ON A DECOMPOSITION THEOREM FOR MEASURES IN EUCLIDEAN n -SPACE⁽¹⁾

BY

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Introduction. It is the purpose of this paper to extend a decomposition theorem of Mickle [1] (square brackets refer to the bibliography at the end of this paper) for $(n-1)$ -dimensional measures in Euclidean n -space R^n , to k -dimensional measures in R^n , for $0 < k < n$, k an integer. In this paper we define a measure μ_n^k on the family $\mathfrak{B}$ of Borel sets of R^n that satisfies the following conditions:

(a) If $B \in \mathfrak{B}$ and $\mu_n^k(B) < +\infty$, then $B = B_1 \cup B_2$, where B_1 is countably k -rectifiable (see 4.1) and $\mu_n^k(B_2) = 0$.

(b) If $B \in \mathfrak{B}$, then

$$(1) \quad F_n^k(B) \leq \mu_n^k(B) \leq H_n^k(B),$$

where F_n^k is the Favard k -dimensional measure in R^n (see 7.3) and H_n^k is the Hausdorff k -measure in R^n (see [2] 2.18). Furthermore (see 7.3),

$$(2) \quad F_n^k(B) = \mu_n^k(B),$$

whenever $\mu_n^k(B) < +\infty$ and

$$(3) \quad F_n^k(B) = \mu_n^k(B) = H_n^k(B),$$

whenever B is countably k -rectifiable. Whether (2) holds for every $B \in \mathfrak{B}$ is an open question.

(c) μ_n^k is the smallest measure on $\mathfrak{B}$ which satisfies a weak projection inequality in the following sense: For almost every R^k (in a sense given in 6.4), the Lebesgue k -dimensional measure of the projection of a Borel set B into R^k is less than or equal to $\mu_n^k(B)$.

(d) If m is a positive integer such that $k < m < n$, and B is a Borel set in R^m , then (see 7.4)

$$\mu_m^k(B) = \mu_n^k(B).$$

While our results are stated in terms of measures on Borel sets, it will be convenient to work with Borel regular Carathéodory outer measures (see

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1.11). The procedures and proofs will follow closely those of Mickle [1] and Federer [2]. However, it should be noted that in the work of Mickle [1] the geometrical arguments needed do not necessitate the use of the group of orthogonal transformations of R^n onto R^n that is used in this paper.

I. Preliminary considerations.

1.1. Let R^n denote Euclidean n -space, $\mathcal{B}$ the Borel sets of R^n , G_n the group of orthogonal $n \times n$ matrices with real entries, and σ_n the Haar measure on G_n . Likewise let G_k denote the group of orthogonal $k \times k$ matrices, and σ_k the Haar measure in G_k . For g , an element of G_n , let g_i be the i th row vector of g . That is,

$$g_i = (g_{i1}, g_{i2}, \dots, g_{in}),$$

where g_{ij} is the entry of g in the i th row and j th column. Let I^k be that $n \times n$ matrix formed from the identity of G_n by setting the last $n - k$ diagonal elements equal to 0. We shall use I^k both for the above matrix and for the mapping effected by the matrix, that is, for the projection of R^n onto the space spanned by the first k basis vectors of R^n . Thus, although I^k considered as a matrix has no inverse, we shall use $(I^k)^{-1}$ for the inverse of the projection mapping. For convenience, we shall set $R^k = I^k(R^n)$. Also let $I_{n-k} = I - I^k$, and $R_{n-k} = I_{n-k}(R^n)$. If y is an element of R^n , and g is an element of G_n , then the length of $I^k g(y)$ and $I_{n-k} g(y)$ are given by

$$\begin{aligned} |I^k g(y)| &= \sum_{i=1}^k [(y \cdot g_i)^2]^{1/2}, \\ |I_{n-k} g(y)| &= \sum_{i=k+1}^n [(y \cdot g_i)^2]^{1/2}, \end{aligned}$$

where " $\cdot$ " denotes inner product.

1.2. Let S be an element of G_{k+1} . Then S is a $k+1$ by $k+1$ orthogonal matrix. We form an $n \times n$ orthogonal matrix S' from S as follows:

$$\begin{aligned} S'_i &= (S_{i1}, S_{i2}, \dots, S_{i,k+1}, 0, \dots, 0) & \text{for } 1 \leq i \leq k+1, \\ S'_i &= (\delta_{i1}, \delta_{i2}, \dots, \delta_{in}) & \text{for } k+2 \leq i \leq n. \\ \delta_{ij} &= \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases} \end{aligned}$$

S' may be considered as a continuous function of S .

1.3. LEMMA. For S in G_{k+1} and y in R^n ,

$$S' I^{k+1}(y) = I^{k+1} S'(y).$$

Proof. Let

$$\begin{aligned}
 u &= (u_1, \dots, u_n) = S' I^{k+1}(y), \\
 v &= (v_1, \dots, v_n) = I^{k+1} S'(y), \\
 I^{k+1}(y) &= (y_1, \dots, y_{k+1}, 0, \dots, 0). \\
 u_i &= \sum_{p=1}^n S'_{ip}(I^{k+1}(y))_p = \sum_{p=1}^{k+1} S'_{ip} y_p, \\
 v_i &= \begin{cases} \sum_{p=1}^n S'_{ip} y_p & \text{if } 1 \leq i \leq k+1, \\ 0 & \text{if } k+2 \leq i \leq n. \end{cases}
 \end{aligned}$$

But if $1 \leq i \leq k+1$ and $k+2 \leq p \leq n$, then $S'_{ip} = 0$; so

$$v_i = \sum_{p=1}^{k+1} S'_{ip} y_p = u_i \quad \text{if } 1 \leq i \leq k+1.$$

Also if $k+2 \leq i \leq n$ and $1 \leq p \leq k+1$, then $S'_{ip} = 0$; so

$$u_i = \sum_{p=1}^{k+1} S'_{ip} y_p = 0 = v_i \quad \text{if } k+2 \leq i \leq n.$$

Hence $u = v$.

1.4. LEMMA. For S in G_{k+1} and y in R^n

$$|I^{k+1} S'(y)| = |I^{k+1}(y)|.$$

Proof. S' is in G_n and hence S' preserves length. Therefore by 1.3,

$$|I^{k+1} S'(y)| = |S' I^{k+1}(y)| = |I^{k+1}(y)|.$$

1.5. LEMMA. For S in G_{k+1} and $\alpha > 0$ let

$$\mathfrak{U}(S, \alpha) = \{x \mid x \in R^n, |I^k S'(x)| < \alpha[S'(x)]k + 1\},$$

$$\mathfrak{V}(S, \alpha) = \{x \mid x \in R^n, |I^{k+1}(x)| < (\alpha^2 + 1)^{1/2}[S'(x)]k + 1\}.$$

Then $\mathfrak{U}(S, \alpha) = \mathfrak{V}(S, \alpha)$.

Proof. First note that for x in either set

$$[S'(x)]_{k+1} = (S'_{k+1} \cdot x) > 0.$$

Now

$$x \in \mathfrak{U}(S, \alpha)$$

if and only if

$$|I^k S(x)|^2 = \sum_{i=1}^k (S'_i \cdot x)^2 < \alpha^2 [(S'(x))_{k+1}]^2$$

if and only if

$$|I^{k+1}S'(x)|^2 = \sum_{i=1}^{k+1} (S'_i \cdot x)^2 < (\alpha^2 + 1)[(S'(x))_{k+1}]^2.$$

By 1.4 then

$$x \in \mathfrak{U}(S, \alpha)$$

if and only if

$$|I^{k+1}(x)|^2 < (\alpha^2 + 1)[(S'(x))_{k+1}]^2$$

if and only if

$$x \in \mathfrak{V}(S, \alpha).$$

1.6. For x a real number, $x > 0$, let

$$y = \beta(x) = \{2[1 - (x^2 + 1)^{-1/2}]\}^{1/2}.$$

Then $\lim_{x \rightarrow 0} \beta(x) = 0$ and

$$x = \beta^{-1}(y) = \frac{y(4 - y^2)^{1/2}}{2 - y^2}, \quad \lim_{y \rightarrow 0} \beta^{-1}(y) = 0,$$

and

$$\lim_{x \rightarrow 0} \frac{\beta(x)}{x} = 1.$$

1.7. Let $C^{k+1} = \{y | y \in R^{k+1}, |y| = 1\}$. For y in C^{k+1} , $0 < \eta < \infty$, let

$$C^0(y, \eta) = \{Z | Z \in C^{k+1}, |Z - y| < \eta\},$$

$$C(y, \eta) = \{Z | Z \in C^{k+1}, |Z - y| \leq \eta\},$$

$$D = \{x | x \in R^n, |I^{k+1}(x)| > 0\}.$$

D is open in R^n .

Let f be the mapping with domain D and range C^{k+1} given by

$$f(x) = \frac{I^{k+1}(x)}{|I^{k+1}(x)|}.$$

f is continuous onto C^{k+1} .

