar over $M - I$.

definiteness that a and d are different from $m(Q^\#)$. Let T' and m' be the mappings described previously and let Q' be a square contained in $Q - Q^\#$ such that $(m')^{-1}(a)$ and $(m')^{-1}(d)$ are contained in $Q' - Q'^\#$. If T is open nondegenerate let $Q' = Q$. Let T'' and m'' be the mappings described previously for the above square Q' .

Let K_1 and K_2 be continua as described in Lemma III.1 with $d=a$, $e=c$ for K_1 and $d=b$, $e=d$ for K_2 . If T is closed nondegenerate $m'|_{Q-Q^\#}$ is a homeomorphism and hence since m'' is monotone $m'(Q^\#) = m'(Q')^\# = m''(Q^\#)$. Then since $(m')^{-1}(a)$ and $(m')^{-1}(d)$ are contained in $Q' - Q'^\#$, $(m'')^{-1}(a)$ and $(m'')^{-1}(d)$ are contained in $Q - Q^\#$. Since K_1 joins $(m'')^{-1}(a)$ and $(m'')^{-1}(c)$ or $(m'')^{-1}(a)$ and $Q^\#$ and a similar statement holds for K_2 , K_1 and K_2 are nondegenerate if T is closed nondegenerate. If T is open nondegenerate K_1 joins $(m'')^{-1}(a)$ and $(m'')^{-1}(c)$ and K_2 joins $(m'')^{-1}(b)$ and $(m'')^{-1}(d)$; hence K_1 and K_2 are nondegenerate in either case.

Let S be the union of all parallels to the coordinate axes on which T is absolutely continuous as a function of one variable. Since S contains almost every parallel to each axis, S separates any pair of points of Q . Hence $K_1 \cap S$ and $K_2 \cap S$ are not empty. Let $r \in K_1 \cap S$, $s \in K_2 \cap S$; then $G(a, m''(r), T) < \infty$ and $G(b, m''(s), T) < \infty$.

If P is a parallel to say the u -axis contained in S such that $T''|_P$ is absolutely continuous then length $T''|_P = \int_0^1 |\partial T'' / \partial u| du < \infty$. Since a similar relationship holds for each parallel contained in S , $G(m''(r), m''(s), T) < \infty$ for $r, s \in S$. Hence $G(a, b, T) \leq G(a, m''(r), T) + G(m''(r), m''(s), T) + G(m''(s), b, T) < \infty$ for $a, b \in M - I$.

Since S is dense in Q , $m''(S)$ is dense in $m''(Q)$. If T is closed nondegenerate let $\{Q_n'\}$ be a sequence of squares contained in $Q - Q^\#$ such that $\bigcup_{n=1}^\infty Q_n' = Q - Q^\#$. If S_n are the corresponding sets of parallels described above $\bigcup_{n=1}^\infty m''(S_n) \subset M - I$ and $(\bigcup_{n=1}^\infty m''(S_n))^* = M$. If T is open nondegenerate $m''(Q) = M$; hence in either case $(M - I)^* = M$.

It follows from Theorem II that if T is an open or closed nondegenerate mapping of finite area there always is a point p as described in the hypothesis of Theorem III. From now on unless stated otherwise the hypothesis of Theorem III will be assumed and T' , m' , Q' , T'' , and m'' will be as defined previously.

A mapping which may be discontinuous but satisfies all the other conditions for an ACT mapping is said to be ACE (absolutely continuous in the sense of Evans).

LEMMA III.2. Let $G(q, T'') = G(m(p), m''(q), T)$. Then $G(q, T'')$ is ACE.

Let Q_{u_0} denote a subset of S of the type $Q_{u_0} = \{(u, v) \in Q: u = u_0\}$. Let $v_1 \leq v_2 \leq \dots \leq v_n$ be points of $[0, 1]$ and $q_i = (u_0, v_i)$ for $i = 1, 2, \dots, n$ be the corresponding points of Q_{u_0} . Since T'' is absolutely continuous as a function of v on Q_{u_0}

$$G(m''(q_{i+1}), m''(q_i), T) \leq \int_{v_i}^{v_{i+1}} \left| \frac{\partial T''}{\partial v} \right| dv.$$

If I is the set described in Theorem II, both $m(p)$ and $m''(S)$ are contained in $M - I$. Hence by Theorem II $G(m(p), m''(q), T) < \infty$ if $q \in S$.

$$\begin{aligned} & \sum_{i=1}^{n-1} |G(q_{i+1}, T'') - G(q_i, T'')| \\ &= \sum_{i=1}^{n-1} |G(m(p), m''(q_{i+1}), T) - G(m(p), m''(q_i), T)| \\ &\leq \sum_{i=1}^{n-1} G(m''(q_{i+1}), m''(q_i), T) \leq \sum_{i=1}^{n-1} \int_{v_i}^{v_{i+1}} \left| \frac{\partial T''}{\partial v} \right| dv. \end{aligned}$$

Since similar considerations hold for analogous sets Q_{v_0} , it follows that $G(q, T'')$ is ACE.

Since T'' is almost conformal and $G(q, T'')$ is ACE there is a measurable set B contained in Q with the following properties:

(a) The partial derivatives of T'' and $G(q, T'')$ exist and $E(q) = G(q)$, $F(q) = 0$ at every point of B .

