FAMILIES OF MEASURES AND REPRESENTATIONS OF ALGEBRAS OF OPERATORS

BY

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Introduction. The purpose of the present paper is most easily explained by examining one part of it in detail. Let $\{B_\omega: \omega \in \Omega\}$ be a set of commuting bounded hermitian operators on a Hilbert space H , and let T be the cartesian product $\times_{\omega \in \Omega} [-\|B_\omega\|, \|B_\omega\|]$. Two of the most useful tools in the study of the operators are the simultaneous spectral resolution of the B_ω by means of a projector-valued measure π on a σ -field $\mathcal{A}$ of subsets of T , and the multiplicative representation. For this last we choose a set $\{x_j: j \in J\}$ of vectors in H such that if $H(x_j)$ is the closed linear extension of the set $\{\pi(A)x_j: A \in \mathcal{A}\}$, the $H(x_j)$ are mutually orthogonal and their closed linear extension is H . Then H is the direct sum of the $H(x_j)$, and for each j in J there is a non-negative measure m_j on $\mathcal{A}$ such that $L_2[T, m_j]$ is isomorphic with $H(x_j)$, and for every (complex) bounded Baire function f on T the operator $f(B_\omega) = \int f(t) d\pi$ corresponds in the direct sum $H_0 = \sum_j \oplus L_2[T, m_j]$ to the multiplication operator that maps $(\phi(t, j): t \in T, j \in J)$ on $(f(t)\phi(t, j): t \in T, j \in J)$. If $\mathfrak{B}$ is the smallest weakly (or strongly) closed commutative self-adjoint algebra of operators that contains all the B_ω , the operators thus defined by multiplication form a subalgebra of $\mathfrak{B}$; but this may be a proper subalgebra and may fail to be weakly closed. On the other hand, if for every bounded complex function $(f(t, j): t \in T, j \in J)$ which is $\mathcal{A}$ -measurable on T for each j in J we construct the operator that maps the vector $(\phi(t, j): t \in T, j \in J)$ of the direct sum H_0 on

$$(f(t, j) \phi(t, j): t \in T, j \in J).$$

we obtain too many operators; there are such operators which, under the isomorphism of H_0 and H , do not correspond to members of $\mathfrak{B}$.

One of our principal results is that if we suitably restrict the functions f of the preceding sentence we obtain the multiplicative representation of the algebra $\mathfrak{B}$ itself. The functions f allowed are those which depend on j only in the following heavily restricted way. To each countable subset J_c of J there corresponds an $\mathcal{A}$ -measurable function $(f(t): t \in T)$ such that for each j in J_c , $f(t) = f(t, j)$ except on a set whose m_j -measure is 0. Such functions on $T \times J$ we call "quasi-functions on T ." For such functions f we can also define $\int_T f d\pi$, by first projecting each vector x of H into the subspaces $H(x_j)$, applying $\int f(t, j) d\pi$ to the projection on $H(x_j)$, and adding the results. The integrals thus formed constitute an algebra of operators, and this algebra is

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exactly the smallest weakly closed self-adjoint algebra of operators containing all the B_ω . It is worth noticing that if J is countable we can replace $f(t, j)$ by the $f(t)$ of the definition (with $J_\epsilon = J$); the "new" integral is then the standard $\int f(t) d\pi$, and we conclude that if the cyclic set $\{x_j: j \in J\}$ is countable the algebra of operators $f(B_\omega)$ defined by the integral is weakly closed, even though Ω may be uncountable and H nonseparable.

The second part of this paper is devoted to showing that the "quasi-functions on T " have this desirable property, and that they also have somewhat similar uses in extending the Radon-Nikodym theorem and the Riesz representation theorem. [Since this was written, A. C. Zaanen has presented to the American Mathematical Society a paper in which the Radon-Nikodym theorem is extended just as it is here. The "quasi-functions" appropriate to this situation are called "cross-sections" by Zaanen. See abstract, Notices Amer. Math. Soc. 7, no. 7 (1960), 986; his paper will appear⁽¹⁾ in the Nederl. Akad. Wetensch. Proc. Ser. A.] The first part of the paper consists largely of the extension (to infinite sets of measure-functions) of a few of the results obtained by I. E. Segal [2] for (single) "localizable" measures. My apology for presenting these extensions is that they are needed in the second part.

1. Measures. Here and henceforth T will denote a set and $\mathcal{A}$ a σ -ring of subsets of T . J will denote a nonempty set, used only for indices to distinguish objects. For each j in J , m_j is a non-negative extended-real-valued function on $\mathcal{A}$, and is countably additive. If $F \subset T$ and $j \in J$, F is said to be m_j -finite if $F \in \mathcal{A}$ and $m_j F < \infty$, and F is said to be m_j -null if $F \in \mathcal{A}$ and $m_j F = 0$. For brevity, we use the expression " F is m_j -finite" to mean that F is m_j -finite for all j in J , while " F is m_J -null" means that F is m_j -null for all j in J . The class of all m_J -finite sets will be denoted by $\mathcal{R}$.

We use a device of I. E. Segal's [2] to eliminate the nuisance of sets of infinite measure that contain no subset of finite positive measure. We define $\mathcal{A}'$ to be the collection of all subsets E of T such that $E \cap F \in \mathcal{A}$ for all F in $\mathcal{R}$. This is a σ -ring, and is in fact a σ -algebra since T is in it. Next, for each j in J and each E in $\mathcal{A}'$ we define $m'_j E$ to be the supremum of $m_j(E \cap F)$ for all F in $\mathcal{R}$. These new measures are countably additive on $\mathcal{A}'$, and $m'_j F = m_j F$ for all j in J and all F in $\mathcal{R}$. If we treat $\mathcal{A}'$ and $\{m'_j: j \in J\}$ as we treated $\mathcal{A}$ and $\{m_j: j \in J\}$ we obtain them back again. It is with $\mathcal{A}'$ and the measures m'_j that we work henceforth. We drop the prime from the notation, and we list the properties henceforth assumed.

(1.1) $\mathcal{A}$ is an algebra of subsets of a set T .

For each j in a set J , m_j is a non-negative extended-real-valued countably additive measure in $\mathcal{A}$.

$\mathcal{R}$ is the set of all m_J -finite sets (i.e., all F in $\mathcal{A}$ with $m_j F < \infty$ for all j in J).

For all subsets S of T , $S \in \mathcal{A}$ if and only if $S \cap F \in \mathcal{A}$ for all F in $\mathcal{R}$, and in that case $m_j S = \sup \{m_j(S \cap F): F \in \mathcal{R}\}$ for every j in J .

⁽¹⁾ See reference [6].

A set S is *measurable* if it belongs to $\mathcal{A}$. An extended-real-valued function f on T is *measurable* if $f^{-1}(I) \in \mathcal{A}$ for every interval I in $[-\infty, \infty]$.

To aid in remembering the domains of functions, we denote the identity function on T by t and the identity function on J by j . Functions will be assumed to be extended-real-valued unless the contrary is stated. Thus $g(t, j)$ will denote a function $g: T \times J \rightarrow [-\infty, \infty]$.

