ORTHONORMAL SETS WITH NON-NEGATIVE DIRICHLET KERNELS. II

BY

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1. **Introduction.** Recently the author has shown [2] that an orthonormal set of functions whose associated Dirichlet kernels are non-negative must be a system of step functions similar in structure to the classical Haar functions. The present paper discusses systems in which infinitely many of the kernels, but not necessarily all, are non-negative. It is shown that such systems also must be composed of step functions of a special type. As an application, a characterization of the Walsh functions is given in §4.

2. **Definitions and preliminaries.** It will be assumed throughout that μ is a totally finite measure on a space S normalized so that $\mu(S)=1$. All sets mentioned will be subsets of S. All functions will be real-valued, bounded, and μ-measurable. For the sake of brevity "almost everywhere" qualifications will be omitted. For example, two functions which differ only on a set of measure zero will be tacitly identified and "essential supremum" will be replaced by "supremum."

A *partition* of S is a finite collection P of disjoint subsets of S whose union is S. Given partitions P_1 and P_2, $P_1 > P_2$ if each element of P_2 is contained in an element of P_1 and if P_1 and P_2 are not identical. A function which is constant on each element of a partition P will be called a *step function* (P).

LEMMA 1. *Let $f(t)$ be a function defined on a set T of measure μ_T. If*

$$\text{(a)} \qquad -M_2 \leqq f(t) \leqq M_1, \qquad t \in T, \ -M_2 < M_1,$$

$$\text{(b)} \qquad \int_T f(t)\,d\mu(t) = I,$$

then

$$\text{(c)} \qquad \int_T f^2(t)\,d\mu(t) \leqq M_1 M_2 \mu_T + (M_1 - M_2) I.$$

Equality holds in (c) *if and only if*

$$\text{(d)} \qquad f(t) = \begin{cases} M_1, & t \in T_1, \\ -M_2, & t \in T_2, \end{cases}$$

where $\{T_1, T_2\}$ *is a partition of* T *such that*

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(e) $$\mu(T_1) = \frac{M_2\mu_T + I}{M_1 + M_2}, \qquad \mu(T_2) = \frac{M_1\mu_T - I}{M_1 + M_2}.$$

The proof of the special case $I=0$ is easy and is given in [2]. We shall omit it here. If $I\neq 0$, the lemma is obtained immediately by applying the special case to $g(t)=f(t)-I\mu_T^{-1}$.

LEMMA 2. *Let $f(t)$ be defined on a set T of measure μ_T. Suppose*

(a) $$-M_2 \leqq f(t) \leqq M_1, \qquad t\in T,\ -M_2 < M_1,$$

(b) $$\int_T f(t)d\mu(t) = 0,$$

(c) $$\int_T f^2(t)d\mu(t) \geqq (M_1 - a/2)M_2\mu_T, \qquad 0 < a < M_1.$$

Then, if $\sigma_a = \{t \mid f(t) \geqq M_1 - a\}$,

(d) $$\mu(\sigma_a) \geqq \frac{1}{2}\,\frac{M_2\mu_T}{M_1 + M_2}.$$

Proof. On σ_a, $-(a-M_1)\leqq f(t)\leqq M_1$, and on the complement σ_a^*, $-M_2 \leqq f(t) < M_1 - a$. Let I and $-I$ denote the integrals of $f(t)$ over σ_a and σ_a^* and let $\mu_a=\mu(\sigma_a)$. Applying Lemma 1 twice,

$$\int_{\sigma_a} f^2(t)d\mu(t) \leqq M_1(a - M_1)\mu_a + (2M_1 - a)I,$$

$$\int_{\sigma_a^*} f^2(t)d\mu(t) \leqq (M_1 - a)M_2(\mu_T - \mu_a) + (M_1 - a - M_2)(-I).$$