(b) $B = \bigcup_{i=1}^{\infty} B_i$ where the B_i are disjoint measurable sets on which T'' and $G(q, T'')$ are continuously differentiable. (A function $f: E^m \rightarrow E^n$ is said to be continuously differentiable on a set $H \subset E^m$ if its restriction to H may be extended to E^m so that the resulting function is continuously differentiable.)

(c) At every point of B , T'' and $G(q, T'')$ have regular approximate differentials, and the same set A of boundaries of squares with sides parallel to the coordinate axes on which the limits are taken may be used for both functions.

(d) For each i let $\phi_i: E^2 \rightarrow E^1$ and $\psi_i: E^2 \rightarrow E^k$ be continuously differentiable functions such that $\phi_i|_{B_i} = G(q, T'')|_{B_i}$ and $\psi_i|_{B_i} = T''|_{B_i}$. Let $D\phi_i(q)$ and $D\psi_i(q)$ denote the differentials of ϕ_i and ψ_i at a point q . Let $DT''(q)$ and $DG(q, T'')$ denote the regular approximate differentials of T'' and $G(q, T'')$ at a point q . Then the linear transformation associated with the Jacobian matrix $J(q)$ of the partial derivatives of T'' , $D\psi_i(q)$, and $DT''(q)$ are equal at every point of B_i and the linear transformation associated with $\text{grad } G(q, T'')$, $D\phi_i(q)$, and $DG(q, T'')$ are equal at every point of B_i .

(e) B is contained in the set S of parallels to the axes on which T'' is absolutely continuous as a function of one variable.

(f) No point of $Q^\#$ is in B .

(g) $|J(q)| \neq 0$ at every point of B .

(h) $L_2((Q - B) \cap \{q \in Q: |J(q)| \neq 0\}) = 0$.

The set B may be obtained as follows: Since T'' is almost conformal and

$G(g, T'')$ is ACE the statement of (a) holds almost everywhere in Q . By Theorem I of [14] the existence of the partial derivatives of T'' and $G(g, T'')$ almost everywhere implies that Q may be written $Q = \bigcup_{i=0}^{\infty} P_i$ where $L_2(P_0) = 0$ and the sets P_i for $i \geq 1$ are a disjoint sequence of closed sets such that $T''|_{P_i}$ and $G(q, T'')|_{P_i}$ are continuously differentiable. By 26.2 (i) and (ii) of [2] the existence of the partial derivatives of T'' and $G(q, T'')$ almost everywhere in Q imply the statement of (c) holds at almost every point of Q .

Theorem 26.2 (i) of [2] implies the equality of the linear transformation associated with $J(q)$ and $DT''(q)$ and the equality of the linear transformation associated with $\text{grad } G(g, T'')$ and $DG(q, T'')$ at almost every point of Q . It is seen that $D\phi_i(q) = DG(q, T'')$ and $D\psi_i(q) = DT''(q)$ at every point of density of P_i where the quantities on the right of the equal sign exist, hence almost everywhere in P_i .

Hence by discarding several sets of measure 0 and the set of points where $|J(q)| = 0$ from the sets P_i for $i \geq 1$ sets B_i are obtained such that $B = \bigcup_{i=1}^{\infty} B_i$ has all the desired properties.

LEMMA III.3. $| \text{Grad } G(q, T'') | = E(q)^{1/2}$ at every point of B .

Consider the regular approximate differential $DT''(q)$ of T'' at a point q of B . Let $w = (a, b)$ be a point of E^2 . From conditions (a) and (d) it is seen that

$$\begin{aligned} |DT''(q)(w)| &= \left| \begin{bmatrix} x_{1u} & x_{1v} \\ \vdots & \vdots \\ x_{ku} & x_{kv} \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right| = \left(\sum_{i=1}^k (x_{iu}a + x_{iv}b)^2 \right)^{1/2} \\ &= (a^2E(q) + b^2G(q) + 2abF(q))^{1/2} = |w| E(q)^{1/2}. \end{aligned}$$

Let $f \in \Gamma(m(p), m''(q), T)$ be such that $\text{length } 1 \circ f = G(m(p), m''(q), T)$. Let K be the corresponding continuum described in Lemma III.1 with $d = m(p)$ and $e = m''(q)$. Let $\{q_n\}$ be a sequence of points approaching q so that $\{q_n\} \subset A \cap K$ where A is the set described in property (c) of the set B . Since $q_n \in K$,

$$\begin{aligned} |G(q, T'') - G(q_n, T'')| &= |G(m(p), m''(q), T) - G(m(p), m''(q_n), T)| \\ &= G(m''(q), m''(q_n), T). \end{aligned}$$

Hence

$$\begin{aligned} \lim_{q_n \rightarrow q} \frac{|G(q, T'') - G(q_n, T'')|}{|q - q_n|} &\geq \lim_{q_n \rightarrow q} \frac{|T''(q_n) - T''(q)|}{|q_n - q|} \\ &= \lim_{q_n \rightarrow q} \left| DT''(q) \left(\frac{q_n - q}{|q_n - q|} \right) \right| = E(q)^{1/2}. \end{aligned}$$