1.8. LEMMA. For S in G_{k+1} , $\alpha > 0$,

$$f^{-1}C^0[S_{k+1}, \beta(\alpha)] = V(S, \alpha).$$

Proof.

$$f^{-1}C^0[S_{k+1}, \beta(\alpha)] \subset D.$$

Also, for x in $\mathfrak{V}(S, \alpha)$, $[S'(x)]_{k+1} > 0$, which implies that $|I^{k+1}S'(x)| > 0$, which

by 1.4 implies that $|I^{k+1}(x)| > 0$. Hence for any x under consideration, $|I^{k+1}(x)| > 0$. Now by 1.3

$$\begin{aligned} [S'(x)]_{k+1} &= [I^{k+1}S'(x)]_{k+1} \\ &= [S'I^{k+1}(x)]_{k+1} = S'_{k+1} \cdot [I^{k+1}(x)] \\ &= S_{k+1} \cdot [I^{k+1}(x)] = (S_{k+1} \cdot f(x))(|I^{k+1}(x)|). \end{aligned}$$

Also

$$|S_{k+1}| = |f(x)| = 1.$$

So

$$|f(x) - S_{k+1}|^2 = 2[1 - S_{k+1} \cdot f(x)].$$

Thus x is an element of $\mathcal{U}(S, \alpha)$ if and only if

$$|I^{k+1}(x)| < (\alpha^2 + 1)^{1/2} [S'(x)]_{k+1},$$

which is true if and only if

$$1 < (\alpha^2 + 1)^{1/2} (S_{k+1} \cdot f(x)),$$

which is true if and only if

$$(S_{k+1} \cdot f(x)) > (\alpha^2 + 1)^{-1/2},$$

which is true if and only if

$$2 - (S_{k+1} \cdot f(x)) < 1 - (\alpha^2 + 1)^{-1/2} = \frac{[\beta(\alpha)]^2}{2},$$

which is true if and only if

$$|f(x) - S_{k+1}|^2 < [\beta(\alpha)]^2.$$

1.9. LEMMA. If $f(x) = S_{k+1}$, then $I^k S'(x) = 0$.

Proof. If $f(x) = S_{k+1}$ then for all $\alpha > 0$, $f(x)$ is an element of $C^0(S_{k+1}, \beta(\alpha))$, which by 1.5 and 1.8 means that x is an element of $U(S, \alpha)$. Thus for all $\alpha > 0$

$$|I^k S'(x)| < \alpha [S'(x)]_{k+1}.$$

Hence

$$|I^k S'(x)| = 0.$$

1.10. By a Carathéodory Outer Measure (abbreviated C.O.M.) on a metric space X , with distance ρ , we shall mean a non-negative set function Λ defined for all subsets of X , such that

- (1) $\Lambda(\emptyset) = 0$ ($\emptyset$ is the empty set),
- (2) $E_1 \subset E_2 \subset X$ implies $\Lambda(E_1) \leq \Lambda(E_2)$,

(3) $E = \bigcup_n E_n$ implies $\Lambda(E) \leq \sum_n \Lambda(E_n)$,

(4) $\rho(E_1, E_2) > 0$ implies $\Lambda(E_1) + \Lambda(E_2) = \Lambda(E_1 \cup E_2)$.

An outer measure is a non-negative set function defined on the subsets of X which satisfies only the first three above conditions. By a Λ measurable subset of X we shall mean a set E such that for all subsets Q of X ,

$$\Lambda(Q) = \Lambda(Q \cap E) + \Lambda(Q \cap CE),$$

where $C(E)$ denotes the complement of E . By a Borel regular C.O.M. we shall mean a C.O.M. such that for any subset E of X there is a Borel set B containing E for which $\Lambda(E) = \Lambda(B)$.

Let A be a closed set in R^n , j a positive integer, Λ an outer measure in R^n . For Y a subset of C^{k+1} , let

$$\Psi(Y) = \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(\bar{O}, r) \cap f^{-1}(Y)]}{r^k},$$

where $\bar{O}$ is the origin of R^n , and $K(x, r)$ is the open sphere of center x and radius r in R^n . Then $\psi(Y)$ is an outer measure in C^{k+1} .

1.11. LEMMA. For A closed $\subset R^n$, j is a positive integer, D as in 1.7, let

$$Z = f(A \cap D \cap K(\bar{O}, 1/j)).$$

Then Z is an analytic subset of C^{k+1} , and

$$\Psi(C^{k+1} - Z) = 0.$$

Proof. Since $A \cap D \cap K(\bar{O}, 1/j)$ is a Borel set of R^n , and f is continuous, Z is analytic. Also,

$$C^{k+1} - Z = C^{k+1} \cap Cf[A \cap D \cap K(\bar{O}, 1/j)].$$

Hence $\Psi(C^{k+1} - Z) =$

$$\text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(\bar{O}, r) \cap f^{-1}\{C^{k+1} \cap Cf[A \cap D \cap K(\bar{O}, 1/j)]\}]}{r^k},$$

which is less than or equal to

$$\text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(\bar{O}, r) \cap D \cap C[A \cap D \cap K(\bar{O}, 1/j)]]}{r^k} = 0.$$

1.12. LEMMA. Let

$$p(y, \eta) = \frac{\Psi C(y, \eta)}{H^k[C(y, \eta)]},$$

where H^k is Hausdorff k measure. Let

$$H = \left\{ y \mid y \in C^{k+1}, \limsup_{\eta \rightarrow 0} p(y, \eta) = +\infty \right\},$$

$$K = \left\{ y \mid y \in C^{k+1}, \limsup_{\eta \rightarrow 0} p(y, \eta) > 0 \right\}.$$

Then $H^k(K - (H \cup Z)) = 0$.

Proof. By 1.11 $\Psi(C^{k+1} - Z) = 0$. So $\Psi[C(y, \eta) - Z] = 0$. Hence

$$\Psi[C(y, \eta) \cap Z] \leq \Psi[C(y, \eta)] \leq \Psi[C(y, \eta) \cap Z] + \Psi[C(y, \eta) - Z].$$

Therefore

$$p(y, \eta) = \frac{\Psi[C(y, \eta) \cap Z]}{H^k[C(y, \eta)]}.$$

Now, by [3; 4],

$$\limsup_{\eta \rightarrow 0} \frac{\Psi[C(y, \eta) \cap Z]}{H^k[C(y, \eta)]} = 0 \quad \text{or} \quad +\infty,$$

H^k almost everywhere on $C^{k+1} - Z$. But on $K - H$

$$0 < \limsup_{\eta \rightarrow 0} p(y, \eta) < +\infty.$$

So $H^k[K - (H \cup Z)] = 0$.

1.13. The following are immediate consequences of 1.6.

$$\lim_{\eta \rightarrow 0} \frac{H^k \left[C \left(y, \frac{\beta(\eta)}{2} \right) \right]}{\alpha(k)\eta^k} = 2^{-k} > 0,$$

$$\lim_{\eta \rightarrow 0} \frac{H^k[C(y, \beta(\eta))]}{\alpha(k)\eta^k} = 1,$$

where $\alpha(k)$ is the Lebesgue k measure of the set

$$\{x \mid x \in R^k, |x| \leq 1\}.$$

1.14. LEMMA. Let

$H^*(\Lambda, A)$

$$= \left\{ S \mid S \in G_{k+1}, \limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/\eta} \frac{\Lambda[A \cap K(\bar{O}, r) \cap u(S, \eta)]}{\alpha(k)r^k\eta^k} = +\infty \right\}.$$

Then S_{k+1} in H implies S is in $H^*(\Lambda, A)$.

Proof. S_{k+1} in H implies

$$\limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(\bar{O}, r) \cap f^{-1}C(S_{k+1}, \eta)]}{r^k H^k[C(S_{k+1}, \eta)]} = +\infty.$$

Hence by 1.6

$$\limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(\bar{O}, r) \cap f^{-1}C(S_{k+1}, \beta(\eta))]}{r^k H^k[C(S_{k+1}, \beta(\eta))]} = +\infty.$$

Hence since

$$f^{-1}[C(S_{k+1}, \beta(\eta))] \subset f^{-1}[C^0(S_{k+1}, 2\beta(\eta))],$$

$$\limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(\bar{O}, r) \cap f^{-1}[C^0(S_{k+1}, 2\beta(\eta))]]}{r^k H^k[C(S_{k+1}, \beta(\eta))]} = +\infty,$$

which implies that

$$\limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(\bar{O}, r) \cap f^{-1}[C^0(S_{k+1}, \beta(\eta))]]}{r^k H^k\left[C\left(S_{k+1}, \frac{\beta(\eta)}{2}\right)\right]} = +\infty.$$

Hence by 1.14,

$$+\infty = \limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(\bar{O}, r) \cap f^{-1}[C^0(S_{k+1}, \beta(\eta))]]}{\alpha(k)r^k \eta^k},$$

which by 1.5 and 1.8 equals

$$\limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(\bar{O}, r) \cap \mathfrak{U}(S, \eta)]}{\alpha(k)r^k \eta^k}.$$

Hence S is in $H^*(\Lambda, A)$.

1.15. LEMMA. Let

$$K^*(\Lambda, A) = \left\{ S \mid S \in G_{k+1}, \limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(\bar{O}, r) \cap \mathfrak{U}(S, \eta)]}{\alpha(k)r^k \eta^k} > 0 \right\}.$$

Then S in $K^*(\Lambda, A)$ implies S_{k+1} is in K .