Adding, and using assumption (c),

$$\left(M_1 - \frac{a}{2}\right)M_2\mu_T \leqq \int_T f^2(t)d\mu(t)$$

$$\leqq (M_1 - a)M_2\mu_T + (M_1 + M_2)(a - M_1)\mu_a + (M_1 + M_2)I.$$

Subtracting $(M_1-a)M_2\mu_T$,

(1) $$\frac{a}{2}M_2\mu_T \leqq (M_1 + M_2)[a\mu_a + (I - M_1\mu_a)].$$

Since $0\leqq I\leqq M_1\mu_a$, and $M_1+M_2>0$ (because of (a) and (b)),

$$\frac{a}{2}M_2\mu_T \leqq (M_1 + M_2)a\mu_a$$

or,

$$\frac{1}{2}\frac{M_2\mu_T}{M_1+M_2} \leqq \mu_a.$$

3. **Main theorem.** With any orthonormal set $\{f_j(s)\}_{j=0}^{\infty}$ in $L^2(S, \mu)$ are associated the Dirichlet kernels

$$D_n(s, t) = \sum_{j=0}^{n-1} f_j(s)f_j(t), \qquad n \geqq 1.$$

The following theorem gives the structure of an orthonormal set when infinitely many of these kernels are non-negative.

THEOREM 1. *Let $\mathfrak{F} = \{f_j(s)\}_{j=0}^{\infty}$ be an orthonormal set in $L^2(S, \mu)$ with $f_0(s) \equiv 1$. Suppose there exists a sequence of integers $1 = n_0 < n_1 < n_2 < \cdots$ such that $D_{n_r}(s, t) \geqq 0$, $r \geqq 0$. Then $\mathfrak{F}$ is a system of step functions of the following type. There exists a sequence $P_0 > P_1 > P_2 > \cdots$ of partitions of S, $P_r = \{S_{r,i}\}_{i=1}^{n_r}$, such that if $0 \leqq j < n_r$, then $f_j(s)$ is a step function (P_r), $r \geqq 0$. Furthermore,*

$$D_{n_r}(s, t) = \begin{cases} p_{r,i}, & (s, t) \in S_{r,i}^2,\ 1 \leqq i \leqq n_r, \\ 0, & \textit{otherwise}, \end{cases} \tag{2}$$

where

$$p_{r,i} = \frac{1}{\mu(S_{r,i})}, \qquad 1 \leqq i \leqq n_r. \tag{3}$$

Proof. It will be shown by induction that for each $k \geqq 0$, there exists a sequence of partitions $P_0 > P_1 > P_2 > \cdots > P_k$ of S such that, if $0 \leqq r \leqq k$,

(i) $f_j(s)$ is a step function (P_r) when $0 \leqq j < n_r$,

(ii) P_r has exactly n_r elements, $\{S_{r,i}\}_{i=1}^{n_r}$,

(iii) (2) and (3) hold.

For $k=0$, the situation is trivial. Since $f_0(s) \equiv 1$, $D_{n_0}(s, t) \equiv 1$. Taking P_0 to be the identity partition $\{S\} = \{S_{0,1}\}$ all assertions are obviously true. Assuming the proposition is true for k, we must construct a partition $P_{k+1} < P_k$ and show (i), (ii), (iii) can be extended to P_{k+1}.

For simplicity, let $D_{n_r}(s, t) = \Delta_r(s, t)$. Define

$$F_k(s, t) = \Delta_{k+1}(s, t) - \Delta_k(s, t) = \sum_{j=n_k}^{n_{k+1}-1} f_j(s)f_j(t). \tag{4}$$

We shall show that $F_k(s, t) = 0$ wherever $\Delta_k(s, t) = 0$, namely outside the set

$$\bigcup_{i=1}^{n_k} S_{k,i}^2.$$

Let t_0 be arbitrary but fixed, say $t_0 \in S_{k,j}$.

$$F_k(s, t_0) \geqq 0, \qquad s \in S^*_{k,j}, \tag{5}$$

where the asterisk denotes complement. This follows from (4) since $\Delta_{k+1}(s, t) \geqq 0$ and $\Delta_k(s, t_0) \equiv 0$, $s \in S^*_{k,j}$, by the induction hypothesis (2). Also by (2), $p^{-1}_{k,j}\Delta_k(s, t_0)$ is the characteristic function of $S_{k,j}$. Hence

$$\begin{aligned} \int_{S^*_{k,j}} F_k(s, t_0)d\mu(s) &= \int_S - \int_{S_{k,j}} \\ &= \int_S F_k(s, t_0)d\mu(s) - \frac{1}{p_{k,j}}\int_S \Delta_k(s, t_0)F_k(s, t_0)d\mu(s). \end{aligned} \tag{6}$$