If $q = (u_0, v_0)$ let q_n^* be a sequence of points approaching q which belong to $Q_{u_0} \cap A$. Then

$$\begin{aligned} |G(q, T'') - G(q_n^*, T'')| &= |G(m(p), m''(q), T) - G(m(p), m''(q_n^*), T)| \\ &\leq G(m''(q), m''(q_n^*), T). \end{aligned}$$

By condition (e) of the set B , $B \subset S$. Hence

$$\begin{aligned} \lim_{q_n^* \rightarrow q} \frac{|G(q, T'') - G(q_n^*, T'')|}{|q - q_n^*|} &\leq \lim_{q_n^* \rightarrow q} \frac{G(m''(q), m''(q_n^*), T)}{|q - q_n^*|} \\ &\leq \lim_{q_n \rightarrow q} \frac{\left| \int_{v_n}^{v_0} E(q)^{1/2} dv \right|}{|v_0 - v_n|} = E(q)^{1/2}. \end{aligned}$$

LEMMA III.4. Let $C_t = \{q \in Q: G(q, T'') = t\}$. Using the preceding definitions and conventions

$$A(T'') = \int_0^\infty \left[\int_{E^*} N(x, T'', B \cap C_t) dH_k^1 \right] dt.$$

The following theorem is a special case of a more general theorem recently found independently by Federer [7] and L. C. Young [13].

Let $F: Q \rightarrow E^1$ be a continuously differentiable real valued function on Q and $h: Q \rightarrow E^1$ a nonnegative Borel measurable real valued function on Q . Let $A_t = \{q \in Q: F(q) = t\}$ and K be measurable subset of Q . Then

$$\int_K h |\text{grad } F| dL_2 = \int_{-\infty}^\infty \left[\int_{A_t \cap K} h dH_2^1 \right] dt.$$

Proof of Lemma III.4. Let ϕ_i be the function described in property (d) of the set B . It follows from property (d) that $B_i \cap \{q \in Q: \phi_i(q) = t\} = B_i \cap C_t$ and that $\text{grad } G(q, T'') = \text{grad } \phi_i(q)$ on B_i . Applying the above theorem with $F = \phi_i$, $h = E^{1/2}$, and $K = B_i$ it follows using Lemma III.3 that

$$\begin{aligned} \int_0^\infty \left[\int_{C_t \cap B_i} E(q)^{1/2} dH_2^1 \right] dt &= \int_{B_i} E(q)^{1/2} |\text{grad } \phi_i(q)| dL_2 \\ &= \int_{B_i} E(q)^{1/2} |\text{grad } G(q, T)| dL_2 = \int_{B_i} E(q) dL_2 = \int_{B_i} |J(q)| dL_2. \end{aligned}$$

Since $|\text{grad } \phi(q)| = E(q)^{1/2} \neq 0$ at every point of B_i and $C_t \cap B_i = B_i \cap \{q \in Q: \phi_i(q) = t\}$ the classical implicit function theorem implies that $C_t \cap B_i$ is the disjoint union of countably many sets C_i each of which lies on a differentiable arc γ_i .

Let $u(q)$ denote a unit tangent vector to γ_i at the point q . Let ψ_i be the

function described in property (d) of the set B and let $D\psi_i(q)$ denote the differential of ψ_i at q . The directional derivative of ψ_i in the direction of the unit tangent vector $u(q)$ at the point q is given by $D\psi_i(q)(u(q))$. By a classical theorem on Hausdorff measures

$$\int_{C_i} D\psi(q)(u(q)) dH_2^1 = \int_{E^k} N(x, \psi, C_i) dH_k^1.$$

Since by property (d) of the set B , $D\psi_i(q) = DT''(q)$ at each point q of B_i and $T''|_{B_i} = \psi|_{B_i}$

$$\int_{C_i} DT''(q)(u(q)) dH_2^1 = \int_{E^k} N(x, T'', C_i) dH_k^1.$$

By the argument used in Lemma III.2 $DT''(q)(u(q)) = E(q)^{1/2} |u(q)| = E(q)^{1/2}$. Hence since the sets C_i are disjoint

$$\int_{C_i \cap B_i} E(q)^{1/2} dH_2^1 = \int_{E^k} N(x, T'', C_i \cap B_i) dH_k^1.$$

Hence

$$\begin{aligned} \int_0^\infty \left[\int_{E^k} N(x, T'', B_i \cap C_i) dH_k^1 \right] dt &= \int_0^\infty \left[\int_{C_i \cap B_i} E(q)^{1/2} dH_2^1 \right] dt \\ &= \int_{B_i} |J(q)| dL_2. \end{aligned}$$

Now

$$\begin{aligned} A(T'') &= \int_Q |J(q)| dL_2 = \int_B |J(q)| dL_2 \\ &= \sum_{i=1}^\infty \int_{B_i} |J(q)| dL_2 = \sum_{i=1}^\infty \int_0^\infty \left[\int_{E^k} N(x, T'', B_i \cap C_i) dH_k^1 \right] dt \\ &= \int_0^\infty \left[\int_{E^k} N(x, T'', B \cap C_i) dH_k^1 \right] dt. \end{aligned}$$

LEMMA III.5. *No point of B is contained in a nondegenerate continuum of constancy of T'' .*

Suppose q were a point of B belonging to such a continuum C . Let A be the set described in property (c) of the set B . Let q_n be a sequence of points belonging to $C \cap A$ such that $q_n \neq q$ and q_n converges to q . There is a point v on the unit circle and subsequence $\{q'_n\}$ of q_n such that $(q'_n - q)/|q'_n - q|$ converges to v . Now

$$\lim_{n \rightarrow \infty} \left| \frac{T''(q'_n) - T''(q) - DT''(q)(q'_n - q)}{|q'_n - q|} \right| = 0.$$

Since $T''(q'_n) = T''(q)$ and $DT''(q)$ is continuous,

$$DT''(q)(v) = \lim_{n \rightarrow \infty} DT''(q) \left(\frac{q'_n - q}{|q'_n - q|} \right) = 0$$

contradicting $|J(q)| \neq 0$.

