CARTAN DECOMPOSITIONS FOR L^* ALGEBRAS

BY

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1. **Introduction.** In the discussion of L^* algebras given in [5] a classification theory was obtained for the separable simple algebras under the assumption of the existence of a Cartan decomposition relative to some Cartan subalgebra. The main result of this paper is a proof that any semi-simple L^* algebra of arbitrary dimension has such a decomposition relative to any Cartan subalgebra.

In the process of proving this several additional results of interest in themselves are obtained, among them one concerning representations of finite-dimensional semi-simple Lie algebras which seems to be new. This is stated in detail in the second corollary of 4.5. The conclusion obtained in 4.5 also adds a new result to the theory of commutators of operators on a Hilbert space.

2. **Continuous decompositions.**

DEFINITIONS AND NOTATION. An L^* algebra is defined as a Lie algebra L over the complex numbers whose underlying vector space is a Hilbert space and such that for each x in L there exists an element x^* with $([x, y], z) = (y, [x^*, z])$ for all y and z. For an x in L, X (occasionally D_x) will denote the linear operator defined by $Xy = [x, y]$ for all y and we will assume that the norm on L is chosen such that $\|X\| \leqq \|x\|$. An L^* algebra is semi-simple if and only if the mapping $x \to X$ is one-one. For the remainder of this paper L will denote an arbitrary (but fixed) semi-simple L^* algebra unless further restrictions are explicitly stated. As shown in [5] this implies the mapping $x \to x^*$ is a Hilbert space conjugation and anti-multiplicative, D_{x^*} is the adjoint of D_x, and that L is a direct sum of simple L^* ideals. A Cartan subalgebra of L is defined as a maximal abelian self-adjoint subalgebra of L. For subsets M, N of L, $\mathrm{Sp}(M)$ will represent the smallest closed linear subspace of L containing M and $[M, N] = \mathrm{Sp}\{[m, n]: m \in M, n \in N\}$. For subspaces S_1, S_2 the notation $S_1 + S_2$ will be used only when S_1 is orthogonal to S_2.

Suppose A is a bounded self-adjoint operator on L. For λ real and $\epsilon > 0$ let $V(\lambda, \epsilon) = \{x: \|(A-\lambda)^n x\| \leqq \epsilon^n \|x\|, n = 1, 2, \cdots\}$. For a Borel set M of the real numbers let $V(M, \epsilon) = \mathrm{Sp}\{V(\lambda, \epsilon): \lambda \in M\}$ and $V(M) = \bigcap_{\epsilon > 0} V(M, \epsilon)$. It is proved in [1, pp. 66–69] that $V(\lambda, \epsilon)$ is a closed subspace and equal to the set of x such that the sequence $\{(\epsilon^{-1}(A-\lambda))^n x\}$ is bounded. Furthermore, if E is the real spectral measure such that $A = \int \lambda dE$ then the range of $E(M)$ is equal to $V(M)$ for M compact. For any Borel set M the range of $E(M)$ will be denoted by $S(M)$. Finally, for Borel sets M and N let $M + N$

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$=\{m+n: m\in M,\ n\in N\}$ and $-M=\{-m: m\in M\}$. Then $M+N$ and $-M$ are also Borel sets.

2.1. Suppose A is a bounded self-adjoint derivation on L and M, N are Borel sets of the real line. Then $[S(M),\ S(N)]\subset S(M+N)$ and $S(M)^* = S(-M)$.

Proof. Suppose first that M and N are compact and $\epsilon>0$. Let $M=\bigcup M_k$ and $N=\bigcup N_k$ where $\{M_k\}$, $\{N_k\}$ are sequences of disjoint Borel sets, each of diameter less than $(1/2)\epsilon$. Let $x\in S(M)$ and $y\in S(N)$. Then $x=\sum x_j$, $y=\sum y_k$ where $x_j\in S(M_j)$, $y_k\in S(N_k)$ and $[x,\ y]=\sum[x_j,\ y_k]$. Suppose $\lambda_j\in M_j$, $\mu_k\in N_k$. Then it follows from the spectral theorem that $\|(A-\lambda_j)^n x_j\| \leqq (2^{-n}\epsilon^n)\|x_j\|$ and $\|(A-\mu_k)^n y_k\|\leqq(2^{-n}\epsilon^n)\|y_k\|$ for each positive integer n. Since A is a derivation it follows by induction on n that $\|(A-(\lambda_j+\mu_k))^n[x_j, y_k]\| =\|\sum_0^n {}_nC_m[(A-\lambda_j)^m x_j,\ (A-\mu_k)^{n-m}y_k]\|\leqq \sum_0^n {}_nC_m\|(A-\lambda_j)^m x_j\|\ \|(A-\mu_k)^{n-m}y_k\| \leqq \sum {}_nC_m 2^{-n}\epsilon^n\|x_j\|\ \|y_k\|=\epsilon^n\|x_j\|\ \|y_k\|$. Hence the sequence

$$\{(\epsilon^{-1}(A - (\lambda_j + \mu_k))^n[x_j,\ y_k]\}$$

is bounded and this implies $[x_j, y_k]\in V(\lambda_j+\mu_k,\ \epsilon)$. Thus $[x, y]\in V(M+N,\ \epsilon)$. Since ϵ was arbitrary, $[x,\ y]\in V(M+N)$. The compactness of M and N implies $M+N$ is also and hence $[x,\ y]\in S(M+N)$. Thus $[S(M),\ S(N)]$ is a subset of $S(M+N)$ for M and N compact.

It is proved in [1] that E is regular, i.e., for any Borel sets M and N, $E(M)=\sup\{E(C): C\subset M,\ C \text{ compact}\}$ and similarly for $E(N)$. For $C\subset M$ and $D\subset N$ with C and D compact we have $[S(C), S(D)]\subset S(C+D) \subset S(M+N)$. Letting C vary gives $[S(M), S(D)]\subset S(M+N)$. Letting D vary gives $[S(M),\ S(N)]\subset S(M+N)$.

Since A is a derivation, $[A, D_x]=D_{Ax}$ for all x and hence $D_{(Ax)}=[A, D_x]^* =[D_x^*, A^*]=-[A, D_{x^*}]=D_{-Ax^*}$ so that $(Ax)^*=-Ax^*$ for all x in L. Using this, a proof like that above can be constructed to prove the second assertion.

Notation. Suppose $\mathfrak{H}$ is a closed self-adjoint abelian subalgebra of L. Let $\mathfrak{A}=\mathfrak{A}(\mathfrak{H})$ be the commutative C^* algebra of bounded operators generated by $\{H: h\in\mathfrak{H}\}$. Since each H is zero on $\mathfrak{H}$ then the identity operator is not in $\mathfrak{A}$. Let $\Delta=\Delta(\mathfrak{H})$ be the set of all homomorphisms of $\mathfrak{A}$ into the complex numbers. For $A\in\mathfrak{A}$ let $\hat{A}$ be the function on Δ defined by $\hat{A}(\alpha)=\alpha(A)$. If Δ is given the weakest topology making these functions continuous then Δ is a compact Hausdorff space and the theory of C^* algebras shows that the mapping $A\rightarrow\hat{A}$ is an algebraic isomorphism mapping $\mathfrak{A}$ isometrically onto the set of all continuous functions on Δ vanishing at the zero homomorphism with A^* corresponding to the complex conjugate of $\hat{A}$. The set Δ with its topology will be called the spectrum of $\mathfrak{H}$. It is also known that the spectrum of an operator $A\in\mathfrak{A}$ is the range of the function $\hat{A}$.

2.2. For each $\alpha\in\Delta$ there is a unique $x_\alpha\in\mathfrak{H}$ such that $\alpha(H)=(h,\ x_\alpha)$ for all $h\in\mathfrak{H}$. Also $\|x_\alpha\|\leqq 1$ and $x_\alpha^*=x_\alpha$. If $\{x_\alpha\}$ is given the induced weak topology of $\mathfrak{H}$ then Δ is homeomorphic to $\{x_\alpha\}$ under the mapping $\alpha\rightarrow x_\alpha$.

Proof. All except the last statement were proved in [5]. The family $\{\hat{H}: h\in\mathcal{H}\}$ separates points of Δ and all members vanish at infinity (zero). By Theorem 5G of [3], the topology of Δ is that generated by the family. But this is clearly equivalent to the weak topology on $\{x_\alpha\}$, finishing the proof.

