ON HOMOLOGICALLY TRIVIAL 3-MANIFOLDS

BY

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1. Introduction. In [6], R. H. Bing proved that a compact, connected absolute 3-manifold M (i.e., 3-manifold without boundary) is topologically a 3-sphere if each polyhedral simple closed curve in M lies in a topological cube in M . He also raised the following question: Is a compact, connected absolute 3-manifold M a topological 3-sphere if each polyhedral simple closed curve in M can be shrunk to a point in a solid torus (of genus one) in M ? This question is answered here in the affirmative by Theorem 2 as an immediate corollary to Theorem 1. The lemmas in the first 4 sections are developed to aid in the proof of Theorem 1. The principal result of this paper, however, is an extension of Theorem 1:

THEOREM 3. *Let M be a compact, connected, absolute 3-manifold such that $H_1(M; Z) = 0$ and each polyhedral simple closed curve lies in a regular free-manifold in M , in a homologically trivial manner. Then, M is homeomorphic to S^3 .*

A regular *free-manifold* is a punctured cube (see [6]) to whose boundary components have been added orientable handles, at most one to each component (see §3). A simple closed curve J is said to lie *trivially* in a regular free-manifold R if it can be shrunk to a point in R . The simple closed curve J is said to lie in R in a *homologically trivial* manner if J circles each handle of R an even number of times. The group $H_1(M; Z)$ is the 1-dimensional simplicial homology group of M with coefficients in the infinite cyclic group Z . It may be obtained by "abelianizing" the fundamental group of M . To be precise, one should say that J lies in R in a homologically trivial manner (mod 2), but the shorter phrase is more convenient. In all other cases, homological terms will refer to the group Z , unless explicitly stated otherwise. Homology groups will be written additively, and fundamental groups multiplicatively.

Both Theorem 1 of [6] and Theorem 3 of this paper are attempts to characterize S^3 by certain of its algebraic properties. For a brief history of the

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Poincaré Conjecture and related questions, see [6]. Some pertinent references are given at the end of this paper.

The overall plan for the proof of Theorem 1 is the same as for the proof of Theorem 1 of [6]. The reader may find it helpful to consult that paper first. All 3-manifolds considered here will have a fixed triangulation. By [3] and [10], there is no loss of generality in assuming this.

All manifolds mentioned in this paper are to be separable metric, and terms such as "simply-connected," "polyhedral," "piecewise linear," etc., are understood in the sense of [3] and [6], as are the abbreviations "Bd M ," "Int M ," etc., for a manifold M . A *punctured disk* is a 2-cell D minus the sum of the interiors of a finite number of disjoint closed polyhedral 2-cells, each contained in Int D . If X is a topological space, the *cone over X* , $C(X)$, is the space formed by identifying the set $X \times \{1\}$ to a point in the product space $X \times [0, 1]$. One usually identifies $X \times \{0\}$ with X .

2. Cellular decompositions of a 3-manifold. Let M be a connected 3-manifold without boundary. A locally-finite collection $T = \{\Delta_j\}$ of polyhedral 3-cells whose union is M constitutes a *proper cellular decomposition* of M if the interiors of distinct Δ_j 's are disjoint, the intersection of $4-i$ distinct Δ_j 's is a closed i -cell or is empty ($i=0, 1, 2$), and no point of M belongs to more than four Δ_j 's. The sum of all points of M which are contained in $4-i$ or more elements of T is called the *i -skeleton* of T , and will be denoted by T_i ($i=0, 1, 2$). Note that T_2 is $\sum \text{Bd } \Delta_j$. The following cellular decomposition will have a 1-skeleton that is better suited to the purposes of this paper than is the 1-skeleton of a triangulation.

LEMMA 1. *Every connected absolute 3-manifold M has a proper cellular decomposition $T = \{\Delta_j\}$ such that each point of T_1 is of order 2 or 4, $T_1 \cdot \Delta_j = T_1 \cdot \text{Bd } \Delta_j$ is a nonempty, connected finite subgraph of the polyhedral graph T_1 , and the boundary of each component of $\text{Bd } \Delta_j - T_1 \cdot \Delta_j$ is a polygonal simple closed curve.*

The decomposition described in the proof of Theorem 2 of [5] for any compact, connected, triangulated 3-manifold without boundary may be used in this case also. In Figure 2 of that paper, the intersection of T_1 with one tetrahedron of a triangulation of M is shown. By [3], every 3-manifold can be triangulated. The decomposition guaranteed by Lemma 1 will be referred to as a *special cellular decomposition* of M .

3. Some 3-manifolds with boundary in E^3 . Let M be a compact, connected, simply-connected 3-manifold. By Lemma 1, M has a special cellular decomposition T . The class of 3-manifolds with boundary to be described here includes the punctured cubes of [6], yet its members retain the property that if one of them lies in M as a polyhedral subset and contains T_1 in its interior, then some other member has the additional property of containing T_2 . It will then be easy to see that M is topologically S^3 .

Let E_1 be a polyhedral cube in E^3 , and $E_2, \dots, E_n$, disjoint polyhedral cubes in $\text{Int } E_1$, where $n \geq 1$. If k is an integer, $1 \leq k \leq n$, there are k polyhedral cubes, $C_1, \dots, C_k$, in E^3 such that C_1 is contained in the closure of the exterior of E_1 , $C_i \subseteq E_i$ for $i \leq k$ and $i \neq 1$, and C_i meets $\text{Bd } E_i$ in 2 disjoint polyhedral disks for $1 \leq i \leq k$. Then, any space homeomorphic to $R = E_1 - \sum_{i=2}^n \text{Int } E_i + \sum_{i=1}^k C_i$ will be called a *regular free-manifold*. In case $k=0$, R is defined to be $E_1 - \sum_{i=2}^n \text{Int } E_i$. When one wishes to be more specific, R may be completely defined by the ordered pair of integers (k, n) , where $0 \leq k \leq n$ and $n \geq 1$. R is said to be obtained by adding orientable handles to a punctured cube. Note that $(1, 1)$ represents a solid torus, and $(0, n)$ represents a punctured cube. In any case, the fundamental group of R is a free group of finite rank k .

The following 3 lemmas are easy consequences of the results found in [2] and [10].