The following can be easily shown to hold by using the usual argument for relative maximum and minimum of functions.

LEMMA III.6. *Let $F: G \rightarrow E^1$ be a function from an open subset G of E^n into the real numbers. Let x be a point of G at which the partial derivatives of F exist. If $F(x) = t$ and there is a neighborhood of x on which $F(x) \geq t$ ($F(x) \leq t$) or there are arbitrarily small neighborhoods of x on whose boundaries $F(x) \geq t$ ($F(x) \leq t$) then $\text{grad } F(x) = 0$.*

LEMMA III.7. *Let $D(t, G, T'') = \{q \in Q: G(q, T'') > t\}$. Then $C_t \cap B \subset D(t, G, T'')^*$.*

If q_0 were a point of $C_t \cap B$ not in $D(t, G, T'')^*$, the interior of $Q - D(t, G, T'')$ would be a neighborhood of q_0 on which $G(q, T'') \leq t$ and hence by Lemma III.6 $\text{grad } G(q_0, T'') = 0$. But since $|\text{gradient } G(q, T'')| = E(q)^{1/2}$ and $E(q) = |J(q)| \neq 0$ at every point of B this is a contradiction.

LEMMA III.8. *The point p does not belong to B .*

Since no point of Q^* belongs to B this follows from Lemma III.6 by an argument entirely analogous to that of Lemma III.7.

The proofs of the following two lemmas are elementary.

LEMMA III.9. *Let X be a locally connected unicoherent Hausdorff space. Let A and B be disjoint, nonempty, closed, connected subsets of X . Let K be the component of $X - A$ containing B . Then $X - K$ is connected and hence by unicoherence $K^* = \overline{K} \cap [X - K]$ is a connected subset of A^* .*

LEMMA III.10. *Let X and Y be compact Hausdorff spaces and $f: X \rightarrow Y$ be a continuous mapping of X onto Y . Let U be an open subset of Y and $V = f^{-1}(U)$. Then $f(V^*) = U^*$.*

LEMMA III.11. *Let $T: Q \rightarrow E^k$ be a continuous mapping. Let $D(t, G, T) = \{a \in M: G(m(p), a, T) > t\}$. Let $\{\beta_i\}$ be the collection of components of $D(t, G, T)$. Then β_i^* is a connected set. If T satisfies the hypothesis of Theorem III and t is greater than 0, β_i^* is a nondegenerate connected set.*

$M - D(t, G, T)$ is closed and connected, for if $c \in M - D(t, G, T)$ there is a function $f \in \Gamma(m(p), c, T)$ for which $\text{length } 1 \circ f = G(m(p), c, T)$ and hence

$f([0, 1])$ is a connected set contained in $M - D(t, G, T)$ joining $m(p)$ and c . Let e be a point of β_i . Applying Lemma III.9 with $A = M - D(t, G, T)$, $B = \{e\}$ it is seen that β_i^* is a connected set.

If T satisfies the hypothesis of Theorem III, M is a 2-sphere or a 2-cell. Since β_i is open β_i^* will be a single point if and only if $M - \beta_i$ is the single point $m(p)$. There is a point $b \in M$ different from $m(p)$ for which $G(m(p), b, T) < \infty$. Let $f \in \Gamma(m(p), b, T)$ be such that $\text{length } 1 \circ f = G(m(p), b, T)$. Let $\tau = \sup \{s: G(m(p), f(s), T) \leq t\}$. Since $t > 0$, $f(0, \tau)$ is a subset of $M - \beta_i$ which is not a single point. Therefore β_i^* is a nondegenerate set.

LEMMA III.12. For $t > 0$ let $D(t, G, T) = \{a \in M: G(m(p), a, T) > t\}$ and $\{\beta_i\}$ be the collection of components of $D(t, G, T)$. Then

$$L(D(t, G, T)) \geq \sum_{i=1}^{\infty} \int_{\beta_i^*} N(x, 1, \beta_i^*) dH_k^1.$$

Let $\{\alpha_i\}$ be the collection of components of $D(t, G, T)$. Since m is monotone and $D(t, G, T) = m^{-1}(D(t, G, T))$ each α_i is of the form $m^{-1}(\beta_j)$ for some j . For convenience it will be supposed that the indices have been chosen such that $\alpha_i = m^{-1}(\beta_i)$. By the additivity of the length

$$L(D(t, G, T)) = \sum_{i=1}^{\infty} L(T, \alpha_i^*).$$

Let $W_i = \{b \in m(\alpha_i^*): \text{dimension } (m(\alpha_i^*), b) > 0\}$. By Lemma I.6 $L(T, \alpha_i^*) \geq \int_{W_i} N(x, 1, W_i) dH_k^1$. By Lemma III.10 $m(\alpha_i^*) = \beta_i^*$. Since by Lemma III.11 β_i^* is a nondegenerate connected set, $\beta_i^* = W_i$. Hence the conclusion of the lemma follows.