There is a unique spectral measure E on the Borel sets of Δ such that $(Ax, y)=\int\hat{A}(\alpha)d(E(\alpha)x, y)$ for all x, y in L. Also it is easily seen that the range of $E(0)$ is $\{x: [\mathcal{H}, x]=0\}$ where $\{0\}$ denotes the Borel set consisting of the zero homomorphism. For an arbitrary Borel set M of Δ let $S(M)$ denote the range of $E(M)$.

Suppose $\alpha, \beta\in\Delta$. If $x_\alpha+x_\beta=x_\gamma$ for some $\gamma\in\Delta$, let γ be denoted by $\alpha+\beta$. If $x_\alpha=-x_\gamma$ for some $\gamma\in\Delta$, let γ be denoted by $-\alpha$. Using this notation we have the following theorem which is the continuous version of the desired composition for L relative to $\mathcal{H}$.

2.3. Suppose M, N are Borel sets of Δ and $M+N=\{m+n: m\in M, n\in N\}$. Then $M+N$ is a Borel set and $[S(M), S(N)]\subset S(M+N)$. If $-M=\{-m: m\in M\}$ then $S(-M)=S(M)^*$.

Proof. Choose a set $\{x_i: i\in I\}$ of elements of $\mathcal{H}$ such that the set spans $\mathcal{H}$ and $x_i^*=x_i$ for each i. Let σ_i be the spectrum of x_i. Then σ_i is compact so that $P=\prod_i\sigma_i$ is compact. For $\alpha\in\Delta$ let $f(\alpha)$ be the element of P whose ith coordinate is (x_i, x_α). Then f is a homeomorphism of Δ onto a compact subset of P. If addition is defined in P (whenever possible) in the obvious coordinatewise fashion then f preserves the algebraic structure of Δ as well as the topology. The spectral measure E can be defined directly on P (hence on $f(\Delta)$ and Δ) by constructing the product measure obtained from the E_i's where $X_i=\int_{\sigma_i}\lambda dE_i$. The measure-theoretic details will be omitted here but a discussion of this type of problem may be found in [2].

A subset M of P will be called a rectangle if and only if $M=\prod M_i$ where M_i is a Borel set of σ_i and $M_i=\sigma_i$ for all but a finite number of indices. Then the Borel sets of P will coincide with the σ-algebra generated by the rectangles. In fact, Chapter 7 of [2] shows that this σ-algebra is obtained as the smallest monotone class containing all finite unions of disjoint rectangles. 2.1 can be used to prove 2.3 for the case when M, N are finite unions of disjoint rectangles. The collection of all sets for which 2.3 holds is clearly a monotone class, hence contains all Borel sets.

2.4. Corollary. *Suppose L is separable and $\mathcal{H}$ is a closed self-adjoint abelian subalgebra of L such that $\{\|x_\alpha\|: \alpha\in\Delta, x_\alpha\neq 0\}$ is bounded away from zero. Then there exist disjoint Borel sets M_k of Δ, $k=0, 1, \cdots$ such that $M_0=\{0\}$, $L=\sum S(M_k)$, and $[S(M_k), S(M_k)^*]\subset\{x: [\mathcal{H}, x]=0\}$.*

Proof. L separable implies $\{x_\alpha: \alpha\in\Delta\}$ is a separable metric space in the norm topology, hence contains a countable dense subset. Suppose $\|x_\alpha\|>c>0$ for all $\alpha\neq 0$. Let $\alpha_0=0$ and let $\{x_{\alpha_k}: k\geqq 1\}$ be a countable dense subset of

$\{x_\alpha\}-\{0\}$. If $N_k=\{\beta: \|x_\beta-x_{\alpha_k}\|\leqq 3^{-1}c\}$ then $N_0=\{0\}$ and $\Delta=\bigcup N_k$. Furthermore, since $\{x_\alpha\}$ is weakly compact, N_k is weakly closed, hence a Borel set. If α, $\beta\in N_k$ the triangle inequality implies $\|x_\alpha-x_\beta\|<c$ so that either $\alpha=\beta$ or $\alpha-\beta\notin\Delta$. Hence $N_k+(-N_k)=\{0\}$. In the usual way it is possible to choose Borel sets M_k such that $M_k\subset N_k$ and Δ is the disjoint union of the M_k's. Then $[S(M_k), S(M_k)^*]\subset S(M_k+(-M_k))\subset S(\{0\})$ and the remaining statements follow easily from the spectral theory.

3. **Nilpotent elements.**

3.1. There exists a nonzero element $a\in L$ such that $(a, a^*)=0$ and $A^3=0$.

Proof. Let x be a self-adjoint element of L with $\|X\|=1$ and let $X=\int\lambda dE$. If M is the real interval $(2/3, 1]$ and V is the range of $E(M)$ then $V\neq 0$ and, since V^* is the range of $E(-M)$, $(V, V^*)=0$. Using 2.1, $[V, [V, [V, L]]]=0$. Thus a may be chosen as any element of V different from zero.

Notation. For this section we choose a fixed a having the properties listed in 3.1 and let $b=a^*$, $c=[a, b]$. Then $C=C^*$ and $A^3=B^3=0$. This section is devoted to an analysis of the L^* subalgebra generated by a, culminating in 3.8. This turns out to be a key result in the general existence proof for Cartan decompositions.

3.2. For any x in L, $A^2XA=AXA^2$ and $B^2XB=BXB^2$.

Proof. $[A, [A, [A, X]]]=0$ and $A^3=0$ together imply $-3A^2XA+3AXA^2=0$. The second equation follows from the first by taking adjoints.

3.3. Suppose x is any element in the closure of the range of A^2. Then $X^3=0$.

Proof. By continuity of the adjoint representation it is sufficient to prove this for the case $x=A^2z$ for some z in L. Then $X=[A, [A, Z]]=A^2Z-2AZA+ZA^2$. Using 3.2, $A^2ZA=AZA^2$ so that $A^2ZA^2=0$. A direct computation then shows that $X^2=A^2Z^2A^2$ and $X^3=0$.

3.4. Suppose n is any positive integer. Then

(a) $C^na=-A^2(BA)^{n-1}b=(AB)^na$,

(b) $C^nb=(-1)^{n-1}B^2(AB)^{n-1}a=(-1)^n(BA)^nb$.

Proof. Since $C=C^*$, $(C^nx)^*=(-1)^nC^nx^*$ for any x. Thus (b) follows from (a) by using adjoints.

The first equation of (a) is proved by induction. For $n=1$, $Ca=[[a, b], a]=-A^2b$. Assuming the result for n, $C^{n+1}a=C(-A^2(BA)^{n-1}b)=-(AB-BA)A^2(BA)^{n-1}b=-ABA^2(BA)^{n-1}b=-A^2(BA)^nb$, after using 3.2.

For the second, repeated application of 3.2 gives $C^na=-(AB)^{n-1}A^2b=(AB)^na$.

Corollary. *For all integers* $n\geqq 0$,

(a) $AC^na=BC^nb=0$,

(b) $ABC^na=C^{n+1}a$, $BAC^nb=-C^{n+1}b$,

(c) $B^2C^na=(-1)^nC^{n+1}b$, $A^2C^nb=(-1)^{n+1}C^{n+1}a$.

3.5. Let $S_0=\mathrm{Sp}\{a, b\}$, $S_n=\mathrm{Sp}\{D_1\cdots D_ns: s=a, b; D_i=A, B\}$ for $n=1, 2, \cdots$. Let $S=\mathrm{Sp}\{S_n: n=0, 1, \cdots\}$.

(a) S is the L^* subalgebra generated by a.

(b) $S_{2n}=\mathrm{Sp}\{C^na, C^nb\}$, $n=0, 1, \cdots$.

(c) $S_{2n+1}=\mathrm{Sp}\{BC^na, AC^nb\}$, $n=0, 1, \cdots$.

(d) $(S_{2m}, S_{2n+1})=0$, $m, n=0, 1, \cdots$.

Proof. (a) It is clear that $S_n=S_n^*$ and hence $S=S^*$. Since $AS_n\subset S_{n+1}$, $BS_n\subset S_{n+1}$, then S is invariant under A and B. Hence S is invariant under X for any x in the L^* subalgebra S' generated by a, i.e., $[S', S]\subset S$. But clearly $S\subset S'$. Hence $[S, S]\subset S$ and therefore $S=S'$.

