VARIATIONAL METHODS FOR FUNCTIONS WITH POSITIVE REAL PART

BY

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1. **Introduction.** M. M. Schiffer [8] has recently derived a formula for the variation of the Green's function of the most general plane domain $\mathfrak{D}$ with boundary $\mathfrak{C}$ due to a small shift

$$(1.1) \quad w^* = w + \rho^2 \phi(w),$$

of the boundary. The variation $\delta g(\zeta, \eta)$ of the Green's function is given by the formula

$$(1.2) \quad \delta g(\zeta, \eta) = \operatorname{Re} \left\{ \frac{\rho^2}{2\pi i} \oint_{\Gamma} p'(w, \eta) p'(w, \zeta) \phi(w) dw \right\} + o(\rho^2)$$

where $p(w, \eta)$ is an analytic function whose real part is the Green function $g(w, \eta)$ of $\mathfrak{D}$, and where Γ is a member of a curve system in $\mathfrak{D}$ homotopic to $\mathfrak{C}$. The function $\phi(w)$ is analytic on Γ and in the ring bounded by $\mathfrak{C}$ and Γ . If $\mathfrak{D}$ is simply-connected and if $z = \psi(w)$ maps $\mathfrak{D}$ on the interior of the unit circle $|z| < 1$, then $g(w, \eta)$ is connected with $\psi(w)$ by the relation

$$(1.3) \quad g(w, \eta) = \log \left| \frac{1 - (\psi(\eta))^{-\psi(w)}}{\psi(w) - \psi(\eta)} \right|.$$

Here and throughout the paper $(\)^{-}$ indicates the complex conjugate. With an appropriate choice of $\phi(w)$ one may then obtain variation formulas for univalent functions $w = f(z)$. J. A. Hummel [5] has recently used this method of interior variations to study the class of univalently star-like functions. The method may also be used to study those functions which are convex-in-one direction [2]. The choice of the shift function $\phi(w)$, however, is not always an obvious one for many special classes of univalent functions, in particular for the class of close-to-convex functions [6]. Many of these special classes, however, have representations of their member functions in terms of functions $P(z)$ with positive real part. It therefore becomes desirable to have a variational formula for $P(z)$ from which one may then easily obtain analogous variational formulas for the special classes of univalent functions.

It is the purpose of this paper first to derive a variational formula for the class $\mathcal{P}$ of normalized regular functions

$$(1.4) \quad P(z) = 1 + p_1 z + p_2 z^2 + \cdots + p_n z^n + \cdots, \quad P(0) = 1,$$

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which have $\operatorname{Re} P(z) > 0$ in $|z| < 1$. Secondly, we shall apply the variational formula to $P(z)$ in order to solve extremal problems for the class $\mathcal{O}$ and in particular to obtain a characterization of the $(n-1)$ Euclidean coefficient space E_{n-1} for the extremal functions $P(z)$ for which $\operatorname{Re} p_n$ is a maximum. Although these results may also be deduced from the Carathéodory-Toeplitz theory [1; 3; 4; 7; 9; 10] it is interesting to see how simply they are derived by the variational method. Thirdly, we shall indicate how the variational formula for $P(z)$ leads to the variational formulas for bounded, regular functions $\omega(z)$ and also for regular functions $F(z) = f\{\omega(z)\}$ which are subordinate to a given univalent function $f(z)$ in $|z| < 1$.

2. A variational formula for the class $\mathcal{O}$. The choice

$$\phi(w) = \frac{e^{i\theta}}{w - w_0}$$

in (1.2), where w_0 is interior to $\mathfrak{D}$, is the Schiffer case for univalent functions [8]. Hummel [5] chose

$$(2.1) \quad \begin{aligned} \phi(w) &= wR[\psi(w)], \\ R(z) &= e^{i\theta} \cdot \frac{1 - \bar{z}_0 z}{z - z_0} + e^{-i\theta} \cdot \frac{z - z_0}{1 - \bar{z}_0 z}, \quad |z_0| < 1, \end{aligned}$$

where $R(z)$ is real and bounded on $|z| = 1$, to obtain the following variational formula for a normalized univalently star-like function $f(z)$ in $|z| < 1, f(0) = 0, f'(0) = 1$:

$$(2.2) \quad f^*(z) = f(z) \left[1 - \rho^2(1 - |z_0|^2) \left(A(z) - \frac{zf'(z)}{f(z)} B(z) \right) \right] + o(\rho^2)$$