LEMMA III.13. Let t be a number greater than 0. Let $\{\alpha_i''\}$ be the collection of components of $D(t, G, T'')$. If $L(D(t, G, T)) < \infty$ then

$$C_i \cap B \cap \left[D(t, G, T'')^* - \bigcup_{i=1}^{\infty} \alpha_i''^* \right]$$

is empty.

(Note that the assertion $L(D(t, G, T)) < \infty$ has to do with the set $D(t, G, T)$ and the mapping T while the conclusion involves the set $D(t, G, T'')$ associated with the mapping T'' .)

Let $\{\gamma_j\}$ denote the collection of components of $m'(Q') \cap D(t, G, T)$. Since $(m'')^{-1}(m'(Q') \cap D(t, G, T)) = D(t, G, T'')$ each γ_j is contained in a component β_i of $D(t, G, T)$ and if γ_j^* denotes the boundary of γ_j relative to $m'(Q')$ it is seen that $\gamma_j^* \subset \beta_i^*$.

By Lemma III.10 $m''(\alpha_j^*) = \gamma_j^* \subset \beta_i^*$. By Lemma III.12

$$\sum_{i=1}^{\infty} \int_{E^k} N(x, 1, \beta_i^*) dH_k^1 < \infty.$$

This implies there is an integer j_n so that

$$\int_{E^k} N\left(x, 1, \bigcup_{j=j_n}^{\infty} \gamma_j^*\right) dH_k^1 < 1/n.$$

Suppose $q \in C_t \cap B \cap [D(t, G, T'')^* - \bigcup_{i=1}^{\infty} \alpha_i'^*]$. Consider the mapping $R: E^k \rightarrow E^1$ defined by $R(x) = |x - T''(q)|$. R is Lipschitzian with Lipschitz constant 1, hence

$$\int_{E^1} N\left(y, R \circ 1, \bigcup_{j=j_n}^{\infty} \gamma_j^*\right) dH_1^1(y) \leq \int_{E^k} N\left(x, 1, \bigcup_{j=j_n}^{\infty} \gamma_j^*\right) dH_k^1(x) < \frac{1}{n}.$$

For each integer n there is a number r_n , $0 < r_n \leq 1/n$ such that the boundary of the sphere $S(T''(q), r_n)$ of center $T''(q)$ and radius r_n does not meet $1(\bigcup_{j=j_n}^{\infty} \gamma_j^*)$. If there were not $R \circ 1(\bigcup_{j=j_n}^{\infty} \gamma_j^*)$ would cover the interval $0 < r_n \leq 1/n$ contradicting

$$\int_{E^1} N\left(y, R \circ 1, \bigcup_{j=j_n}^{\infty} \gamma_j^*\right) dH_1^1(y) < 1/n.$$

Since $q \in B$ by Lemma III.5 q does not belong to a nondegenerate continuum of constancy of T'' . Hence the components G_n of $(T'')^{-1}(S(T''(q), r_n))$ that contain q form a basis of connected neighborhoods of q whose boundaries do not meet $\bigcup_{j=j_n}^{\infty} \alpha_j^*$. If the sets G_n did not form a basis of neighborhoods of q the intersection of their closures would be a nondegenerate continuum of constancy of T'' containing q contradicting Lemma III.5.

Since $q \in B$, $q \notin Q^\#$. Since the sets G_n are a basis of neighborhoods of q it may be assumed by selecting a subsequence that G_n is contained in $S(q, 1/n)$ the sphere of radius $1/n$ about q and that $Q^\#$ and $m'^{-1}(m(p))$ are contained in the same component of the complement of $Q - \bar{G}_n$. Denote this component by K_n . Let $G_n' = Q - \bar{K}_n$. By Lemma III.9 $G_n'^*$ is connected and $G_n'^* \subset G_n^*$. Since $G_n \subset S(q, 1/n)$, $G_n' \subset S(q, 1/n)$ and hence the sets G_n' form a basis of neighborhoods of q with connected boundaries whose boundaries do not meet $\bigcup_{j=j_n}^{\infty} \alpha_j'^*$.

Let $f \in \Gamma(m(p), m''(q), T)$ be such that length $f = G(m(p), m''(q), T)$. Let K be the continuum described in Lemma III.1 connecting $Q^\#$ and q or q and $(m'')^{-1}(m(p))$. Since $G(m(p), m''(q), T) = t$ and $m''(K) \subset f([0, 1])$, $K \subset Q - D(t, G, T'')$. Since $G_n'^*$ separates q and $Q^\#$ or $(m'')^{-1}(m(p))$, $G_n'^* \cap K \neq \emptyset$. Hence $G_n'^* \cap [Q - D(t, g, T'')] \neq \emptyset$. This implies, since $G_n'^* \cap \bigcup_{j=j_n}^{\infty} \alpha_j'^* = \emptyset$ and $G_n'^*$ is connected, that $G_n'^* \cap \bigcup_{j=j_n}^{\infty} \alpha_j'' = \emptyset$.