2. Localizability. A measure m_j will be called *localizable* (by a small modification of the definition in [2]) if

(2.1) *to each nonempty collection $\mathcal{K}$ of measurable sets there corresponds a supremum B in the sense that*

- (i) *for each S in $\mathcal{K}$, $S - B$ is m_j -null;*
- (ii) *if $B' \in \mathcal{A}$ and $S - B'$ is m_j -null for every S in $\mathcal{K}$, so is $B - B'$.*

We shall say that

(2.2) *a set B is a simultaneous supremum of the collection $\mathcal{K}$ of sets with respect to the measures m_j ($j \in J$), or more briefly that B is an m_J -supremum of $\mathcal{K}$, if*

- (i) *for each S in $\mathcal{K}$ and each j in J , $S - B$ is m_j -null;*
- (ii) *for each j in J and each B' in $\mathcal{A}$, if $m_j(S - B') = 0$ for every S in $\mathcal{K}$ then $m_j(B - B') = 0$.*

Likewise,

(2.3) *if $\mathcal{F}$ is a collection of measurable functions on T , a function $b(t)$ is a simultaneous supremum of $\mathcal{F}$ with respect to the measures m_j ($j \in J$), or more briefly an m_J -supremum of $\mathcal{F}$, if*

- (i) *for each f in $\mathcal{F}$ and each j in J , $b(t) \geq f(t)$ except on an m_j -null set;*
- (ii) *if $b'(t)$ is measurable and j is in J , and for each f in $\mathcal{F}$ it is true that $b'(t) \geq f(t)$ except on an m_j -null set, then $b'(t) \geq b(t)$ except on an m_j -null set.*

The measures m_j ($j \in J$) are *simultaneously localizable* if every nonempty collection $\mathcal{K}$ of measurable sets has a simultaneous supremum with respect to the measures m_j ($j \in J$). This immediately implies an analogous property for functions, as follows.

(2.4) COROLLARY. *Let the measures m_j ($j \in J$) be simultaneously localizable. Let $\mathcal{F}$ be a nonempty collection of measurable functions. Then $\mathcal{F}$ has an m_J -supremum, and it is measurable.*

For each rational number r let $\mathcal{K}_r$ consist of the measurable sets $\{t: t \in T, f(t) \geq r\}$ with f in $\mathcal{F}$, and let B_r be the m_J -supremum of the class $\mathcal{K}_r$. If $r' > r$, $B_{r'} - B_r$ is m_J -null, so the set $S_r = \bigcup \{B_{r'}: r' \text{ rational}, r' \geq r\}$ differs from B_r by an m_J -null set. If $r > r'$, $S_{r'} \supset S_r$. Define $b_r(t)$ by setting $b_r(t) = r$ if $t \in S_r$, $b_r(t) = -\infty$ otherwise; and define $b(t) = \sup \{b_r(t): r \text{ rational}\}$. This is measurable. Let f belong to $\mathcal{F}$, and for rational r define $E_r = \{t: f(t) > r > b(t)\}$. Then $E_r \subset \{t: t \in T, f(t) \geq r\} \in \mathcal{K}_r$, so $E_r - B_r$ is m_J -null, and therefore so is $E_r - S_r$. But $E_r \cap S_r$ is empty by definition of $b(t)$, so E_r is m_J -null. The set on which $f(t) > b(t)$ is contained in the m_J -null set $\bigcup \{E_r: r \text{ rational}\}$, and (2.3i) holds.

Now let j be in J and let b' be as in (2.3ii), and for rational r define $B'_r = \{t: t \in T, b'(t) \geq r\}$. For each f in F , B'_r contains all of $\{t: t \in T, f(t) \geq r\}$ except an m_j -null subset. By (2.2ii), $m_j(B_r - B'_r) = 0$, so $m_j(S_r - B'_r) = 0$. Hence $N = \bigcup \{S_r - B'_r: r \text{ rational}\}$ is m_j -null. If $t \in T - N$,

$$\begin{aligned} b(t) &= \sup\{r: r \text{ rational}, t \in S_r\} \\ &\leq \sup\{r: r \text{ rational}, t \in B'_r\} \\ &= b'(t), \end{aligned}$$

and (2.3ii) holds.

3. Simultaneous localizability. We now seek conditions that will ensure that the measures in a set $\{m_j: j \in J\}$ are simultaneously localizable.

(3.1) THEOREM. *If J is countable and T is m_J -finite, the measures m_j ($j \in J$) are simultaneously localizable.*

Let $\mathcal{K}$ be a nonempty collection of measurable sets, $\mathcal{K}'$ the collection of all finite unions of sets in $\mathcal{K}$. We may suppose that J is a subset of the positive integers. For each j in J let $b_j = \sup\{m_j K: K \in \mathcal{K}'\}$. For each positive integer n there is a K_n in $\mathcal{K}'$ such that

$$b_j - 1/n \leq m_j K_n \leq b_j \quad (j \in J, j \leq n);$$

and we may suppose that K_n increases with n . Define $B = \bigcup_n K_n$. If $K \in \mathcal{K}$, $K_n \cup K \in \mathcal{K}'$ for all n , so for each j in J $m_j(K_n \cup K) \leq b_j$, whence $m_j(B \cup K) \leq b_j = m_j B$. Thus $m_j(K - B) = 0$, and (2.2i) holds. If $j \in J$ and B' is measurable and $m_j(K - B') = 0$ for all K in $\mathcal{K}$, then $m_j(K_n - B') = 0$ for all n , and $m_j(B - B') = 0$, so (2.2ii) holds.

If m is a measure on the σ -ring $\mathcal{G}$, and $E \in \mathcal{G}$, we define the "restriction of m to E " to be the measure defined by

$$(3.2) \quad m_E S = m(E \cap S) \text{ for all } S \text{ in } \mathcal{G}.$$

If m has the properties (1.1) so has m_E .

(3.3) THEOREM. *Let (1.1) hold. Assume that there is a disjoint collection $\mathcal{E}$ of measurable subsets of T whose union is T and which satisfy*

- (i) *for each E in $\mathcal{E}$, the restrictions $m_{j,E}$ ($j \in J$) are simultaneously localizable;*
- (ii) *for each subset S of T , $S \in \mathcal{G}$ if and only if $S \cap E \in \mathcal{G}$ for every E in $\mathcal{E}$;*
- (iii) *if $j \in J$ and $S \in \mathcal{G}$ and $m_j S > 0$, there exists an E in $\mathcal{E}$ for which $m_j(E \cap S) > 0$.*

Then the measures m_j ($j \in J$) are simultaneously localizable.

Let $\mathcal{K}$ be a nonempty collection of measurable sets. For each E in $\mathcal{E}$, let B_E be the $m_{j,E}$ -supremum of $\mathcal{K}$; we may assume $B_E \subseteq E$. Let B be the union of the B_E ; by (ii) this is measurable. If $K \in \mathcal{K}$, $m_{j,E}(K - B) = 0$ for all j in J and all E in $\mathcal{E}$, so by (iii) $K - B$ is m_J -null. Thus (2.2i) holds, and (2.2ii) is proved in practically the same way.

(3.4) REMARK. The hypothesis in Theorem 3.3 that $\mathcal{E}$ is a disjoint collec-

tion can be weakened to:

Each set E in $\mathcal{E}$ has points in common with at most countably many other sets in $\mathcal{E}$.

We well-order $\mathcal{E}$ by a relation $<$, and for each E in $\mathcal{E}$ we define $E' = E - \bigcup \{E'' : E'' \in \mathcal{E}, E'' < E\}$. These sets E' are measurable and disjoint, and each E_0 in $\mathcal{E}$ is the union of countably many sets $E_0 \cap E'$, so the collection of E' satisfies the requirements of Theorem 3.3.

(3.5) COROLLARY. *If T is the union of a countable collection $\mathcal{L}$ of measurable sets, and for each L in $\mathcal{L}$ the restrictions $m_{j,L}$ ($j \in J$) are simultaneously localizable, then the m_j ($j \in J$) are simultaneously localizable.*

As a special case, if J is countable and T is the union of countably many m_J -finite sets, the m_j are simultaneously localizable. Still more particularly, every σ -finite measure is localizable.