where the error term $o(\rho^2)$ is an analytic function in z and uniformly bounded in each interior region of $|z| < 1$. $A(z)$ and $B(z)$ are defined by

$$(2.3) \quad A(z) = \frac{ze^{i\theta}}{z_0(z - z_0)} + \frac{ze^{-i\theta}}{1 - \bar{z}_0 z} - \frac{e^{i\theta}f(z_0)}{z_0^2 f'(z_0)}, \quad |z_0| < 1,$$

$$(2.4) \quad B(z) = \frac{e^{i\theta}f(z_0)}{z_0 f'(z_0)(z - z_0)} - e^{-i\theta} \left(\frac{f(z_0)}{z_0 f'(z_0)} \right)^{-1} \frac{z}{1 - \bar{z}_0 z}.$$

Since there is a (1-1) correspondence between the functions $P(z)$ of the class $\mathcal{O}$, given by (1.4), and the univalently star-like functions $f(z), f(0) = 0, f'(0) = 1$, by the relation

$$(2.5) \quad P(z) = \frac{zf'(z)}{f(z)}, \quad \operatorname{Re} P(z) > 0, \quad |z| < 1,$$

we therefore take the logarithmic derivative in (2.2) and replace $z(f^{*'}(z))/f^*(z)$ by $P^*(z)$. After some simplification we obtain

$$(2.6) \quad P^*(z) = P(z) - \rho^2(1 - |z_0|^2)z \\ \cdot \left[\frac{ze^{i\theta}}{z_0(z - z_0)} + \frac{ze^{-i\theta}}{1 - \bar{z}_0 z} - \frac{e^{i\theta}P(z)}{P(z_0)(z - z_0)} + \frac{e^{-i\theta}zP(z)}{(P(z_0))^{-}(1 - \bar{z}_0 z)} \right]' + o(\rho^2),$$

where $[\]'$ denotes differentiation with respect to z . Let $\delta P(z) = P^*(z) - P(z)$, and write (2.6) in the form

$$(2.7) \quad \frac{-\delta P(z)}{\rho^2(1 - |z_0|^2)} = \left(\frac{z_0 P'(z)}{P(z_0)} - 1 \right) \frac{e^{i\theta}z}{z_0(z - z_0)} + \left(\frac{z_0 P(z)}{P(z_0)} - z \right) \frac{e^{i\theta}z^4}{z_0(z - z_0)^2} \\ + \frac{P'(z)}{(P(z_0))^{-}} \left(\frac{z^2 e^{-i\theta}}{1 - \bar{z}_0 z} \right) + \left(\frac{P(z)}{(P(z_0))^{-}} + 1 \right) \frac{e^{-i\theta}z}{(1 - \bar{z}_0 z)^2} + o(1),$$

which is the required variational formula for functions of class $\mathcal{O}$.

If $P(z)$ has the power series expansion (1.4), and if we denote $\delta p_n = p_n^* - p_n$, (2.7) yields

$$(2.8) \quad \frac{-\delta p_n}{\rho^2(1 - |z_0|^2)} = \frac{ne^{i\theta}}{P(z_0)} \sum_{k=0}^n \frac{p_k}{z_0^{n-k+1}} - \frac{ne^{i\theta}}{z_0^{n+1}} + \frac{ne^{-i\theta}}{(P(z_0))^{-}} \sum_{k=0}^{n-1} p_k (\bar{z}_0)^{n-k-1} \\ + ne^{-i\theta} (\bar{z}_0)^{n-1} + o(1)$$

where $p_0 = 1$. Since for any complex number w , $\operatorname{Re} w = \operatorname{Re} \bar{w}$, we have

$$(2.9) \quad \frac{-\operatorname{Re} \delta p_n}{\rho^2(1 - |z_0|^2)} = n \operatorname{Re} \left\{ \frac{e^{i\theta}}{z_0} \left[\frac{1}{P(z_0)} \sum_{k=0}^{n-1} \left(\frac{p_k}{z_0^{n-k}} + \bar{p}_k z_0^{n-k} \right) \right. \right. \\ \left. \left. + \frac{p_n}{P(z_0)} + z_0^n - \bar{z}_0^{-n} \right] \right\} + o(1).$$