Since $q \in [D(t, G, T'')^* - \bigcup_{j=1}^{\infty} \alpha_j'^*]$, $q \in Q - \bigcup_{j=1}^{\infty} \bar{\alpha}_j''$. Let

$$H_n = G_n' \cap \left[Q - \bigcup_{j=1}^{i_n} \bar{\alpha}_j'' \right].$$

Since

$$H_n^* \subset [Q - \bigcup_{j=1}^{i_n} \bar{\alpha}_j'']^*, \quad H_n^* \cap \bigcup_{j=1}^{i_n} \alpha_j'' = 0.$$

Since $H_n^* \subset G_n'^* \cup [Q - \bigcup_{j=1}^{i_n} \bar{\alpha}_j'']^*$ and neither of these meets $\bigcup_{j=1}^{i_n} \alpha_j''$, $H_n^* \cap \bigcup_{j=1}^{i_n} \alpha_j'' = 0$.

Thus the sets H_n are a basis of neighborhoods of q on whose boundaries $G(\cdot, T'') \leq t$, hence by Lemma III.6 $\text{grad } G(q, T'') = 0$ contradicting $q \in B$.

LEMMA III.14. $A(T'') \leq \int_0^\infty L(D(t, G, T)) dt$.

By Lemma III.4 $A(T'') = \int_0^\infty \int_{E^k} N(x, T'', B \cap C_t) dH_k^1 dt$. By Lemma III.12 $L(D(t, G, T)) \geq \sum_{i=1}^\infty \int_{E^k} N(x, 1, \beta_i^*) dH_k^1$. Using Lemma III.10 it is seen that $m''(\alpha_i'^*) \subset \beta_j^*$ for some j . Therefore

$$L(D(t, G, T)) \geq \int_{E^k} N\left(x, 1, m''\left(\bigcup_{j=1}^\infty \alpha_j'^*\right)\right) dH_k^1.$$

By Theorem I, $A(T) \geq \int_0^\infty L(D(t, G, T)) dt$, hence $L(D(t, G, T))$ is finite for almost every t . Therefore by Lemma III.13 and Lemma III.7 $B \cap C_t \subset \bigcup_{j=1}^\infty \alpha_j'^*$ for almost every t . By Lemma III.5 $N(x, T'', B \cap C_t) = N(x, 1, m''(B \cap C_t))$. Hence for almost every t

$$L(D(t, G, T)) \geq \int_{E^k} N\left(x, 1, m''\left(\bigcup_{j=1}^\infty \alpha_j'^*\right)\right) dH_k^1 \geq \int_{E^k} N(x, T'', B \cap C_t) dH_k^1.$$

Therefore

$$A(T'') = \int_0^\infty \left[\int_{E^k} N(x, T'', B \cap C_t) dH_k^1 \right] dt \leq \int_0^\infty L(D(t, G, T)) dt.$$

Proof of Theorem III. If T is open nondegenerate Theorem III follows from Lemma III.14 and Theorem I since T'' and T are Frechet equivalent. If T is closed nondegenerate a succession of squares Q_n may be chosen in a manner similar to the choice of Q' so that the Q_n expand outward to Q . Let $T_n' = T'|_{Q_n}$ and T_n'' be defined as T'' was defined. Then $A(T_n') = A(T_n'')$ and $A(T_n')$ converges to $A(T') = A(T)$. Hence using Lemma III.14

$$\lim_{n \rightarrow \infty} A(T_n'') = A(T) \leq \int_0^\infty L(D(t, G, T)) dt.$$

Part IV.

LEMMA IV.1. Let $T: Q \rightarrow E^k$ be a continuous mapping and $1 \circ m$ a monotone light factorization of T with middle space M . Let β be a component of $D(t, G, T)$. Then either β^* is a single point or β^* belongs to a single proper cyclic element of M .

Suppose not, then there would be a pair of distinct points a and b of β^* and a point c of M such that c separates a and b . By Lemma III.11 β^* is connected, hence c must belong to β^* . Since β is open c must belong to $M - \beta$. Let A be the component of $M - \{c\}$ containing a and $B = M - A \cup \{c\}$. Since c separates a and b and $c \notin \beta$, $A \cap \beta$ and $B \cap \beta$ are nonempty sets which form a separation of β contradicting that β is connected.

If C is a proper cyclic element of M each boundary point of C separates M and there is a monotone retraction $r_C: M \rightarrow C$ of M onto C which takes the part of M separated from the rest of C by a boundary point of C onto the boundary point [9]. Let T_C be defined by $T_C = 1 \circ r_C \circ m$; then $1|_C$ and $r_C \circ m$ are the light and monotone factors and C the middle space in a monotone light factorization of T_C .

LEMMA IV.2. Let $s_C = G(m(p), r_C \circ m(p), T)$. Let t be a number greater than or equal to s_C . Let $D(s, G, T_C) = \{a \in C: G(r_C \circ m(p), a, T_C) > s\}$. Let B_C be the union of the components β_i of $D(t, G, T)$ such that $\beta_i \cap C \neq \emptyset$. Then $B_C = r_C^{-1}(D(t - s_C, G, T_C))$.