Proof. S in $K^*(\Lambda, A)$ implies

$$0 < \limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(\bar{O}, r) \cap \mathfrak{U}(S, \eta)]}{\alpha(k)r^k \eta^k},$$

which by 1.5 and 1.8 is less than or equal to

$$\limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(\bar{O}, r) \cap f^{-1}[C(S_{k+1}, \beta(\eta))]]}{\alpha(k)r^k \eta^k},$$

which by 1.13 is less than or equal to

$$\limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \left\{ \frac{\Lambda[A \cap K(\bar{O}, r) \cap f^{-1}[C(S_{k+1}, \beta(\eta))]]}{\alpha(k)r^k \eta^k} \right\} \\ \cdot \left\{ \frac{\alpha(k)\eta^k}{H^k[C(S_{k+1}, \beta(\eta))]} \right\},$$

which by 1.6 equals $\limsup_{\eta \rightarrow 0} p(S_{k+1}, \eta)$. Hence S_{k+1} is in K .

1.16. LEMMA. *Let*

$$E(S) = \{x \mid x \in R^n, I^k S'(x) = \bar{O}\}.$$

Let

$$L^*(A) = \{S \mid S \in G_{k+1}, (A - \bar{O}) \cap K(\bar{O}, 1/j) \cap E(S) \neq \emptyset\}.$$

Then S_{k+1} in Z (see 1.11) implies that S is in $L^(A)$.*

Proof. S_{k+1} in Z implies that there is an x in $A \cap D \cap K(\bar{O}, 1/j)$, such that $f(x) = S_{k+1}$. Hence by 1.9, $I^k S'(x) = \bar{O}$, which implies that x is in $E(S)$. Now if x is in $A \cap D$, then x is not the origin. So x is in

$$(A - \bar{O}) \cap K(\bar{O}, 1/j) \cap E(S)$$

which is therefore not empty. Hence S is in $L^*(A)$.

1.17. THEOREM.

$$\sigma_{k+1}[K^*(\Lambda, A) - (H^*(\Lambda, A) \cup L^*(A))] = 0.$$

Proof. By 1.15, if S is in $K^*(\Lambda, A)$, then S_{k+1} is in K . By 1.14, if S is not in $H^*(\Lambda, A)$, then S_{k+1} is not in H . By 1.16, if S is not in $L^*(A)$, then S_{k+1} is not in Z . Hence

$$K^*(\Lambda, A) - (H^*(\Lambda, A) \cup L^*(A))$$

is contained in the set of S in G_{k+1} such that S_{k+1} is in $K - (H \cup Z)$. Hence the theorem follows from the fact that $H^*[K - (H \cup Z)] = 0$ (see 1.12).

II. Densities.

2.1. For a fixed $\eta > 0$, a an element of R^n , A a subset of R^n , j a fixed positive integer, g an element of G_n , and Λ a C.O.M., let

$$P(g, \eta, a) = \{x \mid x \in R^n, |I^k g(x - a)| < \eta |I_{n-k} g(x - a)|\}.$$

Let

$$M(\Lambda, A, g, \eta, r, a) = \Lambda[A \cap K(a, r) \cap P(g, \eta, a)] / \alpha(k)\eta^k r^k;$$

$$M_j(\Lambda, A, g, \eta, a) = \text{l.u.b.}_{0 < r \leq 1/j} M(\Lambda, A, g, \eta, r, a);$$

$$U(g, a) = \{x \mid x \in R^n, I^k g(x - a) = \bar{O}\};$$

$$W_j(\Lambda, A) = \left\{ (x, g) \mid (x, g) \in A \times G_n, \limsup_{\eta \rightarrow 0} M_j(\Lambda, A, g, \eta, x) = +\infty \right\};$$

$$X_j(\Lambda, A) = \left\{ (x, g) \mid (x, g) \in A \times G_n, \limsup_{\eta \rightarrow 0} M_j(\Lambda, A, g, \eta, x) = 0 \right\};$$

$$V_j(A) = \left\{ (x, g) \mid (x, g) \in A \times G_n, (A - x) \cap K(x, 1/j) \cap U(g, x) \neq \emptyset \right\}.$$

2.2. LEMMA. If A is closed in R^n , then $V_j(A)$ is a Borel set in the cartesian product space $A \times G_n$.

Proof. For p and q integers, p less than j , and $C(x, r)$ the closed sphere for radius r and center x in R^n , the set $S_{j,p,q}$ of elements of $A \times G_n$ for which

$$A \cap [C(x, 1/j - 1/p) - K(x, (p - j)/(jp + q))] \cap U(g, x) \neq \emptyset$$

is a closed set in $A \times G_n$, and

$$V_j = \bigcup_{p=j+1}^{\infty} \bigcup_{q=1}^{\infty} S_{j,p,q}.$$

2.3. LEMMA. For A a Borel set of R^n , Λ a C.O.M., $W_j(\Lambda, A)$ and $X_j(\Lambda, A)$ are Borel sets in $A \times G_n$.

Proof. It suffices to show that

$$\Lambda[A \cap K(x, r) \cap P(g, \eta, x)]$$

is a lower semi-continuous function of (g, η, r, x) . For

$$(g_0, \eta_0, r_0, x_0),$$

let λ be any number less than

$$\Lambda[A \cap K(x_0, r_0) \cap P(g_0, \eta_0, x_0)].$$

There exists F , a closed set of R^n , such that

$$F \subset K(x_0, r_0) \cap P(g_0, \eta_0, x_0),$$

and

$$\Lambda(A \cap F) > \lambda,$$

and for (g, η, r, x) sufficiently close to (g_0, η_0, r_0, x_0) ,

$$F \subset K(x, r) \cap P(g, \eta, x),$$

and

$$\Lambda[A \cap K(x, r) \cap P(g, \eta, x)] \geq \Lambda(A \cap F) > \lambda.$$

2.4. For A a closed subset of R^n , and x in A , the sets

$$W_j^*(\Lambda, A, x) = \left\{ g \mid g \in G_n, \limsup_{\eta \rightarrow 0} M_j(\Lambda, A, g, \eta, x) = +\infty \right\},$$

$$X_j^*(\Lambda, A, x) = \left\{ g \mid g \in G_n, \limsup_{\eta \rightarrow 0} M_j(\Lambda, A, g, \eta, x) = 0 \right\},$$

$$V_j^*(A, x) = \{ g \mid g \in G_n, (A - x) \cap K(x, 1/j) \cap U(g, x) \neq \emptyset \},$$

are Borel sets in G_n .

2.5. For A a closed subset of R^n , g in G_n , the sets

$$\overline{W}_j(\Lambda, A, g) = \left\{ x \mid x \in R^n, \limsup_{\eta \rightarrow 0} M_j(\Lambda, A, g, \eta, x) = +\infty \right\},$$

$$\overline{X}_j(\Lambda, A, g) = \left\{ x \mid x \in R^n, \limsup_{\eta \rightarrow 0} M_j(\Lambda, A, g, \eta, x) = 0 \right\},$$

$$\overline{V}_j(A, g) = \{ x \mid x \in R^n, (A - x) \cap K(x, 1/j) \cap U(g, x) \neq \emptyset \},$$

$$\overline{W}(\Lambda, A, g) = \bigcap_j \overline{W}_j(\Lambda, A, g), \quad \overline{X}(\Lambda, A, g) = \bigcup_j \overline{X}_j(\Lambda, A, g), \text{ and}$$

$$\overline{V}(A, g) = \bigcap_j \overline{V}_j(A, g)$$

are Borel sets in R^n . Also note that $\overline{V}(A, g)$ is the set of all points x of R^n such that x is an accumulation point of $A \cap U(g, x)$.

2.6. For g in G_n , $\eta > 0$, and a in R^n , let

$$Q_i^+(g, \eta) = \{ x \mid x \in R^n, |I^k g(x - a)| < (n - k)^{1/2} \eta [g(x - a)]_i \},$$

for $k + 1 \leq i \leq n$.

Likewise let

$$Q_i^-(g, \eta) = \{ x \mid x \in R^n, |I^k g(x - a)| < -(n - k)^{1/2} \eta [g(x - a)]_i \},$$

for $k + 1 \leq i \leq n$.

Now let

$$F_i^+ = \left\{ g \mid g \in G_n, \limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(a, r) \cap Q_i^+(g, \eta)]}{\alpha(k)r^k \eta^k} > 0 \right\},$$

and let

$$F_i^- = \left\{ g \mid g \in G_n, \limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(a, r) \cap Q_i^-(g, \eta)]}{\alpha(k)r^k \eta^k} > 0 \right\}.$$

F_i^+ and F_i^- are Borel sets in G . The proof of this is essentially the same as that given in 2.3.

2.7. LEMMA.

$$P(g, \eta, a) \subset \bigcup_{i=k+1}^n [Q_i^+(g, \eta) \cup Q_i^-(g, \eta)].$$

Proof. If not, then there exists x in R^n such that

$$|I^k g(x - a)| < \eta |I_{n-k} g(x - a)|$$

and for all i , $k+1 \leq i \leq n$,

$$|I^k g(x - a)| \geq (n - k)^{1/2} \eta | [g(x - a)]_i |.$$

Hence

$$\begin{aligned} (n - k) |I^k g(x - a)|^2 &\geq (n - k) \eta^2 \left[\sum_{i=k+1}^n [g(x - a)]_i^2 \right] \\ &= (n - k) \eta^2 |I_{n-k} g(x - a)|^2. \end{aligned}$$

This is a contradiction.