3. Extremal functions for the class $\mathcal{O}$. For any positive integer n let $P(z)$ be an extremal function of the class $\mathcal{O}$ for which the coefficient p_n in (1.4) is real, positive and a maximum over all functions of the class. Since $\mathcal{O}$ is compact p_n attains its maximum. Its value is 2 as is well known from the Carathéodory theory, although we do not assume this fact here. Since $\operatorname{Re} \delta p_n \leq 0$ in (2.9) and θ is arbitrary we have from (2.9) on replacing z_0 by z

$$(3.1) \quad \frac{1}{P(z)} \left\{ p_n + \sum_{k=0}^{n-1} \left(\frac{p_k}{z^{n-k}} + \bar{p}_k z^{n-k} \right) \right\} + z^n - \bar{z}^{-n} = 0, \quad p_n > 0, p_0 = 1,$$

$$(3.2) \quad P(z) = \frac{1 + p_1 z + p_2 z^2 + \cdots + p_n z^n + \cdots + p_{2n-1} z^{2n-1} + z^{2n}}{1 - \bar{z}^{2n}} = \frac{Q_{2n}(z)}{1 - \bar{z}^{2n}}$$

where $p_{2n-k} = \bar{p}_k$, $k = 1, 2, \dots, 2n-1$.

Although ordinarily the variational formulas for extremal functions of various classes of functions lead to differential equations we find here for the

class $\mathcal{O}$ that we are led directly to the extremal functions (3.2) without encountering a differential equation. However, the formula (3.2) may be simplified further. We shall presently show that whenever $P(z)$ has the form (3.2) and has a positive real part in $|z| < 1$ then $p_n \leq 2$, and if $p_n = 2$ then $(1+z^n)$ is a factor of both numerator and denominator of (3.2). In this case (3.2) becomes

$$(3.3) \quad P(z) = \frac{1 + p_1 z + p_2 z^2 + \cdots + p_{n-1} z^{n-1} + z^n}{1 - z^n},$$

where $p_{n-k} = \bar{p}_k$, $k = 1, 2, \dots, n-1$.

To obtain (3.3) we place $z = re^{k\pi i/n}$, $0 < r < 1$, $k = \text{odd integer}$, in (3.2). Then

$$P(re^{k\pi i/n}) = \frac{Q_{2n}(re^{k\pi i/n})}{1 - r^{2n}}.$$

Since $\operatorname{Re} P(z) > 0$ it follows that $\operatorname{Re} Q_{2n}(re^{k\pi i/n}) > 0$. Letting $r \rightarrow 1$ we have $\operatorname{Re} Q_{2n}(e^{k\pi i/n}) \geq 0$. However, $Q_{2n}(e^{k\pi i/n})$ is real. This follows since $p_r z^r + p_{2n-r} z^{2n-r} = p_r z^r + \bar{p}_r z^{-r}$ is a real number when $z = e^{k\pi i/n}$, and because $p_n z^n$ and z^{2n} are also real for this choice of z . Thus

$$(3.4) \quad Q_{2n}(e^{k\pi i/n}) \geq 0.$$

Let

$$(3.5) \quad P_{n-1}(z) = \sum_{s=1}^{n-1} p_s z^s,$$

$$(3.6) \quad Q_{2n}(z) = 1 + p_n z^n + z^{2n} + P_{n-1}(z) + z^{2n} \cdot \left(P_{n-1} \left(\frac{1}{\bar{z}} \right) \right)^-.$$

For k odd, $z = e^{k\pi i/n}$, we have $z^{2n} = 1$, $z^n = -1$, so that

$$(3.7) \quad 0 \leq Q_{2n}(e^{k\pi i/n}) = (2 - p_n) + P_{n-1}(e^{k\pi i/n}) + (P_{n-1}(e^{k\pi i/n}))^-,$$

$$(3.8) \quad 0 \leq (2 - p_n) + 2 \operatorname{Re} P_{n-1}(e^{k\pi i/n}) = (2 - p_n) + 2 \operatorname{Re} \left(\sum_{s=1}^{n-1} p_s e^{ks\pi i/n} \right).$$

By virtue of the identity

$$(3.9) \quad \sum_{r=1}^n e^{(2r-1)s\pi i/n} = 0, \quad s = 1, 2, \dots, n-1,$$

it follows that

$$(3.10) \quad \operatorname{Re} \sum_{r=1}^n P_{n-1}(e^{(2r-1)\pi i/n}) = 0.$$

From (3.8) and (3.10) we then have

$$(3.11) \quad 0 \leq \sum_{r=1}^n [2 - p_n + 2 \operatorname{Re} P_{n-1}(e^{(2r-1)\pi i/n})] = n(2 - p_n).$$

Thus $p_n \leq 2$. But since $(1 + 2 \sum_1^\infty z^n)$ is a member of class $\mathcal{O}$ and p_n is maximal we must have $p_n = 2$. In this case (3.8) reduces to

$$(3.12) \quad \operatorname{Re} P_{n-1}(e^{k\pi i/n}) \geq 0, \quad k \text{ odd.}$$

However, only equality can hold in (3.12) since otherwise (3.10) would be contradicted. (3.7) now becomes

$$(3.13) \quad Q_{2n}(e^{k\pi i/n}) = 2 \operatorname{Re} P_{n-1}(e^{k\pi i/n}) = 0, \quad k \text{ odd.}$$

It follows at once that $(1 + z^n)$ is a factor of $Q_{2n}(z)$.

