VITALI'S THEOREM FOR INVARIANT MEASURES

BY

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1. Introduction. If $\mathfrak{U}$ is a family of closed sets each with positive k -dimensional Lebesgue measure, and if the subset A of the Euclidean space $X = R^k$ has the property that for each neighborhood V of each point $x \in A$ there exists $U \in \mathfrak{U}$ with $x \in U \subset V$, then $\mathfrak{U}$ is said to be a Vitali cover for A . The classical Vitali theorem for the case $k=1$ asserts that if $\mathfrak{U}$ is a Vitali cover for the set $A \subset X = R$, then there is in $\mathfrak{U}$ a sequence of pairwise disjoint elements whose union exhausts all of A but a Lebesgue null set. The conclusion follows also if $k > 1$, provided one assumes that the Vitali cover $\mathfrak{U}$ of A is regular in the following sense: For each $x \in A$ there is a number $\alpha > 0$ and a sequence U_n of elements of $\mathfrak{U}$ and a sequence S_n of spheres (i.e., "balls") for which $x \in U_n \subset S_n$, $\lim_{n \rightarrow \infty} \lambda_k S_n = 0$, and $\lambda_k U_n / \lambda_k S_n \geq \alpha$. (λ_k denotes Lebesgue measure in the space $X = R^k$. The number α is called a parameter of regularity at x .)

The invariance under translation of the set-function λ_k suggests the point of view adopted in the present generalization of Vitali's theorem. We consider a group G acting transitively on a Hausdorff space X , the latter endowed with a measure μ for which $\mu(gB) = \mu B$ whenever $g \in G$ and B is a μ -measurable subset of X . A Vitali covering for a subset A of X is defined in the obvious way. Noticing that if S_α denotes the sphere of radius α centered at the origin ϕ in R^k , then $S_\alpha + S_\alpha \subset S_{2\alpha}$ and $\lambda_k S_\alpha / \lambda_k S_{2\alpha} = 1/2^k$, we define regularity of a Vitali cover in terms of a set-theoretic "multiplication" defined between subsets of X . We replace the spheres S_α by what we call quasispheres. As a conceptual aid, the reader may regard the quasisphere S_n of Definition 2.7, as corresponding to a sphere centered at ϕ of radius $1/2^n$. (See §4 for a precise treatment of this matter.)

The upshot of these considerations is our main result, Theorem 3.5, which may be informally stated as follows: If μ is G -invariant on X , and if $\mathfrak{U}$ is a regular Vitali cover for $A \subset X$, then $\mathfrak{U}$ admits a sequence of pairwise disjoint elements whose union exhausts all of A but a μ -null set. Our theorem includes the classical Vitali theorem in R^k , but the first example of §4 shows that even in R^2 the conventional regularity restrictions imposed upon the cover $\mathfrak{U}$ are unnecessarily stringent. This example shows that Vitali's conclusion may hold, in nontrivial circumstances, even when $\mathfrak{U}$ admits no positive parameter of regularity at any point $x \in A$.

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The question as to whether regular Vitali covers can exist other than in the case $X = R^k$ is settled affirmatively in §4. There we assert their existence whenever G satisfies at some point a "local equi-Lipschitz condition" (see §4) and X is locally Euclidean. A simple instance in which these hypotheses are satisfied is the case in which $G = O_3$ is the orthogonal group of rotations acting on the 2-sphere $X = S^2 \subset R^3$, μ being taken as 2-dimensional Lebesgue measure in X ; many other examples will occur to the reader.

Our theorem is readily applicable to the instance in which a locally compact Hausdorff group acts by left-translation on itself, μ being taken as Haar measure. We show that any Lie group G acting upon itself satisfies the "local equi-Lipschitz condition", and thus that every subset of G admits numerous Vitali covers which are regular relative to a naturally defined collection of quasispheres.

There exist many generalizations of the Vitali theorem; the interested reader is referred to [2, pp. 267-268] for a bibliography on this topic. None of these generalizations seems to be relevant to our present undertaking.