2.8. LEMMA. Let $I(p, q)$ be the identity matrix with the p th row replaced by the q th row, and the q th row replaced by the negative of the p th row. Then, for $k+1 \leq p, q \leq n$,

$$I(p, q) \cdot [F_p^+ - (V_j^*(A, a) \cup W_j^*(\Lambda, A, a))] = F_q^- - (V_j^*(A, a) \cup W_j^*(\Lambda, A, a)),$$

where “ $\cdot$ ” denotes cosetting with respect to $I(p, q)$.

Proof. For g in G_n , and $i \neq p, q$,

$$[I(p, q)g]_i = g_i; \quad [I(p, q)g]_p = g_q; \quad [I(p, q)g]_q = -g_p.$$

Thus, since $k+1 \leq p, q \leq n$, we have

$$|I^k(I(p, q)g)(x - a)| = |I^k g(x - a)|,$$

and

$$|I_{n-k}(I(p, q)g)(x - a)| = |I_{n-k} g(x - a)|.$$

Thus $Q_p^+(g, \eta) = Q_p^-(I(p, q)g, \eta)$, and hence

$$F_q^- = I(p, q) \cdot F_p^+.$$

Also $P(g, \eta, a) = P(I(p, q)g, \eta, a)$, and hence

$$W_j^*(\Lambda, A, a) = I(p, q) \cdot W_j^*(\Lambda, A, a).$$

Also $U(g, a) = U(I(p, q)g, a)$, and hence

$$V_j^*(A, a) = I(p, q) \cdot V_j^*(A, a).$$

Hence

$$I(p, q) \cdot [F_p^+ - (V_j^*(A, a) \cup W_j^*(\Lambda, A, a))] = F_j^- - (V_j^*(A, a) \cup W_j^*(\Lambda, A, a)).$$

2.9. LEMMA. For $k+1 \leq p, q \leq n$,

$$\sigma_n[F_p^+ - (V_j^*(A, a) \cup W_j^*(\Lambda, A, a))] = \sigma_n[F_q^- - (V_j^*(A, a) \cup W_j^*(\Lambda, A, a))].$$

Proof. By 2.8

$$F_q^- - (V_j^*(A, a) \cup W_j^*(\Lambda, A, a)) = I(p, q) \cdot [F_p^+ - (V_j^*(A, a) \cup W_j^*(\Lambda, A, a))],$$

and the lemma follows since σ_n is the Haar measure in G_n .

III. Further density considerations.

3.1. Let A be a closed subset of R^n , g an element of G_n , and a an element of A . Let

$$t(x) = g(x - a), \quad \text{for } x \text{ an element of } R^n.$$

Then t is a distance preserving homeomorphism of R^n onto R^n such that $t(a) = \bar{O}$, $tK(a, r) = K(\bar{O}, r)$, and $t(A)$ is a closed subset of R^n .

3.2. LEMMA. For s in G_{k+1} , $\eta > 0$,

$$Q_{k+1}^+(S'g, \eta) = t^{-1}[\mathfrak{U}(S, (n-k)^{1/2}\eta)].$$

Proof. $x \in Q_{k+1}^+(S'g, \eta)$ if and only if

$$|I^k S'g(x - a)| < (n-k)^{1/2}\eta [S'g(x - a)]_{k+1},$$

which is true if and only if

$$|I^k S'g(x - a)| < (n-k)^{1/2}\eta [S'(t(x))]_{k+1},$$

which is true if and only if $t(x)$ is in $\mathfrak{U}(S, (n-k)^{1/2}\eta)$.

3.3. LEMMA. For g in G_n , a in R^n , S in G_{k+1} , $\eta > 0$,

$$t^{-1}(\mathfrak{U}(S, \eta)) \subset P(S'g, \eta, a).$$

Proof. If x is in $t^{-1}(\mathfrak{U}(S, \eta))$, then $t(x)$ is in $\mathfrak{U}(S, \eta)$, and therefore

$$|I^k S'g(x - a)| < \eta [S'g(x - a)]_{k+1},$$

and hence

$$|I^k S'g(x-a)|^2 < \eta^2 [S'g(x-a)]_{k+1}^2 \leq \eta^2 |I_{n-k} S'g(x-a)|^2,$$

and therefore x is in $P(S'g, \eta, a)$.

3.4. LEMMA. For g in G_n , a in R^n , S in G_{k+1} ,

$$t^{-1}(E(S)) = \mathfrak{U}(S'g, a).$$

Proof. x is in $t^{-1}(E(S))$ if and only if $t(x)$ is in $E(S)$, which is true if and only if

$$|I^k S'g(x-a)| = 0,$$

which is true if and only if x is in $\mathfrak{U}(S'g, a)$.

3.5. LEMMA. For A a closed subset of R^n , a an element of A , g in G_n , S in G_{k+1} ; S is in $L^*(t(A))$ if and only if $S'g$ is in $V_j^*(A, a)$.

Proof. By 3.4

$$(A-a) \cap K(a, 1/j) \cap \mathfrak{U}(S'g, a) = t^{-1}[(t(A) - \bar{O}) \cap K(0, 1/j) \cap E(S)].$$

So S is in $L^*(t(A))$ if and only if

$$(t(A) - \bar{O}) \cap K(\bar{O}, 1/j) \cap E(S) \neq \emptyset,$$

which is true if and only if

$$t[(A-a) \cap K(a, 1/j) \cap \mathfrak{U}(S'g, a)] \neq \emptyset,$$

which is true if and only if

$$(A-a) \cap K(a, 1/j) \cap \mathfrak{U}(S'g, a) \neq \emptyset,$$

which is true if and only if $S'g$ is in $V_j^*(A, a)$.

3.6. For Λ an outer measure, and E a subset of R^n , let

$$\Lambda^*(E) = \Lambda[t^{-1}(E)].$$

Then Λ^* is an outer measure in R^n .

3.7. LEMMA. For Λ an outer measure, A a closed subset of R^n , g in G_n , S in G_{k+1} ; if S is in $H^*(\Lambda^*, t(A))$, then $S'g$ is in $W_j^*(\Lambda, A, a)$.

Proof. If S is in $H^*(\Lambda^*, t(A))$, then

$$+\infty = \limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda^*[t(A) \cap K(\bar{O}, r) \cap \mathfrak{U}(S, \eta)]}{\alpha(k)r^k\eta^k},$$

which by 3.3 is less than or equal to

$$\limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(a, r) \cap P(S'g, \eta, a)]}{\alpha(k)r^k\eta^k}.$$

Hence $S'g$ is in $W_j^*(\Lambda, A, a)$.

3.8. LEMMA. If $S'g$ is in F_{k+1}^+ , then S is in $K^*(\Lambda^*, t(A))$.

Proof. If $S'g$ is in F_{k+1}^+ , then

$$0 < \limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/\eta} \frac{\Lambda[A \cap K(a, r) \cap Q_{k+1}^+(S'g, \eta)]}{\alpha(k)r^k\eta^k}$$

which by 3.2 equals

$$\limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/\eta} \frac{\Lambda^*[t(A) \cap K(\bar{O}, r) \cap \mathfrak{U}(S, (n-k)^{1/2}\eta)]}{\alpha(k)r^k\eta^k},$$

and hence S is in $K^*(\Lambda^*, t(A))$.

3.9. THEOREM. For any g in G_n , and a in A , a closed subset of R^n , and Λ an outer measure,

$$\sigma_{k+1}\{S \mid S'g \in [F_{k+1}^+ - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))]\} = 0.$$

Proof. By 3.5, 3.7, and 3.8,

$$\begin{aligned} \{S \mid S'g \in [F_{k+1}^+ - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))]\} \\ \subset K^*[\Lambda^*, t(A)] - \{H^*[\Lambda^*, t(A)] \cup L^*[t(A)]\}. \end{aligned}$$

Since Λ^* is an outer measure, and $t(A)$ is closed in R^n , we may apply 1.17.

3.10. THEOREM. For $a \in A$, a closed subset of R^n , and Λ a C.O.M.

$$\sigma_n[F_{k+1}^+ - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))] = 0.$$

Proof. Let $\alpha: (G_{k+1} \times G_n) \rightarrow G_n$ be a mapping defined by $\alpha(S, g) = S'g$. α is a continuous mapping. Hence the set

$$\begin{aligned} \{(S, g) \mid (S, g) \in G_{k+1} \times G_n, S'g \in [F_{k+1}^+ - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))]\} \\ = \alpha^{-1}[F_{k+1}^+ - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))] \end{aligned}$$

is a Borel set in $G_{k+1} \times G_n$.