LEMMA 2. *Suppose that R_1 and R_2 are regular free-manifolds, where $R_1 \cdot R_2 = \text{Bd } R_1 \cdot \text{Bd } R_2$ is a 2-sphere. Then $R_1 + R_2$ is a regular free-manifold or a 3-sphere.*

LEMMA 3. *Suppose that R is a regular free-manifold, and S is a polyhedral 2-sphere in $\text{Int } R$. Then, R is the sum of two polyhedral subsets R_1 and R_2 which are regular free-manifolds, where $R_1 \cdot R_2 = \text{Bd } R_1 \cdot \text{Bd } R_2 = S$.*

LEMMA 4. *Let R be a regular free-manifold and J a polygonal simple closed curve in $\text{Bd } R$ which separates the component of $\text{Bd } R$ containing it and which bounds a polyhedral disk D such that $\text{Int } D \subseteq \text{Int } R$. Then, R is the sum of two polyhedral subsets R_1 and R_2 which are regular free-manifolds, where $R_1 \cdot R_2 = \text{Bd } R_1 \cdot \text{Bd } R_2 = D$.*

The following shows that if a regular free-manifold is cut along a handle, then the result is also a regular free-manifold.

LEMMA 5. *Let R' be a compact, connected 3-manifold with nonempty boundary and let D_1^* and D_2^* be disjoint polyhedral disks in the same component of $\text{Bd } R'$. Denote by R the 3-manifold with boundary obtained by identifying D_1^* with D_2^* in an orientable fashion. Then, if R is a regular free-manifold, so is R' .*

Proof. There is a homeomorphism h taking R onto a polyhedral regular free-manifold in E^3 , and by [10], h may be supposed to be piecewise linear with respect to some triangulation of R . Let $J_1, J_2, \dots, J_n$, be polygonal simple closed curves on $h(\text{Bd } R)$ such that J_i separates no component of $h(\text{Bd } R)$, J_i bounds a polyhedral disk D_i , where $\text{Int } D_i \subseteq \text{Int } h(R)$, no two D_i 's intersect, and each D_i is in general position relative to $h(D_1^*) = h(D_2^*)$. It is further required that each component of $h(\text{Bd } R)$ of genus 1, except the one containing $h(\text{Bd } D_1^*) = h(\text{Bd } D_2^*)$, contain exactly 1 of the J_i , and that no J_i lie in the component containing $h(\text{Bd } D_1^*)$.

It will be shown first that the disks D_i can be chosen so that no one of them meets $h(D_1^*)$. Denote by $n(D_i)$ the number of components of $D_i \cdot h(D_1^*)$. Each is a polygonal simple closed curve. If $\sum_{i=1}^n n(D_i) = 0$, the desired property holds. Suppose that the D_i have been chosen as above so as to minimize $\sum_{i=1}^n n(D_i)$. If this number is not 0, then for some j there is a polygonal simple closed curve K bounding a disk E in $\text{Int } h(D_1^*)$ and a disk F in $\text{Int } D_j$, where $\text{Int } E$ misses $\sum_{i=1}^n D_i$. A disk D_j' could then be found so that the collection $D_1, \dots, D_{j-1}, D_j', D_{j+1}, \dots, D_n$, satisfies the requirements of the above paragraph, yet $\sum_{i \neq j} n(D_i) + n(D_j') < \sum_{i=1}^n n(D_i)$. The disk D_j' is constructed by adjusting the polyhedral disk $(D_j - \text{Int } F) + E$ in a small neighborhood of E . Hence, $\sum_{i=1}^n n(D_i) = 0$.

If $h(R)$ is cut along each of the disks $D_1, \dots, D_n$, $h(D_1^*)$, a compact, connected 3-manifold H in E^3 is obtained, each component of whose boundary is a 2-sphere. H is a punctured cube and R' is obtained by adding handles to $\text{Bd } H$. It follows that R' is a regular free-manifold.

If M is a compact, connected 3-manifold with boundary, M is said to be *simply-connected mod Bd M* if $M + C(\text{Bd } M)$ is simply-connected, where $C(\text{Bd } M)$ is the cone over $\text{Bd } M$ as defined at the end of §1 (a set X is closed in $M + C(\text{Bd } M)$ if and only if each of the sets $X \cdot M$, $X \cdot C(\text{Bd } M)$ is closed, in M and $C(\text{Bd } M)$, respectively). This is equivalent to the condition that for each polygonal simple closed curve J in $\text{Int } M$, there is a mapping f of a punctured disk D into M such that one component of $\text{Bd } D$ maps homeomorphically onto J under f , while every other component of $\text{Bd } D$ is mapped by f into $\text{Bd } M$. Note that every 3-manifold with boundary which can be embedded in a simply-connected 3-manifold, and in particular every regular free-manifold, is simply-connected modulo its boundary. The converse is not true. That is, there is an M such that $M + C(\text{Bd } M)$ is simply-connected, but M can be embedded in no simply-connected 3-manifold (see §7).

The above definition, plus a brief discussion of "circling," will be needed for Lemma 6. Let R be a regular free-manifold, and $D_1, \dots, D_n$, a collection of mutually exclusive polyhedral disks in R each of which meets $\text{Bd } R$ in its boundary and such that the 3-manifold obtained by cutting R along these disks is a punctured cube. The D_i form a collection of "handles" for R . If f is a mapping of a simple closed curve J into R , then f can be approximated with any desired accuracy by a piecewise linear homeomorphism h of J into R such that $h(J) \subseteq \text{Int } R$ and $h(J)$ pierces a D_i whenever it meets one. Then, if h is sufficiently close to f , the algebraic linking number of $h(J)$ with $\text{Int } D_i$ will be independent of the choice of h and of D_i , and will depend only upon the component of $\text{Bd } R$ containing $\text{Bd } D_i$. If this number is k , the mapping f will be said to *circle R k times longitudinally in C_i* , where C_i is the component of $\text{Bd } R$ (necessarily of genus 1) containing $\text{Bd } D_i$.

The following shows how one may attach cubes to a regular free-manifold in such a way that the result is a regular free-manifold.

LEMMA 6. *Let R_1 be a regular free-manifold, and R_2 a punctured cube, where $R_1 \cdot R_2$ is an annulus ring A in $\text{Bd } R_1 \cdot \text{Bd } R_2$. Suppose that $M = R_1 + R_2$ is simply-connected mod $\text{Bd } M$. Then, M is a regular free-manifold.*

Proof. Consider the following properties of a simple closed curve J in $\text{Bd } R_1$:

- (a) J lies in a simply-connected component of $\text{Bd } R_1$;
- (b) J lies in a component of $\text{Bd } R_1$ of genus 1 and bounds a disk in this component;
- (c) J lies in a component C of $\text{Bd } R_1$ of genus 1 and circles R_1 once longitudinally in C .