2. Preliminary discussion and definitions.

2.1. **STANDING CONVENTIONS.** X will denote a Hausdorff space, G a group which acts transitively on X . By this we mean that G is a subgroup of the group of all one-to-one mappings of X onto itself such that, whenever $x \in X$ and $y \in X$, there is a $g \in G$ for which $gx = y$. We consider on X a countably additive, non-negative measure μ which is G -invariant, in the sense that gU is μ -measurable and $\mu gU = \mu U$ whenever U is a μ -measurable subset of X and $g \in G$. We suppose also that for each $x \in X$ and each $\epsilon > 0$ there exists a μ -measurable neighborhood V of x for which $\mu V < \epsilon$. The outer measure determined by μ will be denoted μ^* .

2.2. **REMARK.** The requirements of 2.1 are satisfied, for example, if μ is Haar measure on a locally compact nondiscrete Hausdorff group $G = X$ which acts on itself by left-translation. We do not specifically suppose μ to be a regular Borel measure which assigns measure zero to each point of X , since such an assumption does not shorten our discussion.

2.3. **DEFINITION.** ϕ will denote a point of X , arbitrary but henceforth fixed.

2.4. **DEFINITION.** For $B \subset X$ and $C \subset X$, let $B^{-1}C = \bigcup \{gC \mid g \in G, \phi \in gB\}$.

2.5. **REMARK.** If ϕ' had replaced ϕ , and if h is any element of G for which $h\phi = \phi'$, then $B^{-1}C$ is replaced by the set $D = \bigcup \{gC \mid g \in G, \phi' \in gB\} = \bigcup \{hgC \mid g \in G, \phi' \in hgB\} = \bigcup \{hgC \mid g \in G, \phi \in gB\} = hB^{-1}C$. Thus a change in the choice of ϕ results simply in a translation of the set $B^{-1}C$.

In case $X = G$ is a group acting by translation upon itself, $B^{-1}C$ has its usual meaning provided ϕ is chosen to be the identity element; that is, $B^{-1}C$ represents the set of all elements of the form $g^{-1}h$ with $g \in B$ and $h \in C$. In the discussion which follows we shall be concerned chiefly with the situation where $B = C$; $B^{-1}B$ may be described as the union of all translates of B which contain ϕ .

Purely as a notational convenience, we introduce the following definition.

2.6. DEFINITION. For $B \subset X$ and $C \subset X$, let

$$BC = \bigcup \{gC \mid g \in G, g\phi \in B\}.$$

The following two definitions describe the covers to which our theorem applies.

2.7. DEFINITION. A sequence S_n of subsets of X will be called a sequence of quasispheres if, for some $\epsilon > 0$ and each positive integer n :

- (1) $S_{n+1}^{-1}S_{n+1} \subset S_n$;
- (2) $\mu^*S_{n+1} > \epsilon\mu^*S_n$.

2.8. DEFINITION. Let $A \subset X$, and let $\mathfrak{U}$ be a family of closed, μ -measurable subsets of X each with positive μ -measure. Then $\mathfrak{U}$ is said to be a regular Vitali cover for A if there exists a sequence S_n of quasispheres and a real-valued function M on A such that, for each $x \in A$ and each neighborhood V of x , there is a $U \in \mathfrak{U}$, a $g \in G$ and an integer n for which $x \in U \subset V$, $gU \subset S_n$ and $\mu^*S_n \leq M(x)\mu U$.

2.9. REMARK. The concept of a regular Vitali cover has been defined in terms of the set-theoretic multiplication of Definition 2.4, which depends in turn upon the choice of $\phi \in X$. But from Remark 2.5 and the fact that $(hB)^{-1}(hC) = B^{-1}C$ for each $h \in G$, it is easy to see that if a cover $\mathfrak{U}$ is a regular Vitali cover for A for a particular choice of ϕ , then it is a regular Vitali cover for every choice of ϕ .

The function M of Definition 2.8 corresponds to the parameter of regularity of the classical theory. This observation motivates the following definition.

2.10. DEFINITION. A family $\mathfrak{U}$ of closed, μ -measurable subsets of X is said to be a regular Vitali cover with constant parameter of regularity if it satisfies the conditions of the preceding definition with some constant function M .