Let $c(S, g)$ equal 1 if $S'g$ is in $F_{k+1}^+ - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))$, and 0 otherwise. Then, since σ_n is a Haar measure in G_n , for any S in G_{k+1} ,

$$\begin{aligned} \sigma_n[F_{k+1}^+ - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))] \\ = \sigma_n\{g \mid S'g \in [F_{k+1}^+ - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))]\} = \int_{G_n} c(S, g) d\sigma_n. \end{aligned}$$

So by 3.9

$$\begin{aligned} \sigma_n[F_{k+1}^+ - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))] \\ = \int_{G_{k+1}} \int_{G_n} c(S, g) d\sigma_n d\sigma_{k+1} = \int_{G_n} \int_{G_{k+1}} c(S, g) d\sigma_{k+1} d\sigma_n = 0. \end{aligned}$$

3.11. LEMMA. For $k+1 \leq p, q \leq n, a, A$, and Λ as in 3.10,

$$\sigma_n[F_p^+ - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))] = \sigma_n[F_q^- - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))] = 0.$$

Proof. By 2.8 and 3.10,

$$\begin{aligned} \sigma_n[F_p^+ - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))] &= \sigma_n[F_q^- - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))] \\ &= \sigma_n[F_{k+1}^+ - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))] = 0. \end{aligned}$$

3.12. LEMMA. For a, A , and Λ as in 3.10

$$G_n - X_j^*(\Lambda, A, a) \subset \bigcup_{i=k+1}^n (F_i^+ \cup F_i^-).$$

Proof. If not, then there exists a g in G_n , such that

$$\limsup_{\eta \rightarrow 0} M_j(\Lambda, A, g, \eta, a) > 0,$$

and for all $i, k+1 \leq i \leq n$,

$$\begin{aligned} 0 &= \limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(a, r) \cap Q_i^+(g, \eta)]}{\alpha(k)r^k\eta^k} \\ &= \limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(a, r) \cap Q_i^-(g, \eta)]}{\alpha(k)r^k\eta^k}. \end{aligned}$$

Hence by 2.7,

$$\begin{aligned} 0 &< \limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(a, r) \cap P(g, \eta, a)]}{\alpha(k)r^k\eta^k} \\ &\leq \sum_{i=k+1}^n \left[\limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(a, r) \cap Q_i^+(g, \eta)]}{\alpha(k)r^k\eta^k} \right. \\ &\quad \left. + \limsup_{\eta \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(a, r) \cap Q_i^-(g, \eta)]}{\alpha(k)r^k\eta^k} \right] = 0. \end{aligned}$$

This is a contradiction.

3.13. LEMMA. For a, A and Λ as in 3.10

$$\sigma_n[G_n - (X_j^*(\Lambda, A, a) \cup W_j^*(\Lambda, A, a) \cup V_j^*(A, a))] = 0.$$

Proof. By 3.12

$$G_n - (X_j^*(\Lambda, A, a) \cup W_j^*(\Lambda, A, a) \cup V_j^*(A, a)) \\ \subset \bigcup_{i=k+1}^n [F_i^+ - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))] \cup [F_i^- - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))],$$

and the result follows from 3.11.

3.14. LEMMA. For A , a closed subset of R^n , j an integer, Λ a C.O.M.,

$$\Lambda[A - (\overline{W}_j(\Lambda, A, g) \cup \overline{X}_j(\Lambda, A, g) \cup \overline{V}_j(A, g))] = 0$$

for σ_n almost every g in G_n .

Proof. By 3.13 for any a in A ,

$$\sigma_n[G_n - (W_j^*(\Lambda, A, a) \cup X_j^*(\Lambda, A, g) \cup V_j^*(A, a))] = 0.$$

Apply Fubini's theorem to the characteristic function of

$$(A \times G_n) - (W_j(\Lambda, A) \cup X_j(\Lambda, A) \cup V_j(A)).$$

3.15. THEOREM. For A a closed subset of R^n , Λ a C.O.M.

$$\Lambda[A - (\overline{W}(\Lambda, A, g) \cup \overline{X}(\Lambda, A, g) \cup \overline{V}(A, g))] = 0,$$

for σ_n almost every g in G_n .

Proof. By 3.14, for every integer j , there is a subset Z_j of G_n , such that $\sigma_n(Z_j) = 0$, and for g in $G_n - Z_j$

$$\Lambda[A - (\overline{W}_j(\Lambda, A, g) \cup \overline{X}_j(\Lambda, A, g) \cup \overline{V}_j(A, g))] = 0.$$

Let $Z = \bigcup_{j=1}^{\infty} Z_j$. Then $\sigma_n(Z) = 0$, and for g in $G_n - Z$, and for all integers j ,

$$\Lambda[A - (\overline{W}_j(\Lambda, A, g) \cup \overline{X}_j(\Lambda, A, g) \cup \overline{V}_j(A, g))] = 0,$$

and thus, by 2.5,

$$\Lambda[A - (\overline{W}(\Lambda, A, g) \cup \overline{X}(\Lambda, A, g) \cup \overline{V}(A, g))] = 0.$$

Also, $\overline{W}_j(\Lambda, A, g)$ and $\overline{V}_j(A, g)$ are monotone decreasing sequences of sets. So

$$\begin{aligned} A - (\overline{W}(\Lambda, A, g) \cup \overline{X}(\Lambda, A, g) \cup \overline{V}(A, g)) \\ = [A - \overline{X}(\Lambda, A, g)] \cap \left[\bigcup_j C\overline{W}_j(\Lambda, A, g) \right] \cap \left[\bigcup_j C\overline{V}_j(A, g) \right] \\ \subset [A - \overline{X}(\Lambda, A, g)] \cap \bigcup_j [C\overline{W}_j(\Lambda, A, g) \cap C\overline{V}_j(A, g)] \\ = \bigcup_j [A - \overline{X}(\Lambda, A, g) \cup \overline{W}_j(\Lambda, A, g) \cup \overline{V}_j(A, g)], \end{aligned}$$

and the result follows.

IV. (Λ, k) Unrectifiable sets.

4.1. By a Lipschitz mapping T from a metric space X , with metric ρ , to a metric space Y , with metric δ , we shall mean a mapping such that there exists a constant M such that for all x, y in X , $\delta[T(x), T(y)] \leq M\rho(x, y)$.

A subset of Euclidean n -space will be said to be k -rectifiable if it is the image under a Lipschitz mapping of a bounded subset of Euclidean k -space.

If Λ is an outer measure on Euclidean n -space, a subset E of n -space will be said to be (Λ, k) unrectifiable if every k -rectifiable subset of E is of Λ measure 0.

LEMMA. Let Λ be an outer measure in R^n . Let A be a subset of R^n such that $d(A)$, the diameter of A is less than $1/j$, for a fixed integer j . Let g be an element of G_n , $0 < \delta < 1$, $0 < \gamma < +\infty$, and assume that

$$(1) \quad M_j(\Lambda, A, g, \eta, x) < \gamma \quad \text{for } x \text{ in } A \text{ and } 0 < \eta < \delta.$$

Let

$$(2) \quad B = \{x \mid x \in A, A \cap P(g, \eta, x) \cap K(x, r) \neq \emptyset \text{ for } r > 0, 0 < \eta < 1\}.$$

For x in R^n , let

$$x' = I_{n-k}g(x);$$

and let

$$x'' = I^k g(x).$$

Also let

$$K''(x'', r) = I^k g(K(x, r)).$$

Then for a in B , we have

$$(3) \quad \Lambda[B \cap (I^k g)^{-1}(K''(a'', r))] < 2^{k+1} 10^{2k} \gamma \alpha(k) r^k,$$

for $0 < r < \delta/12j$.

Proof. Since $d(A) < 1/j$, (1) implies that

$$(4) \quad M(\Lambda, A, g, \eta, r, x) < \gamma \quad \text{for } x \text{ in } A, 0 < \eta < \delta, r > 0.$$

Now for x, y in R^n

$$\begin{aligned} |y'' - x''| &= |I^k g(y) - I^k g(x)| = |I^k g(y - x)|, \\ |y' - x'| &= |I_{n-k} g(y) - I_{n-k} g(x)| = |I_{n-k} g(y - x)|. \end{aligned}$$

With this remark, the proof proceeds exactly as in ([1], 5.2). For a in B , $0 < r < \delta/12j$, let $\eta = 12rj$, $\epsilon = \eta/12 = rj$. Set

$$E = B \cap (I^k g)^{-1}(K''(a'', r)).$$

For x in E , set $h(x) = \text{l.u.b. } |y' - x'|$ for y in

$$A \cap P(g, \epsilon, x) \cap (I^k g)^{-1}(K''(a'', r)).$$

Then $0 < h(x) \leq 1/j$. For x in E , let $\bar{x}$ in $A \cap P(g, \epsilon, x) \cap (I^k g)^{-1}(K''(a'', r))$ be such that

$$(5) \quad 12 |\bar{x} - x'| > 11h(x).$$

Since $\bar{x}$ is in $P(g, \epsilon, x)$,

$$(6) \quad |\bar{x}'' - x''| < \epsilon |\bar{x} - x'|.$$

For x in E , set

$$(7) \quad Q(x) \text{ equal the set of } y \text{ in } E \text{ such that} \\ |y'' - x''| < 5\epsilon h(x).$$

The remainder of the proof is divided into five parts.

PART 1. For x in E , $Q(x) \subset P(g, \eta, x) \cup P(g, \eta, \bar{x})$.