Both integrals on the right side of (6) vanish because $F_k(s, t_0)$ is orthogonal to $f_0(s) \equiv 1$ and to $\Delta_k(s, t_0)$. Therefore,

$$\int_{S^*_{k,j}} F_k(s, t_0)d\mu(s) = 0. \tag{7}$$

It follows from (5) and (7) that $F_k(s, t_0) \equiv 0$ on $S^*_{k,j}$. Since t_0 was arbitrary, $F_k(s, t) \equiv 0$ outside the sets $S^2_{k,i}$, $1 \leqq i \leqq n_k$.

We now begin the construction of the partition P_{k+1}. This will be done by partitioning separately each of the sets $S_{k,i}$, $1 \leqq i \leqq n_k$.

The quantity

$$F_k(t, t) = \sum_{j=n_k}^{n_{k+1}-1} f_j^2(t)$$

does not vanish identically on S. It is no loss of generality to assume $\sup_{S_{k,1}} F_k(t, t) = m_1 > 0$.

Let us carry out the partitioning of $S_{k,1}$ in detail. Henceforth, all points mentioned, unless otherwise stated, will be understood to belong to $S_{k,1}$.

The first step is to show that $F_k(t, t) \equiv m_1$ on a set T_1 of positive measure. Choose a sequence $\{t_n\}_{n=1}^{\infty}$ such that $F_k(t_n, t_n) \geqq m_1(1 - 1/2n)$. Then

$$-p_{k,1} \leqq F_k(s, t_n) \leqq m_1, \tag{8a}$$

$$\int_{S_{k,1}} F_k(s, t_n)d\mu(s) = 0, \tag{8b}$$

$$\int_{S_{k,1}} F_k^2(s, t_n)d\mu(s) \geqq m_1\left(1 - \frac{1}{2n}\right). \tag{8c}$$

The upper bound in (8a) is a consequence of the Schwarz inequality,

$$F_k(s, t) = \sum_{j=n_k}^{n_{k+1}-1} f_j(s)f_j(t) \leqq (F_k(s, s)F_k(t, t))^{1/2}. \tag{9}$$

The lower bound in (8a) follows from (4) and the facts that $\Delta_k(s, t_n) \leqq p_{k,1}$ by the induction hypothesis and that $\Delta_{k+1}(s, t_n) \geqq 0$. (8b) is obtained from (7) since $F_k(s, t_n)$ is orthogonal (over S) to $f_0(s) \equiv 1$. Finally, using orthonormality, one obtains the general identity

$$(10) \qquad \int_{S_{k,1}} F_k(s, t_\alpha) F_k(s, t_\beta) d\mu(s) = F_k(t_\alpha, t_\beta).$$

Putting $t_\alpha = t_\beta = t_n$ yields (8c).

Now (8a, b, c) are precisely the conditions needed to apply Lemma 2 to $F_k(s, t_n)$ with $T = S_{k,1}$, $\mu_T = p_{k,1}^{-1}$, $M_1 = m_1$, $M_2 = p_{k,1}$ and $a = m_1/n$. We obtain

$$\mu(\sigma_n) \geqq \frac{1}{2} \frac{1}{m_1 + p_{k,1}} = c_1$$

where $\sigma_n = \{ s \mid F_k(s, t_n) \geqq m_1(1 - 1/n) \}$. Now let $\tau_n = \{ s \mid F_k(s, s) \geqq m_1(1 - 1/n)^2 \}$. Then $\sigma_n \subset \tau_n$. For, by (9), if $s \in \sigma_n$

$$F_k(s, s) m_1 \geqq F_k(s, s) F_k(t_n, t_n) \geqq F_k^2(s, t_n) \geqq m_1^2 (1 - 1/n)^2.$$

Consequently $\mu(\tau_n) \geqq \mu(\sigma_n) \geqq c_1 > 0$, $n \geqq 1$. If $T_1 = \{ s \mid F_k(s,s) = m_1 \}$ then $T_1 = \bigcap_{n=1}^\infty \tau_n$. But $\tau_1 \supset \tau_2 \supset \tau_3 \supset \cdots$. Therefore, $\mu(T_1) \geqq c_1 > 0$.