3. The main theorem.

3.1. LEMMA. Let $A \subset X$, and let $\mathfrak{U}$ be a collection of closed μ -measurable subsets of X each of which has positive μ -measure. Let K be an open subset of X for which $A \subset K$ and $\mu^*K < \infty$, and suppose that for each $x \in A$ and each neighborhood V of x there exists $U \in \mathfrak{U}$ for which $x \in U \subset V$. Suppose there is a real number P and a sequence C_n of subsets of X for which

- (1) $n\mu^*(C_nC_n) < P$ for each n ; and
- (2) if $U \in \mathfrak{U}$ and $g \in G$ with $\phi \in gU$ and $\mu U < 1/n$, then $gU \subset C_n$.

Then there is a (possibly finite) sequence U_k of pairwise disjoint elements of $\mathfrak{U}$ for which $\mu^*(A \setminus \bigcup_{k=1}^{\infty} U_k) = 0$. The sequence may be chosen so that $\bigcup_{k=1}^{\infty} U_k \subset K$.

Proof. Insofar as the proof of Banach (see [1]) is applicable in our gener-

alized context, we present it here as reported in [3, §39] by Munroe.

We may as well suppose, discarding certain elements of $\mathfrak{U}$ if necessary, that $U \subset K$ whenever $U \in \mathfrak{U}$.

The sequence U_k is defined recursively. Choose any $U_1 \in \mathfrak{U}$, and suppose that U_k has been chosen for $1 \leq k \leq n$. If $A \subset \bigcup_{k=1}^n U_k$, the construction terminates. Otherwise, since the elements of $\mathfrak{U}$ are closed, there are a point $x \in A \setminus \bigcup_{k=1}^n U_k$ and a $U \in \mathfrak{U}$ for which $x \in U$ and $U \cap \bigcup_{k=1}^n U_k = \emptyset$. Then, with $\delta_n = \sup \{ \mu U \mid U \in \mathfrak{U}, U \cap \bigcup_{k=1}^n U_k = \emptyset \}$, we have $\delta_n > 0$. We select $U_{n+1} \in \mathfrak{U}$ with $U_{n+1} \cap \bigcup_{k=1}^n U_k = \emptyset$ and $\mu U_{n+1} > \delta_n/2$.

To show the sequence U_k is as required, we set $T = \limsup T_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} T_n$ where, for each positive integer n , we define

$$T_n = \bigcup \{ U \in \mathfrak{U} \mid \mu U < 2\mu U_n, U \cap U_n \neq \emptyset \}.$$

We will show (1) $A \setminus \bigcup_{k=1}^{\infty} U_k \subset T$ and (2) $\mu^* T = 0$.

Let $x \in A \setminus \bigcup_{k=1}^{\infty} U_k$, and choose a positive integer m . Then $x \notin \bigcup_{k=1}^m U_k$, so there is a $U \in \mathfrak{U}$ with $x \in U$ and $U \cap \bigcup_{k=1}^m U_k = \emptyset$. Since the measurable sets U_k are pairwise disjoint subsets of K , it follows from our hypothesis $\mu^* K < \infty$ that $\mu U_k \rightarrow 0$. If $U \cap U_k = \emptyset$ for each $k > m$, then $\mu U \leq \delta_k < 2\mu U_{k+1}$ for each $k > m$, so that $\mu U = 0$. From this contradiction it follows that there is a smallest integer N for which $U \cap U_N \neq \emptyset$. Then $\mu U \leq \delta_{N-1} < 2\mu U_N$, so that $x \in U \subset T_N$. Since $N > m$, we have $x \in \bigcup_{n=m}^{\infty} T_n$. Thus (1) is established.