Proof. If not, then there is an x in E , such that

$$|y'' - x''| < 5\epsilon h(x), |y'' - x''| \geq \eta |y' - x'|, |y'' - \bar{x}''| \geq \eta |y' - \bar{x}'|.$$

Then

$$0 < 11\epsilon h(x) < 12\epsilon |\bar{x} - x'| = \eta |\bar{x} - x'| \leq \eta |\bar{x} - y'| + \eta |y' - x'| \\ \leq |y'' - \bar{x}''| + |y'' - x''| \leq |x'' - \bar{x}''| + 2|y'' - x''| < \epsilon |x' - \bar{x}'| \\ + 10\epsilon h(x) \leq 11\epsilon h(x).$$

This is a contradiction.

PART 2. For x in E , $K''(x'', \epsilon h(x)) \subset K''(a'', 2r)$.

Proof. If $|y'' - x''| < \epsilon h(x)$, then, since $|x'' - a''| < r$,

$$|y'' - a''| < |y'' - x''| + |x'' - a''| < \epsilon h(x) + r < 2r.$$

PART 3. If y is in $Q(x)$, then $|y' - x'| < 5h(x)$.

Proof. Assume that there is a y in $Q(x)$ such that $|y' - x'| \geq 5h(x)$. Then $|y'' - x''| < 5\epsilon h(x) \leq \epsilon |y' - x'|$. Thus y is in $A \cap P(g, \epsilon, x) \cap (I^k g)^{-1}(K(a'', r))$, and hence $1/j \geq h(x) \geq 5h(x) > 0$. This is a contradiction.

PART 4. If x is in E , then $\Lambda(Q(x)) \leq 2(84)^k \gamma \alpha(k) [\epsilon h(x)]^k$.

Proof. By Part 3, if y is in $Q(x)$, then

$$|y - x| \leq |y' - x'| + |y'' - x''| < 5h(x) + 5\epsilon h(x) = 5(\epsilon + 1)h(x) < 6h(x).$$

Also

$$|y - \bar{x}| \leq |y - x| + |x - \bar{x}| \leq 5(\epsilon + 1)h(x) + |x' - \bar{x}'| + |x'' - \bar{x}''| \\ < 5(\epsilon + 1)h(x) + (\epsilon + 1)|x' - \bar{x}'| < 6(\epsilon + 1)h(x) \\ < (13/12) \cdot 6h(x) < 7h(x).$$

Hence

$$Q(x) \subset K(x, 7h(x)) \cap K(\bar{x}, 7h(x)).$$

Hence, by Part 1,

$$Q(x) \subset [A \cap P(g, \eta, x) \cap K(x, 7h(x))] \cup [A \cap P(g, \eta, \bar{x}) \cap K(\bar{x}, 7h(x))].$$

Thus by (4) and since

$$\eta = 12\epsilon, \Lambda[Q(x)] \leq 2\gamma\alpha(k)(7h(x)\eta)^k = 2(84)^k\gamma\alpha(k)(\epsilon h(x))^k.$$

PART 5. $\Lambda(E) \leq 2^{k+1}10^{2k}\alpha(k)\gamma r^k$.

Proof. Since, for x in E , $I^k g(E) \subset \bigcup K''(x'', \epsilon h(x))$, by a covering theorem of Morse, [5], there are $x_1, x_2, \dots$ in E such that $I^k g(E) \subset \bigcup_{i=1}^{\infty} K''(x_i'', 5\epsilon h(x_i))$, and for $p \neq q$

$$K''(x_p'', \epsilon h(x_p)) \cap K''(x_q'', \epsilon h(x_q)) = \emptyset.$$

Now for x in E , there is an x_i such that $|x'' - x_i''| < 5\epsilon h(x)$, and hence x is in $Q(x_i)$. Thus

$$E \subset \bigcup_{i=1}^{\infty} Q(x_i).$$

From Parts 2 and 4,

$$\begin{aligned} \Lambda(E) &\leq \sum_{i=1}^{\infty} \Lambda(Q(x_i)) \leq 2(84)^k\alpha(k)\gamma \sum_{i=1}^{\infty} (\epsilon h(x_i))^k \\ &\leq 2(84)^k(2)^k\gamma\alpha(k)r^k \leq 2^{k+1}10^{2k}\gamma\alpha(k)r^k. \end{aligned}$$

4.2. THEOREM. Let Λ be a C.O.M. in R^n , g in G_n , and let A be a (Λ, k) unrectifiable subset of R . Then (see 2.5)

$$\Lambda[\overline{X}(\Lambda, A, g)] = 0.$$

Proof. It suffices to show that for j an integer $\Lambda[\overline{X}_j(\Lambda, A, g)] = 0$. Also, since R^n is separable, we may assume $d(\overline{X}_j(\Lambda, A, g)) < 1/j$. Now $\limsup_{\eta \rightarrow 0} M_j(\Lambda, \overline{X}_j, g, \eta, x) = 0$ for x in $\overline{X}_j$. Assume $\Lambda[\overline{X}_j(\Lambda, A, g)] > 0$. Hence by 2.5, there is a set $\overline{A}_j$ contained in $\overline{X}_j$, and a sequence of positive n numbers $\delta_1, \delta_2, \dots$, such that

$$\Lambda(\overline{A}_j) > 0, \overline{M}_j(\Lambda, \overline{A}_j, g, \eta, x) < 1/i, \quad \text{for } 0 < \eta < \delta_i$$

and x in $\overline{A}_j$. Let B_j be the set of x in $\overline{A}_j$, such that for $r > 0, 0 < \eta < 1$,

$$\overline{A}_j \cap P(g, \eta, x) \cap K(x, r) \neq \emptyset.$$

Since $\overline{A}_j$ is (Λ, k) unrectifiable, by Federer ([2] 4.3 Theorem),

$$\Lambda(\overline{A}_j - B_j) = 0, \quad \Lambda[B_j] = \Lambda[\overline{A}_j] > 0.$$

Thus there is a point a in B , such that

$$(1) \quad \limsup_{r \rightarrow 0} \Lambda[B \cap (I^k g)^{-1}(K''(a'', r))]/\alpha(k)r^k > \lambda > 0,$$

for some λ . Choose i such that $1/i < \lambda/2^{k+1}10^{2k}$. By 4.1,

$$(2) \quad \Lambda[B \cap (I^k g)^{-1}(K''(a'', r))] < \alpha(k)r^k \lambda,$$

for $0 < r < \delta_i/12j$.

Since (1) contradicts (2), it follows that

$$\Lambda(\overline{X}_j(\Lambda, A, g)) = 0.$$

V. Lebesgue measure of projected sets.

5.1. LEMMA. If Λ is a C.O.M. in R^n , g in G_n , and A is a subset of R^n , with $\Lambda(A) < +\infty$, then,

$$L_k[I^k g(\overline{W}(\Lambda, A, g))] = 0,$$

where L_k denotes Lebesgue measure in R^k .

Proof. For E in R^k , let

$$\Lambda_g(E) = \Lambda[A \cap (I^k g)^{-1}(E)].$$

Then Λ_g is finite-valued C.O.M. in R^k , and for x'' in R^k

$$(1) \quad \limsup_{t \rightarrow 0} \Lambda_g(K''(x'', t))/\alpha(k)t^k < +\infty,$$

for L_k almost every x'' in R^k . Now for x in $\overline{W}(\Lambda, A, g)$, and $0 < \eta < 1$, $0 < r < +\infty$, the set $P(g, \eta, x) \cap K(x, r)$ is contained in the set of y in R^n such that y'' is an element of $K''(x'', \eta r)$. Thus there is a sequence

$$(\eta_i, r_i) \rightarrow (0, 0) \quad \text{for } i \rightarrow +\infty,$$

such that

$$+\infty = \lim_{i \rightarrow \infty} M(\Lambda, A, g, \eta_i, r_i, x) = \lim_{i \rightarrow \infty} \Lambda_g(K''(x'', \eta_i r_i))/\alpha(k)\eta_i^k r_i^k,$$

and therefore the result follows from (1).

5.2. LEMMA. Let Λ be a C.O.M. in R^n , g in G_n , and A a Borel subset of R^n such that

$$\Lambda(A) < +\infty$$

and

$$\Lambda(A) \geq \int_{R^k} N(x'', I^k g, A) dL_k,$$

where $N(x'', I^k g, A)$ is the number of points (possibly ∞) in $A \cap (I^k g)^{-1}(x'')$. Then

$$L_k(I^k g(\bar{V}(A, g))) = 0.$$

Proof. $N(x'', I^k g, A)$ is finite for L_k almost every x'' in R^k . But x in $\bar{V}(A, g)$ implies $N(x'', I^k g, A)$ is infinite. Hence the result follows.

VI. The μ_n^k measure.

6.1. Let $\mathcal{Z}$ denote the family of subsets Z of G_n , such that $\sigma_n(Z) = 0$. For B a Borel subset of R^n , and Z in $\mathcal{Z}$, let

$$\lambda_Z^*(B) = \text{l.u.b. } L_k[I^k g(B)], \quad \text{for } g \text{ in } G_n - Z.$$

Let

$$\lambda(B) = \text{g.l.b. } \lambda_Z^*(B), \quad \text{for } Z \text{ in } \mathcal{Z}.$$

For E a subset of R^n , and $\epsilon > 0$, let $\mu_\epsilon(E) = \text{g.l.b. } \sum \lambda(B_i)$, the sum being taken over a countable covering of E by Borel sets B_i such that $d(B_i) < \epsilon$, and the greatest lower bound being taken over the family of all such coverings.

