A NOTE ON THE MEAN VALUE PROPERTY

BY

ADRIANO M. GARSIA

Introduction. In a recent paper⁽¹⁾ A. Friedman and W. Littman have considered the following problem. In n -dimensional space is given a mass distribution μ with compact support S_μ of total mass one. Then, a function u defined in a domain D is said to satisfy the mean value property in D with respect to μ if

$$(1.1) \quad u(x) = \int u(x + ty) d\mu(y)$$

for all $x \in D$ and sufficiently small $t > 0$.

In the above mentioned paper the authors show that if μ is positive and S_μ is not flat, then all functions satisfying the mean value property with respect to μ are analytic. If μ in addition satisfies certain restrictions, then the functions satisfying the mean value property with respect to μ fill only a finite dimensional vector space of polynomials.

In particular the latter result is shown to hold if μ is positive, S_μ is not flat and contains only a finite number of points. It is also shown that when μ is positive and concentrates equal masses at N points, then the polynomials satisfying the mean value property with respect to μ are all of degree less than or equal to $N(N-1)/2$.

It develops that the last mentioned result holds under lesser conditions on the mass μ .

It will be the purpose of this note to show that the following theorem holds.

THEOREM. *If μ concentrates all its mass on a finite set of points $S = P_1 \cup P_2 \cup \dots \cup P_N$ whose convex hull is not flat and if in addition, setting $\mu_i = \int_{(P_i)} d\mu$ we have*

$$\mu_{i_1} + \mu_{i_2} + \dots + \mu_{i_k} \neq 0 \quad \text{for all } i \leq i_1 < i_2 < \dots < i_k \leq N$$

then the functions satisfying the mean value property with respect to μ fill only a space of polynomials of degree less than or equal to $N(N-1)/2$.

1.1. An auxiliary result. To establish the theorem we shall need the following

Received by the editors April 13, 1961.

⁽¹⁾ *Editorial note.* See *Functions satisfying the mean value property*, immediately preceding this paper.

LEMMA. Let $\sigma_p(y_1, y_2, \dots, y_N)$ for $p=1, 2, \dots, N$ be polynomials in the indeterminates $y_1, y_2, \dots, y_N$ defined as follows

$$(1.11) \quad \sigma_p(y_1, y_2, \dots, y_N) = \sum_{1 \leq i_1 < \dots < i_p \leq N} c_{i_1, i_2, \dots, i_p} y_{i_1} y_{i_2} \dots y_{i_p}.$$

Then if $c_{i_1, i_2, \dots, i_p} \neq 0$ for all $1 \leq i_1 < i_2 < \dots < i_p \leq N$, all monomials of the type

$$\alpha = y_1^{k_1} y_2^{k_2} \dots y_N^{k_N}, \quad k_1 + k_2 + \dots + k_N > \frac{N(N-1)}{2}$$

belong to the ideal generated by $\sigma_1, \sigma_2, \dots, \sigma_N$.

Proof. Let $K = (k_1, k_2, \dots, k_N)$ denote an N -tuple of non-negative integers such that $k_1 \geq k_2 \geq \dots \geq k_N$. We shall set $d(K) = k_1 + k_2 + \dots + k_N$. If α is a monomial in the y_i 's, we shall write $\exp \alpha = K = (k_1, k_2, \dots, k_N)$ if and only if

$$\alpha = y_{i_1}^{k_1} y_{i_2}^{k_2} \dots y_{i_N}^{k_N}$$

where $(i_1, i_2, \dots, i_N)$ is some permutation of $(1, 2, \dots, N)$. The set of all K such that $d(K) = M$ (for a fixed M) will be ordered by setting

$$K = (k_1, k_2, \dots, k_N) < H = (h_1, h_2, \dots, h_N)$$

if and only if

$$k_1 = h_1, \dots, k_\nu = h_\nu, \text{ but } k_{\nu+1} < h_{\nu+1} \quad \text{for some } 1 \leq \nu \leq N-1.$$

We shall also adopt the following convention: for two polynomials $A(y_1, y_2, \dots, y_N)$ and $B(y_1, y_2, \dots, y_N)$ the relation $A \equiv B$ is to mean that the polynomial $A - B$ belongs to the ideal generated by $\sigma_1, \sigma_2, \dots, \sigma_N$.