(3.6) COROLLARY. *If J is countable, and there exists a collection $\mathcal{E}$ of m_J -finite sets whose union is T and no one of which has points in common with more than countably many other members of $\mathcal{E}$, and for each j in J and each S in $\mathcal{A}$ such that $m_j S > 0$ there is an E in $\mathcal{E}$ such that $m_j(S \cap E) > 0$, then the measures m_j ($j \in J$) are simultaneously localizable.*

By the proof of (3.4) we may assume that $\mathcal{E}$ is disjoint, and by Theorem 3.1, (3.3i) holds. For each m_J -finite set F let $\mathcal{E}[F]$ be the (countable) set of E in $\mathcal{E}$ such that $m_j(E \cap F) > 0$ for at least one j in J , and let D_F be the (measurable) set $F - \bigcup \{F \cap E : E \in \mathcal{E}[F]\}$. Clearly $D_F \cap E$ is m_J -null for all E in $\mathcal{E}$, so D_F is m_J -null. If $S \subset T$ and $S \cap E \in \mathcal{A}$ for all E in $\mathcal{E}$, then whenever F is m_J -finite $S \cap F$ is the union of the countably many measurable sets $S \cap E \cap F$ ($E \in \mathcal{E}[F]$) and the m_J -null set $S \cap D_F$, so by (1.1) $S \in \mathcal{A}$. In particular, $T - \bigcup \mathcal{E}$ is m_J -null, and we may adjoin it to any one set of $\mathcal{E}$ without disturbing the requirements. Now by Theorem 3.3 the m_j ($j \in J$) are simultaneously localizable.

4. A decomposition theorem. The results of §3 suggest the following question. If a measure m is localizable, can T be subdivided into a disjoint collection of sets of finite measure in such a way that (3.3ii, iii) are satisfied? We partially answer this in Theorem 4.2. We consider countably many measures m_j simultaneously, since this creates no added difficulty.

(4.1) LEMMA. *Let J be countable. Let $\mathcal{E}^*$ be a maximal family of m_J -finite subsets of T , such that no set in $\mathcal{E}^*$ is m_J -null and each pair of distinct members of $\mathcal{E}^*$ has m_J -null intersection. Then*

- (i) $T - \bigcup \mathcal{E}^*$ is m_J -null;
- (ii) if $S \subset T$, $S \in \mathcal{A}$ if and only if $S \cap E \in \mathcal{A}$ for every E in $\mathcal{E}^*$;
- (iii) for each j in J and each S in $\mathcal{A}$ such that $m_j S > 0$, there exists an E in $\mathcal{E}^*$ such that $m_j(S \cap E) > 0$.

For each m_J -finite set F define $\mathcal{E}^*[F]$ to be the (countable) set of E in $\mathcal{E}^*$ with $F \cap E$ not m_J -null; then $D_F = F - \bigcup \{E \cap F: E \in \mathcal{E}^*[F]\}$ is measurable. Then for E in $\mathcal{E}^*$, $D_F \cap E$ is m_J -null; and by the maximality of $\mathcal{E}^*$, D_F must be m_J -null. Now let S be a subset of T such that $S \cap E \in \mathcal{A}$ for all E in $\mathcal{E}^*$. For every m_J -finite set F , $S \cap F$ is the union of countably many measurable sets $S \cap F \cap E$ ($E \in \mathcal{E}^*[F]$) and the m_J -null set $S \cap D_F$, so $S \cap F \in \mathcal{A}$. By (1.1), $S \in \mathcal{A}$. In particular, if $j \in J$ and $m_j S > 0$, there is an m_J -finite set F for which $m_j(S \cap F) > 0$, and one of the countably many sets $S \cap F \cap E$ ($E \in \mathcal{E}^*[F]$) must have positive m_j -measure; then $m_j(S \cap E) > 0$. This implies that $T - \bigcup \mathcal{E}^*$ is m_J -null, completing the proof.

The Hausdorff maximality principle allows us to prove easily that sets $\mathcal{E}^*$ satisfying the hypotheses of (4.1) exist. Segal uses this [2, p. 287] to show that there exists a measure-preserving map of the measurable sets in T onto measurable sets in another measure-space under which images of sets in $\mathcal{E}^*$ are disjoint. However, this does not provide a decomposition of T itself. If $\mathcal{E}^*$ is countable it is trivial to construct another family whose members are disjoint. We now show that such a construction is possible whenever there are at most continuum-many sets in $\mathcal{E}^*$.

(4.2) THEOREM. *Let the measures m_j ($j \in J$) be simultaneously localizable. If J is countable, and there exists a family $\mathcal{E}^*$ satisfying the hypotheses of Lemma 4.1 and having cardinal number not greater than that of the continuum, there exists a decomposition $\mathcal{E}$ of T with the properties (ii) and (iii) of (3.3), the sets in $\mathcal{E}$ being m_J -finite and mutually disjoint.*

There is a mapping r of $\mathcal{E}^*$ onto a subset R_0 of the open interval $(0, 1)$. Let $\mathcal{F}$ be the set of functions f_{E^*} ($E^* \in \mathcal{E}^*$), where f_{E^*} has value $r(E^*)$ on E^* and value 0 on $T - E^*$. By Corollary 2.4, $\mathcal{F}$ has a supremum $b(t)$. Then $b(t) \geq 0$ except on an m_J -null set, and we may suppose it non-negative on T . Let E' be in $\mathcal{E}^*$; define $h(t)$ by

$$h(t) = r(E')(t \in E'); \quad h(t) = b(t)(t \in T - E').$$

For each E^* in $\mathcal{E}^*$ let $D_{E^*} = \{t: t \in T, f_{E^*}(t) > h(t)\}$. If $E^* = E'$ this is empty; if $E^* \neq E'$, $D_{E^*} \cap [T - E']$ is contained in the m_J -null set $\{t: t \in T, f_{E^*}(t) > b(t)\}$ and $D_{E^*} \cap E'$ is contained in the m_J -null set $E^* \cap E'$, so in all cases D_{E^*} is m_J -null. By definition of m_J -supremum, $h(t) \geq b(t)$ except on an m_J -null set. In particular, the subset of E' on which $h(t) \geq b(t)$ fails to hold is m_J -null. The reversed inequality is obvious, so $b(t) = r(E')$ on all of E' except an m_J -null subset.

Now for each r in R_0 we define $E(r) = b^{-1}(r)$, and we define $\mathcal{E}$ to be $\{E(r): r \in R_0\}$. The mapping $E^* \rightarrow E(r(E^*))$ is a one-to-one correspondence between $\mathcal{E}^*$ and $\mathcal{E}$, and corresponding sets differ by m_J -null sets. Thus $\mathcal{E}$ is a disjoint collection of sets none of which is m_J -null. The set $\mathcal{E}$ satisfies the hypotheses placed upon $\mathcal{E}^*$ in Lemma 4.1. Thus $T - \bigcup \mathcal{E}$ is m_J -null; we adjoin

it to any one set of $\mathcal{E}$, and the amended $\mathcal{E}$ has union T . The other conclusions hold by Lemma 4.1.

5. Definition and lattice properties. We continue to assume that (1.1) is satisfied.

(5.1) A function $f(t, j)$ will be called a "quasi-function on T " with respect to the measures m_j ($j \in J$) if to each countable subset J_c of J there corresponds a function $f(t)$ such that $m_j\{t: t \in T, f(t, j) \neq f(t)\} = 0$ for every j in J_c .