If we set

$$(3.14) \quad 1 + z^{2n} + \sum_{s=1}^{2n-1} p_s z^s = Q_{2n}(z) = (1 + z^n) \sum_0^n q_s z^s$$

where $p_{2n-s} = \bar{p}_s$, $s = 1, \dots, 2n-1$, and $p_n = 2$, we find that

$$q_s = p_s, \quad s = 1, 2, \dots, n-1;$$

$$q_0 = q_n = 1;$$

$$\bar{p}_s = p_{2n-s} = q_{n-s} = p_{n-s}, \quad s = 1, 2, \dots, n-1.$$

Thus we have shown that (3.3) follows from (3.2).

It is interesting to observe also that the extremal functions of (3.3) satisfy the identity

$$(3.15) \quad P(z) + \left(P\left(\frac{1}{\bar{z}}\right) \right)^- = 0.$$

Consequently, the real part of $P(z)$ vanishes identically on $|z| = 1$.

If we let $\omega_k = e^{2k\pi i/n}$, $k = 1, 2, \dots, n$, we may write $P(z)$ of (3.3) in the form

$$(3.16) \quad P(z) = \sum_{k=1}^n \lambda_k \left(\frac{1 + \omega_k z}{1 - \omega_k z} \right), \quad 0 \leq \lambda_k \leq 1, \quad \sum_1^n \lambda_k = 1.$$

$\operatorname{Re} P(z) \equiv 0$ on $|z| = 1$. But if we let $z = e^{i\theta} \rightarrow \bar{\omega}_r$, we find that the real part of the right-hand side of equation (3.16) is unbounded unless λ_r is real. Moreover $\lambda_r \geq 0$. For if we assume $\lambda_r \neq 0$ and let $z = r\bar{\omega}_r$, $0 < r < 1$, as r approaches 1 we find that $\operatorname{Re} P(z)$ must coincide in sign with that of λ_r . Furthermore

$$\sum_1^n \lambda_k = P(0) = 1.$$

It follows from (3.3) and (3.16) that the coefficients p_n in (3.3) must be expressible in terms of the barycentric coordinates λ_k as follows:

$$(3.17) \quad \begin{cases} p_\nu = 2 \sum_{k=1}^n \lambda_k e^{2\nu k \pi i / n}, & 0 \leq \lambda_k \leq 1, \sum_1^n \lambda_k = 1, \quad \nu = 1, 2, \dots, n; \\ p_{n-\nu} = \bar{p}_\nu, & \nu \leq n - \nu, 1 \leq \nu < n. \end{cases}$$

Conversely, if $0 \leq \lambda_k \leq 1$, $\sum_1^n \lambda_k = 1$, then $P(z)$ given by (3.16) has $\operatorname{Re} P(z) > 0$, $|z| < 1$.

Let $p_k = x_k + iy_k$. Since $p_{n-k} = \bar{p}_k$, it is seen that the coefficient space E_{n-1} of the extremal functions $P(z)$, for which $p_n = 2$, depends upon $(n-1)$ real variables x_k, y_k .

$$(3.18) \quad \begin{aligned} E_{n-1} &= E_{n-1}(p_1, p_2, \dots, p_{(n-1)/2}) \\ &= E_{n-1}(x_1, y_1, x_2, y_2, \dots, x_{(n-1)/2}, y_{(n-1)/2}) \end{aligned}$$

for n odd > 1 , and

$$(3.19) \quad \begin{aligned} E_{n-1} &= E_{n-1}(p_1, p_2, \dots, p_{n/2}) \\ &= E_{n-1}(x_1, y_1, \dots, x_{(n-2)/2}, y_{(n-2)/2}, x_{n/2}) \end{aligned}$$

for n even. We define E_0 to be the point $p_1 = 2$ corresponding to the extremal $P(z) = (1+z)/(1-z)$.