If $m(p) \notin C$, $r_C \circ m(p)$ is a boundary point of C and hence separates $m(p)$ and $C - \{r_C \circ m(p)\}$. If $m(p) \in C$, $G(m(p), r_C \circ m(p), T) = 0$. Hence if $a \in C$, $G(m(p), a, T) = G(m(p), r_C \circ m(p), T) + G(r_C \circ m(p), a, T)$. Since if $a, b \in C$, $G(a, b, T) = G(a, b, T_C)$, $G(m(p), a, T) = s_C + G(r_C \circ m(p), a, T_C)$ for $a \in C$.

Since each β_i is connected, $\beta_i \cap C \neq \emptyset$, and r_C has the properties described above, $r_C(\beta_i) \subset \beta_i$. Hence if $b \in \beta_i$

$$s_C + G(r_C \circ m(p), r_C(b), T_C) = G(m(p), r_C(b), T_C) > t$$

which implies $\beta_i \subset r_C^{-1}(D(t - s_C, G, T_C))$ for each β_i .

If $a \in r_C^{-1}(D(t - s_C, G, T_C))$, $G(r_C \circ m(p), r_C(a), T) > 0$. This implies $m(p)$ and a do not belong to the same component of the complement of C , hence either $a \in C$ or $r_C(a)$ separates $m(p)$ and a in M . Therefore $G(m(p), a, T) \geq G(m(p), r_C(a), T) = s_C + G(r_C \circ m(p), r_C(a), T_C) > t$. Hence $a \in D(t, G, T)$.

Since r_C is a monotone retraction each component of $r_C^{-1}(D(t - s_C, G, T_C))$ meets C and hence is contained in a component β_i of B_C . Therefore $B_C = r_C^{-1}(D(t - s_C, G, T_C))$.

LEMMA IV.3. If C is a cyclic element for which $s_C \leq t$ and there is a component β of $D(t, G, T)$ which meets C , β^* meets C . Conversely if β^* meets C , $s_C \leq t$.

Suppose C was a cyclic element for which $s_C \leq t$ and a component β of $D(t, G, T)$ met C and β^* did not meet C . Since $s_C \leq t$, $C - \beta$ is not empty. Since β^* separates M and $\beta^* \cap C = \emptyset$, $\beta \cap C$ and $C - \beta$ would be a separation of C contradicting that C is connected.

If $b \in \beta^* \cap C$, $G(m(p), b, T) \leq t$. If $m(p) \in C$, $s_C = G(m(p), r_C \circ m(p), T) = 0$.

If $m(p) \notin C$, $r_C \circ m(p)$ separates $m(p)$ and the rest of C hence in either case if $\beta^* \cap C \neq 0$, $s_C \leq t$.

COROLLARY. *There is only one cyclic element C for which $\beta \cap C \neq 0$ and $s_C \leq t$.*

By Lemma IV.1 β^* is contained in a single proper cyclic element of M .

LEMMA IV.4. *Let $F = \{a \in M : G(m(p), a, T) < \infty\}$. For a proper cyclic element C of M let*

$$L_C(t) = \begin{cases} L(D(t - s_C, G, T_C)) & \text{if } s_C \leq t, \\ 0 & \text{if } s_C > t. \end{cases}$$

Then $L(D(t, G, T)) = \sum_{C \in \bar{F}} L_C(t)$.

For each cyclic element C of M for which $s_C \leq t$, let B_C be the union of the components of $D(t, G, T)$ which meet C and let $\{\beta\}$ be the collection of components not meeting any cyclic element C for which $s_C \leq t$. Let $A_C = m^{-1}(B_C)$ and $\{\alpha\} = \{m^{-1}(\beta)\}$. By Lemma IV.3 and its corollary $D(t, G, T)$ is a disjoint union of the sets of $\{\alpha\}$ and the sets A_C . By the additivity properties of the length

$$L(D(t, G, T)) = \sum_{\alpha \in \{\alpha\}} L(T, \alpha^*) + \sum_{\{C: s_C \leq t\}} L(T, A_C^*).$$

By Lemma IV.1 β^* is either a single point or is contained in a proper cyclic element of M . From Lemma IV.3 if β^* meets a proper cyclic element C , $s_C \leq t$. Since $\beta \cap C = 0$ for each C for which $s_C \leq t$, $\beta \subset M - C$. Then β^* must be contained in $C \cap (M - C)^*$ which is a single point. Hence β^* is always a single point of M . Therefore Lemma III.10 implies that T is constant on α^* , and hence $L(T, \alpha^*) = 0$ for every $\alpha \in \{\alpha\}$.

By Lemma IV.2 $r_C \circ m^{-1}(D(t - s_C, G, T_C)) = m^{-1}(B_C) = A_C$. Since $m(A_C^*) \subset C$, it follows from Lemma III.10 that $r_C \circ m(A_C^*) = m(A_C^*) = B_C^*$. Hence $T|A_C^* = T_C|A_C^*$ which implies by Lemma I.3 that $L(D(t - s_C, G, T_C)) = L(T, A_C^*)$.

For the mapping T_C let I_C be the set I described in Theorem II. If $C \notin \bar{F}$ either s_C is infinite or $r_C \circ m(p)$ belongs to I_C . If this were not so, by Theorem II and the fact that if $b \in C$, $G(m(p), b, T) = s_C + G(r_C \circ m(p), b, T_C)$, F would be dense in C contradicting $C \notin \bar{F}$.