(b) and (c) are true for $n=0$. Suppose they hold for some n. Then

$$\begin{aligned} S_{2n+2} &= \mathrm{Sp}\{AS_{2n+1}, BS_{2n+1}\} = \mathrm{Sp}\{ABC^na, A^2C^nb, B^2C^na, BAC^nb\} \\ &= \mathrm{Sp}\{C^{n+1}a, C^{n+1}b\}, \end{aligned}$$

using the corollary of 3.4. Hence

$$S_{2n+3} = \mathrm{Sp}\{AS_{2n+2}, BS_{2n+2}\} = \mathrm{Sp}\{AC^{n+1}b, BC^{n+1}a\}$$

since $AC^{n+1}a=BC^{n+1}b=0$. Thus, by induction on n, (b) and (c) are true for all n.

(d) $(C^ma, BC^na)=(AC^ma, C^na)=0$ and

$$\begin{aligned} (C^ma, AC^nb) &= (-1)^n(C^ma, A(BA)^nb) = (-1)^n(C^ma, (AB)^nAb) \\ &= (-1)^{n+1}(C^{n+m}a, Ba) = (-1)^{n+1}(AC^{n+m}a, a) = 0. \end{aligned}$$

Similarly $(C^mb, AC^nb)=(C^mb, BC^na)=0$, completing the proof of (d).

3.6. Letting n range over the non-negative integers, let $V_0=\mathrm{Sp}\{S_{2n+1}\}$, $V_1=\mathrm{Sp}\{C^na\}$, $V_2=\mathrm{Sp}\{C^nb\}$.

(a) $S=V_0+V_1+V_2$.

(b) $V_0=V_0^*$, $V_1^*=V_2$, $V_2^*=V_1$.

(c) $[V_1, V_1]=[V_2, V_2]=0$.

(d) $[V_1, V_1^*]=[V_2, V_2^*]=V_0$.

Proof. (a) It only remains to prove that V_1 is orthogonal to V_2. Now $(a, b)=0$ and, if either n or m is nonzero, then $(C^na, C^mb)=-(A^2(BA)^{n+m-1}b, b)$ $=-((BA)^{n+m-1}b, B^2b)=0$. Hence $(V_1, V_2)=0$.

(b) $V_1^*=\mathrm{Sp}\{(C^na)^*\}=\mathrm{Sp}\{C^nb\}=V_2$. Similarly $V_2^*=V_1$. Since $S^*=S$, $V_0^*=V_0$.

(c) It is sufficient to prove $[V_1, V_1]=0$ or $[C^na, C^ma]=0$ for all m and n. This is done by induction on n. The case $n=0$ is given by the corollary of 3.4. Suppose $[C^{n-1}a, C^ma]=0$ for all m. Then $C^n[a, C^ma]=C^n0=0$. By Leibniz's rule, $0=[C^na, C^ma]+$terms of the form $[C^pa, C^qa]$ where $p<n$. Each of these latter terms is zero by the induction hypothesis.

(d) $V_0=\mathrm{Sp}\{[a, C^nb], [b, C^na]\}$ implies $V_0\subset[V_1, V_1^*]$. But $(V_1, [V_1, V_1^*])$ $=([V_1, V_1], V_1)=0$ implies $[V_1, V_1^*]$ is orthogonal to V_1. Similarly, since $[V_1, V_1^*]=[V_2, V_2^*]$, $[V_1, V_1^*]$ is also orthogonal to V_2, hence must be a subset of V_0.

3.7. $[c, V_0]=0$.

Proof. By using adjoints it is sufficient to prove $CAC^nb=0$ for all n. Since C is self-adjoint it is sufficient to prove $C^2AC^nb=0$. Using the appropriate cases of the corollary of 3.4,

$$\begin{aligned} CAC^nb &= (AB-BA)AC^nb = A(BAC^nb) - B(A^2C^nb) \\ &= -AC^{n+1}b + (-1)^nBC^{n+1}a. \end{aligned}$$

Hence

$$\begin{aligned} C^2AC^nb &= -A(BAC^{n+1}b) + B(A^2C^{n+1}b) + (-1)^nAB^2C^{n+1}a + (-1)^{n+1}BABC^{n+1}a \\ &= AC^{n+2}b + (-1)^nBC^{n+2}a + (-1)^{2n+1}AC^{n+2}b + (-1)^{n+1}BC^{n+2}a = 0. \end{aligned}$$

Corollary 1. *$[C^na, C^mb]=(-1)^n[a, C^{n+m}b]$ for all n, m.*

Proof. $[C^na, C^mb]\in V_0$ by 3.6. Also we may assume n is positive. Then

$$0 = C[C^{n-1}a, C^mb] = [C^na, C^mb] + [C^{n-1}a, C^{m+1}b]$$

so that $[C^na, C^mb]=-[C^{n-1}a, C^{m+1}b]$. Applying this repeatedly gives the result in general.

Corollary 2. *$[V_0, V_i]=V_i$ for $i=1, 2$.*

Proof. It is sufficient to prove this for $i=1$. Now $[V_0, C^na]=C^n[V_0, a]$ and V_1 is invariant under C so it is enough to prove $[V_0, a]\subset V_1$ in order to prove $[V_0, V_1]\subset V_1$. But $AV_0=\mathrm{Sp}\{A^2C^nb, ABC^ma\}\subset V_1$ by the corollary of 3.4.

For the reverse inclusion, suppose $x\in V_1$ and $(x, [V_0, V_1])=0$. Then $([x, x^*], V_0)=0$ so that $[x, x^*]=0$ and X is normal. To prove x must be zero it is sufficient to show that X is nilpotent. In fact, for future reference, we will prove that $X^3=0$ for all $x\in V_1$. Since $V_1=\mathrm{Sp}\{C^na: n\geqq 0\}$, it is clear that $\mathrm{Sp}\{C^na: n\geqq 1\}$ is either all of V_1 or a hyperplane in V_1. In the first case V_1 is contained in the closure of the range of A^2 (see 3.4) and the assertion follows from 3.3. In the other case an element x of V_1 must be of the form $x=\mu a+y$ where μ is a scalar and y is in the closure of the range of A^2. A proof like that of 3.3 then shows that $A^2Y=AY^2=0$. Since $A^3=Y^3=0$, the binomial theorem gives $X^3=0$.

Corollary 3. *$[V_0, V_0]=0$.*

Proof. Suppose $x\in V_0$ and $(x, AC^nb)=0$ for all n. Then $([b, x], C^nb)=0$ for all n implies $[b, x]=0$ since $[b, x]\in V_2$. But then $0=C^nXb=XC^nb$ and this implies X is zero on V_2 so that $(x, V_0)=(x, [V_2, V_2^*])=0$ and hence x is zero. Thus $V_0=\mathrm{Sp}\{AC^nb\}$. Now $0=[C^nb, C^mb]$ implies $A^2[C^nb, C^mb]=0$ so that $[A^2C^nb, C^mb]+2[AC^nb, AC^mb]+[C^nb, A^2C^mb]=0$. The sum of the first and last terms on the left side is $(-1)^{n+1}[C^{n+1}a, C^mb]+(-1)^{m+1}[C^nb, C^{m+1}a]$ which is zero by Corollary 1. Thus $[AC^nb, AC^mb]=0$ and, since $V_0=\mathrm{Sp}\{AC^nb\}$, V_0 is abelian.

Corollary 4. *S is semi-simple with V_0 as a Cartan subalgebra.*

Proof. Suppose $x \in S$ and $[x, V_0]=0$. Then $0=([x, V_0], V_i)=(x, V_i)$ for $i=1, 2$ implies $x \in V_0$ and V_0 is maximal abelian. To show that S is semi-simple suppose that $x \in S$ and $[x, S]=0$. Then $x \in V_0$ and $[x, b]=0$. The proof of Corollary 3 shows that x must be zero.

3.8. Suppose S is a semi-simple L^* algebra and $S=V_0+V_1+V_2$ with V_0 as a Cartan subalgebra and that the relations of 3.6 hold. Then S is a direct sum of three-dimensional ideals I_j where $I_j=\mathrm{Sp}\{e_j, e_j^*, [e_j, e_j^*]\}$ for some nonzero $e_j \in V_1$.

Proof. The decomposition theorem of [5] for semi-simple algebras shows that S can be written as the direct sum of simple ideals I_j where $I_j=H_j+[H_j, S]$ for some closed self-adjoint subspace H_j of V_0. We choose a fixed I_j and let $U_0=H_j$, $U_1=[H_j, V_1]$, $U_2=[H_j, V_2]$. Then $I_j=U_0+U_1+U_2$ and it is clear that $[U_i, U_i]=0$ for each i, $[U_0, U_i]=U_i$ for $i=1, 2$, while $[U_i, U_i^*]=U_0$ for $i=1, 2$.