A simple closed curve J with one of these properties is unique in the sense that if J' is another simple closed curve with the same property as J , then there is a homeomorphism of R_1 onto itself taking J onto J' . In the case of property (c), this homeomorphism can be realized by cutting R_1 along one of its handles, twisting one of the resulting "ends" the proper integral number of times, and then rejoining these two ends. It follows that if J is a polygonal simple closed curve in $\text{Int } A$ which divides A into 2 annulus rings, and J' is such a simple closed curve in $\text{Int } A'$, where A' is also an annulus ring in $\text{Bd } R_1$, and if each of J, J' satisfies the same one of the conditions (a), (b), (c), then there is a homeomorphism of R_1 onto itself taking A onto A' . Hence, it will suffice to show that such a simple closed curve J in $\text{Int } A$ has one of the properties (a), (b), or (c).

Suppose first that J circles R_1 n times longitudinally in C , where C is a component of $\text{Bd } R_1$ of genus 1, and $n > 1$. Let $C_1, \dots, C_k, C$, be the distinct components of $\text{Bd } R_1$ that are of genus 1. There is a collection of mutually exclusive polyhedral 2-spheres $S_1, \dots, S_k$, in $\text{Int } R_1$ such that for each i , S_i separates C_i from $\text{Bd } R_1 - C_i$ in R_1 . Now, if H_i is the component of $R_1 - S_i$ that contains C_i , then $T = R_1 - \sum H_i$ is homeomorphic to a solid torus of genus 1 from which has been removed the sum of the interiors of at least k mutually exclusive polyhedral 3-cells, each 3-cell lying in the interior of this torus. Also, $T + R_2$ is simply-connected mod its boundary, since it is contained in $R_1 + R_2$, and each component of $\text{Bd}(T + R_2)$ is a 2-sphere. Hence, $T + R_2$ is simply-connected, and J circles T n times longitudinally in C .

Let K be a polygonal simple closed curve in $\text{Int } T$ which circles T once longitudinally. Since J bounds a disk in R_2 , K can be shrunk to a point in $(T + R_2) - J$. Hence, there is a map f of a punctured disk D into $T - J$ such that the "outer" component of $\text{Bd } D$ maps onto K homeomorphically under f , while the "inner" components of $\text{Bd } D$ are mapped by f into $C - J$. If F is one of these components, then $f|F$ is a mapping of a simple closed curve into T which circles T a number of times longitudinally in C which is a multiple of n . It follows that the number of times that K circles T longitudinally is a multiple of n , where $n > 1$, a contradiction. The reason for this last statement is that the mapping f induces a homomorphism f^* from the group

$H_1(D; Z)$ into the group $H_1(T; Z)$ (see §1), where these groups are, respectively, the free abelian groups on m generators and on 1 generator. The elements $z_1, z_2, \dots, z_m$, of $H_1(D; Z)$ corresponding to the oriented "inner" components of $\text{Bd } D$ generate $H_1(D; Z)$, and the element z of $H_1(T; Z)$ corresponding to the oriented simple closed curve K generates $H_1(T; Z)$. Since, for $1 \leq j \leq m$, $f^*(z_j)$ is some multiple of nz , and since f^* is a homomorphism, $z^{\pm 1} = f^*(z_1 + z_2 + \dots + z_m)$ is some multiple of nz . This is equivalent to the last assertion about "circling." Hence, J can circle R_1 not more than once in C .

Lastly, suppose that J does not circle R_1 longitudinally in C , yet fails to bound a disk on C , where C is a component of $\text{Bd } R_1$ of genus 1. Then, J bounds a polyhedral disk E such that $\text{Int } E \subseteq \text{Int } R_1$ and a polyhedral disk E' such that $\text{Int } E' \subseteq \text{Int } R_2$. Now, $E + E'$ is a polyhedral 2-sphere S in $\text{Int } M$, and there is a polygonal simple closed curve L in $\text{Int } R_1$ which meets S in a single point and pierces it there. L would then be a simple closed curve that could not be shrunk to a point in $M + C(\text{Bd } M)$, since the algebraic linking number of L and S is $+1$ or -1 . This proves Lemma 6.

4. Adjusting 3-manifolds in M . The proof of Theorem 1 follows the pattern of the proof of Theorem 1 of [6]. The reader should compare Lemma 7 of this paper with Lemma 4 of that paper, noting particularly that the proof of the present lemma requires the assumption that M be simply-connected, while that of Lemma 4 of [6] does not.

LEMMA 7. *Let M be a compact, connected, simply-connected 3-manifold with a special cellular decomposition $T = \{\Delta_j\}$. Suppose that M contains a polyhedral subset R which is a regular free-manifold, and that $\text{Int } R$ contains T_1 . Then, M is homeomorphic to S^3 .*

Proof. Note that T_1 is a connected, polyhedral finite graph. It may be assumed, since $T_1 \subseteq \text{Int } R$, that $\text{Bd } R$ is in general position with respect to each component of $T_2 - T_1$, so that each component of $T_2 \cdot \text{Bd } R$ is a polygonal simple closed curve. The proof is by induction on n , the number of components of $T_2 \cdot \text{Bd } R$.

If $n = 0$, then $T_2 \subseteq \text{Int } R$. Then, according to Lemma 3, $R - R \cdot \text{Int } \Delta_1$ is a regular free-manifold, and by Lemma 2, $R_1 = (R - R \cdot \text{Int } \Delta_1) + \Delta_1$ is a regular free-manifold whose interior contains Δ_1 or a 3-sphere (the latter in case $\text{Bd } R \subseteq \text{Int } \Delta_1$). Continuing in this manner, there is R_2 , a regular free-manifold or a 3-sphere, such that $\text{Int } R_2 \supseteq \Delta_1 + \Delta_2$. Finally, after a finite number of steps, there is a 3-sphere which completely fills M . This completes the proof in case $n = 0$.

Now suppose that $T_2 \cdot \text{Bd } R$ consists of n polygonal simple closed curves, where $n \geq 1$. It suffices to find R' , a polyhedral regular free-manifold, such that $T_1 \subseteq \text{Int } R'$ and $T_2 \cdot \text{Bd } R'$ consists of fewer than n polygonal simple closed curves. Let J be a curve in $T_2 \cdot \text{Bd } R$ which bounds a disk D , $D \subseteq T_2 - T_1$,

such that $\text{Int } D \cdot \text{Bd } R = \emptyset$. There are now two cases to consider: $R \cdot \text{Int } D = \emptyset$ or $\text{Int } D \subseteq \text{Int } R$.