Occasionally we shall use the symbol $f(t, *)$ for a quasi-function; the use of the asterisk in place of j shall connote the relation between the various $f(t, j)$ specified in the definition of quasi-function. A quasi-function $f(t, *)$ is called *measurable* if all the functions $f(t, j)$ ($j \in J$) are measurable.

Each function $f(t)$ defines a quasi-function $f(t, *)$ such that $f(t, j) = f(t)$ for all t in T and all j in J . Conversely, if $f(t, j)$ is independent of j it defines a function on T . By a familiar abuse of language we then say that the quasi-function $f(t, *)$ is the function on T that it defines.

Algebraic and lattice operations on quasi-functions need no discussion, since each quasi-function is an extended-real-valued function on $T \times J$. Two quasi-functions $f(t, *)$, $g(t, *)$ are *equivalent* if for each j in J we have $m_j\{t: t \in T, f(t, j) \neq g(t, j)\} = 0$. Clearly, if J is countable every quasi-function is equivalent to a function on T .

Let $\mathfrak{F}$ be a nonempty set of quasi-functions on T , with respect to the m_j ($j \in J$). Then:

(5.2) A quasi-function $b(t, *)$ is called an m_J -supremum of $\mathfrak{F}$ if

- (i) for each j in J and each f in $\mathfrak{F}$, $b(t, j) \geq f(t, j)$ except on an m_j -null set;
- (ii) for each j in J , if $b'(t)$ is a function on T such that for every f in $\mathfrak{F}$ the inequality $b'(t) \geq f(t, j)$ holds except on an m_j -null set, then $b'(t) \geq b(t, j)$ except on an m_j -null set.

Evidently any two m_J -suprema of a set $\mathfrak{F}$ are equivalent.

Quasi-functions have lattice properties stronger than those of ordinary functions, as the following theorem shows.

(5.3) **THEOREM.** Let (1.1) hold, and for each countable subset J_c of J let the measures m_j ($j \in J_c$) be simultaneously localizable. Then each nonempty collection $\mathfrak{F}$ of measurable quasi-functions on T (with respect to the measures m_j , $j \in J$) has an m_J -supremum, and this is a measurable quasi-function.

By Corollary 2.4, for each j in J there is a function $b(t, j)$ satisfying (5.2i, ii). It remains to show that $b(t, j)$ is a quasi-function. Let J_c be a countable subset of J . For each $f(t, j)$ in $\mathfrak{F}$ we can and do choose an $f(t)$ such that for all j in J_c , $f(t) = f(t, j)$ except on an m_j -null set. Denote by $\mathfrak{F}_c$ the set of $f(t)$ thus chosen. By Corollary 2.4, $\mathfrak{F}_c$ has a supremum $b(t)$ with respect to the measures m_j ($j \in J_c$). For each j in J_c we have $b(t, j) \geq f(t, j)$ except on an m_j -null set whenever $f(t, j) \in \mathfrak{F}$, so $b(t, j) \geq f(t)$ except on an m_j -null set whenever $f(t) \in \mathfrak{F}_c$, and so $b(t, j) \geq b(t)$ except on an m_j -null set. On the other hand,

if $f(t, j) \in \mathfrak{F}$, we have $b(t) \geq f(t) = f(t, j)$ except on an m_j -null set, whence by (5.2ii) $b(t) \geq b(t, j)$ except on an m_j -null set. Thus for all j in J_c we have $b(t, j) = b(t)$ except on an m_j -null set, and the proof is complete.

6. Summable quasi-functions. Suppose that we have a system satisfying (1.1) with J a singleton; we then drop the subscript j from m_j . Let $\mathfrak{R}$ be the collection of m -finite sets, and $\mathfrak{L}$ the collection of all measurable subsets L of T such that the restriction of m to L is localizable (cf. (3.2)). Then the system $\{T, \mathfrak{R}, \{m_R: R \in \mathfrak{R}\}\}$ satisfies (1.1), and so does $\{T, \mathfrak{R}, \{m_L: L \in \mathfrak{L}\}\}$. If we define L and R to be the identity functions on $\mathfrak{L}$ and $\mathfrak{R}$ respectively, we can construct two classes of quasi-functions, $f(t, L)$ and $f(t, R)$. Since $\mathfrak{R} \subset \mathfrak{L}$, if $f(t, L)$ is a quasi-function with respect to the measures m_L ($L \in \mathfrak{L}$), its restriction $f(t, R)$ to $T \times \mathfrak{R}$ is a quasi-function with respect to the measures m_R ($R \in \mathfrak{R}$). Conversely, every quasi-function with respect to the m_R can be extended to a quasi-function with respect to the m_L , as we now show.

(6.1) **LEMMA.** *If $f(t, R)$ is a quasi-function with respect to the measures m_R ($R \in \mathfrak{R}$), it has an extension $f(t, L)$ to $T \times \mathfrak{L}$ which is a quasi-function with respect to the measures m_L ($L \in \mathfrak{L}$). In particular, if m is localizable $f(t, R)$ is equivalent to a measurable function $f(t)$ on T .*

For each R in $\mathfrak{R}$ let g_R be the function coinciding with $f(t, R_R)$ on R and identically $-\infty$ on $T - R$. By Corollary 3.5 the hypotheses of Theorem 5.3 are satisfied with $J = \mathfrak{L}$, so the set $\{g_R: R \in \mathfrak{R}\}$ has a supremum $b(t, L)$ among the measurable quasi-functions with respect to the m_L ($L \in \mathfrak{L}$). Let R_1 and R_2 be in $\mathfrak{R}$. Then there is a measurable function $f(t)$ such that $f(t) = f(t, R_i)$ except on an m_{R_i} -null set ($i = 1, 2$). Hence $g_{R_1}(t) = g_{R_2}(t)$ at all points of $R_1 \cap R_2$ except those of an m -null subset. This implies $g_{R_1}(t) \geq g_{R_2}(t)$ at all points of T except an m_{R_1} -null set, and since R_2 is any member of $\mathfrak{R}$, $g_{R_1}(t) \geq b(t, R_1)$ except on an m_{R_1} -null set. The reversed inequality is an immediate consequence of the definition of b , so $f(t, R_1) = g_{R_1}(t) = b(t, R_1)$ except on an m_{R_1} -null set. Without loss of generality we may assume that $b(t, R_1)$ was already defined so as to be identical with $f(t, R_1)$ for all R_1 in $\mathfrak{R}$. Then $b(t, L)$ is an extension of $f(t, R)$ to $T \times \mathfrak{L}$, and is a quasi-function with respect to the measures m_L ($L \in \mathfrak{L}$). In particular, if m is localizable, T is in $\mathfrak{L}$ and $m = m_T$; and $b(t, T)$ is a function on T equivalent to $b(t, L)$. This completes the proof.

7. An extension of the Radon-Nikodym theorem. Let $f(t, R)$ be a quasi-function with respect to the m_R ($R \in \mathfrak{R}$); and suppose that for each R in $\mathfrak{R}$, $f(t, R)$ is m_R -summable over T (or, equivalently, m -summable over R). Then $\int f(t, R') dm_R(t)$ has the same value for all R' in $\mathfrak{R}$ that contain R . This integral is said to have a limit k (in $[-\infty, \infty]$) as R expands if to each neighborhood U of k there corresponds an R_U in $\mathfrak{R}$ such that whenever $R \in \mathfrak{R}$ and $R \supset R_U$, $\int f(t, R) dm_R(t) \in U$. In this case we write

$$\int f(t, *) dm(t) = k.$$

(All integrals are over T .) If f is non-negative the limit will surely exist; and in fact we can relax the requirement that $f(t, R)$ be m_R -summable and require that it have an integral, finite or infinite. It is easy to see that on the class of summable quasi-functions (those having finite integrals) the integral is linear, and the class is a vector lattice.