For E a subset of R^n , let

$$\mu_n^k(E) = \lim_{\epsilon \rightarrow 0} \mu_\epsilon(E).$$

μ_n^k is a Borel regular C.O.M. in R^n .

6.2. LEMMA. Let B_i be Borel sets for $i = 0, 1, 2, \dots$, and assume $B_0 \subset \bigcup_i B_i$, and Z is in $\mathcal{Z}$. Then

$$\lambda_Z^*(B_0) \leq \sum_i \lambda_Z^*(B_i).$$

Proof.

$$\begin{aligned} \text{l.u.b. } L_k[I^k g(B_0)] &\leq \text{l.u.b. } L_k \left[\bigcup_i [I^k g(B_i)] \right] \leq \text{l.u.b. } \sum_i L_k[I^k g(B_i)] \\ &\leq \sum_i \text{l.u.b. } L_k[I^k g(B_i)], \end{aligned}$$

all least upper bounds being taken over g in $G_n - Z$.

6.3. LEMMA. For B a Borel set in R^n , there is a set Z in $\mathcal{Z}$ such that $\lambda(B) = \lambda_Z^*(B)$.

Proof. For any integer i , there is a set Z_i in $\mathcal{Z}$ such that $\lambda_{Z_i}^*(B) < \lambda(B) + 1/i$. Let $Z = \bigcup_i Z_i$. Z is in $\mathcal{Z}$, and $\lambda(B) \leq \lambda_Z^*(B) < \lambda_{Z_i}^*(B) < \lambda(B) + 1/i$. Hence for all i , $\lambda(B) \leq \lambda_Z^*(B) < \lambda(B) + 1/i$. Therefore $\lambda(B) = \lambda_Z^*(B)$.

6.4. LEMMA. For $E \subset R^n$, there is a set Z in $\mathcal{Z}$ such that for g in $G_n - Z$, $\mu_n^k(E) \geq L_k I^k g(E)$.

Proof. Since μ_n^k is Borel regular, it suffices to work with Borel sets. Let B be a Borel set, $\epsilon > 0$, and i an integer. There is a countable covering of B by Borel sets B_j , $d(B_j) > \epsilon$, such that $\mu_n^k(B) + 1/i > \sum_{j=1}^{\infty} \lambda(B_j)$. By 6.3 for each j there is a Z_j in $\mathcal{Z}$ such that

$$\lambda(B_j) = \lambda^* Z_j(B_j).$$

Set $Z^i = \bigcup_{j=1}^{\infty} Z_j$. Then Z^i is in $\mathcal{Z}$ and, (by 6.2)

$$\mu_n^k(B) + 1/i > \sum_{j=1}^{\infty} \lambda_{Z^i}^* (B_j) \geq \lambda_{Z^i}^* (B).$$

Now set $Z = \bigcup_{i=1}^{\infty} Z^i$. Then Z is in $\mathcal{Z}$, and

$$\mu_n^k(B) + 1/i > \lambda_Z^* (B) \geq L_k I^k g(B),$$

for g in $G_n - Z$, and for all i . Hence $\mu_n^k(B) \geq L_k I^k g(B)$ for g in $G_n - Z$.

6.5. LEMMA. Let Λ be a Borel regular C.O.M. in R^n . Assume that for any subset E of R^n , there is a set Z in $\mathcal{Z}$ such that for g in $G_n - Z$, $\Lambda(E) \geq L_k I^k g(E)$. Then $\Lambda(E) \geq \mu_n^k(E)$.

Proof. Again it suffices to work only with Borel sets B . For $\epsilon > 0$, B may be written as a countable union of Borel sets B_i , where $d(B_i) < \epsilon$, and for $i \neq j$, $B_i \cap B_j = \emptyset$. Further for all integers i , there is a set Z_i in $\mathcal{Z}$ such that for g in $G_n - Z_i$, $\Lambda(B_i) \geq L_k I^k g(B_i)$. Set $Z = \bigcup_{i=1}^{\infty} Z_i$. Then Z is in $\mathcal{Z}$, and $\Lambda(B_i) \geq \text{l.u.b. } [L_k I^k g(B_i)] \geq \lambda(B_i)$ for g in $G_n - Z$. Hence $\Lambda(B) \geq \sum_{i=1}^{\infty} \lambda(B_i) \geq \mu_n^k(B)$.

The result follows since ϵ was arbitrary.

6.6. LEMMA. Let Λ be a Borel regular C.O.M. in R^n . Let Γ be a Borel regular C.O.M. in R^k such that for all $E \subset R^n$, such that $\Lambda(E) < +\infty$, there is a set Z_E in $\mathcal{Z}$ such that for g in $G - Z_E$, $\Lambda(E) \geq \Gamma I^k g(E)$. Then for all $E \subset R^n$, such that $\Lambda(E) < +\infty$, there is a set Z in $\mathcal{Z}$ such that for all subsets $E^* \subset E$, and g in $G_n - Z$,

$$\Lambda(E^*) \geq \Gamma [I^k g(E^*)].$$

Proof. Again it suffices to consider only Borel sets B in R^n , and Borel subsets $B^* \subset B$. First, let θ be the family of sets C such that C is the union of a finite number of spheres of rational center and rational radius in R^n . θ is a countable family. If F is a compact subset of R^n , then there exists $C_1 \supset C_2 \supset \dots$, $C_i \in \theta$, $F = \bigcap_{i=1}^{\infty} C_i$. We may further assume that B is bounded. Let

$$\theta_B = \{E \mid E = B \cap C, C \in \theta\}.$$

θ_B is countable. For each E in θ_B , there is a Z_E in $\mathcal{Z}$ such that, for g in $G_n - Z_E$,

$\Lambda(E) \geq \Gamma[I^k g(E)]$. Set $Z = \bigcup Z_B$, for E in θ_B . Z is in $\mathcal{Z}$, and for E in θ_B and g in $G_n - Z$,

$$\Lambda(E) \geq \Gamma[I^k g(E)].$$

Now take F closed with respect to B . Then there is a compact set $F^* \subset R^n$, such that $F = F^* \cap B$. Hence there is a sequence of sets $C_1 \supset C_2 \supset \dots$ such that $F^* = \bigcap_{i=1}^{\infty} C_i$. Set $E_i = C_i \cap B$. $F = \bigcap_{i=1}^{\infty} E_i$, and $E_1 \supset E_2 \supset \dots$. Also for all integers i , and for g in $G_n - Z$,

$$\Gamma[I^k g(F)] \leq \Gamma[I^k g(E_i)] \leq \Lambda(E_i).$$

Since $\lim_{i \rightarrow \infty} \Lambda(E_i) = \Lambda(F)$, we have, for all g in $G_n - Z$,

$$\Gamma[I^k g(F)] \leq \Lambda(F).$$

Now let O be open with respect to B . Then there is a bounded open set O^* in R^n , such that $O = O^* \cap B$, and a sequence of compact sets $F_1^* \subset F_2^* \subset \dots$, such that $O^* = \bigcup_{i=1}^{\infty} F_i^*$. Set $F_i = F_i^* \cap B$. F_i is closed relative to B , and $O = \bigcup_{i=1}^{\infty} F_i$. For all integers i , and g in $G_n - Z$,

$$\Gamma[I^k g(F_i)] \leq \Lambda(F_i) \leq \Lambda(O).$$

Now $I^k g(F_i) \subset I^k g(F_{i+1})$, and $I^k g(F_i)$ is Γ measurable, and $I^k g(O) = \bigcup_{i=1}^{\infty} I^k g(F_i)$. So we have that $\Gamma[I^k g(F_i)] \rightarrow \Gamma[I^k g(O)]$ as $i \rightarrow +\infty$, and thus $\Gamma[I^k g(O)] \leq \Lambda(O)$. Now take B^* a Borel subset of B . Fix $\epsilon > 0$. Then there is a set O open relative to B , such that $O \supset B^*$, and

$$\Lambda(O) < \Lambda(B^*) + \epsilon.$$

Then for g in $G_n - Z$,

$$\Gamma[I^k g(B^*)] \leq \Gamma[I^k g(O)] \leq \Lambda(O) < \Lambda(B^*) + \epsilon.$$

Thus

$$\Gamma[I^k g(B^*)] \leq \Lambda(B^*).$$

6.7. LEMMA. For E a subset of R^n , such that $\mu_n^k(E) < +\infty$, there is a set Z in $\mathcal{Z}$ such that for g in $G_n - Z$,

$$\mu_n^k(E) \geq \int_{R^k} N(x'', I^k g, E) dL_k.$$

Proof. Again it suffices to work with Borel sets $B \subset R^n$. By 6.6 there is a set Z in $\mathcal{Z}$ such that for all Borel subsets $B^* \subset B$, and g in $G_n - Z$, $\mu_n^k(B^*) \geq L_k I^k g(B^*)$. For i an integer, R^n may be written as the union of a countable family of disjoint Borel sets B_{ij} , where $d(B_{ij}) < 1/i$. For x'' in R^k , let $f_{ij}(x'')$ be the characteristic function of $I^k g(B \cap B_{ij})$, and let $f_i(x'') = \sum_{j=1}^{\infty} f_{ij}(x'')$. Then

$$\begin{aligned}\int_{R^k} f_i(x'') dL_k &= \sum_{j=1}^{\infty} \int_{R^k} f_{ij}(x'') dL_k = \sum_{j=1}^{\infty} L_k[I^k g(B \cap B_{ij})] \\ &\leq \sum_{j=1}^{\infty} \mu_n^k(B \cap B_{ij}) = \mu_n^k(B).\end{aligned}$$

Since $f_i(x'') \rightarrow N(x'', I^k g, B)$ as $i \rightarrow +\infty$, the result follows from the lemma of Fatou.