From (3.17) it is readily seen that $E_1(p_1)$ is the closed 1-simplex, or line segment $-2 \leq p_1 \leq 2$, corresponding to

$$(3.20) \quad P(z) = \frac{1 + p_1 z + z^2}{1 - z^2}, \quad p_1 \text{ real}, -2 \leq p_1 \leq 2.$$

$E_2(p_1)$ is a closed 2-simplex consisting of an equilateral triangle with vertices $(2, 0)$, $(-1, 3^{1/2})$ and $(-1, -3^{1/2})$ and E_2 corresponds to the extremal functions of the form

$$(3.21) \quad P(z) = \frac{1 + p_1 z + \bar{p}_1 z^2 + z^3}{1 - z^3}, \quad p_1 \in E_2.$$

$E_3(p_1, p_2)$ is a tetrahedron with vertices $(0, 2, -2)$, $(0, -2, -2)$, $(-2, 0, 2)$ and $(2, 0, 2)$. E_3 corresponds to the extremal function

$$(3.22) \quad P(z) = \frac{1 + p_1 z + p_2 z^2 + \bar{p}_1 z^3 + z^4}{1 - z^4},$$

where p_2 is real.

In general E_{n-1} is the closed $(n-1)$ -simplex with the n vertices:

$$(3.23) \quad \left(2 \cos \frac{2\nu\pi}{n}, 2 \sin \frac{2\nu\pi}{n}, 2 \cos \frac{4\nu\pi}{n}, 2 \sin \frac{4\nu\pi}{n}, \dots, \right. \\ \left. 2 \cos (n-2) \frac{\nu\pi}{n}, 2 \sin (n-2) \frac{\nu\pi}{n}, 2 \cos \nu\pi \right),$$

$\nu = 1, 2, \dots, n$, when n is even, or

$$(3.24) \quad \left(2 \cos \frac{2\nu\pi}{n}, 2 \sin \frac{2\nu\pi}{n}, 2 \cos \frac{4\nu\pi}{n}, 2 \sin \frac{4\nu\pi}{n}, \dots, \right. \\ \left. 2 \cos (n-1) \frac{\nu\pi}{n}, 2 \sin (n-1) \frac{\nu\pi}{n} \right)$$

if n is odd.

The boundary hyperplanes of E_{n-1} (corresponding to a $\lambda_k = 0$) for n odd, $n > 1$, have the equations, for $k = 0, 1, \dots, n-1$,

$$(3.25) \quad 1 + \sum_{m=1}^{(n-1)/2} \left(x_m \cos \frac{2km\pi}{n} - y_m \sin \frac{2km\pi}{n} \right) = 0;$$

for n even, $n > 2$, the equations of the hyperplanes are

$$(3.26) \quad 1 + \frac{(-1)^k}{2} x_{n/2} + \sum_{m=1}^{(n-2)/2} \left(x_m \cos \frac{2km\pi}{n} - y_m \sin \frac{2km\pi}{n} \right) = 0$$

where $p_m = x_m + iy_m$.

It is seen that the hyperplanes (3.25) and (3.26) are tangent to the spheres

$$(3.27) \quad \sum_{m=1}^{(n-1)/2} (x_m^2 + y_m^2) = \frac{4}{2n-2}, \quad x_{n/2}^2 + \sum_{m=1}^{(n-2)/2} (x_m^2 + y_m^2) = \frac{4}{2n-3},$$

respectively.

We summarize these results in the following theorem and corollaries.

THEOREM 1. *Let the function*

$$P(z) = 1 + p_1 z + \dots + p_n z^n + \dots$$

be regular and have a positive real part in $|z| < 1$. Then $|p_n| \leq 2$, and $p_n = 2$ for a given n when, and only when, $P(z)$ is of the form

$$P(z) = \frac{1 + p_1 z + p_2 z^2 + \dots + p_{n-1} z^{n-1} + z^n}{1 - z^n}, \quad p_{n-k} = \bar{p}_k, 0 < k < n,$$

and the coefficient space E_{n-1} of $P(z)$ is the closed $(n-1)$ -simplex determined by the equations (3.17). The vertices of the $(n-1)$ -simplex are given by (3.23) and (3.24) and the boundary hyperplanes by the equations (3.25) and (3.26).