To prove (2), we first find an integer n_0 for which $2\mu U_n < 1$ whenever $n \geq n_0$. Let $n \geq n_0$ be fixed, and let z be a fixed point in U_n . We will show that if N is the largest integer for which $2\mu U_n < 1/N$, and if h is any element of G for which $hz = \phi$, then $T_n \subset h^{-1}C_N C_N$. For each $x \in T_n$ there exist $U \in \mathfrak{U}$ and $y \in U \cap U_n$ for which $x \in U$ and $\mu U < 2\mu U_n$. Choosing a point $g \in G$ for which $gy = \phi$, we see from hypothesis (2) of the present lemma that $gU \subset C_N$ and $hU_n \subset C_N$, so that $gx \in C_N$ and $hy \in C_N$. Then $\phi = (gh^{-1})hy \in gh^{-1}C_N$, so that $hg^{-1}C_N \subset C_N C_N$ by 2.6. Hence $hx \in C_N C_N$ for each $x \in T_n$. Now since $2\mu U_n \geq 1/(2N)$, we have $\mu^* T_n \leq \mu^* C_N C_N < P/N \leq 4P\mu U_n$ for each $n \geq n_0$, so that $\sum_{n=n_0}^{\infty} \mu^* T_n \leq 4P \sum_{n=n_0}^{\infty} \mu U_n \leq 4P\mu^* K < \infty$. That $\mu^* T = 0$ now follows from the relations $\mu^* T \leq \mu^* \bigcup_{n=k}^{\infty} T_n \leq \sum_{n=k}^{\infty} \mu^* T_n$, valid for each k . This concludes the proof of the lemma.

The next two lemmas concern the properties of the product $B^{-1}C$.

3.2. LEMMA. *If $g \in G$ and $U \subset X$, then $g\phi \in U^{-1}U$ if and only if $g^{-1}\phi \in U^{-1}U$.*

Proof. If $g\phi \in U^{-1}U$, then there is an $h \in G$ for which $g\phi \in hU$ and $\phi \in hU$. But then $\phi \in g^{-1}hU$ and $g^{-1}\phi \in g^{-1}hU$, so that $g^{-1}\phi \in U^{-1}U$ by 2.4. The replacement of g by g^{-1} in this argument proves the converse.

3.3. LEMMA. *Let $U_{\iota} \subset X$ for each $\iota \in I$, and let $C = \bigcup_{\iota \in I} U_{\iota}^{-1}U_{\iota}$. Then $CD = C^{-1}D$ for each $D \subset X$.*

Proof. We have, using 3.2,

$$\begin{aligned} C^{-1}D &= \left(\bigcup_{i \in I} U_i^{-1}U_i \right)^{-1}D = \bigcup \left\{ gD \mid g \in G, g^{-1}\phi \in \bigcup_{i \in I} U_i^{-1}U_i \right\} \\ &= \bigcup \left\{ gD \mid g \in G, g\phi \in \bigcup_{i \in I} U_i^{-1}U_i \right\} = \left(\bigcup_{i \in I} U_i^{-1}U_i \right) D = CD. \end{aligned}$$

We now prove as a lemma a special case of our theorem.

3.4. LEMMA. *Let $\mathfrak{U}$ be a regular Vitali cover for A with constant parameter of regularity M . Let K be an open subset of X for which $A \subset K$ and $\mu^*K < \infty$. Then there exists a (possibly finite) sequence U_k of pairwise disjoint elements of $\mathfrak{U}$ for which $\mu^*(A \setminus \bigcup_{k=1}^{\infty} U_k) = 0$. The sequence may be chosen so that $\bigcup_{k=1}^{\infty} U_k \subset K$.*

Let ϵ and S_n be as in Definition 2.8. We may clearly suppose, discarding certain elements from $\mathfrak{U}$ if necessary, that for each $U \in \mathfrak{U}$ there are a $g \in G$ and an S_k with $gU \subset S_k$ and $\mu^*S_k \leq M\mu U$.

Since $\phi \in B^{-1}B$ for each nonempty subset B of X , we have $\phi \in S_{k+1}^{-1}S_{k+1} \subset S_k$ for each k . Denoting by e the identity element of the group G , we have $S_{k+1} = eS_{k+1} \subset \bigcup \{gS_{k+1} \mid g \in G, \phi = g\phi\} = \{\phi\}^{-1}S_{k+1} \subset S_{k+1}^{-1}S_{k+1} \subset S_k$, so that the real sequence μ^*S_k is monotone decreasing. Since $\mathfrak{U}$ is a Vitali cover for the set A , each point of which admits neighborhoods of arbitrarily small μ -measure, we see from the conclusion of the preceding paragraph that $\mu^*S_k \rightarrow 0$.