Assume now that $R \cdot \text{Int } D = \emptyset$. There is a polyhedral cube H in M such that $\text{Int } D \subseteq \text{Int } H$, $H \cdot T_2 = D$, and $H \cdot R = \text{Bd } H \cdot \text{Bd } R$ is an annulus ring A such that $J \subseteq \text{Int } A$. H is obtained by thickening D slightly. Now, $R + H$ is a polyhedral 3-manifold with boundary in M whose interior contains T_1 and which is simply-connected mod its boundary, since M is simply-connected. By Lemma 6, $R + H$ is a regular free-manifold, and clearly $T_2 \cdot \text{Bd}(R + H)$ has one less component than does $T_2 \cdot \text{Bd } R$.

Suppose next that $\text{Int } D \subseteq \text{Int } R$. If $J = \text{Bd } D$ separates the component of $\text{Bd } R$ containing it, then Lemma 4 states that $R = R_1 + R_2$, where R_1, R_2 are regular free-manifolds such that $R_1 \cdot R_2 = D$. Now, either $T_1 \subseteq \text{Int } R_1$ or $T_1 \subseteq \text{Int } R_2$, say $T_1 \subseteq \text{Int } R_1$. There is a piecewise linear homeomorphism, h , of M onto M which is fixed except near D , and which pushes $\text{Bd } R_1$ slightly to one side of D in such a manner that $T_1 \subseteq \text{Int } h(R_1)$ and $T_2 \cdot \text{Bd}(h(R_1))$ has fewer components than does $T_2 \cdot \text{Bd } R$.

Finally, it might happen that $\text{Int } D \subseteq \text{Int } R$, but $J = \text{Bd } D$ does not separate the component of $\text{Bd } R$ containing it. Let H be a polyhedral cube in R such that $H \cdot T_2 = D$, $\text{Int } D \subseteq \text{Int } H$, and $(\text{Bd } H) \cdot (\text{Bd } R)$ is an annulus ring A containing J in its interior. By Lemma 5, the closure of $R - H$ is the required regular free-manifold R' . This completes Lemma 7.

The following will permit a relaxation of certain polyhedral assumptions in Theorems 1, 2, and 3. Theorem 1 is proved in §5, immediately after Lemma 8. If P is a topological polyhedron in a 3-manifold M , then P is *tame* if there is a homeomorphism of M onto M taking P onto a polyhedral subset of M . If P is a topological 2-manifold which separates M and U is a component of $M - P$, then P is *tame from the U side* if $\bar{U}$ is a 3-manifold with boundary.

LEMMA 8. *Let J be a polyhedral simple closed curve in the interior of a 3-manifold M , and suppose that there is a regular free-manifold R (not necessarily tame) in $\text{Int } M$ which contains J in such a way that at least one of the following holds:*

- (a) J lies trivially in R ;
- (b) J lies in R in a homologically trivial manner.

Then, there is a polyhedral regular free-manifold R' containing J in its interior so that J lies in R' in the same manner as it did in R , with respect to the properties (a) and (b).

Proof.

1. Exactly as in [6], one may suppose that $\text{Bd } R$ is locally polyhedral mod $J \cdot \text{Bd } R$ (i.e., except possibly at these points). For, by [10], there is a homeomorphism h of R onto a polyhedron in E^3 such that h is locally piecewise linear at each point of $\text{Int } R$. One may then shrink $h(\text{Bd } R)$ slightly into

$h(\text{Int } R)$ at each point of $h(\text{Bd } R - J)$, moving no point of $h(J \cdot \text{Bd } R)$, to obtain a regular free-manifold R^* in E^3 such that $h(J) \subseteq R^* \subseteq h(R)$, $\text{Bd } R^* \cdot h(\text{Bd } R) = h(J \cdot \text{Bd } R) = h(J) \cdot \text{Bd } R^*$, and $\text{Bd } R^*$ is locally polyhedral mod $h(J \cdot \text{Bd } R)$. Then, $h^{-1}(R^*)$ will have the desired properties.

2. Suppose now that J meets $\text{Int } R$. There is a polyhedral solid torus T of genus 1 such that $J \subseteq \text{Int } T$, J circles T once longitudinally, and $\text{Bd } T$ and $\text{Bd } R$ are in general position. One may regard T as a closed ϵ -neighborhood of J , for sufficiently small ϵ . Since there are uncountably many such ϵ 's from which to choose, and since $\text{Bd } T$ and $\text{Bd } R$ will fail to be in general position for only countably many of these, the last requirement is permissible. Further, since $J \cdot \text{Int } R \neq \emptyset$, one can choose ϵ so small that some handle of T lies in $\text{Int } R$. Hence, each of the components of $\text{Bd } T \cdot \text{Bd } R$ (these are polygonal simple closed curves) circles T longitudinally no times. Also, it may be supposed that each of these curves bounds a disk on $\text{Bd } R$.

3. No component of $\text{Bd } T \cdot \text{Bd } R$ circles T meridinally (once) in $\text{Bd } T$. For, suppose this were the case for some component K , and let J^* be a polygonal simple closed curve in $\text{Int } R$ which is close to J in the sense that there is a homeomorphism of J^* onto J which moves no point of J^* more than some small positive number δ . One can choose δ so small that J^* circles each handle of R the same number of times as does J , and so that J^* lies in $\text{Int } T$ and circles T once longitudinally. Now, J^* bounds a compact, polyhedral 2-manifold D such that $D \subseteq \text{Int } R$. Since $K \subseteq \text{Bd } R$, D misses K . Also, a slight adjustment of T suffices to bring $\text{Bd } T$ and $\text{Int } D$ into general position, while preserving the other positional properties of $\text{Bd } T$. The simple closed curve K is deformed slightly by this adjustment onto another polygonal simple closed curve, which also circles T once meridinally and lies in $\text{Bd } T \cdot \text{Bd } R$. Thus, it may be assumed that no such adjustment of T is necessary. Now, each component of $\text{Bd } T \cdot \text{Int } D$ is a polygonal simple closed curve which does not circle T longitudinally, and so bounds a disk in T . Hence, J^* bounds a 2-complex in T , which is impossible.

Note that each component of $\text{Bd } T \cdot \text{Bd } R$ bounds a disk in $\text{Bd } T$ and a disk in $\text{Bd } R$.