Now consider a system satisfying (1.1) with $J = \{1, 2\}$. To save subscripts we write m for m_1 and n for m_2 . (The assumption that n is non-negative is not vital, but saves some bother.) $\mathfrak{L}$ will be the class of sets L in $\mathfrak{Q}$ such that m_L is localizable.

(7.1) THEOREM. *Let m and n be as just described. Assume that for each R in $\mathfrak{Q}$, if $mR=0$ then $nR=0$. Then there exists a non-negative finite valued measurable quasi-function $f(t, L)$ such that for all S in $\mathfrak{Q}$,*

$$nS = \int f(t, *)K_S(t)dm(t).$$

If m is localizable $f(t, *)$ can be taken to be a measurable function $f(t)$ on T .

If S is σ -finite, by the standard form of the Radon-Nikodym theorem ([1], p. 128) there is a measurable real-valued function f on S such that for every measurable subset E of S ,

$$n(E) = \int_E f(t)dm(t).$$

For each σ -finite set S in $\mathfrak{Q}$ we choose such an integrand, and we define $f_S(t)$ to be $f(t)$ for t in S and to be 0 for t in $T-S$. These f_S may be regarded as quasi-functions on T with respect to the m_L ($L \in \mathfrak{L}$), and as such they have an $m_{\mathfrak{L}}$ -supremum $f(t, L)$, which we may suppose to be everywhere non-negative.

If R is in $\mathfrak{Q}$ it is in $\mathfrak{L}$, so the inequality $f(t, R) \geq f_R(t)$ holds except on an m_R -null set. If S is σ -finite, f_S and f_R have the same integral $n(E)$ over every measurable subset E of $R \cap S$, so they coincide on all of $R \cap S$ except an m -null set. But f_S vanishes on $R-S$, so $f_R \geq f_S$ except on an m_R -null set. By definition of supremum, $f_R \geq f(t, R)$ except on an m_R -null set. With the previous inequality, this proves that $f_R(t) = f(t, R)$ except on an m_R -null set. So for all S in $\mathfrak{Q}$

$$n(R \cap S) = \int f(t, R)K_S(t)dm_R(t).$$

The integral increases as R expands, so for fixed S in $\mathfrak{Q}$ its limit as R expands is the same as its supremum for all R in $\mathfrak{Q}$, whence

$$nS = \int f(t, *)K_S(t)dm(t).$$

The last conclusion follows readily. (Cf. [2].)

8. The Riesz representation theorem. A function $f(t, j)$ on $T \times J$ (not necessarily a quasi-function with respect to the measures $m_j, j \in J$) will be called *measurable* if $f(t, j)$ is measurable on T for each j in J ; it is *essentially bounded* if there is a real number r such that for each j in J , $|f(t, j)| \leq r$ except on an m_j -null set. The infimum of r is the *essential supremum* of $|f(t, j)|$. If $f(t, j)$ and $f'(t, j)$ are equivalent quasi-functions and one is essentially bounded, so is the other, and the essential suprema of their absolute values are equal. The set of all equivalence classes of essentially bounded quasi-functions will be denoted by $L_{\infty*}[T, \{m_j: j \in J\}]$, or by $L_{\infty*}$ when this will not cause confusion. If $\phi \in L_{\infty*}$ and $f(t, j) \in \phi$, we define $\|\phi\|_{\infty}$ to be the essential supremum of $|f(t, j)|$. As usual, if $f(t, j)$ is essentially bounded we shall (incorrectly, but conveniently) say that f belongs to $L_{\infty*}$ and has norm $\|f(t, j)\| = \text{ess. sup. } |f(t, j)|$.

If $b(t, *)$ is an essentially bounded quasi-function with respect to the measures m_R ($R \in \mathcal{R}$) or the measures m_L ($L \in \mathcal{L}$), it is easy to see that for every m -summable function $f(t)$ the integral

$$l_b(f) = \int f(t)b(t, *)dm(t)$$

exists and is finite. This defines a continuous linear functional l_b over $L_1[T, m]$, and $\|l_b\| = \|b(t, *)\|$. We now prove the converse of this, which is a sort of extension of the Riesz representation theorem. (For a different kind of extension see [3].)

(8.1) **THEOREM.** *Let l be a continuous linear real-valued functional over $L_1[T, m]$. Then there exists an essentially bounded measurable quasi-function $b(t, L)$ with respect to $\{m_L: L \in \mathcal{L}\}$ such that $\|b(t, *)\|_{\infty} = \|l\|$ and for every function f in $L_1[T, m]$*

$$l(f) = \int f(t)b(t, *)dm(t).$$

The quasi-function $b(t, *)$ is unique up to equivalence.

By a well-known device we can represent l as the difference of two non-negative functionals, so without loss of generality we may assume l non-negative. For each S in $\mathcal{A}$ we define n_S to be the supremum of $l(K_{R \cap S})$ for all R in $\mathcal{R}$; this is a countably additive measure on $\mathcal{A}$. By Theorem 7.1 there is a quasi-function $b(t, L)$ with respect to $\{m_L: L \in \mathcal{L}\}$ such that for each S in $\mathcal{A}$

$$n_S = \int b(t, *)K_S(t)dm(t).$$

It is easy to see that the essential supremum of $|b(t, L)|$ is $\|l\|$. If f is a "simple function," meaning that f is a finite linear combination of characteristic functions of sets of finite measure, this implies

$$l(f) = \int b(t, *)f(t)dm(t).$$

Both members of this equation define continuous linear functions on $L_1[T, m]$, and the simple functions are dense in that space, so the equation holds for all f in $L_1[T, m]$. The uniqueness of b , up to equivalence, is easy to establish.

9. Decomposition of a Hilbert space by means of a cyclic set. Let T be a set and $\mathfrak{A}$ a σ -algebra of subsets of T . Let H be a complex Hilbert space (not necessarily separable) and π a resolution of the identity assigning to each set S in $\mathfrak{A}$ a projector $\pi(S)$ in H such that $\pi(T)$ is the identity 1 , $\pi(\emptyset)$ is the projector 0 on the origin, $\pi(S_1)\pi(S_2) = \pi(S_1 \cap S_2)$ and π is countably additive. For each x in H let $H(x)$ be the closure of the set of all linear combinations of projectors $\pi(S)x$ ($S \in \mathfrak{A}$). If y is orthogonal to $H(x)$, for all S and S' in $\mathfrak{A}$ we have $(\pi(S)y, \pi(S')x) = (y, \pi(S \cap S')x) = 0$, so $H(y)$ is orthogonal to $H(x)$. From the Hausdorff maximality principle we deduce that there is a set $\{x_j: j \in J\}$ of unit vectors in H such that $H(x_j)$ and $H(x_k)$ are orthogonal whenever j and k are distinct members of J , and such also that the closed linear span of the $H(x_j)$ ($j \in J$) is H .

For all j in J and S in $\mathfrak{A}$ we define

$$m_j S = \pi(S)x_j, \quad m_j S = \|\pi(S)x_j\|^2.$$

If z is a simple function, with values $z_1, \dots, z_n$ on disjoint measurable sets $S_1, \dots, S_n$, we define

$$\begin{aligned} \int z(t)dm_j(t) &= \sum_{i=1}^n z_i m_j S_i, \\ \int z(t)dm_j(t) &= \sum_{i=1}^n z_i m_j S_i. \end{aligned}$$

Then

$$\begin{aligned} \left\| \int z(t)dm_j(t) \right\|^2 &= \sum_i \|z_i m_j S_i\|^2 \\ &= \sum_i |z_i|^2 m_j S_i \\ &= \int |z|^2 dm_j(t). \end{aligned}$$

This isometry can be extended to all of (complex) $L_2[T, m_j]$; to each $z(t)$ in

$L_2[T, m_j]$ there corresponds a unique vector-integral $\int z(t) dm_j(t)$, and its length is the norm of $z(t)$ in $L_2[T, m_j]$.