Suppose $U_1=P+Q$ where P and Q are closed subspaces invariant under U_0. Then $([[U_1, U_1^*], P], Q)=0$ implies $([U_1, U_1^*], [Q, P^*])=0$ so that $[Q, P^*]=0$. Since $[Q, P]$ is also zero, it follows that $U_0=[P+Q, P^*+Q^*]$ $=[P, P^*]+[Q, Q^*]$ and, furthermore, that $XY=0$ (on S) for all $x \in [P, P^*]$ and $y \in [Q, Q^*]$. Referring to the proof in [5] of the decomposition theorem and using the simplicity of I_j we must have $[P, P^*]=0$ or $[Q, Q^*]=0$. But every element of U_1 is nilpotent on S so that necessarily either $P=0$ or $Q=0$. Hence U_1 contains no nontrivial closed subspaces invariant under U_0. By the spectral theorem U_1 must be one-dimensional and this completes the proof.

COROLLARY 1. *The L^* algebra generated by a is a direct sum of three-dimensional ideals.*

COROLLARY 2. *There exists a nonzero element $x \in L$ such that $X^3=0$ and $[[x, x^*], x]=\lambda x$ with λ positive. In fact $\lambda\|x\|^2=\|[x, x^*]\|^2$.*

Proof. Let I_j be a simple ideal of S as above and let $x=e_j$. Then $[[x, x^*], x]$ $=\lambda x$ for some λ. Hence $\lambda\|x\|^2=([[x, x^*], x], x)=\|[x, x^*]\|^2$. Since $x \in V_1$, the proof of Corollary 2 of 3.7 shows that $X^3=0$. Thus $[x, x^*]\neq 0$ and λ must be different from zero.

4. **A commutator equation.** For this section, A will denote a fixed nonzero bounded operator on a Hilbert space such that $[[A, A^*], A]=\lambda A$ for some $\lambda\neq 0$. From this considerable information about the spectra of AA^*, A^*A, and $[A, A^*]$ can be obtained. Also there are some interesting consequences for representations of Lie algebras as bounded operators on a Hilbert space. By assuming 4.1 and Corollary 1 of 4.2 much of what is done here is valid for elements A, A^* of an arbitrary algebra with identity over a field of characteristic zero.

4.1. A is nilpotent.

Proof. The mapping $B\to[[A, A^*], B]$ is a derivation on the set of all bounded operators and has norm not exceeding $2\|[A, A^*]\|$. Hence $[[A, A^*], A^n]=n\lambda A^n$ for n a positive integer. Then $n\lambda A^n=[[A, A^*], A^n]$ implies $|n|\ |\lambda|\ \|A^n\|\leqq 2\|[A, A^*]\|\ \|A^n\|$ so that A^n is zero for some n.

4.2. $[A^*, A^n]=nA^{n-1}[A^*, A]-(\lambda/2)(n)(n-1)A^{n-1}$ for $n=1, 2, \cdots$.

Proof. The case $n=1$ is trivial. Assuming the equation for n gives

$$\begin{aligned}[A^*, A^{n+1}] &= A^n[A^*, A] + [A^*, A^n]A \\ &= A^n[A^*, A] + nA^{n-1}[A^*, A]A - (\lambda/2)(n)(n-1)A^n \\ &= A^n[A^*, A] + nA^n[A^*, A] - n\lambda A^n - (\lambda/2)(n)(n-1)A^n \\ &= (n+1)A^n[A^*, A] - (\lambda/2)(n)(n+1)A^n.\end{aligned}$$

COROLLARY 1. *λ is real and positive and* $[[A^*, A], A^*]=\lambda A^*$.

Proof. Choose n such that $A^n\neq 0$ but $A^{n+1}=0$. Then $0=[A^*, A^{n+1}]=(n+1)A^n[A^*, A]-(\lambda/2)(n)(n+1)A^n$ implies $A^n(A^*A-n(\lambda/2))=0$. Since $A^n\neq 0$, $A^*A-n(\lambda/2)$ does not have a bounded inverse, hence $n(\lambda/2)$ is in the spectrum of the positive operator A^*A and λ must be positive. Since λ is real, taking adjoints of both sides of the equation $[[A, A^*], A]=\lambda A$ gives the second assertion.

COROLLARY 2. $[A^{*n}, A]=nA^{*n-1}[A^*, A]+(\lambda/2)(n)(n-1)A^{*n-1}$.

Proof. Taking adjoints of both sides of the equation in 4.2 gives

$$\begin{aligned}[A^{*n}, A] &= n[A^*, A]A^{*n-1} - (\lambda/2)(n)(n-1)A^{*n-1} \\ &= nA^{*n-1}[A^*, A] + \lambda n(n-1)A^{*n-1} - (\lambda/2)(n)(n-1)A^{*n-1}.\end{aligned}$$

COROLLARY 3. *AA^* commutes with A^*A.*

Proof. Since $[A, A^*]=AA^*-A^*A$, it is sufficient to prove AA^* commutes with $[A, A^*]$. But

$$[[A, A^*], AA^*] = [[A, A^*], A]A^* + A[[A, A^*], A^*] = \lambda AA^* - \lambda AA^* = 0.$$

4.3. For each non-negative integer n let $B_n=A^nA^{*n}$ and $D_n=A^{*n}A^n$.

(a) $B_nAA^*=(1/n+1)B_{n+1}+(n/n+1)B_nA^*A+n(\lambda/2)B_n$.

(b) $D_nA^*A=(1/n+1)D_{n+1}+(n/n+1)D_nAA^*+n(\lambda/2)D_n$.

(c) For all $n, m\geqq 0$, B_n and D_n commute with B_m and D_m.

Proof. (a)

$$\begin{aligned}B_nAA^* &= A^nA^{*n}AA^* = A^{n+1}A^{*n+1} + A^n[A^{*n}, A]A^* \\ &= B_{n+1} + A^n(nA^{*n-1}[A^*, A] + (\lambda/2)(n)(n-1)A^{*n-1})A^* \\ &= B_{n+1} + (\lambda/2)(n)(n-1)B_n + nA^nA^{*n-1}[A^*, A]A^* \\ &= B_{n+1} + (\lambda/2)(n)(n-1)B_n + n\lambda A^nA^{*n} + nA^nA^{*n}[A^*, A] \\ &= B_{n+1} + (\lambda/2)(n)(n+1)B_n + nB_nA^*A - nB_nAA^*.\end{aligned}$$

Solving for B_nAA^* gives the assertion in (a).

(b) Because of the symmetry between A and A^* the proof is like that for (a).

(c) If either n or m is zero the result is immediate. By Corollary 3 of 4.2, B_1 commutes with D_1. Using equations (a) and (b) an induction on n shows that B_n and D_n are polynomials in B_1 and D_1 and this gives (c).

4.4. Let p, q be non-negative integers and $n=p+q$. Then

$$B_pD_qAA^* = (q + 1/n + 1)B_{p+1}D_q + (p/n + 1)B_pD_{q+1} + (\lambda/2)(p)(q + 1)B_pD_q.$$

Proof. The case $q=0$ is given by equation (a) of 4.3 and the case $p=0$ reduces to $D_nAA^*=AA^*D_n$. Thus we may assume both p and q are positive. Now $B_pD_qAA^*=(B_pAA^*)D_q=((1/p+1)B_{p+1}+(p/p+1)B_pA^*A+p(\lambda/2)B_p)D_q$ implies

$$\text{(1)}\quad B_pD_qAA^* = (1/p + 1)B_{p+1}D_q + p(\lambda/2)B_pD_q + (p/p + 1)B_pD_qA^*A.$$

But, using equation (b) of 4.3,

$$B_p(D_qAA^*) = B_p((q + 1/q)D_qA^*A - (1/q)D_{q+1} - (\lambda/2)(q + 1)D_q)$$

which gives

$$\text{(2)}\quad B_pD_qAA^* = - (1/q)B_pD_{q+1} - (\lambda/2)(q + 1)B_pD_q + (q + 1/q)B_pD_qA^*A.$$

Using (1) and (2) to eliminate the term $B_pD_qA^*A$ gives the conclusion.