Now, suppose that $H = (h_1, h_2, \dots, h_N)$ is such an N -tuple that

- (1) $d(H) = M$.
- (2) For every monomial α the relations $d(\exp \alpha) = M$, $\exp \alpha < H$ imply $\alpha \equiv 0$.
- (3) For some permutation $(j_1, j_2, \dots, j_N)$ of $(1, 2, \dots, N)$

$$(1.12) \quad \alpha_0 = y_{j_1}^{h_1} y_{j_2}^{h_2} \dots y_{j_N}^{h_N} \neq 0.$$

Then one of the following must hold: either

$$(a) \quad h_\lambda > h_{\lambda+1} + 1 \quad \text{for some } 1 \leq \lambda \leq N-1$$

or

$$(b) \quad h_\lambda \leq h_{\lambda+1} + 1 \quad \text{for all } 1 \leq \lambda \leq N-1.$$

In the case that (a) holds we express $y_{j_1} y_{j_2} \dots y_{j_\lambda}$ by means of (1.11) written for $p=\lambda$ and obtain

$$\begin{aligned} y_{j_1} y_{j_2} \cdots y_{j_\lambda} &\equiv \frac{-1}{c_{j_1, j_2, \dots, j_\lambda}} \sum^{(i_1, i_2, \dots, i_\lambda)} c_{i_1, i_2, \dots, i_\lambda} y_{i_1} y_{i_2} \cdots y_{i_\lambda} \\ &= c(y_{j_1}, y_{j_2}, \dots, y_{j_\lambda}) \end{aligned}$$

where the sum is to include all the terms in (1.11) with the exception of the term in $y_{j_1} y_{j_2} \cdots y_{j_\lambda}$. Hence we get

$$(1.13) \quad \alpha_0 \equiv y_{j_1}^{h_1-1} y_{j_2}^{h_2-1} \cdots y_{j_\lambda}^{h_\lambda-1} y_{j_{\lambda+1}}^{h_{\lambda+1}} \cdots y_{j_N}^{h_N} C(y_{j_1}, y_{j_2}, \dots, y_{j_\lambda}).$$

It is clear that the expression on the right-hand side of (1.13) when expanded will yield a linear combination of monomials of degree M . We claim that for each monomial α that will be produced we shall have $\exp \alpha < H$. In fact, each monomial appearing in $C(y_{j_1}, y_{j_2}, \dots, y_{j_\lambda})$ contains at most $\lambda - 1$ of the variables $y_{j_1} y_{j_2} \cdots y_{j_\lambda}$. Thus for each α appearing on the right-hand side of (1.13) $\exp \alpha$ will not have all the first λ components as large as $h_1, h_2, \dots, h_\lambda$. For, every exponent h_i with $i \geq \lambda + 1$ even if increased by one unit, can never become as large as one of the $h_1, h_2, \dots, h_\lambda$.

From these considerations, in view of the hypothesis (2) we obtain $\alpha_0 \equiv 0$, a contradiction. Thus only (b) is possible. However, (b) implies that

$$d(H) \leq N h_N + 1 + 2 + \cdots + N - 1.$$

Now, if $d(H) > 1 + 2 + \cdots + N - 1$, we must have $N h_N \geq 1$; but this (in view of (1.11) written for $p = N$) would again imply $\alpha_0 \equiv 0$. Thus the only hypothesis compatible with $\alpha_0 \neq 0$ is $d(H) \leq N(N-1)/2$.

To complete the proof of the lemma, we observe that for a given M either all monomials of degree M are congruent to zero or there is a set of exponents H such that the conditions (1), (2), and (3) are satisfied.

1.2. Proof of the theorem. Let the point P_i on which μ concentrates the mass μ_i have coordinates

$$(x_{i,1}, x_{i,2}, \dots, x_{i,n}).$$

Denote by y_i and L_p the differential operators

$$(1.21) \quad y_i = \sum_{j=1}^n x_{i,j} \frac{\partial}{\partial x_j} \quad (i = 1, 2, \dots, N)$$

and

$$(1.22) \quad L_p = \sum_{i=1}^N \mu_i y_i^p \quad (p = 1, 2, \dots, \text{etc.}).$$

A. Friedman and W. Littman have shown that every function $u(x_1, x_2, \dots, x_n)$ which satisfies the mean value property with respect to μ is a weak solution of the differential operators in (1.22).

There is a short and elementary proof of this result which proceeds as follows. We multiply the equation (1.1) by a C^∞ function $\phi(x)$ whose support is compact and contained in D and obtain

$$(1.23) \quad \int u(x)\phi(x)dx = \int \phi(x) \left[\int u(x+ty)d\mu(y) \right] dx.$$

Changing the order of integration and making the substitution $x+ty=z$ we can write (1.23) in the form

$$(1.24) \quad \int u(x)\phi(x)dx = \int u(z) \left[\int \phi(z-ty)d\mu(y) \right] dz.$$

We then observe that, since ϕ is C^∞ and has compact support, the right-hand side of (1.24) can be differentiated under the integral sign as many times as we please. Since the left-hand side is constant, all the derivatives of the right-hand side must vanish. In particular, (for $t=0$) we must have

$$(1.25) \quad \sum_{k_1+k_2+\dots+k_n=p} \int y_1^{k_1} y_2^{k_2} \dots y_n^{k_n} d\mu \int u(z) \frac{\partial^p \phi}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}} dz = 0$$

for $p=1, 2, \dots$, etc. This implies the assertion⁽²⁾.