Now for each positive integer n we set $C_n = \bigcup \{U^{-1}U \mid U \in \mathfrak{U}, \mu U < 1/n\}$. The sequence C_n clearly satisfies hypothesis (2) of 3.1, so by 3.1 the proof of the present lemma may be completed by showing that the real sequence $n\mu^*C_n$ is bounded. To this end we restrict our attention to those positive integers n for which $n > M/\mu S_2$, and we associate with each such n the smallest integer k_n for which $\mu^*S_{k_n} \leq M/n$. Then always $k_n \geq 3$ and $S_k \subset S_{k_n}$ whenever $\mu^*S_k \leq M/n$. Thus for each $U \in \mathfrak{U}$ with $\mu U < 1/n$ there is a $g \in G$ for which $gU \subset S_{k_n}$. Hence $U^{-1}U = (gU)^{-1}(gU) \subset S_{k_n}^{-1}S_{k_n} \subset S_{k_n-1}$ whenever $U \in \mathfrak{U}$ and $\mu U < 1/n$. Thus $C_n \subset S_{k_n-1}$, and from 3.3 we have $C_n C_n = C_n^{-1}C_n \subset S_{k_n-1}^{-1}S_{k_n-1} \subset S_{k_n-2}$. Hence $\mu^*C_n C_n \leq \mu^*S_{k_n-2} < \mu^*S_{k_n-1}/\epsilon < \mu^*S_{k_n}/\epsilon^2 \leq M/(\epsilon^2 n)$ whenever $n > M/\mu S_2$, and the proof is complete.

The next result, our main theorem, is obtained by removing some of the hypotheses of the preceding lemma.

3.5. THEOREM. *Let $\mathfrak{U}$ be a regular Vitali cover for A , and suppose that there is a sequence V_k of open subsets of X for which $A \subset \bigcup_{k=1}^{\infty} V_k$ and always $\mu^*V_k < \infty$. Then there is a (possibly finite) sequence U_k of pairwise disjoint elements of $\mathfrak{U}$ for which $\mu^*(A \setminus \bigcup_{k=1}^{\infty} U_k) = 0$.*

Proof. Let S_n be the sequence of quasispheres, M the (not necessarily constant) function given by 2.8. We suffer no loss in generality in supposing that $V_k \subset V_{k+1}$ for each positive integer k . For each positive integer k , we define

$$A_k = A \cap M^{-1}(0, k] \cap V_k$$

and $\mathfrak{U}_k = \{U \in \mathfrak{U} \mid gU \subset S_n \text{ for some } g \in G \text{ and some integer } n \text{ with } \mu^* S_n < k\mu U\}$. Then $\mathfrak{U}_k$ is a regular Vitali cover for A_k with constant parameter of regularity k , so by 3.4 we can find pairwise disjoint elements $U_1, U_2, \dots, U_{n_1}$ of $\mathfrak{U}_1$ for which $\mu^*(A_1 \setminus \bigcup_{k=1}^{n_1} U_k) < 1$. Proceeding inductively on the assumption that pairwise disjoint elements $U_1, \dots, U_{k_n}$ of $\mathfrak{U}_n$ have been chosen so that $\mu^*(A_n \setminus \bigcup_{k=1}^{k_n} U_k) < 1/n$, we notice that $W_n = V_n \setminus \bigcup_{k=1}^{k_n} U_k$ is an open subset of X for which $A_n \setminus \bigcup_{k=1}^{k_n} U_k \subset W_n$ and $\mu^* W_n < \infty$. Thus there are by Lemma 3.4 pairwise disjoint elements $U_{k_{n+1}}, \dots, U_{k_{n+1}}$ of $\mathfrak{U}_{n+1}$ for which

$$\mu^* \left[\left(A_n \setminus \bigcup_{k=1}^{k_n} U_k \right) \setminus \bigcup_{k=k_{n+1}}^{k_{n+1}} U_k \right] < \frac{1}{n+1}$$

and

$$\bigcup_{k=k_{n+1}}^{k_{n+1}} U_k \subset W_n = V_n \setminus \bigcup_{k=1}^{k_n} U_k.$$

From these computations it follows that

$$\mu^* \left(A_n \setminus \bigcup_{k=1}^{\infty} U_k \right) \leq \mu^* \left(A_n \setminus \bigcup_{k=1}^{k_n} U_k \right) < 1/n \quad \text{for each positive integer } n,$$

which together with the relations $A = \bigcup_{n=1}^{\infty} A_n$ and $A_n \subset A_{n+1}$ ($n \geq 1$) yields the conclusion $\mu^*(A \setminus \bigcup_{k=1}^{\infty} U_k) = 0$.