COROLLARY. *For n a positive integer $(AA^*)^n$ is a linear combination of the B_pD_q where $1\leqq p\leqq n$, $0\leqq q\leqq n$.*

Proof. For $n=1$, $AA^*=B_1D_0$. 4.4 and an induction on n gives the result in general.

4.5. Let n be the greatest integer such that $A^n\neq 0$. Then

$$(AA^*)^n \prod (AA^* - (\lambda/2)(p)(q + 1)) = 0$$

where the product is taken over all pairs p, q with $1\leqq p\leqq n$, $0\leqq q\leqq n$.

Proof. $(AA^*)^n$ is a linear combination of terms of the form B_pD_q with $1\leqq p\leqq n$, $0\leqq q\leqq n$. Thus it is sufficient to show that for each such pair p, q, $B_pD_q\prod(AA^*-(\lambda/2)(k)(m+1))=0$ where the product is taken over all pairs k, m with $p\leqq k\leqq n$, $q\leqq m\leqq n$. If we define the degree of B_pD_q as $p+q$ then $B_pD_q(AA^*-(\lambda/2)(p)(q+1))=(q+1/p+q+1)B_{p+1}D_q+(p/p+q+1)B_pD_{q+1}$ (by 4.4) and hence is a sum of terms of degree greater than that of B_pD_q. If the degree of B_pD_q is n each of the terms on the right is zero since $A^{n+1}=A^{*n+1}=0$. In general an induction on the terms of higher degree will yield the conclusion.

COROLLARY. *AA^* and A^*A have finite spectra contained in the set $\{k(\lambda/2): k=0, 1, \cdots, n(n+1)\}$. $[A, A^*]$ has spectrum contained in this set and its negatives.*

Proof. Let $\mathfrak{A}$ be the commutative C^* algebra generated by AA^* and A^*A.

For any homomorphism α of $\mathfrak{A}$ onto the complex numbers the value of α at AA^* must satisfy the same polynomial relation as AA^*, proving the first part. Because of the symmetry between A and A^*, A^*A must also satisfy a polynomial identity like that of 4.5. Since $[A, A^*]=AA^*-A^*A$, a similar argument applies here.

COROLLARY. *Suppose L is a finite-dimensional semi-simple complex Lie algebra with $\mathfrak{H}$ as a Cartan subalgebra and $\{h_\alpha, e_\alpha: \alpha \text{ a root}\}$ is a Weyl basis of L relative to $\mathfrak{H}$. Let σ be the associated involution and $x^*=-\sigma(x)$ for all x in L. Suppose ϕ is a representation of L as bounded operators on a Hilbert space with $\phi(x^*)=\phi(x)^*$ for all x.*

(a) *If $[x, x^*]=0$ then $\phi(x)$ is diagonalizable with finite spectrum.*

(b) *The eigenvalues of $\phi(h_\alpha)$ are integer multiples of $(1/2)\alpha(h_\alpha)$.*

(c) *$\phi(\mathfrak{H})$ is diagonalizable.*

(d) *$\phi(e_\alpha)$ is nilpotent.*

Proof. For each α, $h_\alpha=[e_\alpha, e_\alpha^*]$ and $[h_\alpha, e_\alpha]=\alpha(h_\alpha)e_\alpha$ together with the first corollary give (b). (d) is a consequence of this and 4.1. (c) is true since $\phi(\mathfrak{H})$ is spanned by the finitely many diagonalizable operators $\phi(h_\alpha)$ which are mutually commutative. If $[x, x^*]=0$ then x is contained in some Cartan subalgebra of the L^* algebra L and this subalgebra is spanned by elements of the form $[f_\beta, f_\beta^*]$ where f_β is a root vector relative to it so that the arguments used in (a) and (b) can be used to prove $\phi(x)$ is diagonalizable.

A slightly improved version of 4.5 for a special case will be needed later and this is proved below.

4.6. Suppose A is equal to D_x for some a in L and $A^3=0$. Then the spectrum of $[A, A^*]$ lies in the set $\{k(\lambda/2): k=0, 1, 2, -1, -2\}$.

Proof. By 3.2, $A^2A^*A=AA^*A^2$ and, using the argument in the proof of the first corollary of 4.2, each of these is equal to λA^2. Because of symmetry similar relations hold with A and A^* interchanged.

Now $(AA^*)^2 = A^2A^{*2}+A[A^*,A]A^* = A^2A^{*2}+AA^*[A^*,A]+\lambda AA^*$ which implies $2(AA^*)^2=A^2A^{*2}+AA^{*2}A+\lambda(AA^*)^2$. From these two relations a direct computation shows that $2(AA^*)^3-3\lambda(AA^*)^2+\lambda^2AA^*=0$ and (by symmetry) that the same relation holds for A^*A. Then an argument like that used in proving the corollary of 4.5 will finish the proof.

5. Reduction to the separable case.

DEFINITION. Let $\mathfrak{H}$ be a Cartan subalgebra of L. If L' is a semi-simple subalgebra of L, L' will be called regular (with respect to $\mathfrak{H}$) if and only if L' is separable and $\mathfrak{H}'=\mathfrak{H}\cap L'$ is a Cartan subalgebra of L'. It will be proved here that if each regular L' has a Cartan decomposition with respect to the accompanying $\mathfrak{H}'$ then L has a decomposition with respect to $\mathfrak{H}$. For x in L let $M(x)$ denote the smallest closed subspace of L containing x and invariant under $\mathfrak{H}$. Then $M(x)=\mathrm{Sp}\{V_n: n=0, 1, \cdots\}$ where $V_0=\mathrm{Sp}\{x\}$, $V_n=[\mathfrak{H}, V_{n-1}]$ for $n\geqq 1$.

5.1. Let x be fixed and let B be the bounded operator on $\mathfrak{H}$ defined by

$(Bh, h') = (Hx, H'x)$ for h, h' in $\mathfrak{H}$. Then B is self-adjoint and completely continuous.

Proof. $(Bh, h') \geqq 0$ implies $B = B^*$. Let E be the spectral measure on the spectrum of $\mathfrak{H}$ such that $(Hy, z) = \int (h, x_\alpha) d(E(\alpha)y, z)$ for h in $\mathfrak{H}$ and y,z in L. Then $(Bh, h') = \int (h, x_\alpha)(x_\alpha, h') d(E(\alpha)x, x)$. If $\{h_n\}$ converges weakly to h and $\{h_n'\}$ to h' then both sequences are bounded and the Lebesgue dominated convergence theorem implies (Bh_n, h_n') converges to (Bh, h'). By [4, Definition 2, p. 206], B is completely continuous.

5.2. For x in L let $\mathfrak{H}'(x) = \{h: h \in \mathfrak{H}, [h, x] = 0\}$ and let $\mathfrak{H}(x)$ be the orthogonal complement of $\mathfrak{H}'(x)$ in $\mathfrak{H}$. If x is self-adjoint then so are $\mathfrak{H}(x)$ and $M(x)$ and both are separable.

Proof. It is clear that $\mathfrak{H}'(x)$ is self-adjoint and hence the same is true of $\mathfrak{H}(x)$. Since V_0 is self-adjoint, induction on n proves that each V_n is also and hence $M(x)$ is.

Let $h' \in \mathfrak{H}$. By the definition of the operator B in 5.1, $(Bh, h') = 0$ for all h if and only if $H'x = 0$. Hence $\mathfrak{H}'(x)$ is the null-space of B and, since B is self-adjoint, $\mathfrak{H}(x)$ is the closure of the range of B. Since B is completely continuous, the reference in [4] shows that $\mathfrak{H}(x)$ must be separable. Then an induction on n proves that $V_n = [H(x), V_{n-1}]$ and that each V_n is separable so that $M(x)$ is separable.

5.3. Suppose x is self-adjoint, nonzero, and orthogonal to $\mathfrak{H}$. Let L' be the L^* algebra generated by $\mathfrak{H}(x) + M(x)$. Then $\mathfrak{H}(x) = \mathfrak{H} \cap L'$, L' is regular, $[\mathfrak{H}'(x), L'] = 0$, and $(\mathfrak{H}'(x), L') = 0$.

Proof. Since the orthogonal complement of $\mathfrak{H}$ is invariant under $\mathfrak{H}$ and contains x it also contains $M(x)$ so that the indicated sum is actually direct. Since $\mathfrak{H}(x)$ and $M(x)$ are separable and self-adjoint it is possible to choose a countable (or finite) orthogonal basis of the space $\mathfrak{H}(x) + M(x)$, say $\{e_n\}$, such that each e_n is self-adjoint. Then a proof like that for 3.5 (a) shows that L' is spanned by products of the form $E_{i_1} \cdots E_{i_k} e_n$ and, since the set of these is countable, L' is separable.