(9.1) LEMMA. Let H_0 be the direct sum of the spaces $L_2[T, m_j]$ ($j \in J$). For each function $z(t, j)$ in H_0 define

$$\psi(z) = \sum_{j \in J} \int z(t, j) dm_j(t).$$

Then ψ is an isometric linear mapping of H_0 onto H .

The linearity of ψ is obvious. If $z(t, j) \in H_0$, for each j in J , $z(t, j)$ is in $L_2[T, m_j]$ and therefore has an integral y_j with respect to m_j , and

$$\|y_j\|^2 = \int |z(t, j)|^2 dm_j(t).$$

The y_j are mutually orthogonal, and the sum of the right members is $\|z(t, j)\|^2$, so $\sum_j y_j$ exists and has length $\|z(t, j)\|$. Thus the mapping ψ is defined on H_0 and is an isometry. The range of ψ includes all vectors $\pi(S)x_j$ ($S \in \mathfrak{A}, j \in J$), and the linear combinations of these are dense in H , so the range of ψ is H .

For each bounded linear operator B_0 on H_0 we define

$$\psi_0(B_0) = \psi B_0 \psi^{-1}.$$

Then ψ_0 is an isometric isomorphism of the algebra of all bounded linear operators on H_0 onto the algebra of all bounded linear operators on H , and is weakly continuous.

10. Multiplication operators. Each essentially bounded measurable function $f(t, j)$ on $T \times J$ (not necessarily a quasi-function on T) defines a bounded linear operator $f^\times$ on H_0 by the relation

$$f^\times(z) = f(t, j)z(t, j) \quad (z(t, j) \in H_0).$$

When $f(t, j)$ is a quasi-function on T with respect to the measures m_j ($j \in J$) the operator $f^\times$ will be called a *multiplication operator*; the set of all multiplication operators will be denoted by $\mathfrak{M}$. It is obvious that $\mathfrak{M}$ is a commutative $*$ -algebra over the complex field.

The algebra $\mathfrak{M}$ is mapped isometrically and isomorphically onto an algebra $\psi_0(\mathfrak{M})$ of operators in H . One special case is worth noting. Let S be measurable, and let q be the characteristic function of $S \times J$. Then $\psi_0(q^\times) = \pi(S)$. For if $j \in J$ and z is the characteristic function of a set $E \times \{j\}$ with E in $\mathfrak{A}$, we have

$$\psi(z) = \int z(t, j) dm_j(t) = m_j E = \pi(E)x_j,$$

so

$$\pi(S)\psi(z) = \pi(E \cap S)x_j = m_j(E \cap S) = \psi(q(t, j)z(t, j)) = \psi(q^\times(z)).$$

This extends to linear combinations of such z , and by continuity to the closure of the set of linear combinations of such z , which is H_0 . If $y \in H$ then there exists a z in H_0 such that $y = \psi(z)$, so

$$\pi(S)y = \psi(q^\times(\psi^{-1}(y))) = \psi_0(q^\times)y.$$

11. The multiplicative representation. We can now state our principal theorem on algebras of operators.

(11.1) THEOREM. *With the notation of the two preceding sections,*

(i) *every bounded linear operator C that commutes with every projector $\pi(S)$ ($S \in \mathfrak{A}$) also commutes with all operators in $\psi_0(\mathfrak{M})$;*

(ii) *the algebra $\psi_0(\mathfrak{M})$ is weakly closed, and is the smallest weakly closed algebra of linear operators on H that contains all the projectors $\pi(S)$ ($S \in \mathfrak{A}$).*

For any set $\mathfrak{N}$ of bounded linear operators on H (or on H_0) let $\mathfrak{N}'$ denote the set of all bounded linear operators on H (or on H_0 , respectively) that commute with every operator in $\mathfrak{N}$. Let $\Pi = \{\pi(S) : S \in \mathfrak{A}\}$. Since $\Pi \subset \psi_0(\mathfrak{M})$, it is clear that $\Pi' \supset [\psi_0(\mathfrak{M})]'$. Conclusion (i) is the assertion $\Pi' \subset [\psi_0(\mathfrak{M})]'$, and implies the equality of the two sets.

We shall use certain kinds of characteristic functions often enough to warrant abbreviation. If $j \in J$ and $S \subset T$, we shall use the symbols $K_{S,j}$, K_j , K_S to denote the characteristic functions of the respective sets $S \times \{j\}$, $T \times \{j\}$, $S \times J$.

By §10, conclusion (i) is equivalent to $\{K_S^\times : S \in \mathfrak{A}\}' \subset \mathfrak{M}'$. Let then C_0 be an operator on H_0 that commutes with $K_S^\times$ for all S in $\mathfrak{A}$; we must prove that C_0 commutes with all $q^\times$ in $\mathfrak{M}$. So consider any $q(t, j)$ in L_∞^* . Let $z(t, j)$ belong to H_0 , and let $y = C_0 z$. Then $\int |z(t, j)|^2 dm_j$ and $\int |y(t, j)|^2 dm_j$ vanish except for a countable subset J_e of J . There is a bounded measurable function $q_0(t)$ such that for every j in J_e , $q_0(t) = q(t, j)$ except on an m_j -null set. This $q_0(t)$ is the uniform limit of simple functions $q_n(t)$ ($n = 1, 2, \dots$). Each q_n is a finite linear combination

$$q_n = \sum_{k=1}^{m_n} c_{n,k} K_{S_{n,k}} \quad (S_{n,k} \in \mathfrak{A}),$$

so $q_n^\times$ is a sum of operators each of which commutes with C_0 . Hence

$$C_0[q_n(t)z(t, j)] = q_n(t)C_0[z(t, j)] = q_n(t)y(t, j),$$

so that

$$C_0[q_0(t)z(t, j)] = q_0(t)y(t, j).$$

If $j \in J_e$, the equation

$$q_0(t)z(t, j) = q(t, j)z(t, j)$$

holds except on an m_j -null set, by definition of q_0 ; if $j \in J - J_e$, it holds except

on the m_j -null set on which $z(t, j) \neq 0$. So the two members of the equation represent the same point of H_0 . Similarly, $q_0(t)y(t, j)$ and $q(t, j)y(t, j)$ represent the same point of H_0 , and the last equation of the preceding paragraph implies

$$C_0 q^\times z = q^\times C_0 z.$$

This holds for all z in H_0 , and (i) is established.

If $\mathfrak{N}$ is any set of bounded linear operators on H such that for all A in $\mathfrak{N}$, A^* is also in $\mathfrak{N}$, the smallest weakly closed algebra containing $\mathfrak{N}$ is $\mathfrak{N}''$ [5, p. 44]. Since (i) holds, $\Pi'' = [\psi_0(\mathfrak{N})]''$, and (ii) will be established when we prove $[\psi_0(\mathfrak{N})]'' = \psi_0(\mathfrak{N})$, or (more conveniently) $\mathfrak{N}'' = \mathfrak{N}$. Let A be any operator in $\mathfrak{N}''$; we shall prove $A \in \mathfrak{N}$.