4. EXAMPLES. The examples presented here are designed to indicate the applicability of our theorem to numerous instances in which a group acts transitively on a measure space. For the sake of simplicity we consider cases where each U in the cover $\mathfrak{U}$ is a translate of a quasisphere. It will be clear how more complicated examples can be constructed. We begin by stating a corollary to our theorem.

4.1. COROLLARY. *Let S_n be a sequence of closed, measurable quasispheres, and let A be contained in a countable union of open sets each of which has finite outer measure. Suppose that for each neighborhood V of each point x in A there exists an element $g \in G$ and a positive integer n for which $x \in gS_n \subset V$. Then there are a sequence g_k of elements of G and a sequence n_k of positive integers for which the sets $g_k S_{n_k}$ are pairwise disjoint and $\mu^*(A \setminus \bigcup_{k=1}^{\infty} g_k S_{n_k}) = 0$.*

From this corollary it follows, speaking loosely, that Theorem 3.5 will be applicable to any situation in which quasispheres can be introduced. Euclidean k -space R^k is one setting in which this can be done in such a way that the cover which results from a translation of the quasispheres to the points of a nonempty subset A of R^k is nowhere regular in the classical sense. By using a sequence of solid ellipses with increasing eccentricity, we give in Example 4.2 a possible construction for the case $k = 2$.

4.2. EXAMPLE. Let the topological group $G = X = R^2$ act on itself by trans-

lation, and let μ denote 2-dimensional Lebesgue measure. Choosing ϕ to be the identity element in R^2 we define

$$S_n = \{(x_1, x_2) \mid (x_1, x_2) \in R^2 \text{ and } (2^n x_1)^2 + (2^{2n} x_2)^2 \leq 1\},$$

for each positive integer n . To see that the closed sets S_n form a sequence of quasispheres in the sense of Definition 2.7, we observe that $S_{n+1}^{-1} S_{n+1} = 2S_{n+1} = \{(x_1, x_2) \mid (x_1, x_2) \in R^2 \text{ and } (2^{n+1} x_1)^2 + (2^{2n+2} x_2)^2 \leq 1\} \subset S_n$, and that $\mu S_{n+1} = \pi/2^{3n+2} > \epsilon \pi/2^{3n} = \epsilon \mu S_n$ if $\epsilon < 1/8$. The sequence S_n being fundamental at ϕ , we see that the hypotheses of Corollary 4.1 are satisfied for an arbitrary subset A of R^2 . Since the smallest disk D_n in R^2 containing S_n has area $\pi/2^{2n}$, however, the sequence $\mu S_n / \mu D_n$ has limit zero and the classical Vitali theorem is not applicable.

We next show how to find quasispheres in a more general situation.

4.3. EXAMPLE. Suppose that the space X admits a point, which we may as well choose for ϕ , some neighborhood V of which admits a homeomorphism onto a neighborhood of the origin in R^k . If η is such a homeomorphism, arbitrary but henceforth fixed, then $\eta^{-1}(\eta x + \eta y)$, whenever it is defined, will be denoted by $x + y$; $x - y$ is defined similarly. When $x \in V$, we write $|x|$ for the distance from ηx to the origin. Then quasispheres can be defined as the image under η^{-1} of spheres in R^k , provided that the elements of G satisfy the following condition E-L, a sort of local equi-Lipschitz condition.

CONDITION E-L. There is an open subset W of X and a real number P for which $\phi \in W \subset V$ and for which $gy \in V$ and $|gx - gy| < P|x - y|$ whenever $x \in W$ and $y \in W$ and $gx = \phi$.

Indeed, let r be a positive number for which $x \in W$ whenever $|x| < r$, and for which the sphere of radius r about the origin in R^k is contained in ηV . Now define

$$S_n = \{x \mid |x| \leq r/(2P)^n\},$$

for each positive integer n . Our discussion will be complete if we can show that the sequence S_n satisfies conditions (1) and (2) of Definition 2.7.