COROLLARY 1. *If n is an odd positive integer > 1 and*

$$|p_1|^2 + \dots + |p_{(n-1)/2}|^2 \leq \frac{2}{n-1},$$

or if n is an even integer ≥ 2 and

$$|p_1|^2 + \cdots + |p_{n/2}|^2 \leq \frac{4}{2n-3},$$

then the function

$$P(z) = (1 + p_1 z + p_2 z^2 + \cdots + p_{n-1} z^{n-1} + z^n) \div (1 - z^n),$$

$p_{n-k} = \bar{p}_k$, $2k \leq n$, has $\operatorname{Re} P(z) > 0$ in $|z| < 1$.

COROLLARY 2. With the notation of Theorem 1 the boundary of the $(n-1)$ -simplex E_{n-1} given by (3.17) is determined from the equation $\Delta_n = 0$, $n > 1$, where Δ_n is the determinant

$$\Delta_n = \begin{vmatrix} 2 & p_1 & p_2 & \cdots & p_{n-1} \\ \bar{p}_1 & 2 & p_1 & \cdots & p_{n-2} \\ \bar{p}_2 & \bar{p}_1 & 2 & \cdots & p_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{p}_{n-1} & \bar{p}_{n-2} & \bar{p}_{n-3} & \cdots & 2 \end{vmatrix} = 2^n (n!)^2 \sin^2 \frac{\pi}{n} \cdot \prod_{k=1}^n \lambda_k.$$

Proof of Corollary 2. Using the representation (3.17) for the coefficients p , we write Δ_n as the product of two determinants, $\Delta_n = 2^n A_n B_n$, where

$$A_n = \begin{vmatrix} \lambda_1, \lambda_1 e^{-2\pi i/n}, \lambda_1 e^{-4\pi i/n}, \dots, \lambda_1 e^{-2(n-1)\pi i/n} \\ \lambda_2, \lambda_2 e^{-4\pi i/n}, \lambda_2 e^{-8\pi i/n}, \dots, \lambda_2 e^{-4(n-1)\pi i/n} \\ \lambda_3, \lambda_3 e^{-6\pi i/n}, \lambda_3 e^{-12\pi i/n}, \dots, \lambda_3 e^{-6(n-1)\pi i/n} \\ \vdots \\ \lambda_n, \lambda_n e^{-2n\pi i/n}, \lambda_n e^{-4n\pi i/n}, \dots, \lambda_n e^{-2n(n-1)\pi i/n} \end{vmatrix}$$

and B_n is the determinant A_n with the λ 's replaced by 1's and i replaced by $-i$. Thus $\prod_1^n \lambda_k$ is a factor of A_n and we obtain

$$\Delta_n = 2^n \left(\prod_1^n \lambda_k \right) |B_n|^2 \geq 0.$$

In the determinant B_n except for the first column, the elements in the columns add up to zero, so that the order of B_n is easily reduced a step at a time. Thus

$$\begin{aligned} \Delta_n &= 2^n \left(\prod_1^n \lambda_k \right) n^2 (n-1)^2 \cdots 3^2 \left\| \begin{array}{cc} 1 & 1 \\ 1 & e^{2\pi i/n} \end{array} \right\|^2 \\ (3.28) \quad &= 2^n (n!)^2 \sin^2 \frac{\pi}{n} \left(\prod_1^n \lambda_k \right) \quad n > 1, 0 < \lambda_k < 1. \\ &> 0, \end{aligned}$$

Since a boundary point of E_{n-1} corresponds to a λ_k having the values 0 or 1,

$\Delta_n=0$ defines the boundary of E_{n-1} for $n>1$. When $n=1$, $\Delta_1=2$ and E_0 is the point $p_1=2$.

Although we have confined our attention to extremal functions for which $\operatorname{Re} p_n$ is a maximum, the formula (2.8) for δp_n may be used to determine the extremal functions $P(z)$ which maximize $\operatorname{Re} F(p_1, p_2, \dots, p_n)$ where $F(p_1, \dots, p_n)$ is any continuous function having continuous partial derivatives in an open set containing the coefficient space $V_n(p_1, \dots, p_n)$ of the class $\mathcal{O}$ and for which the partial derivatives $\lambda_k = \partial F / \partial p_k$, $k=1, \dots, n$, are not all zero at the point $(p_1, \dots, p_n)$ determined by the extremal function. We find that the extremal function maximizes $\operatorname{Re}(\sum_1^n \lambda_k p_k)$ and is the function

$$(3.29) \quad P(z) = \frac{\sum_{k=1}^n k \left(\lambda_k \sum_{s=0}^k \frac{p_s}{z^{k-s}} + \bar{\lambda}_k \sum_{s=0}^k \bar{p}_s z^{k-s} \right)}{\sum_{k=1}^n k \left(\frac{\lambda_k}{z^k} - \bar{\lambda}_k z^k \right)}, \quad p_0 = 1.$$

It is seen from (3.29) that the extremal functions $P(z)$ maximizing $\operatorname{Re} F$ have the property that the real part of $P(z)$ vanishes identically on $|z|=1$.