For $h' \in \mathfrak{H}'(x)$, H' is zero on $\mathfrak{H}(x) + M(x)$, hence on a set of generators of L'. Since H' is a derivation, H' is zero on L', proving $[\mathfrak{H}'(x), L'] = 0$. Now $(\mathfrak{H}'(x), e_n) = 0$ for each n and, since $[\mathfrak{H}'(x), L'] = 0$, it follows readily that $\mathfrak{H}'(x)$ is orthogonal to each of the finite products of generators, hence is orthogonal to L'. This implies $\mathfrak{H}(x) = \mathfrak{H} \cap L'$.

Finally, if $y \in L'$ and $[y, \mathfrak{H}(x)] = 0$ then $[y, \mathfrak{H}] = 0$ so that $y \in \mathfrak{H}$ and hence $y \in \mathfrak{H}(x)$. Thus $\mathfrak{H}(x)$ is a maximal abelian subalgebra of L'. If $y \in L'$ and $[y, L'] = 0$ then $y \in \mathfrak{H}(x) \cap \mathfrak{H}'(x)$ implies y is zero. Thus L' is semi-simple and $\mathfrak{H}(x)$ is a Cartan subalgebra of L'.

5.4. Suppose every regular subalgebra L' of L has a Cartan decomposition with respect to $\mathfrak{H}' = \mathfrak{H} \cap L'$. Then L has a decomposition with respect to $\mathfrak{H}$.

Proof. Let K be the L^* subalgebra of L obtained by letting $K = \mathfrak{H} + V$ where V is the span of all the nonzero root vectors of L relative to $\mathfrak{H}$. It is sufficient to prove $K = L$. Now K is invariant under $\mathfrak{H}$ so that K', the orthogonal complement of K in L, is also invariant. Furthermore, $K'^* = K'$ and if $K' \neq 0$ there is a nonzero self-adjoint element $x \in K'$. Then $M(x) \subset K'$. Let L' be the L^* subalgebra generated by $\mathfrak{H}(x) + M(x)$. By 5.3 L' is regular so that the hypothesis here implies L' has a Cartan decomposition with respect to $\mathfrak{H}(x)$. Since $M(x)$ is invariant under $\mathfrak{H}(x)$ it will be spanned by root vectors of $\mathfrak{H}(x)$ and hence there is a nonzero v in $M(x)$ which is a common eigenvector for all H, $h \in \mathfrak{H}(x)$. But if $h' \in \mathfrak{H}'(x)$ then $H'v = 0$ so that it follows immediately that v is a common eigenvector for $\mathfrak{H}$. Since $v \in M(x) \subset K'$ this gives the desired contradiction.

6. Existence of Cartan decompositions.

REMARK. It will be proved here that if L is simple and separable there is a Cartan subalgebra $\mathfrak{H}$ of L such that L as a decomposition with respect to $\mathfrak{H}$. Hence L must be one of the five types A, A', B, C, D obtained in [5]. Since each of these is a Lie subalgebra of an H^* algebra, Theorem 2 of [5] shows that L has a decomposition with respect to any Cartan subalgebra. From this it is clear that any separable semi-simple L^* algebra has a decomposition with respect to any Cartan subalgebra. Finally, 5.4 shows that this is true with no restriction on the dimension of L.

6.1. Suppose a_1, a_2 are self-adjoint elements of L and $A_1A_2 = 0$. Then either $a_1 = 0$ or $a_2 = 0$.

Proof. Since A_i is self-adjoint, A_2A_1 is also zero. Let C_i be the null-space of A_i and R_i the closure of the range of A_i, $i = 1, 2$. Then $L = C_i + R_i$ and both C_i and R_i are self-adjoint. Let $I_i = \mathrm{Sp}\{R_i, [R_i, R_i]\}$. Since $A_1A_2 = 0$ then $(R_1, R_2) = 0$. Also $[[a_1, L], [a_2, L]] = [[[a_1, L], a_2], L] + [a_2, [[a_1, L], L]] = [a_2, [[a_1, L], L]] \subset R_2$. Similarly $[[a_1, L], [a_2, L]] \subset R_1$. From this we have $[R_1, R_2] \subset R_1 \cap R_2 = 0$. Hence the Jacobi identity gives $[I_1, I_2] = 0$. From the above it is easy to see that I_1 is orthogonal to I_2. Let W be the orthogonal complement of $I_1 + I_2$. Then $(W, R_i) = 0$ implies $W \subset C_1 \cap C_2$. This in turn implies that R_i is invariant under W. But $([W, R_i], R_i) = (W, [R_i, R_i]) = 0$ so that $[W, R_i] = 0$ and hence $[W, I_i] = 0$. Since $L = W + I_1 + I_2$ it follows immediately that I_i is an ideal of L. By the simplicity of L either I_1 or I_2 must be zero. Now $A_iL \subset R_i \subset I_i$ so that either A_1 or A_2 is zero.

NOTATION. By Corollary 2 of 3.8 there exists an element a of L such that $A^3 = 0$, $\|a\| = 1$, and $[[a, a^*], a] = \lambda a$ where $\lambda = \|[a, a^*]\|^2 \neq 0$. Thus $[[A, A^*], A] = \lambda A$ and 4.6 implies $L = V_0 + V_{\lambda/2} + V_{-\lambda/2} + V_\lambda + V_{-\lambda}$ where V_μ is the eigenspace for $[A, A^*]$ with the indicated subscript as eigenvalue. The usual relations hold between these subspaces, i.e., $[V_\mu, V_\nu] \subset V_{\mu+\nu}$ and $V_\mu^* = V_{-\mu}$ for each μ. In particular, $X^3 = 0$ for all $x \in V_\lambda$.

6.2. Let $S = [V_\lambda, V_\lambda^*] + V_\lambda + V_\lambda^*$. Then S is a semi-simple L^* algebra.

Proof. Clearly $S^* = S$ and $[V_\lambda, V_\lambda] = [V_\lambda^*, V_\lambda^*] = 0$. Since V_λ, V_λ^* are both invariant under $[V_\lambda, V_\lambda^*]$, so is $[V_\lambda, V_\lambda^*]$. From this it follows that S is a subalgebra. If $x \in S$ and $[x, S] = 0$ then $[A, A^*]x = 0$ implies $x \in [V_\lambda, V_\lambda^*]$. But $(x, [V_\lambda, V_\lambda^*]) = 0$ since $[x, V_\lambda] = 0$. Thus x is zero and S is semi-simple.

DEFINITION. By Zorn's Lemma it is possible to choose a subset $\mathfrak{F}$ of V

which is maximal with respect to the following properties:

(i) $b \in \mathfrak{F}$ implies $\|b\|=1$ and $[[b, b^*], b]=\lambda_b b$ ($\lambda_b=\|[b, b^*]\|^2$).

(ii) $b, c \in \mathfrak{F}$ implies $[[b, b^*], [c, c^*]]=0$.

Necessarily a is in $\mathfrak{F}$ since $[V_\lambda, V_\lambda^*] \subset V_0$. Let $M=\mathrm{Sp}\{[b, b^*]: b \in \mathfrak{F}\}$. Then M is self-adjoint and abelian. Let $C(M)=\{x: x \in S, [x, M]=0\}$.

6.3. Let Δ be the spectrum of M (acting on S) and suppose $\alpha \in \Delta$ with α nonzero. Then $\|x_\alpha\| \geqq (1/2)\lambda^{1/2}$.

Proof. For any b in $\mathfrak{F}$, $([b, b^*], [a, a^*])=\lambda$ implies $\lambda \leqq \|[a, a^*]\| \, \|[b, b^*]\| =\lambda^{1/2}\lambda_b^{1/2}$ so that $\lambda \leqq \lambda_b$. Since α is not zero there is a $b \in \mathfrak{F}$ such that $(x_\alpha, [b, b^*]) \neq 0$. By the first corollary of 4.5 the spectrum of $[B, B^*]$ consists of integer multiples of $(1/2)\lambda_b$. Thus we must have $\|x_\alpha\| \, \|[b, b^*]\| \geqq (1/2)\lambda_b$ which gives $\|x_\alpha\| \geqq (1/2)\lambda_b^{1/2}$.