To show $S_{n+1}^{-1} S_{n+1} \subset S_n$, which is (1), we notice that the constant P of condition E-L exceeds 1, so that always $S_n \subset W$. If $x \in S_{n+1}^{-1} S_{n+1}$, so that there exist points y and z in S_{n+1} and $g \in G$ for which $gy = x$ and $gz = \phi$, then we have

$$|x| = |gy - gz| < P|y - z| \leq P(|y| + |z|) \leq 2Pr/(2P)^{n+1} = r/(2P)^n;$$

thus $x \in S_n$ and (1) of Definition 2.7 is established.

To prove (2), we notice that for each positive integer n the sets $\eta S_n, \eta S_{n+2}$ are spheres in R^k , the radius of the former being $(2P)^2$ times the radius of the latter. Thus there is an integer m , dependent only on k and P , for which $\eta S_n \subset \bigcup_{i=1}^m \eta x_i + \eta S_{n+2}$ for appropriately chosen points $x_i \in S_n$. Now if $y \in S_{n+2}$, then the distance in R^k from the origin to $\eta x_i + \eta y$ does not exceed $r/(2P)^n + r/(2P)^{n+2} < r$, so that $\eta^{-1}(\eta x_i + \eta y) = x_i + y$ exists and lies in V . Choosing $g_i \in G$ so that $g_i x_i = \phi$, we have

$$\begin{aligned} |g_i(x_i + y)| &= |g_i(x_i + y) - g_i x_i| < P |(x_i + y) - x_i| = P |y| \\ &\leq Pr/(2P)^{n+2} < r/(2P)^{n+1}. \end{aligned}$$

Thus $g_i(x_i + S_{n+2}) \subset S_{n+1}$, and from the relation $S_n \subset \bigcup_{i=1}^m x_i + S_{n+2}$ it follows that

$$\mu^* S_n \leq \sum_{i=1}^m \mu^*(x_i + S_{n+2}) \leq \sum_{i=1}^m \mu^* S_{n+1} = m\mu^* S_{n+1}.$$

4.4. EXAMPLE. We consider finally the case in which a Lie group $G = X$ acts upon itself by translation. Retaining the notation introduced in 4.3, we recall from Smith [4] that the homeomorphism η may be chosen to introduce on X a coordinate system which is right regular, in the sense that there is a function F on $V \times V$, jointly continuous at (ϕ, ϕ) , for which

$$uv = u + v + |v| F(u, v),$$

whenever each of the points, $u, v, uv, F(u, v)$ and $|v| F(u, v)$ lies in V . (By $|v| F(u, v)$, we mean $\eta^{-1}[|v| \eta F(u, v)]$.) The fact that our theorem is applicable in this situation will be proved by showing that if η is so chosen, then condition E-L is satisfied. The condition now reads as follows: There is an open subset W of X and a real number P for which $\phi \in W \subset V$ and for which $x^{-1}y \in V$ and $|x^{-1}y| < P|x - y|$ whenever $x \in W$ and $y \in W$.

Now if x and y are arbitrary points of V and if $z \in X$, we could write

$$(1) \quad y = xx^{-1}y = x + x^{-1}y + |x^{-1}y| z,$$

then

$$(2) \quad y - x - |x^{-1}y| z = x^{-1}y,$$

whence $|x^{-1}y| \leq |y - x| + |x^{-1}y| |z|$ and $(1 - |z|)|x^{-1}y| < |y - x|$, provided that each of the points appearing in equations (1) and (2) is well-defined (i.e., lies in V). In the present case, a real number $P > 1$ being given, we choose W so small that (1) and (2) are meaningful for $z = F(x, x^{-1}y)$, and so small that $|F(x, x^{-1}y)| < (P - 1)/P$ whenever $x \in W$ and $y \in W$. With this choice of W we have $(1/P)|x^{-1}y| < (1 - |z|)|x^{-1}y| < |y - x| = |x - y|$ whenever $x \in W$ and $y \in W$, so that condition E-L is satisfied.

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