If s_C is finite then $r_C \circ m(p) \in I_C$ and hence for every $t > 0$, $D(t, G, T_C)$ consists of the complement of the single point p . Hence $L(D(t, G, T_C)) = 0$ for every t in this case.

Combining the preceding remarks it is seen that

$$L(D(t, G, T)) = \sum_{\{C: s_C \leq t \text{ and } C \in \bar{F}\}} L(D(t - s_C, G, T_C)) = \sum_{C \in \bar{F}} L_C(t).$$

THEOREM IV. *Let $T: Q \rightarrow E^k$ be a continuous mapping of finite area. Let $T = 1 \circ m$ be a monotone light factorization of T with middle space M . Let C denote the collection of proper cyclic elements of M . Let $D(t, G, T)$, T_C , and F be as defined previously. Then*

$$\int_0^\infty L(D(t, G, T))dt = \sum_{C \subset \bar{F}} A(T_C).$$

From Lemma IV.4, $L(D(t, G, T)) = \sum_{C \subset \bar{F}} L_C(t)$. Integrating and using Lebesgue's convergence theorem

$$\int_0^\infty L(D(t, G, T))dt = \sum_{C \subset \bar{F}} \int_0^\infty L_C(t)dt = \sum_{C \subset \bar{F}} \int_0^\infty L(D(t - s_C, G, T_C))d(t - s_C).$$

Since $C \subset \bar{F}$ the mappings T_C satisfy the conditions of Theorem III hence

$$\int_0^\infty L(D(t, G, T))dt = \sum_{C \subset \bar{F}} A(T_C).$$

If C is a proper cyclic element of M and $b \in C$, $G(m(p), b, T) = s_C + G(r_C \circ m(p), b, T_C)$, hence by Theorem II $F \cap C$ is either empty, the single point $r_C \circ m(p)$, or dense in C . Since F is connected this implies that $\bar{F}$ is either a single point or an A -set in the terminology of [9]. By [9] there is a monotone retraction $r_F: M \rightarrow \bar{F}$ retracting M onto $\bar{F}$. Let $T_F = 1 \circ r_F \circ m$. Then $r_F \circ m$ is the monotone factor, 1 the light factor and $\bar{F}$ the middle space in a monotone light factorization of T_F .

THEOREM V. *Let T be as in Theorem IV and suppose the hypothesis of Theorem IV holds. Let F and T_F be as described above. Then*

$$A(T_F) = \int_0^\infty L(D(t, G, T_F))dt = \int_0^\infty L(D(t, G, T))dt.$$

If $\bar{F}$ is not a single point, $\bar{F}$ is a A -set and each boundary point of $\bar{F}$ separates M . Since $m(p) \in F$ and r_F is a retraction $r_F \circ m(p) = m(p)$. These two statements imply that if $b \in \bar{F}$

$$G(r_F \circ m(p), b, T_F) = G(m(p), b, T).$$

Hence $\{b \in \bar{F}: G(r_F \circ m(p), b, T_F) < \infty\} = F$.

Since $r_C \circ r_F = r_C$ if $C \subset \bar{F}$, by applying Theorem IV to the mapping T_F it is seen that

$$\int_0^\infty L(D(t, G, T_F))dt = \sum_{C \subset \bar{F}} A(T_C).$$

Hence by the cyclic additivity theorem of [9] it is seen that $\int_0^\infty L(D(t, G, T_F))dt = A(T_F)$.

REFERENCES

1. J. Cecconi, *La disuguaglianza di Cavalieri per la k -area secondo Lebesgue in un n -spazio*, Ann. Mat. Pura Appl., Ser. 4 vol. 42 (1956) p. 189.
2. L. Cesari, *Surface area*, Annals of Mathematics Studies, Princeton University Press, 1956.
3. ———, *Contours of a Frechet surface*, Riv. Mat. Univ. Parma vol. 4 (1953) p. 173.
4. H. Federer, *Surface area* I, II, Trans. Amer. Math. Soc. vol. 55 (1944) p. 420.
5. ———, *Measure and area*, Bull. Amer. Math. Soc. vol. 58 (1952) p. 306.
6. ———, *On Lebesgue area*, Ann. of Math. vol. 61 (1955) p. 289.
7. ———, *Curvature measures*, Trans. Amer. Math. Soc. vol. 93 (1959) p. 418.
8. M. Frechet, *Sur quelques points du calcul fonctionnel*, Rend. Circ. Mat. Palermo vol. 22 (1906) p. 1.
9. T. Radó, *Length and area*, Amer. Math. Soc. Colloquium Publications, vol. 30, 1948.
10. E. Silverman, *An intrinsic property for Lebesgue area*, Riv. Mat. Univ. Parma vol. 2 (1951) p. 193.
11. ———, *An intrinsic inequality for Lebesgue area*, Pacific J. Math. vol. 6 (1956) 363.
12. S. Saks, *The theory of the integral*, Warsaw, 1937.
13. L. C. Young, *Partial area*. I, to appear in Riv. Mat. Univ. Parma.
14. H. Whitney, *On totally differentiable smooth functions*, Pacific J. Math. vol. 1 (1951) p. 143.

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