COROLLARY. *There exist subspaces V_k of S, invariant under $C(M)$, such that $S=C(M)+\sum V_k$ and $[V_k, V_k^*] \subset C(M)$.*

Proof. The existence of the V_k's is implied by 2.4. Since they are spectral subspaces they are invariant under all operators commuting with $\{X: x \in M\}$, hence invariant under $C(M)$.

6.4. $M=C(M)$ and M is a Cartan subalgebra of S.

Proof. For x in $C(M)$, $[A, A^*]x=0$ implies $x \in V_0$, hence $x \in [V_\lambda, V_\lambda^*]$. Thus V_λ is invariant under $C(M)$. Then if $W_k=V_\lambda \cap V_k$ we have W_k is invariant under X for any x commuting with M, $V_\lambda=\sum W_k$, and $[W_k, W_k^*] \subset C(M)$.

Suppose $c \in W_k$. Then W_k is invariant under $[C, C^*]$. Hence $\mathrm{Sp}\{X^n c: X=[C, C^*], n=0, 1, \cdots\} \subset W_k$. Since $C^3=0$, the proof of 3.8 shows that there exists an orthonormal set $\{e_i\} \subset W_k$ such that $[[e_i, e_i^*], e_i] =\lambda_i e_i$, $[e_i, e_j]=[e_i, e_j^*]=0$ for $i \neq j$, and $c=\sum c_i e_i$. Then $[c, c^*]=\sum |c_i|^2 [e_i, e_i^*]$. By the maximality of $\mathfrak{F}$, each $e_i \in \mathfrak{F}$ since $[e_i, e_i^*] \in C(M)$. Hence $[e_i, e_i^*] \in M$ so that $[c, c^*] \in M$.

For future reference we will now prove that if x is any element of L with $[x, M]=0$ and $(x, M)=0$ then $[x, V_\lambda]=0$. To see this note first that $x \in V_0$ (since $[a, a^*] \in M$) and this implies V_λ and V_k are invariant under X so that W_k is also invariant. If $c \in W_k$ then $[c, c^*] \in M$ so that $0=(x, [c, c^*])=(Xc, c)$. Since the operator X on W_k is completely determined by the quadratic form (Xc, c) this gives X is zero on W_k for each k so that X is zero on V_λ.

The preceding paragraph shows that if $x \in C(M)$ and $(x, M)=0$ then $[x, V_\lambda]=0$. Now M is self-adjoint which implies the same for $C(M)$ and thus $x^* \in C(M)$, $(x^*, M)=0$ so that we also have $[x^*, V_\lambda]=0$ and this gives $[x, V_\lambda^*]=0$. But then $[x, [V_\lambda, V_\lambda^*]]$ is also zero and this implies X is zero on S so that $x=0$. Hence $M=C(M)$ so that M is maximal abelian in S, hence a Cartan subalgebra of S.

6.5. S has a Cartan decomposition with respect to M.

Proof. Using the notation of 6.4 we now have $V_\lambda=\sum W_k$ with $[W_k, W_k^*]$

$\subset M$. Also $[W_k, W_k]=0$. Let k be fixed and let $S_1=[W_k, W_k^*]+W_k+W_k^*$. Then it is easily seen that S_1 is an L^* subalgebra since W_k is invariant under M. Let P be the projection of S onto S_1 and $a_0=P[a, a^*]$. Then for $z\in S_1$, $[a_0, z]=[P[a, a^*], z]=P[[a, a^*], z]=[[a, a^*], z]$ since Z and Z^* leave S_1 invariant. Hence if $[z, S_1]=0$ then $[z, [a, a^*]]=0$ so that $z\in[W_k, W_k^*]$. But $(z, [W_k, W_k^*])=0$ since $[z, W_k]=0$, hence z is zero and S_1 is semi-simple. Now $[W_k, W_k^*]\subset M$ implies $[W_k, W_k^*]$ is abelian and the proof above shows that it is maximal abelian in S_1, thus a Cartan subalgebra. By 3.8, S_1 is a direct sum of simple ideals I_j where $I_j=\mathrm{Sp}\{e_i, e_i^*, [e_i, e_i^*]\}$ for some e_i in W_k. If $x\in M$ then $[x, S_1]\subset S_1$ and X is zero on S_1 if and only if it is zero on W_k which is equivalent to $(x, [W_k, W_k^*])=0$. From this it follows readily that each e_i is a common eigenvector for all X, $x\in M$. Thus each W_k, and therefore V_λ, is spanned by root vectors for M. By symmetry the same is true of V_λ^*. Since S is generated by V_λ and V_λ^*, and since each finite product of eigenvectors for M is again an eigenvector for M, S is spanned by eigenvectors and this completes the proof.

Definition. Let $\mathfrak{F}_1$ be a set in $V_{\lambda/2}$ which is maximal with respect to the following properties:

(i) $b\in\mathfrak{F}_1$ implies $\|b\|=1$, $[[b, b^*], b]=\lambda_b b$,

(ii) $b, c\in\mathfrak{F}_1$ implies $[[b, b^*], [c, c^*]]=0$,

(iii) $b\in\mathfrak{F}_1$ implies $[[b, b^*], M]=0$.

Let $\mathcal{H}_0=\mathrm{Sp}\{[b, b^*]: b\in\mathfrak{F}_1\cup\mathfrak{F}\}$. Then $M\subset\mathcal{H}_0$ and $\mathcal{H}_0$ is abelian and self-adjoint. Let $\mathcal{H}=\{x: [x, \mathcal{H}_0]=0\}$. Since $[a, a^*]\in\mathcal{H}_0$, $\mathcal{H}\subset V_0$. We will show that $\mathcal{H}$ is the desired Cartan subalgebra. Note that V_μ is invariant under $\mathcal{H}$ for each eigenspace V_μ of $[A, A^*]$.

6.6. Let Δ be the spectrum of $\mathcal{H}_0$ and suppose α is a nonzero element of Δ. Then $\|x_\alpha\|\geqq(1/4)\lambda^{1/2}$. Hence $L=\mathcal{H}+\sum V_k$ where $[V_k, V_k^*]\subset\mathcal{H}$ and V_k is invariant under $\mathcal{H}$.

Proof. For $b\in\mathfrak{F}_1$, $([b, b^*], [a, a^*])=(1/2)\lambda$ implies $\lambda_b^{1/2}\lambda^{1/2}\geqq(1/2)\lambda$ so that $\lambda_b\geqq(1/4)\lambda$. The remainder of the argument is like that of 6.3 and the corollary.

6.7. $\mathcal{H}$ is abelian and is a Cartan subalgebra of L.

Proof. If $\mathcal{H}$ is abelian it is necessarily maximal abelian so that we need only prove the first assertion.

Let $W_k=V_{\lambda/2}\cap V_k$. Then W_k is invariant under $\mathcal{H}$ and $V_{\lambda/2}=\sum W_k$. Also $[W_k, W_k^*]\subset\mathcal{H}$. Suppose $\mathcal{H}$ is not abelian. Then $[\mathcal{H}, \mathcal{H}]$ is a semi-simple L^* subalgebra and $(\mathcal{H}_0, [\mathcal{H}, \mathcal{H}])=0$. Thus there exists a $w\in[\mathcal{H}, \mathcal{H}]$ with $\|w\|=1$ and $[[w, w^*], w]=\mu w$. Also $([w, w^*], \mathcal{H}_0)=0$.

Now $[W, W^*]$ has spectrum contained in the set $\{r\mu\}$ where r is a half-integer (by the first corollary of 4.5). Let $T_{r\mu}$ be the eigenspace associated with the value $r\mu$. Choose a fixed W_k. Since W_k is invariant under $[W, W^*]$, $W_k=\sum Z_{r\mu}$ where $Z_{r\mu}$ is the intersection of W_k with $T_{r\mu}$.

Suppose $r\neq 0$. Then $[Z_{r\mu}, Z_{r\mu}]\subset T_{2r\mu}\subset\mathrm{Range}\ [W, W^*]$. But $[Z_{r\mu}, Z_{r\mu}]$

$\subset V_\lambda$. Since $([w, w^*], M)=0$, the remark made in the proof of 6.4 shows that $[W, W^*]$ is zero on V_λ and thus V_λ is contained in the null-space of $[W, W^*]$. Hence $[Z_{r\mu}, Z_{r\mu}]=0$.