4. A slice can be removed from T along a handle of T contained in $\text{Int } R$, so as to obtain a polyhedral cube H such that $J \cdot \text{Bd } H$ is 2 points a and b , an arc from a to b in J contains $J \cdot \text{Bd } R$, $\text{Bd } H \cdot \text{Bd } R = \text{Bd } T \cdot \text{Bd } R$, and no simple closed curve in $\text{Bd } H \cdot \text{Bd } R$ separates a from b in $\text{Bd } H$. If there is more than 1 simple closed curve in this last set, the excess ones may be removed by a sequence of operations, each operation consisting of attaching a cube to R or cutting R in half (see the proof of Lemma 7). Note that neither of these destroys properties (a) or (b). If $\text{Bd } H \cdot \text{Bd } R$ has only one component, then $R + H$ is the required R' . This completes the proof in case $J \cdot \text{Int } R \neq \emptyset$.

5. Suppose now that $J \subseteq \text{Bd } R$, where $\text{Bd } R$ is locally polyhedral mod J , and J is polyhedral. It will be shown in step 5 that $\text{Bd } R$ is, in fact, tame.

There are annulus rings A_i ($i=1, 2$) in $\text{Bd } R$ such that $A_1 \cdot A_2 = \text{Bd } A_1 \cdot \text{Bd } A_2 = J$ and A_i is locally polyhedral mod J . By Lemma 2.1 of [11], A_i is tame. By Lemma 5.2 of [11], $A_1 + A_2$ is tame, so that $\text{Bd } R$ is locally tame at each of its points and, by Theorem 8.1 of [11], is tame. Although the 2 lemmas of [11] used here are stated for the case where M is E^3 , they hold also for the more general situation.

The truth of Lemma 8 is now evident.

QUESTION. It seems likely that the 3-manifold R obtained in step 1 of the proof of Lemma 8 is tame. If so, some of the rather artificial restrictions of that lemma could be dropped. In particular, suppose that S is a 2-sphere in E^3 , U is a component of $E^3 - S$, and S is tame from the U side. If G is a polyhedral finite graph in $\bar{U}$ such that S is locally polyhedral mod $G \cdot S$, is S tame?

5. Proof of Theorem 1.

THEOREM 1. *Let M be a compact, connected 3-manifold such that each polyhedral simple closed curve can be shrunk to a point in a regular free-manifold in M . Then, M is homeomorphic to S^3 .*

FIG. 1(a)

Proof. According to Lemma 1, M has a special cellular decomposition $T = \{\Delta_j\}$, so that each point of the connected, polyhedral finite graph T_1 is of order two or four. The proof is broken into nine steps.

1. One proceeds exactly as in [6] to approximate T_1 with a polyhedral simple closed curve J . That is, if p_j is a point of T_1 or order 4, there is a polyhedral 3-cell C_j containing p_j in its interior, meeting T_1 in 4 points a_j, b_j, c_j, d_j ,

FIG. 1(b)

FIG. 1(c)

and such that no 2 C_j 's intersect. The required simple closed curve J contains $T_1 - \sum \text{Int } C_i$, is contained in $T_1 + \sum \text{Int } C_i$ and meets C_j in arcs $a_j b_j$ and $c_j d_j$ (see Figures 1(a), 1(b), and 1(c)). The arc $a_j b_j$ is knotted here exactly as in [6], while the arc $c_j d_j$ is unknotted, and the two are linked in such a way that $c_j d_j$ is *not* homotopic in $C_j - a_j b_j$ to any arc in $\text{Bd } C_j$. (See Lemma 6 of [6].) Note, however, that $a_j b_j$ is homotopic in $C_j - c_j d_j$ to an arc in $\text{Bd } C_j$.

2. Let R be a polyhedral regular free-manifold containing J in its interior, such that J can be shrunk to a point in R , and such that $\text{Bd } R$ and $\sum \text{Bd } C_i$ are in general position. Such exists by Lemma 8. Each component of $\text{Bd } R \cdot \sum \text{Bd } C_i$ is a polygonal simple closed curve. It will be assumed that if R' is a polyhedral regular free-manifold in M containing J in its interior such that J can be shrunk to a point in R' , and $\text{Bd } R'$, $\sum \text{Bd } C_i$ are in general position, then $\text{Bd } R' \cdot \sum \text{Bd } C_i$ has no fewer components than does $\text{Bd } R \cdot \sum \text{Bd } C_i$. This last set is assumed to be nonempty. The case where it is empty is covered in step 9.

3. There is no curve K in $\text{Bd } R \cdot \sum \text{Bd } C_i$ which bounds a disk D in one of the surfaces $\text{Bd } C_j$ so that D misses J . For, if so, K could be chosen so that $\text{Bd } R \cdot \text{Int } D = \emptyset$. One then proceeds as in the proof of Lemma 7 to reduce the number of components of $\text{Bd } R \cdot \sum \text{Bd } C_i$. That is, in case $R \cdot \text{Int } D = \emptyset$ one adds a cube to R ; in case $\text{Int } D \subseteq \text{Int } R$ and K separates some component of $\text{Bd } R$, one cuts R into 2 pieces, keeping the half that contains J and adjusting it slightly near D ; and in case $\text{Int } D \subseteq \text{Int } R$ but K separates no component of $\text{Bd } R$, one removes a handle from R . In any event, the resulting regular free-manifold would contradict the assumption in step 2.

4. Let j be any integer for which $\text{Bd } R \cdot \text{Bd } C_j \neq \emptyset$. There is a simple closed curve K_j in this intersection which bounds a disk D_j in $\text{Bd } C_j$ for which $\text{Bd } R \cdot \text{Int } D_j = \emptyset$. Since J can be shrunk to a point in R , D_j must contain an even number of a_j, b_j, c_j, d_j , and by step 3 this number must be 2. Then, $\text{Int } D_j \subseteq \text{Int } R$. There is another simple closed curve K'_j in $\text{Bd } R \cdot \text{Bd } C_j$ which bounds a disk D'_j in $\text{Bd } C_j$ such that $\text{Int } D'_j \subseteq \text{Int } R$, $\text{Int } D_j \cdot \text{Int } D'_j = \emptyset$, and D'_j contains the other two of a_j, b_j, c_j, d_j . Note that, by step 3, each component of $\text{Bd } R \cdot \text{Bd } C_j$ separates $J \cdot D_j$ from $J \cdot D'_j$ in $\text{Bd } C_j$.

5. Let j be as in step 4, and suppose that there is a polygonal arc α_j from a point x_j in $\text{Int } D_j$ to a point y_j in $\text{Int } D'_j$ such that $\text{Int } \alpha_j \subseteq \text{Int } C_j$, $\alpha_j \subseteq \text{Int } R$, and α_j is unknotted in C_j (i.e., α_j is contained in a polyhedral disk in C_j whose intersection with $\text{Bd } C_j$ is its boundary). Then, there is a polyhedral regular free-manifold R' whose boundary is in general position with respect to $\sum \text{Bd } C_i$, whose interior contains $J + C_j$ and is such that, if $i \neq j$, then $\text{Bd } C_i \cdot \text{Bd } R'$ has the same number of components as does $\text{Bd } C_i \cdot \text{Bd } R$. The manifold R' is obtained by deforming R as explained in the next paragraph.