Let now t be a dummy variable and note that

$$(1.26) \quad \sum_{p=1}^{\infty} L_p t^p = \sum_{i=1}^N \frac{\mu_i y_i t}{1 - y_i t} = t \frac{Q(t)}{P(t)},$$

where we have set

$$P(t) = (1 - y_1 t)(1 - y_2 t) \dots (1 - y_N t)$$

and

$$Q(t) = \sum_{i=1}^N \mu_i y_i Q_i(t), \quad Q_i(t) = \prod_{j \neq i} (1 - y_j t).$$

We can write (1.26) in the form

$$P(t) \sum_{p=1}^{\infty} L_p t^{p-1} = Q(t).$$

Since $Q(t)$ is a polynomial of degree $N-1$ we deduce that each of its coeffi-

⁽²⁾ As was pointed out by the above mentioned authors, when μ is a positive distribution, the infinite system (1.25) is equivalent to the mean value property. For then, the equation one obtains from (1.25) for $p=2$ is elliptic and by Weyl's lemma μ must necessarily be analytic. Thus the right-hand side of (I.1) is analytic in t and it is a constant (in t) if and only if all its derivatives vanish for $t=0$.

cients belongs to the ideal generated by $L_1, L_2, \dots, L_N$. To calculate these coefficients we write $Q_i(t)$ in the form

$$Q_i(t) = \sum_{\nu=0}^{N-1} (-1)^\nu \sigma_{i,\nu} t^\nu,$$

where $\sigma_{i,\nu}$ denotes the elementary symmetric function of degree ν in the $N-1$ variables $y_1, y_2, \dots, y_{i-1}, y_{i+1}, \dots, y_N$. We then have

$$Q(t) = \sum_{i=1}^N \mu_i y_i \sum_{\nu=0}^{N-1} (-1)^\nu \sigma_{i,\nu} t^\nu = \sum_{\nu=0}^{N-1} (-1)^\nu \sigma_{\nu+1}^\mu t^\nu,$$

where we have set

$$(1.27) \quad \sigma_{\nu+1}^\mu = \sum_{i=1}^N \mu_i y_i \sigma_{i,\nu}.$$

It can be readily seen that each σ_ν^μ is of the type

$$(1.28) \quad \sigma_\nu^\mu = \sum_{1 \leq i_1 < \dots < i_\nu \leq N} c_{i_1, i_2, \dots, i_\nu} y_{i_1} y_{i_2} \dots y_{i_\nu}.$$

But the actual expressions for the coefficients $c_{i_1, i_2, \dots, i_\nu}$ can also be obtained. In fact, because of the symmetry of the problem, we need only to calculate the coefficient of $y_1, y_2, \dots, y_{\nu+1}$ in (1.27). However, a monomial such as $y_1, y_2, \dots, y_{\nu+1}$ can only be produced by the first $\nu+1$ terms in the sum (1.27). On the other hand, $\sigma_{i,\nu}$ for $1 \leq i \leq \nu+1$ contains exactly once the monomial $y_1 y_2 \dots y_{i-1} y_{i+1} \dots y_{\nu+1}$. The coefficient of $y_1 y_2 \dots y_{\nu+1}$ in (1.27) is therefore $\mu_1 + \mu_2 + \dots + \mu_{\nu+1}$. Consequently we must also have

$$c_{i_1, i_2, \dots, i_\nu} = \mu_{i_1} + \mu_{i_2} + \dots + \mu_{i_\nu}.$$

In view of the hypothesis of the theorem the lemma can be applied to deduce that the function $u(x_1, x_2, \dots, x_n)$ must also be a weak solution of all differential operators of the type

$$(1.29) \quad \alpha = y_1^{k_1} y_2^{k_2} \dots y_N^{k_N} \quad k_1 + k_2 + \dots + k_N > \frac{N(N-1)}{2}.$$

The assumption that the support $S = P_1 \cup P_2 \cup \dots \cup P_N$ is not flat assures that the differential operators

$$(1.210) \quad \beta = \frac{\partial^{k_1+k_2+\dots+k_N}}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_N^{k_N}} \quad k_1 + k_2 + \dots + k_N > \frac{N(N-1)}{2}$$

can be expressed as linear combinations of the differential operators in (1.28). We thus obtain that $u(x_1, x_2, \dots, x_n)$ is a weak solution of all operators in

(1.210). In other words, $u(x_1, x_2, \dots, x_n)$ is a polynomial in its arguments of degree not exceeding $N(N-1)/2$.

Perhaps we should add the remark that from (1.26) it can be deduced that the finite system

$$(1.211) \quad L_1 u = L_2 u = \dots = L_N u = 0$$

is equivalent to the infinite system $L_p u = 0$ ($p = 1, 2, \dots$, etc.). Therefore each polynomial u satisfying (1.211) will also satisfy the mean value property with respect to μ .

UNIVERSITY OF MINNESOTA,
MINNEAPOLIS, MINNESOTA