We may now obtain some precise information about $F_k(s, t)$ as follows. Using (8a, b), we may apply Lemma 1 to $F_k(s, t)$ with t fixed and obtain

$$\int_{S_{k,1}} F_k^2(s, t) d\mu(s) \leqq m_1.$$

On the other hand, if $t \in T_1$, we have from (10)

$$\int_{S_{k,1}} F^2(s, t) d\mu(s) = F_k(t, t) = m_1.$$

Thus, $F_k(s, t)$ is an extremal function in the sense of Lemma 1 when $t \in T_1$. Therefore, for each $t \in T_1$, there exists a set $V(t)$ such that

$$(11) \qquad F_k(s, t) \equiv \begin{cases} m_1, & s \in V(t), \\ -p_{k,1}, & \text{otherwise}, \end{cases}$$

where

$$(12) \qquad \mu(V(t)) = \frac{1}{m_1 + p_{k,1}} \cdot$$

We are going to show that there are only a finite number of distinct sets $V(t)$, that these form a partition $P(T_1)$ of T_1, and that the functions $f_j(s)$, $0 \leqq j < n_{k+1}$, are step functions $(P(T_1))$.

Let $t \in T_1$ and $s \in V(t)$. From (9) and (11),

$$m_1 = F_k(s, t) \leqq (F_k(s, s)F_k(t, t))^{1/2} \leqq m_1. \tag{13}$$

Hence, $F_k(s,s) = m_1$ which shows that $V(t) \subset T_1$. Since $t \in V(t)$, $T_1 = \bigcup_{t\in T_1} V(t)$.

Also from (13),

$$\sum_{j=n_k}^{n_{k+1}-1} f_j(s)f_j(t) = \left(\sum_{j=n_k}^{n_{k+1}-1} f_j^2(s) \sum_{j=n_k}^{n_{k+1}-1} f_j^2(t) \right)^{1/2}.$$

Consequently, for t fixed, there exists a proportionality factor $\lambda(s)$ such that $f_j(s) = \lambda(s)f_j(t)$ for all $s \in V(t)$, $n_k \leqq j < n_{k+1}$. Therefore, $F_k(s, t) = \lambda(s)F_k(t, t)$. But $F_k(s, t) \equiv F_k(t, t) = m_1$, $s \in V(t)$, by (11). It follows that $\lambda(s) \equiv 1$ on $V(t)$. Therefore, the functions $f_j(s)$, $n_k \leqq j < n_{k+1}$, are constant on $V(t)$. (By the induction hypothesis, this is also true for $0 \leqq j < n_k$ since $f_j(s)$ is then constant on the superset $S_{k,1}$.)

It is now clear that $F_k(s, \tau) \equiv F_k(t, t) = m_1$, $(s, \tau) \in V^2(t)$. Hence, if $t' \in V(t)$, $F_k(s, t') \equiv m_1$ for $s \in V(t)$. Since $V(t')$ is that set of measure $(m_1 + p_{k,1})^{-1}$ on which $F_k(s, t') \equiv m_1$, we have that $V(t') = V(t)$. It follows that if $t_1, t_2 \in T_1$, then $V(t_1)$ and $V(t_2)$ are either disjoint or identical. In other words, the sets $V(t)$ form a partition $P(T_1)$ of T_1. Since they all have the same positive measure and $\mu(T_1)$ is finite, $P(T_1)$ is a finite partition. Let us call the elements of this partition $S_{k+1,i}$, $1 \leqq i \leqq q_1$. Define $p_{k+1,i} = \mu(S_{k+1,i})^{-1} = m_1 + p_{k,1}$.