There is a polyhedral cube H in C_j such that $\text{Int } \alpha_j \subseteq \text{Int } H$, $\text{Bd } H \cdot \text{Bd } C_j$ consists of 2 disjoint disks E and F , where $E \subseteq \text{Int } D_j$, $\text{Int } E \supseteq x_j + J \cdot D_j$ and $F \subseteq \text{Int } D'_j$, $\text{Int } F \supseteq y_j + J \cdot D'_j$, and $H \subseteq \text{Int } R$. There is a piecewise linear homeomorphism h of M onto M which is fixed on $E + F + \alpha_j$ and outside a small neighborhood of C_j and which takes H onto C_j . The required R' is $h(R)$. By applying, if necessary, a finite sequence of such homeomorphisms to R , it may be assumed that R^* is a polyhedral regular free-manifold containing J in its interior such that $\text{Bd } R^*$ and $\sum \text{Bd } C_i$ are in general position and if $\text{Bd } R^* \cdot \text{Bd } C_j \neq \emptyset$, then each component of this intersection separates $c_j + d_j$ from $a_j + b_j$ in $\text{Bd } C_j$. Note that it is not required that J lie trivially in R^* . The notation is chosen so that $c_j + d_j \subseteq D'_j$ and $a_j + b_j \subseteq D_j$.

6. It is shown in this step that if K is a component of $\text{Bd } R^* \cdot \sum \text{Bd } C_i$, then K does not separate any component of $\text{Bd } R^*$. For, if so, K could be chosen to bound a disk D in $\text{Bd } R^*$ such that $\text{Int } D \cdot \sum \text{Bd } C_i = \emptyset$. Suppose $K \subseteq \text{Bd } C_j$.

The first possibility is that $\text{Int } D \subseteq \text{Int } C_j - J$. But D must separate the arc $c_j d_j$ from the arc $a_j b_j$ in C_j , since $K = \text{Bd } D$ separates $c_j + d_j$ from $a_j + b_j$

in $\text{Bd } C_j$. Then C_j is the sum of polyhedral cubes H_1 and H_2 such that $H_1 \cdot H_2 = D$, $\text{Int } a_j b_j \subseteq \text{Int } H_1$, and $\text{Int } c_j d_j \subseteq \text{Int } H_2$. Hence, $c_j d_j$ is homotopic in $C_j - a_j b_j$ to an arc in $\text{Bd } C_j$. This is in contradiction to Lemma 6 of [6].

Finally, it might happen that $\text{Int } D \subseteq M - (J + \sum C_i)$. Let D' be a disk in $\text{Bd } C_j$ such that $K = \text{Bd } D'$ and $D' \cdot J = c_j + d_j$. Then, $S = D + D'$ is a polyhedral 2-sphere in M which is pierced by J at two points and otherwise misses J . The component of $J - (c_j + d_j)$ contained in C_j lies in one component of $M - S$, while the other component of $J - (c_j + d_j)$ must lie in the other component of $M - S$. But this is clearly not the case, since any arc in $\text{Int } C_j$ from $\text{Int } c_j d_j$ to $\text{Int } a_j b_j$ will miss S . The assertion at the beginning of step 6 follows. In particular, no simply-connected component of $\text{Bd } R^*$ can intersect $\sum \text{Bd } C_i$.

7. It will be assumed from this point that no component of $\text{Bd } R^*$ lies in any of the sets $\text{Int } C_j$. This is justified by filling the holes in each C_j . That is, suppose S is a component of $\text{Bd } R^*$ such that $S \subseteq \text{Int } C_j$. Let M^* be the closure of the component of $C_j - S$ not containing $\text{Bd } C_j$. Then M^* is a 3-cell or a cube with a tubular hole, $M^* \cdot R^* = \text{Bd } M^* \cdot \text{Bd } R^* = S$ and $M^* + R^*$ is a regular free-manifold.

This last assertion is clear if M^* is a 3-cell. If S is not a 2-sphere, let D be a polyhedral disk in M which forms a handle for R^* , whose boundary is in S , and such that $\text{Int } D$ and $\text{Bd } C_j$ are in general position. It suffices to show that $M^* + D$ can be embedded in E^3 . One does this by using standard techniques to construct a polyhedral punctured cube about $M^* + D$ in M , beginning with C_j .

Here and in step 8, consider a fixed j such that $\text{Bd } R^* \cdot \text{Bd } C_j \neq \emptyset$ and let C be a component of $\text{Bd } R^*$ for which $C \cdot \text{Bd } C_j \neq \emptyset$. Since no component of $C \cdot \text{Bd } C_j$ bounds a disk in C and since $\text{Bd } C_j$ separates M , $C \cdot \text{Bd } C_j$ has at least two components and each component of $C_j \cdot \text{Bd } R^*$ is an annulus ring. Since each component of $\text{Bd } R^* \cdot \text{Bd } C_j$ can be shrunk to a point in $C_j - a_j b_j$ and in $C_j - c_j d_j$, the following holds: any simple closed curve in $C_j \cdot \text{Bd } R^*$ can be shrunk to a point in $C_j - a_j b_j$ and in $C_j - c_j d_j$.

8. If it can now be shown that the arc α_j described at the beginning of step 5 exists for this j , it will follow as at the end of step 5 that there is a polyhedral regular free-manifold whose interior contains J and whose boundary misses $\sum \text{Bd } C_i$. Then, step 9 will apply and the proof will be complete. The existence of such an arc is shown now.

Let $c_j d_j$ be a polygonal arc from c_j to d_j in $\text{Int } D_j'$. It would be convenient if this arc were in the position shown in Figure 1(a). The general situation is indicated in Figure 1(b), where $c_j d_j$ spirals around $\text{Bd } C_j$ several times. However, one may simplify the position of $c_j d_j$ if he is willing to complicate the arcs $a_j b_j$ and $c_j d_j$ somewhat.

Consider now just the cube C_j of Figure 1(b). There is a homeomorphism h of C_j onto itself which wraps the left end of $c_j d_j$ around the left end of $a_j b_j$

as in Figure 1(c), leaves c_j , d_j , and $a_j b_j$ fixed, and takes $c_j d_j$ into the position also indicated in Figure 1(c).