In particular, one may obtain the extremal functions which minimize the Toeplitz form

$$F = \sum p_{\mu-\nu} X_\nu \bar{X}_\mu.$$

F then turns out to be non-negative in accordance with the Carathéodory-Toeplitz theory. We omit the details.

Turning to another problem we shall now see how the variational technique easily leads to the well-known inequality

$$(3.30) \quad \operatorname{Re} P(z) \geq \frac{1-r}{1+r}, \quad |z| = r < 1,$$

for functions of class $\mathcal{O}$.

Let $P_0(z)$ be an extremal function for which, when z is fixed in the unit circle, $\operatorname{Re} P_0(z)$ is a minimum for the class $\mathcal{O}$. By a rotation in the z -plane we may assume z to be a positive number r . Since $\operatorname{Re} \delta P_0(r) \geq 0$ in (2.7) we have $\operatorname{Re} e^{i\theta} A \leq 0$ for all θ where

$$(3.31) \quad \begin{aligned} A = A(r) = & \left(\frac{z_0 P'_0(r)}{P_0(z_0)} - 1 \right) \frac{r}{z_0(z_0 - r)} + \left(\frac{z_0 P_0(r)}{P_0(z_0)} - r \right) \frac{r}{z_0(z_0 - r)^2} \\ & + \frac{(P'_0(r))^-}{P_0(z_0)} \cdot \frac{r^2}{1 - z_0 r} + \left(\frac{(P_0(r))^-}{P_0(z_0)} + 1 \right) \frac{r}{(1 - z_0 r)^2}. \end{aligned}$$

Since θ is arbitrary it follows that $A=0$. Replacing z_0 by z and solving the equation $A=0$ for $P_0(z)$ we obtain

$$\begin{aligned}
 (1-r^2)(1-z^2)P_0(z) &= A_3z^3 + A_2z^2 + A_1z + A_0, \\
 A_0 &= -rP'_0(r) + r^2(P'_0(r))^- + P_0(r) + r^2(P_0(r))^- = (1-r^2)P_0(0) \\
 &= 1-r^2, \\
 (3.32) \quad A_1 &= (1+2r^2)P'_0(r) - (2r^2+r^4)(P'_0(r))^- - 2r\{P_0(r) + (P_0(r))^- \}, \\
 A_2 &= (2r^3+r)(P'_0(r))^- - (2r+r^3)P'_0(r) + r^2P_0(r) + (P_0(r))^-, \\
 A_3 &= r^2\{P'_0(r) - (P'_0(r))^- \}.
 \end{aligned}$$

Since $\operatorname{Re} P(r) \geq \operatorname{Re} P_0(r)$ for all $P(z) \in \mathcal{O}$, and since $P(z) = P_0(ze^{i\theta}) \in \mathcal{O}$, we have $\operatorname{Re} P_0(re^{i\theta}) \geq \operatorname{Re} P_0(r)$ for all θ . Thus $\operatorname{Re} P_0(r)$ is a minimum value of $\operatorname{Re} P_0(re^{i\theta})$ as a function of θ . It follows from the Cauchy-Riemann equations that

$$IP'_0(r) = \frac{\partial}{\partial r} IP_0(re^{i\theta}) \Big|_{\theta=0} = -\frac{1}{r} \frac{\partial}{\partial \theta} \operatorname{Re} P_0(re^{i\theta}) \Big|_{\theta=0} = 0.$$

Thus $P'_0(r)$ is real. Then $A_3 = 0$. Since A_0 is real it follows that $\{P_0(r) + r^2(P_0(r))^- \}$ and $\{P_0(r) + (P_0(r))^- \}$ are both real. This implies that $P_0(r)$ is real. Since $A_2 - A_0 = (1-r^2)\{(P_0(r))^- - P_0(r)\}$, we have $A_2 = A_0 = 1-r^2$. Also A_1 is seen to be real. We have now seen that $P_0(z)$ is of the form

$$(3.33) \quad P_0(z) = \frac{1 + kz + z^2}{1 - z^2}, \quad k \text{ real.}$$

Since $P_0(r) \geq 0$, we have $k \geq -2$. Since $P_0(r)$ is minimal for the class $\mathcal{O}$ we must take $k = -2$. In this case

$$(3.34) \quad P_0(z) = \frac{1-z}{1+z}$$

so that (3.30) follows with equality holding only for the function $P_0(\epsilon z)$, $|\epsilon| = 1$.