To summarize, we have established the following facts. There exists a set of positive measure T_1 on which $F_k(t, t) \equiv m_1$. There is a partition $P(T_1) = \{S_{k+1,i}\}_{i=1}^{q_1}$ of T_1 into sets of equal measure such that $f_j(s)$ is a step function $(P(T_1))$ for $0 \leqq j < n_{k+1}$. Furthermore,

$$F_k(s, t) \equiv \begin{cases} m_1 = p_{k+1,i} - p_{k,1}, & (s, t) \in S_{k+1,i}^2,\ 1 \leqq i \leqq q_1, \\ -p_{k,1}, & \text{elsewhere on } S_{k,1} \times T_1 \cup T_1 \times S_{k,1}. \end{cases} \tag{14}$$

Now $\Delta_{k+1}(s, t) = \Delta_k(s, t) + F_k(s, t)$. We obtain from (14) and the induction hypothesis (2) that

$$\Delta_{k+1}(s, t) \equiv \begin{cases} p_{k+1,i}, & (s, t) \in S_{k+1,i}^2,\ 1 \leqq i \leqq q_1, \\ 0, & \text{elsewhere on } S_{k,1} \times T_1 \cup T_1 \times S_{k,1}, \end{cases} \tag{15}$$

where

$$p_{k+1,i} = \frac{1}{\mu(S_{k+1,i})}. \tag{16}$$

(15) and (16) represent a partial extension of (2) and (3) to the case $r = k+1$. If $T_1 = S_{k,1}$ we have the desired partition of $S_{k,1}$.

Suppose then, that T_1 is a proper subset of $S_{k,1}$. Let $m_2 = \sup F_k(t, t)$, $t \in T_1^*$. m_2 must be positive. For otherwise $f_j(t) \equiv 0$ on T_1^*, $n_k \leqq j < n_{k+1}$. Hence $F_k(s, t) \equiv 0$ outside of T_1^2 which contradicts (14). (Note that $m_2 \leqq m_1$.)

Using the above arguments, we can easily establish the following. There exists a set $T_2 \subset T_1^*$, $\mu(T_2) \geqq c_2 = (1/2)(m_2 + p_{k,1})^{-1}$, such that $F_k(t, t) \equiv m_2$ on T_2. There is a partition $P(T_2) = \{S_{k+1,i}\}_{i=q_1+1}^{q_2}$ of T_2 into sets of equal measure such that the functions $f_j(s)$, $0 \leqq j < n_{k+1}$ are step functions $(P(T_2))$ and the analogues of (15) and (16) hold.

Continuing in this way, we obtain sets $T_1, T_2, \cdots$ such that $F_k(t, t) \equiv m_\nu > 0$ on T_ν and $\mu(T_\nu) \geqq c_\nu = 2^{-1}(m_\nu + p_{k,1})^{-1}$. The process terminates after a finite number of steps since $\mu(T_\nu) \geqq 2^{-1}(m_\nu + p_{k,1})^{-1} \geqq 2^{-1}(m_1 + p_{k,1})^{-1} = c_1$ while $\mu(S_{k,1})$ is finite. The sets T_ν form a finite partition of $S_{k,1}$. Each of these is partitioned in the same way as T_1. The result is a partition of $S_{k,1}$ possessing all the required properties.

(We now drop the convention that all points named belong to $S_{k,1}$.) Each of the sets $S_{k,i}$, $2 \leqq i \leqq n_k$, can be partitioned in the same way provided $\sup_{S_{k,i}} F_k(t, t) > 0$. If $\sup_{S_{k,i}} F_k(t, t) = 0$, then $f_j(s) \equiv 0$ on $S_{k,i}$, $n_k \leqq j < n_{k+1}$. In this case $\Delta_{k+1}(s, t) \equiv \Delta_k(s, t)$ on $S_{k,i}^2$ and (2) and (3) trivially carry over if we take the identity partition $\{S_{k,i}\}$.