Consider this last figure. There is a polyhedral disk D^* whose interior is in $\text{Int } C_j$, whose boundary is $h(c_j d_j + c_j d_j)$, and which is pierced by $a_j b_j$ as in this figure. One such intersection point, indicated by p , is at the extreme bottom point of $a_j b_j$, and the rest are near $\text{Bd } C_j$. Note that $h^{-1}(\text{Bd } D^*) \subseteq \text{Int } R^*$, so it may be assumed that $h^{-1}(\text{Int } D^*)$ and $\text{Bd } R^*$ are in general position. It will be shown that no component K of $\text{Int } D^* \cdot h(C_j \cdot \text{Bd } R^*)$ separates p from $\text{Bd } D^*$ in D^* . From this, it will be immediate that the required arc can be found in $h^{-1}(D^*) + a_j b_j$. Suppose that one of these curves K bounds a disk D in $\text{Int } D^*$ that contains p . According to step 7, K can be shrunk to a point in $C_j - a_j b_j$. The next paragraph completes step 8 by showing that this is impossible.

It is convenient to consider C_j in E^3 and to note that a loop in $C_j - a_j b_j$ can be shrunk to a point in $C_j - a_j b_j$ if and only if it can be shrunk to a point in $E^3 - J^*$ (see Figure 1(a)). The fundamental group G of $E^3 - J^*$ has generators x and z with the single relation $xzx = zxz$. These generators are indicated in Figure 1(a). If K were trivial in $E^3 - J^*$, then for some integer n the element $x^n z$ would be the identity of G . That this is not the case is shown by the representation ϕ of G defined by:

$$\phi(x) = (12)(3),$$

$$\phi(z) = (13)(2).$$

9. If $\text{Bd } R^* \cdot \sum \text{Bd } C_i = \emptyset$, then, by Lemmas 2 and 3,

$$[R^* - \sum (R^* \cdot \text{Int } C_i)] + \sum C_i$$

is a 3-sphere or a polyhedral regular free-manifold whose interior contains T_1 , and the proof of Theorem 1 is completed by Lemma 7.

As a special case of Theorem 1, the following holds:

THEOREM 2. *Let M be a compact, connected absolute 3-manifold such that each polyhedral simple closed curve in M lies trivially in a solid torus (of genus 1) in M . Then M is topologically S^3 .*

6. Extension of results to homologically trivial 3-manifolds. Theorem 1 was proved under the assumption that the 3-manifold M was simply-connected in a strong sense. Actually, more was assumed than was necessary in the proof of this theorem. This permitted a few simplifications in the proof, however. If M is a compact, connected, 3-manifold for which $H_1(M; Z) = 0$, then by Poincaré duality, $H_2(M; Z) = 0$. It follows that compact, polyhedral 2-manifolds separate M .

The result analogous to Lemma 6 is:

LEMMA 6'. *Let R_1 be a regular free-manifold and R_2 a punctured cube, where*

$R_1 \cdot R_2$ is an annulus ring A in $\text{Bd } R_1 \cdot \text{Bd } R_2$. If $R_1 + R_2$ can be embedded in a 3-manifold M for which $H_1(M; Z) = 0$, then $R_1 + R_2$ is a regular free-manifold.

Note that in the proof of Theorem 1, one needs only Lemma 6', the assumption that J circles each handle of R an even number of times (see step 4 of Theorem 1), and the fact that compact, polyhedral 2-manifolds separate M . Hence, the following holds:

THEOREM 3.⁽²⁾ *Let M be a compact, connected, absolute 3-manifold for which $H_1(M; Z) = 0$ and such that each polyhedral simple closed curve lies in a regular free-manifold in M in a homologically trivial manner. Then, M is homeomorphic to S^3 .*

The interested reader should have no trouble in supplying the details omitted here.

EXAMPLE. It is of interest to note that the condition that $H_1(M; Z)$ be trivial is essential in Theorem 3. That is, there is a compact, connected 3-manifold M such that each polyhedral simple closed curve lies in the interior of a solid torus of genus 1 in M in a homologically trivial manner, yet $H_1(M; Z)$ does not vanish. The following M is a lens space, and is one of many examples with the above properties that could be given.

Let T_i ($i = 1, 2$) be a solid torus of genus 1. Let M be the 3-manifold obtained by identifying $\text{Bd } T_2$ with $\text{Bd } T_1$ in such a way that the meridional simple closed curve J_1 on $\text{Bd } T_2$ circles T_1 three times longitudinally and once meridionally, and the longitudinal simple closed curve J_2 on $\text{Bd } T_2$ circles T_1 twice longitudinally and once meridionally. Note that $H_1(M; Z)$ is Z_3 , the cyclic group of order three. Consider M as having a fixed triangulation.

It will be shown that if K is a simple closed curve in M , then there is a homeomorphism of M onto itself which throws K onto a simple closed curve K_3 in $\text{Int } T_1$ which circles T_1 an even number of times. This homeomorphism will be the composition of three homeomorphisms of M onto M . The first of these is fixed outside a small neighborhood of T_1 , and pushes K onto a simple closed curve K_1 in $\text{Int } T_2$.

Now let N be a small polyhedral tubular neighborhood in M of the longitudinal curve J_2 (N is topologically a solid torus of genus 1). The second homeomorphism is the identity on T_1 and takes K_1 onto K_2 in $\text{Int } N$.

Lastly, there is a homeomorphism of M which is fixed outside a small neighborhood of T_2 and throws N onto a torus in $\text{Int } T_1$ which circles T_1 twice longitudinally. This homeomorphism takes K_2 onto a simple closed curve K_3 in $\text{Int } T_1$ which circles T_1 an even number of times. The property of M asserted above follows.

7. An example. Let M be a compact, connected 3-manifold with boundary. If M can be embedded in a simply-connected 3-manifold, then M is

⁽²⁾ *Added in proof.* J. J. Andrews has pointed out that, except in Lemma 8, one needs only that the simple closed curve circles no handle of the regular free-manifold exactly once.

simply-connected mod its boundary. If each component of $\text{Bd } M$ is a 2-sphere and M is simply-connected mod $\text{Bd } M$, then M can be embedded in a compact, connected simply-connected 3-manifold. Lemma 6 concerns a situation in which a 3-manifold with boundary which is simply-connected mod its boundary can be embedded in E^3 . An example M_1 is given here to show that if one component of $\text{Bd } M_1$ fails to be simply-connected, then M_1 may not be embeddable in any simply-connected 3-manifold, even though M_1 is simply-connected mod $\text{Bd } M_1$ and its boundary is connected and of genus 1. Further, there is a compact, connected 3-manifold M_2 whose boundary is of genus 1, such that M_2 can be embedded in E^3 and the fundamental group of M_2 is isomorphic to the fundamental group of M_1 . The manifold M_2 is a cube with a knotted tubular hole. In [1], Alexander has given examples of topologically different compact, connected absolute 3-manifolds with the same fundamental group.