Now suppose some $Z_{r\mu}\neq 0$ for $r\neq 0$. For this r let $S_1=[Z_{r\mu}, Z^*_{r\mu}]+Z_{r\mu}+Z^*_{r\mu}$. Since $[Z_{r\mu}, Z^*_{r\mu}]\subset \mathfrak{K}\cap$ null-space $[W, W^*]$ then $Z_{r\mu}=W_k\cap T_{r\mu}$ is invariant under $[Z_{r\mu}, Z^*_{r\mu}]$ so that it is easy to see that S_1 is an L^* subalgebra. Using the technique of projecting $[w, w^*]$ onto S_1, the proof of 6.5 can be used to show that S_1 is semi-simple. Let $c\in S_1$. Then $C^3=0$ on S_1. Hence it follows as in 6.4 that $c=\sum c_i e_i$ where $\{e_i\}$ is an orthonormal set in $Z_{r\mu}\subset W_k$ with $[[e_i, e_i^*], e_i]=\lambda_i e_i$ and $[c, c^*]=\sum |c_i|^2[e_i, e_i^*]$. Now $[e_i, e_i^*]\in[W_k, W_k^*]\subset\mathfrak{K}$. By the maximality of $\mathfrak{F}_1$ this implies $[e_i, e_i^*]\in\mathfrak{K}_0$. Thus $[c, c^*]\in\mathfrak{K}_0$ so that $0=([w, w^*], [c, c^*])=([[w, w^*], c], c)$. Since $Z_{r\mu}$ is invariant under the operator $[W, W^*]$ we must have $[W, W^*]$ is zero on $Z_{r\mu}$. But this implies $Z_{r\mu}$ is zero for all $r\neq 0$ and hence $[W, W^*]$ is zero on $V_{\lambda/2}$ as well as on V_λ. Using the self-adjointness of $[W, W^*]$ this implies $[W, W^*]$ is zero on $V^*_{\lambda/2}$ and V^*_λ so that $[W, W^*][A, A^*]=0$. By 6.1 we must have $[W, W^*]=0$ and, since w is an eigenvector for $[W, W^*]$, this gives $w=0$, finishing the proof.

6.8. $\mathfrak{K}$ has at least one nonzero root.

Proof. $[\mathfrak{K}, V_\lambda]\subset V_\lambda$ and $M\subset\mathfrak{K}$. If $x\in\mathfrak{K}$ and x is orthogonal to M the remark in the proof of 6.4 shows that X is zero on V_λ. Now, using 6.5, V_λ is spanned by eigenvectors for M and it is immediate that each of these will be a root vector for $\mathfrak{K}$, corresponding to a nonzero root.

6.9. Suppose L' is one of the simple algebras A, A', B, C, D discussed in §3 of [5]. If e_α is a normalized root vector corresponding to some root α then $E_\alpha^3=0$. There exists an e_α in L' such that the eigenspace corresponding to the maximal eigenvalue of $[E_\alpha, E_\alpha^*]$ is finite-dimensional.

Proof. It is easily seen that $\beta+3\alpha$ is never a root for any root β in the same root set as α and this implies $E_\alpha^3=0$. Hence the spectrum of $[E_\alpha, E_\alpha^*]$ is contained in the set $\{r(\alpha, \alpha): r=0, 1/2, -1/2, 1, -1\}$. Now the eigenspace in question is spanned by root vectors e_β with $(\alpha, \beta)=(\alpha, \alpha)$. Using the notation of [5], the roots are obtained from the set $\{\lambda_i-\lambda_j, \lambda_i+\lambda_j, 2\lambda_i, \lambda_i: i, j \text{ integers}\}$ with $(\lambda_i, \lambda_j)=\delta_{i,j}$. Furthermore, L' must contain a root of the form $\lambda_i-\lambda_j$. Taking $\alpha=\lambda_i-\lambda_j$, $(\alpha, \beta)=(\alpha, \alpha)$ implies β is contained in the finite set $\{\lambda_i-\lambda_j, 2\lambda_i\}$.

6.10. L has a Cartan decomposition with respect to $\mathfrak{K}$.

Proof. Let R be the (nonempty) set of all nonzero roots of L relative to $\mathfrak{K}$ and choose a normalized root vector e_α for each $\alpha\in R$. Let $V'=\mathrm{Sp}\{e_\alpha: \alpha\in R\}$, $\mathfrak{K}'=\mathrm{Sp}\{[e_\alpha, e_\alpha^*]: \alpha\in R\}$, and $S'=\mathfrak{K}'+V'$. Then S' is semi-simple, $\mathfrak{K}'$ is a Cartan subalgebra of S', and S has a Cartan decomposition with respect to $\mathfrak{K}'$ with R as a complete set of roots. If $x\in\mathfrak{K}$ and $(x, \mathfrak{K}')=0$ then $[x, e_\alpha]=0$ for all α so that $[x, S']=0$. Let W be the orthogonal complement of $\mathrm{Sp}\{\mathfrak{K}, S'\}$. Then W is invariant under both S' and $\mathfrak{K}$. If $[S', W]=0$ then it is immediate that $[S', L]\subset S'$ so that S' is a nonzero ideal, $S'=L$, and the existence theorem is proved.

By the remarks above it is enough to prove $[S', W]=0$. Now W is invariant under $\mathfrak{H}$ so that $W=\sum W_k$ where W_k is invariant under $\mathfrak{H}$ and $[W_k, W_k^*]\subset\mathfrak{H}$.

Let S'' be any simple ideal of S'. By the classification theory of [5] and 6.9, there is an e_α in S'' such that the eigenspace associated with the maximal eigenvalue of $[E_\alpha, E_\alpha^*]$ (restricted to S'') is finite-dimensional. We will show that $[E_\alpha, E_\alpha^*]$ is zero on W.

Since $[[e_\alpha, e_\alpha^*], e_\alpha]=\mu e_\alpha$, L is spanned by subspaces T_r where T_r is the eigenspace for $[E_\alpha, E_\alpha^*]$ associated with the value $r\mu$, r being a half-integer. Suppose $W_k\cap T_r$ contains a nonzero subspace U invariant under $\mathfrak{H}$ with $[U, U]=0$ for some k. If $S_1=[U, U^*]+U+U^*$ it is easy to see that S_1 is a semi-simple L^* subalgebra (see the proof of 6.5) with $[U, U^*]$ as a Cartan subalgebra. By 3.8, S_1 is a direct sum of three-dimensional ideals I_j with $I_j=\mathrm{Sp}\{e_j, e_j^*, [e_j, e_j^*]\}$ for some e_j in U. Thus each e_j is a root vector for $[U, U^*]\subset\mathfrak{H}$. Since S_1 is invariant under $\mathfrak{H}$ it follows as in the proof of 6.8 that each e_j is a root vector for $\mathfrak{H}$, a contradiction of the definition of S'.

Thus if $r_0\mu$ is the maximal eigenvalue for $[E_\alpha, E_\alpha^*]$, since T_{r_0} is abelian, we must have $W\cap T_{r_0}=0$. But then $(T_{r_0}, W)=0$ and this implies T_{r_0} is entirely contained in S''. Since $E_\alpha^3=0$ on S'', r_0 must equal one. By the choice of α, T_1 is finite-dimensional, say of dimension m.

Suppose $[E_\alpha, E_\alpha^*]\neq 0$ on W. Since W is self-adjoint we must have $Z = T_{1/2}\cap W_k \neq 0$ for some k. Then Z is invariant under $\mathfrak{H}$. Suppose $Z=Z_1+\cdots+Z_{m+1}$ with each Z_i invariant under $\mathfrak{H}$. Then $([Z_i, Z_j^*], \mathfrak{H})=(Z_i, [\mathfrak{H}, Z_j])=0$ for $i\neq j$. Hence $[Z_i, Z_j^*]=0$ for $i\neq j$ and this implies $([Z_i, Z_i], [Z_j, Z_j])=0$. Since $[Z_i, Z_i]\subset T_1$ for each i, at least one Z_i must be abelian, so that the remarks above imply that Z contains no more than m mutually orthogonal subspaces invariant under $\mathfrak{H}$. From this it is a simple consequence of the spectral theorem that Z must contain a root vector for $\mathfrak{H}$; a contradiction. Hence $[E_\alpha, E_\alpha^*]$ must be zero on W. Using the simplicity of S'', this implies the representation of S'' on W obtained by restricting the adjoint representation is trivial. Since S'' was an arbitrary simple component of S' this implies $[S', W]=0$.

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