Combining these partitions we obtain a partition $P_{k+1} = \{S_{k+1,i}\}_{i=1}^{N}$ of S such that $P_{k+1} < P_k$, $f_j(s)$ is a step function (P_{k+1}) if $0 \leqq j < n_{k+1}$, and

$$\Delta_{k+1}(s, t) \equiv \begin{cases} p_{k+1,i}, & (s, t) \in S_{k+1,i}^2,\ 1 \leq i \leq N, \\ 0, & \text{otherwise}, \end{cases} \tag{17}$$

where

$$p_{k+1,i} = \frac{1}{\mu(S_{k+1,i})}. \tag{18}$$

To complete the induction, it remains only to show that $N = n_{k+1}$. By orthonormality,

$$\int_S \left(\int_S \Delta_{k+1}^2(s, t) d\mu(s) \right) d\mu(t) = n_{k+1}.$$

On the other hand this integral is easily computed as a double integral from (17) and (18). Its value is

$$\sum_{i=1}^{N} p_{k+1,i}^2 \mu(S_{k+1,i})^2 = \sum_{i=1}^{N} 1 = N.$$

Hence, $N = n_{k+1}$.

It is worth noting the following facts, all of which follow directly from the proof of Theorem 1 but are not given in the statement of the theorem. P_{k+1} is obtained from P_k by partitioning each set $S_{k,i}$ into two or more subsets unless $F_k(t, t) \equiv 0$ on $S_{k,i}$. In particular, if $F_k(t, t) > 0$ for all t and $n_{k+1} = 2n_k$, then each element of P_k splits into exactly two parts. If $n_{k+1} < 2n_k$, then

$F_k(t, t) \equiv 0$ on a set of positive measure. Finally we observe that if $F_k(t, t)$ is constant on $S_{k,i}$, then $S_{k,i}$ is partitioned into sets of equal measure.

4. **An application to Walsh functions.** In this section the unit interval $\{x \mid 0 \leqq x < 1\}$ will be denoted by I, the dyadic interval

$$\{x \mid r \cdot 2^{-k} \leqq x < (r+1)2^{-k}\} \text{ by } I(r, k),$$

and the dyadic partition $\{I(r, k)\}_{r=0}^{2^k-1}$ of I by J_k.

The Walsh functions(²) are step functions related to the sequence of partitions $J_0 > J_1 > J_2 > \cdots$ in the sense of Theorem 1. This suggests a characterization of the Walsh system by its Dirichlet kernels.

THEOREM 2. *Let $\mathfrak{F} = \{f_n(x)\}_{n=0}^{\infty}$ be an orthonormal set on I with the following properties.*

(a) $f_0(x) \equiv 1$.

(b) $D_{2^k}(x, y) \geqq 0$, $k \geqq 0$.

(c) *For each $n \geqq 0$, there is a partition $Q_n = \{Q_{n,j}\}_{j=0}^{n}$ of I into $n+1$ sub-intervals on which $f_n(x)$ is alternately non-negative and nonpositive. ($f_n(x)$ is non-negative on the sub-interval containing* 0.)

(d) *For each n,* $\sup_{Q_{n,j}} |f_n(x)|$ *is independent of j.*

Then $\mathfrak{F}$ is the set of Walsh functions.

Proof. By Theorem 1, assumptions (a) and (b) imply that $\mathfrak{F}$ is a system of step functions relative to a sequence of partitions $P_0 > P_1 > P_2 > \cdots$ of I, P_k having 2^k elements. It follows from (d) and the fact that $f_n(x)$ is normalized that $|f_n(x)| \equiv 1$. Consequently,

$$F_k(x, x) \equiv \sum_{n=2^k}^{2^{k+1}-1} f_n^2(x) \equiv 2^k.$$

By the remarks following Theorem 1, P_{k+1} arises by splitting each element of P_k into two subsets of equal measure. These must be intervals because of (c). Therefore, $\{P_k\}_{k=0}^{\infty}$ is the sequence of dyadic partitions $\{J_k\}_{k=0}^{\infty}$.

To complete the proof of Theorem 2, it suffices to prove the following assertion. If $\{f_n(x)\}_{n=0}^{2^k-1}$ is an orthonormal set of step functions (J_k) satisfying (c) such that $|f_n(x)| \equiv 1$, $0 \leqq n \leqq 2^k - 1$, then the given set is the set of Walsh functions $\{\psi_n(x)\}_{n=0}^{2^k-1}$ (in some order).