It should be noticed that the equation $A_0 = 1-r^2$ may not be treated as a differential equation for finding $P_0(r)$ unless it is first shown that the extremal function P_0 does not vary with r .

4. Interior variations for subordinate functions. Let the analytic function

$$(4.1) \quad f(z) = A_1z + A_2z^2 + \cdots + A_nz^n + \cdots, \quad f(0) = 0, A_1 \neq 0,$$

be regular and univalent in $|z| < 1$. Let

$$(4.2) \quad F(z) = a_1z + a_2z^2 + \cdots + a_nz^n + \cdots, \quad F(0) = 0,$$

be regular and subordinate to $f(z)$ in $|z| < 1$. Then

$$(4.3) \quad F(z) = f(\omega(z))$$

where $\omega(z)$ is regular in $|z| < 1$, $\omega(0) = 0$, $|\omega(z)| < 1$ for $|z| < 1$. We may write $\omega(z)$ in the form

$$(4.4) \quad \omega(z) = \frac{P(z) - 1}{P(z) + 1}$$

where $P(z)$ is a member of class $\mathcal{P}$. We also denote by $z = \phi(w)$ the inverse function of $w = f(z)$. If $P^*(z)$ is given by (2.6) upon varying $P(z)$, and if $\omega^*(z)$ corresponds by (4.4) to $P^*(z)$, we easily obtain

$$(4.5) \quad \begin{aligned} \omega^*(z) &= \frac{P^*(z) - 1}{P^*(z) + 1} = \frac{P(z) - 1}{P(z) + 1} + \rho^2 \lambda(z) + o(\rho^2) \\ &= \omega(z) + \rho^2 \lambda(z) + o(\rho^2), \end{aligned}$$

where

$$(4.6) \quad \lambda(z) = -\frac{1}{2} (1 - |z_0|^2)(1 - \phi(F))^2 A(z)$$

and where $A(z)$ is defined as the right-hand side of equation (2.7) omitting the term $o(1)$. We also have

$$(4.7) \quad P(z) = \frac{1 + \phi(F)}{1 - \phi(F)}, \quad (P(z) + 1)^2 = 4(1 - \phi(F))^{-2},$$

$$(4.8) \quad P'(z) = \frac{2\phi'(F)F'(z)}{(1 - \phi(F))^2}.$$

We now write $A(z)$ in the form

$$(4.9) \quad \begin{aligned} A(z) &= \frac{2\phi'(F)F'(z)}{(1 - \phi(F))^2} \left\{ \frac{e^{i\theta}z}{P(z_0)(z_0 - z)} + \frac{z^2 e^{-i\theta}}{(P(z_0))^{-1}(1 - \bar{z}_0 z)} \right\} \\ &+ \frac{1 + \phi(F)}{1 - \phi(F)} \left\{ \frac{e^{i\theta}z}{P(z_0)(z_0 - z)^2} + \frac{e^{-i\theta}z}{(P(z_0))^{-1}(1 - \bar{z}_0 z)^2} \right\} \\ &+ \left\{ \frac{e^{-i\theta}z}{(1 - \bar{z}_0 z)^2} - \frac{e^{i\theta}z^2}{z_0(z_0 - z)^2} - \frac{e^{i\theta}z}{z_0(z_0 - z)} \right\} \end{aligned}$$

where

$$(4.10) \quad \begin{aligned} P(z_0) &= \frac{1 + \phi(F(z_0))}{1 - \phi(F(z_0))}, & |z_0| &< 1. \\ F^*(z) &= f(\omega(z) + \rho^2 \lambda(z) + o(\rho^2)) \\ &= F(z) + f'(\omega(z)) \lambda(z) \rho^2 + o(\rho^2) \\ &= F(z) + \frac{\lambda(z)}{\phi'(F)} \rho^2 + o(\rho^2). \end{aligned}$$

Thus the variational formula for analytic functions $F(z)$ subordinate to a given univalent function $f(z)$ in $|z| < 1$ is given by

$$(4.11) \quad F^*(z) = F(z) - \rho^2(1 - |z_0|^2) \frac{(1 - \phi(F))^2}{2\phi'(F)} A(z) + o(\rho^2)$$

where $A(z)$ is defined by (4.9) and ϕ is the inverse of f .

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