For $j \in J$, the characteristic function K_j of $T \times \{j\}$ is in H_0 , so AK_j is an element $r_j(t, j)$ of H_0 . Since $K_j^\times$ is in $\mathfrak{N}'$,

$$K_j^\times r_j = AK_j^\times(K_j) = AK_j = r_j.$$

We may therefore suppose that the chosen representation r_j of AK_j already has the property of vanishing except on $T \times \{j\}$, and we define

$$r(t, j) = \sum_{j \in J} r_j(t, j);$$

at each point in $T \times J$ at most one term in the sum is different from 0.

If $j \in J$ and $S \in \mathfrak{A}$ we have $K_{S,j} = K_{S,j}K_j$, so

$$A(K_{S,j}) = AK_{S,j}^\times(K_j) = K_{S,j}^\times A(K_j) = K_{S,j}(t, j)r_j(t, j) = K_{S,j}(t, j)r(t, j).$$

Hence if $z(t, j)$ is a function assuming finitely many values, each on a set of the form $S \times \{j\}$ with $j \in J$ and $S \in \mathfrak{A}$, we obtain

$$A(z) = z(t, j)r(t, j) = r^\times(z).$$

By the usual device we deduce that the essential supremum of $|r(t, j)|$ is $\|A\|$. Furthermore, the simple functions of the type just described are dense in H_0 , and A and $r^\times$ are continuous, so the last equation implies $A = r^\times$. It remains to prove that r is a quasi-function on T with respect to the m_j ($j \in J$), so that $r^\times$ will belong to $\mathfrak{N}$.

Let J_e be a countable subset of J ; it can be mapped one-to-one on a set of positive integers by a function $(n(j): j \in J)$. For each S in $\mathfrak{A}$ define

$$mS = \sum_{j \in J_e} 2^{-n(j)} m_j S.$$

This is a countably additive non-negative measure on $\mathfrak{A}$, with $m(T) \leq 1$; and for each j in J and each S in $\mathfrak{A}$, $m_j S \leq 2^{n(j)} mS$. By the Radon-Nikodym theorem, for each j in J there is a measurable function $f_j(t)$ on T , with $0 \leq f_j \leq 2^{n(j)}$, such that for all S in $\mathfrak{A}$

$$\int_S f_j(t) dm(t) = m_j S.$$

With the standard definition of the signum function (so that $\text{sgn } f_j(t)$ is 1 if $f_j(t) > 0$ and is 0 if $f_j(t) = 0$), for all t in T we define

$$r(t) = \sup \{ r_j(t, j) \text{sgn } f_j(t) : j \in J_c \}.$$

This is measurable and essentially bounded. We shall now prove

(*) for each j in J_c , $r(t) = r(t, j)$ except on an m_j -null set.

Suppose this false; for some k in J_c , $r(t) \neq r(t, k)$ on a set S_1 with $m_k S_1 > 0$. Since $m_k S_1$ is the limit as n increases of $m_k \{t \in S_1, f_k(t) \geq 1/n\}$, there is a subset S_2 of S_1 with $m_k S_2 > 0$ on which f_k has a positive lower bound. By definition of r , this implies $r(t) \geq r_k(t, k)$ on S_2 , and they are unequal on S_1 , so for each t in S_2 there is a j in J_c for which $r_j(t, j) > r_k(t, k)$ and $f_j(t) > 0$. Since J_c is countable and $m S_2 > 0$, there is an h in J_c for which the set $S_3 = \{t \in S_2, f_h(t) > 0, r_h(t, h) > r_k(t, k)\}$ has positive m -measure. Since $f_k > 0$ on S_3 , $m_k S_3 > 0$. As in defining S_2 , we find that there is a subset S_4 of S_3 with $m_h S_4 > 0$ on which $f_h(t)$ has a positive lower bound. Summarizing, $m_h S_4 > 0$ and $m_k S_4 > 0$; on S_4 , both f_h and f_k have positive lower bounds and finite upper bounds; and on S_4 , $r_h(t, h) > r_k(t, k)$.

Define C on H_0 by setting $Cz(t, j) = y(t, j)$, where

$$\begin{aligned} y(t, j) &= z(t, j) \text{ unless } t \in S_4 \text{ and } j \in \{h, k\}, \\ y(t, h) &= z(t, k) \text{ and } y(t, k) = z(t, h) \text{ } (t \in S_4). \end{aligned}$$

This is a bounded linear operator which is easily seen to commute with $K_S^\times$ for every S in $\mathfrak{A}$, so by conclusion (i) C is in $\mathfrak{M}'$. Now let z be the characteristic function of $S_4 \times \{h\}$. Then at each point (t, k) of $S_4 \times \{k\}$ we have

$$\begin{aligned} Cr^\times(z)(t, k) &= r_h(t, h), \\ r^\times Cz(t, k) &= r_k(t, k), \end{aligned}$$

and the right members differ on all of S_4 . So $r^\times$ does not commute with the element C of $\mathfrak{M}'$, and is therefore not in $\mathfrak{M}''$. But $r^\times = A$, and by hypothesis A is in $\mathfrak{M}''$. This contradiction establishes (*), so $r(t, j)$ is a quasi-function and A is in $\mathfrak{M}$. Thus $\mathfrak{M}'' \subset \mathfrak{M}$, and the proof is complete.

12. The algebra of quasi-functions of operators. Consider now a weakly closed commutative $*$ -algebra $\mathfrak{B}$ of bounded operators on a Hilbert space H (a von Neumann algebra, in Dixmier's terminology). Let $(B_\omega : \omega \in \Omega)$ be a set of hermitian operators in $\mathfrak{B}$ such that the smallest weakly closed algebra containing the set is $\mathfrak{B}$ itself. Let T_ω be the interval $[-\|B_\omega\|, \|B_\omega\|]$, and let T be the cartesian product $\times_\omega T_\omega$. By a proof in [4, pp. 125 et seq.] we know that there exists a σ -field $\mathfrak{A}$ of subsets of T including all Borel subsets of T , and on $\mathfrak{A}$ a projector-valued measure $(\pi(S) : S \in \mathfrak{A})$ which is a resolution of the identity, such that for each ω in Ω the integral of the coordinate-function

t_ω with respect to π is B_ω . Whenever $q(t)$ is bounded and measurable the integral $\int q(t)d\pi(t)$ exists and is a bounded operator on H , which we denote by $q(B_\omega)$ to indicate its dependence on all the B_ω ; these operators obey the usual rules of an operational calculus. As we shall shortly prove, the set of all such $q(B_\omega)$ is contained in $\mathfrak{B}$. However, it may be a proper subset of $\mathfrak{B}$ and may fail to be weakly closed, as we now show.