FIG. 2(a)

FIG. 2(b)

Let T_i ($i=1, 2$) be a double torus (cube with 2 handles). Each of M_1 and M_2 will be obtained by attaching a cube C_i to T_i so that $T_i \cdot C_i = \text{Bd } T_i \cdot \text{Bd } C_i$ is an annulus ring A_i . Let J_i be one component of $\text{Bd } A_i$. The simple closed curve J_i circles T_i as shown in Figures 2(a) and 2(b). The fundamental group of M_i is generated by elements a and b subject to the relation $a^4 b^5 = 1$, since the fundamental group of T_i is the free group on 2 generators a and b .

First, consider Figure 2(b). One can attach a cube C' to M_2 along an annulus ring in such a way that $M_2 + C'$ is a simply-connected manifold

bounded by a 2-sphere, $T_2 \cdot C' = \text{Bd } T_2 \cdot \text{Bd } C'$ is an annulus ring, and $T_2 + C'$ is a solid torus. By Lemma 6, $T_2 + C' + C_2$ is a cube, and $T_2 + C_2 = M_2$ can be embedded in E^3 .

Now, suppose that $M_1 = T_1 + C_1$ is contained in a simply-connected 3-manifold M as a polyhedral subset. It follows from Theorem 1 of [13] and Dehn's Lemma of [12] that there is a polygonal simple closed curve J in $\text{Bd } M_1$ which does not separate $\text{Bd } M_1$ but which bounds a polyhedral disk D in M such that $\text{Bd } M_1 \cdot \text{Int } D = \emptyset$. In case M is S^3 , this is shown in [8]. Now, either $\text{Int } D \subseteq \text{Int } M_1$ or $M_1 \cdot \text{Int } D = \emptyset$. If the former were the case, then M_1 would be obtained by adding a handle to a simply-connected 3-manifold bounded by a 2-sphere, and the fundamental group of M_1 would be infinite cyclic. It is easy, however, to construct examples of nonabelian groups generated by 2 elements a and b satisfying $a^4b^5 = 1$. Hence, $M_1 \cdot \text{Int } D = \emptyset$.

If D is thickened slightly to obtain a polyhedral cube C such that $\text{Int } D \subseteq \text{Int } C$ and $C \cdot M_1 = \text{Bd } C \cdot \text{Bd } M_1 = A$, an annulus ring, then $M_1 + C$ is a compact, connected, simply-connected 3-manifold bounded by a 2-sphere. Hence, the assumption that M_1 can be piecewise linearly embedded in M will be shown to be incorrect if it can be shown that it is impossible to attach a cube to M_1 in this manner so that the resulting manifold is simply-connected.

If this were possible, then there would be a polygonal simple closed curve J in $\text{Bd } T_1 - A_1$ such that $\text{Bd } T_1 - (J + J_1)$ is connected and, if the element of the fundamental group of T_1 corresponding to J is $W(a, b)$, then the only group with generators a and b satisfying $a^4b^5 = 1 = W(a, b)$ is the trivial group. It will be shown that there is no such J .

There are polygonal simple closed curves B and C in $\text{Bd } T_1 - A_1$ such that B circles the left handle of T_1 once, the right handle of T_1 3 times, and C circles the left handle of T_1 4 times, and the right handle not at all. Further, B and C meet at only 1 point, where they cross. Then, B and C generate the fundamental group of $\text{Bd } M_1$, where this last set is a 2-dimensional torus. Recalling that the number of times that a simple closed curve in $\text{Bd } M_1$ circles longitudinally is relatively prime to the number of times it circles meridionally, one finds that $W(a, b)$ can, by use of the relation $a^4b^5 = 1$, be put into the form $(ab^3)^i a^{4j}$, where i and j are relatively prime integers. Hence, if the simple closed curve J described in the previous paragraph were to exist, there would have to be relatively prime integers i and j such that if the relations $a^4b^5 = 1 = (ab^3)^i a^{4j}$ are added to the free group on generators a and b , then the resulting group is trivial.

If the relation that a and b commute is added to the above relations, an abelian group on generators a and b is obtained, where $a^4b^5 = 1 = a^{4j+i}b^{3i}$. This group is trivial if and only if

$$\begin{vmatrix} 4 & 5 \\ 4j+i & 3i \end{vmatrix} = 7i - 20j = \pm 1.$$

Hence, i is of the form $3 + 20x$ and j is of the form $1 + 7x$, where x is an integer. It is not necessary to consider separately the case $i = -3 + 20x$ and $j = -1 + 7x$.

In [7], examples are given to show that if k and m are relatively prime integers, and $k, m, n \neq 1$, then there exist nontrivial groups on elements u and v , where $u^k = v^m = (uv)^n = 1$. Let G_x be such a group for $k=4$, $m=5$, and $n=3+20x$, so that $u^4 = v^5 = (uv)^{3+20x} = 1$. The group G_x is also generated by the elements $a=u$ and $b=v^2$, where $a^4 = b^5 = (ab^3)^{3+20x} = 1$. Thus, for each integer x there is a nontrivial group G_x generated by elements a and b such that $a^4 b^5 = 1 = (ab^3)^{3+20x} a^{4(1+7x)}$. This completes the proof that there is no piecewise linear homeomorphism taking M_1 into a simply-connected 3-manifold M . According to Theorem 10 of [3], there can be no homeomorphism taking M_1 into M .

The 3-manifold M_1 is simply-connected mod $\text{Bd } M_1$, since if G is a group generated by elements x and y such that $x^4 y^5 = xy^3 = x^4 = 1$, then $G = \{1\}$.

QUESTION. The preceding example shows how little can be said of a compact, connected, orientable 3-manifold with boundary M from a knowledge of its fundamental group. However, if this group is a free group and if every compact, polyhedral 2-manifold separates M , then the structure of M is determined mod the Poincaré Conjecture. Perhaps the following can be proved without the Poincaré Conjecture: Let M_1 be a cube with handles and M_2 a cube, where $M_1 \cdot M_2 = \text{Bd } M_1 \cdot \text{Bd } M_2$ is an annulus ring. Suppose that the 3-manifold with boundary $M' = M_1 + M_2$ has a free fundamental group, and that every compact, polyhedral 2-manifold separates M' . Then, M' can be embedded in E^3 .

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