6.8. LEMMA. *Let E be a k -rectifiable set in R^n . Then*

$$H^k(E) = \mu_n^k(E).$$

Proof. Again it suffices to work with Borel sets B . Let Q denote the unit cube in R^k , that is, for x in R^n , and x in Q , and $x = (x_1, x_2, \dots, x_n)$, we have

$$0 \leq x_i \leq 1, \quad \text{for } 1 \leq i \leq k,$$

and

$$x_i = 0, \quad \text{for } k+1 \leq i \leq n.$$

Now B may be taken as the image under a Lipschitz mapping of a subset of Q . Also it follows immediately from 6.5 that (*) $H^k(B) \geq \mu_n^k(B)$. By [6], this Lipschitz mapping may be extended to all of Q , with the same Lipschitz constant. Hence, in view of (*) without loss of generality, we may assume that

$$B = T(Q), \quad T \text{ a Lipschitz mapping.}$$

Now let u be any point of Q , and g any element of G_n . Let

$J(u)$ be the Jacobian of T at u ;

$J_g^*(u)$ be the Jacobian of gT at u ;

$J_g(u)$ be the Jacobian of $I^k gT$ at u .

Now by ([7], 4.2), $J(u) = J_g^*(u)$. Also all of the above Jacobians exist L_k almost everywhere on Q . Hence there is a Borel subset E of Q such that

(1) T is univalent on E ,

(2) All of the Jacobians exist and have positive, finite absolute values everywhere on E .

$$(3) \quad \mu_n^k(B - T(E)) = H^k(B - T(E)) = 0.$$

Now let u be any point in E . By (2) there is a k -plane π which is spanned by the k column vectors of the differential matrix of T at u . Select g in G_n such that $g(\pi) = R^k$. Then for this g ,

$$J(u) = J_g^*(u) = J_g(u).$$

Thus for $\epsilon > 0$ there can be selected a sequence $g_1, g_2, \dots$ in G_n such that the sets

$$(4) \quad E_i^* = \{u \mid u \in E, 0 \leq |J(u) - J_{g_i}(u)| < \epsilon\}.$$

cover E . Let

$$E_1 = E_1^*, \quad E_i = E_i^* - \bigcup_{j=1}^{i-1} E_j^*.$$

Now, from 6.7 if necessary a second sequence $g'_1, g'_2, \dots$ can be selected from G_n , such that for u in E_i ,

$$(5) \quad 0 \leq |J(u) - J_{g'_i}(u)| < \epsilon,$$

and

$$(6) \quad \mu_n^k(T(E_i)) \geq \int_{R^k} N(x'', I^k g'_i, T(E_i)) dL_k.$$

Hence, from ([7], 4.5 Theorem), we have

$$\begin{aligned} H^k(B) &= \int_E J(u) dL_k = \sum_{i=1}^{\infty} \int_{E_i} J(u) dL_k \leq \sum_{i=1}^{\infty} \int_{E_i} J_{g'_i}(u) dL_k + \epsilon \\ &= \sum_{i=1}^{\infty} \int_{R^k} N(x'', I^k g'_i, T(E_i)) dL_k + \epsilon \leq \sum_{i=1}^{\infty} \mu_n^k(T(E_i)) + \epsilon = \mu_n^k(B) + \epsilon. \end{aligned}$$

Hence

$$H^k(B) \leq \mu_n^k(B).$$

VII. The decomposition theorem.

7.1. LEMMA. Let A be a closed (μ_n^k, k) unrectifiable set in R^n , such that $\mu_n^k(A) < +\infty$. Then $\mu_n^k(A) = 0$.

Proof. For any g in G_n ,

$$\begin{aligned} A &= [A - (\overline{W}(\mu_n^k, A, g) \cup \overline{X}(\mu_n^k, A, g) \cup \overline{V}(A, g))] \\ &\quad \cup [\overline{W}(\mu_n^k, A, g) \cup \overline{X}(\mu_n^k, A, g) \cup \overline{V}(A, g)]. \end{aligned}$$

By 3.15

$$\mu_n^k[A - (\overline{W}(\mu_n^k, A, g) \cup \overline{X}(\mu_n^k, A, g) \cup \overline{V}(A, g))] = 0,$$

for σ_n almost every g in G_n . By 6.4 and 6.6,

$$L_k[I^k g(A - (\overline{W}(\mu_n^k, A, g) \cup \overline{X}(\mu_n^k, A, g) \cup \overline{V}(A, g)))] = 0,$$

for σ_n almost every g in G_n . Likewise by 4.2 and 6.6 and 6.4, we have that

$$L_k[I^k g(X(\mu_n^k, A, g))] = 0,$$

for σ_n almost every g in G_n . Likewise by 5.1,

$$L_k[I^k g(\overline{W}(\mu_n^k, A, g))] = 0,$$

for every g in G_n . Likewise by 5.2 and 6.7, we have that

$$L_k[I^k g(\overline{V}(A, g))] = 0,$$

for σ_n almost every g in G_n . Hence, for σ_n almost every g in G_n ,

$$L_k[I^k g(A)] = 0,$$

and therefore

$$\mu_n^k(A) = 0.$$

7.2. THEOREM. *Let A be a Borel set in R^n such that $\mu_n^k(A) < +\infty$. Then $A = A_1 \cup A_2$, where A_1, A_2 are Borel sets. $A_1 \cap A_2 = \emptyset$. A_1 is countably k -rectifiable, and $\mu_n^k(A_2) = 0$.*

Proof. Let

$$\gamma = \text{l.u.b. } \mu_n^k(A'),$$

for A' a k -rectifiable subset of A . For each positive integer i , there is a k -rectifiable set $B_i \subset A$ such that

$$\mu_n^k(B_i) > \gamma - 1/i.$$

Then there is a Lipschitz transformation from a bounded set E_i in R^k onto B_i . By [6] this transformation can be extended to the whole of R^k with the same Lipschitz constant. If F_i is a bounded closed set in R^k containing E_i , then the intersection of A and the image of F_i under this extended transformation is a Borel k -rectifiable subset of A containing B_i . Hence we may assume that B_i is a Borel set for each i . Set

$$A_1 = \bigcup_{i=1}^{\infty} B_i.$$

Then A_1 is a Borel and countably k -rectifiable set. The set $A_2 = A - A_1$ is Borel and (μ_n^k, k) unrectifiable, $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$. Since A_2 is the union of a countable number of closed (μ_n^k, k) unrectifiable sets and a set of μ_n^k measure 0, it follows from 7.1 that $\mu_n^k(A_2) = 0$.

7.3. THEOREM. *If A is a Borel subset of R^n and $\mu_n^k(A) < +\infty$, then $\mu_n^k(A) = F_n^k(A)$, where F_n^k is the integralgeometric Favard k -measure of A , in R^n . (See [2], 2.18, 5.11)*

Proof. By 7.2, $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$, A_1, A_2 Borel, A_1 is countably k -rectifiable, and $\mu_n^k(A_2) = 0$. Thus

$$F_n^k(A_2) = 0.$$

Also by 6.8 and ([2], 5.14 Theorem),

$$\mu_n^k(A_1) = H^k(A_1) = F_n^k(A_1).$$

7.4. THEOREM. Let m, k and n be positive integers, $k < n < m$, and let A be a Borel subset of R^n , which may be thought of as a subspace of R^m . Then

$$\mu_n^k(A) = \mu_m^k(A).$$

Proof. Under the above assumptions Federer has shown ([8], 7) that

$$F_m^k(A) = F_n^k(A).$$

To begin, we assume $\mu_n^k(A) < +\infty$. Then, by 6.8 and 7.2, $A = A_1 \cup A_2$, A_i Borel for $i = 1, 2$, $A_1 \cap A_2 = \emptyset$. Also A_1 is a countably k -rectifiable subset of both R^n and R^m . So

$$\mu_n^k(A_1) = H^k(A_1) = \mu_m^k(A_1), \quad \text{and} \quad \mu_n^k(A_2) = F_n^k(A_2) = 0.$$

Hence $F_m^k(A_2) = 0 = \mu_m^k(A_2)$. Therefore $\mu_n^k(A) = \mu_m^k(A)$. We now wish to show that if $\mu_n^k(A) = +\infty$, then $\mu_m^k(A) = +\infty$. To show this it suffices to show that if $\mu_m^k(A) < +\infty$, then $\mu_m^k(A) = \mu_n^k(A)$. But this follows from exactly the same type of argument as used above.

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