The proof is by induction. When $k = 0$, the assumptions imply $f_0(x) \equiv 1 \equiv \psi_0(x)$ and the assertion is true. Assuming it true for k, consider a set $\{f_n(x)\}_{n=0}^{2^{k+1}-1}$ satisfying the given conditions.

Let $2^k \leqq n \leqq 2^{k+1} - 1$. We claim that on two successive intervals of the form $I(2r, k+1)$ and $I(2r+1, k+1)$, $f_n(x)$ takes values ϵ and $-\epsilon$ respectively ($\epsilon = \pm 1$). To see this let $\chi(x)$ be the characteristic function of $I(r, k) = I(2r, k+1) \cup I(2r+1, k+1)$. Since $\{f_j(x)\}_{j=0}^{2^k-1}$ is clearly a basis for the space of step functions (J_k),

(²) For particulars on the Walsh functions see [1].

$$\chi(x) = \sum_{j=0}^{2^k-1} a_j f_j(x)$$

for appropriate coefficients a_j. Suppose $f_n(x)$ takes the values ϵ and ϵ' on $I(2r, k+1)$ and $I(2r+1, k+1)$ respectively. Then by orthogonality,

$$0 = \sum_{j=0}^{2^k-1} a_j \int_I f_j(x)f_n(x)dx = \int_I \chi(x)f_n(x)dx = \int_{I(r,k)} f_n(x)dx = \frac{\epsilon+\epsilon'}{2^{k+1}} \cdot$$

Hence $\epsilon' = -\epsilon$.

The above property is also possessed by the Rademacher function $\phi_k(x)$ defined by $\phi_k(x) \equiv (-1)^r$ on $I(r, k+1)$. Thus while $f_n(x)$ is a step function (P_{k+1}), the product $\phi_k(x)f_n(x)$ is a step function (P_k).

Consider the functions $\{g_{n'}(x) = \phi_k(x)f_{2^k+n'}(x)\}_{n'=0}^{2^k-1}$. We claim they satisfy all the conditions of our assertion. Since $|\phi_k(x)| \equiv 1$, it is clear that $|g_{n'}(x)| \equiv 1$ and that the $g_{n'}(x)$ form an orthonormal set of step functions (P_k). It remains to show that (c) holds.

Let $n = 2^k + n'$ where $0 \leqq n' \leqq 2^k - 1$. From our assumptions $f_n(x)$ has $n+1$ intervals of constancy, or equivalently, n discontinuities. The latter occur among the dyadic rationals $r \cdot 2^{-(k+1)}$, $1 \leqq r \leqq 2^{k+1}-1$. Multiplication by $\phi_k(x)$ removes these discontinuities (since $|f_n(x)| \equiv 1$), but introduces new ones at the remaining dyadic rationals $r \cdot 2^{-(k+1)}$. Therefore, $g_{n'}(x) = \phi_k(x)f_n(x)$ has exactly $2^{k+1} - 1 - n = 2^k - 1 - n'$ jumps. Hence, the set $\{g_{n'}(x)\}_{n'=0}^{2^k-1}$ can be reordered so that (c) is satisfied (set $h_{n'}(x) = g_{2^k-1-n'}(x)$). By the induction hypothesis, this set as well as $\{f_j(x)\}_{j=0}^{2^k-1}$ is the set $\{\psi_j(x)\}_{j=0}^{2^k-1}$.

Since $\phi_k^{-1}(x) = \phi_k(x)$, $f_n(x) = \phi_k(x)g_{n'}(x)$. Thus, the set $\{f_j(x)\}_{j=2^k}^{2^{k+1}-1}$ is obtained when the Walsh functions $\{\psi_j(x)\}_{j=0}^{2^k-1}$ are multiplied by $\phi_k(x)$. But this is precisely the definition of the Walsh functions $\{\psi_j(x)\}_{j=2^k}^{2^{k+1}-1}$. Therefore $\{f_j(x)\}_{j=0}^{2^{k+1}-1}$ is the set of Walsh functions $\{\psi_j(x)\}_{j=0}^{2^{k+1}-1}$ (in some order) which completes the induction.

References

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