Let $U = [-1, 1]$, $U^2 = U \times U$. Let m_1 be one-dimensional Lebesgue measure, and let $\mathfrak{A}$ be the class of all subsets of U^2 that meet each ordinate and each abscissa in an m_1 -measurable set. For each S in $\mathfrak{A}$ we define

$$mS = \sum_{u \in U} m_1(S \cap [\{u\} \times U]) + \sum_{u \in U} m_1(S \cap [U \times \{u\}]).$$

This system satisfies (1.1). Let H be $L_2[U^2, m]$. The operator B that transforms z into $(u_1 z(u_1, u_2): (u_1, u_2) \in U^2)$ is hermitian and has bound 1, so its spectrum is contained in $T = [-1, 1]$. Let π be the resolution of the identity for B , and let $\mathfrak{B}$ be the smallest weakly closed algebra of operators on H that contains B . For each τ in T define

$$f_{n,\tau}(t) = [1 - (t - \tau)^2/5]^n \quad (t \in T).$$

As n increases this converges monotonically to the characteristic function of $\{\tau\}$. Hence $\pi(\{\tau\})$ is the weak limit of the $f_{n,\tau}(B)$ and is therefore in $\mathfrak{B}$. So too is $\pi(F)$ for every finite subset F of T , and by [4, p. 108] so is

$$C = \sup\{\pi(F): F \text{ finite}, F \subset T\}.$$

It is easy to show that $\pi(\{\tau\})$ is the operation of multiplying by the characteristic function of $\{\tau\} \times U$, so for each z in H the transform $w = Cz$ is defined by

$$\begin{aligned} w(u_1, u_2) &= z(u_1, u_2) \text{ for all } u_1 \text{ such that } \int_U |z(u_1, v)|^2 dv > 0, \\ &= 0 \text{ for all other } u_1. \end{aligned}$$

This operator cannot be represented in the form $\int f(t)d\pi(t)$ with any function f . For suppose C thus represented. For each τ in T , let z be the characteristic function of $\{\tau\} \times U$. This is in H , and $\pi(\{\tau\})z = z$. Hence $\pi(T')z = 0$ whenever $T' \subset T - \{\tau\}$, and

$$z = Cz = \left[\int_T f(t)d\pi(t) \right] z = f(\tau)z,$$

whence $f(\tau) = 1$. This holds for all τ in T , so C is the identity operator. But if z is the characteristic function of $U \times \{0\}$, $Cz = 0 \neq z$. So C is not represented as an integral, and the class of functions $q(B)$ is not weakly closed.

We correct this inadequacy by enlarging the class of integrands. Let the cyclic set $\{x_j: j \in J\}$, the subspaces $H(x_j)$ and the measures m_j be as in §9

and let P_j be the projector on $H(x_j)$. If $q(t, j)$ is measurable and essentially bounded (cf. §8) the expression

$$q(B_w) = \sum_{j \in J} \int q(t, j) d\pi(t) P_j$$

defines a bounded linear operator on H ; for if $y \in H$, $P_j y = 0$ except for countably many j , and the sum

$$\sum_{j \in J} \int q(t, j) d\pi(t) P_j y$$

converges in norm to a limit whose norm is at most $\text{ess. sup. } |q(t, j)| \cdot \|y\|$, and the linearity is evident. However, we now have too many operators. Easy examples, even with two-dimensional H , show that $q(B_w)$ may fail to belong to $\mathfrak{B}$. This excess is remedied by restricting the $q(t, j)$ to be essentially bounded quasi-functions on T , that is to belong to $L_{\infty*} = L_{\infty*}[T, \{m_j: j \in J\}]$. It is trivially easy to prove all but the last conclusion of the following theorem.

(12.1) THEOREM. *For each quasi-function $q(t, j)$ in $L_{\infty*}$, the operator $q(B_w)$ just defined is linear and has bound $\text{ess. sup. } |q(t, j)|$; and $\bar{q}(B_w) = [q(B_w)]^*$. If $q_1(t, j)$ and $q_2(t, j)$ are in $L_{\infty*}$ and c_1 and c_2 are complex numbers, and $q_3(t, j) = c_1 q_1(t, j) + c_2 q_2(t, j)$ and $q_4(t, j) = q_1(t, j) q_2(t, j)$, then*

$$q_3(B_w) = c_1 q_1(B_w) + c_2 q_2(B_w) \quad \text{and} \quad q_4(B_w) = q_1(B_w) q_2(B_w).$$

If $q(t, j)$ is equivalent to $q_0(t)$, then $q(B_w)$ is the same as $q_0(B_w)$ as ordinarily defined. The set of all operators $q(B_w)$ ($q(t, j) \in L_{\infty}$) is the algebra $\mathfrak{B}$.*

Let $q(t, j)$ belong to $L_{\infty*}$. For each C in $\mathfrak{B}'$, to each y in H there corresponds a countable subset J_c of J such that $P_j y = P_j C y = 0$ for all j in $J - J_c$. There is a measurable $f(t)$ such that $f(t) = q(t, j)$ except on an m_j -null set for all j in J_c , and by classical theory $f(B_w)$ commutes with C . Thus $C q(B_w) y = C f(B_w) y = f(B_w) C y = q(B_w) C y$. Since y is arbitrary in H , $q(B_w)$ commutes with the arbitrary member C of $\mathfrak{B}'$, and is therefore in $\mathfrak{B}''$. It remains to prove that every member of $\mathfrak{B}$ can be written in the form $q(B_w)$ for some q in $L_{\infty*}$.

Let K_S be the characteristic function of a measurable subset S of T . By §10, $\psi_0(K_S^\times) = \pi(S)$, so when $f = K_S$ the equation

$$(**) \quad \int f(t) d\pi(t) = \psi_0(f^\times)$$

is valid. By linearity it holds for all simple functions. If $f(t)$ is bounded and measurable it is the uniform limit of a sequence $s_1, s_2, \dots$ of simple functions. Then $s_n^\times$ tends to $f^\times$ in the uniform topology of operators, so $\psi_0(s_n^\times)$ tends in

the same manner to $\psi_0(f^\times)$. Also, $\int s_n(t) d\pi(t)$ tends in the same manner to $\int f(t) d\pi(t)$, so (**) holds for all bounded measurable functions $f(t)$.

Let $q(t, j)$ belong to $L_{\infty*}$. For each $z(t, j)$ in H_0 , let z_j be the function on $T \times J$ defined by $z_j(t, k) = z(t, j)$ if $k = j$, $z_j(t, k) = 0$ otherwise. By Lemma 9.1, the vector $y = \psi(z)$ of H is expressed as the sum of integrals, each of which is a vector in the correspondingly labeled subspace $H(x_j)$. Hence

$$P_j y = \int z(t, j) d\mathbf{m}_j(t) = \psi(z_j) \quad (j \in J).$$

From this and (**), with the definition of ψ_0 ,

$$\begin{aligned} \left[\int q(t, j) d\pi \right] P_j y &= \psi_0(q(t, j)^\times) \psi(z_j) \\ &= \psi(q(t, j)^\times z_j) \quad (j \in J). \end{aligned}$$

Summing over all j in J yields

$$q(B_w)y = \psi(q^\times z),$$

or

$$\psi^{-1}(q(B_w)\psi(z)) = q^\times z;$$

that is, $q(B_w) = \psi_0(q^\times)$. This last is an arbitrary member of the algebra $\psi_0(\mathfrak{M})$. By Theorem 11.1 this algebra is weakly closed, and it clearly contains all the B_w , so it contains all of $\mathfrak{B}$. Thus each operator in B can be represented in the form $q(B_w)$ for some q in $L_{\infty*}$, and the proof is complete.

As a corollary, if the spectrum has countable multiplicity (that is, if J is countable) the set of operators $f(B_w)$ with f essentially bounded, measurable and complex-valued on T is identical with the algebra $\mathfrak{B}$.

REFERENCES

1. Paul Halmos, *Measure theory*, Van Nostrand, New York, 1950.
2. I. E. Segal, *Equivalence of measure spaces*, Amer. J. Math. **73** (1951), 275-313.
3. J. Schwartz, *A note on the spaces L_p^** , Proc. Amer. Math. Soc. **2** (1951), 270-275.
4. E. J. McShane, *Order-preserving maps and integration processes*, Princeton Univ. Press, Princeton, N. J., 1954.
5. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, Gauthier-Villars, Paris, 1957.

Added in proof:

6. A. C. Zaanen, *The Radon-Nikodym theorem*, Nederl. Akad. Wetensch. Proc. Ser. A **64** (1961), 156-187.

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