SOME PROBABILISTIC THEOREMS ON DIOPHANTINE APPROXIMATIONS⁽¹⁾

BY
HARRY KESTEN

1. Introduction. Let

$$\langle \xi \rangle = \min_{k \text{ integer}} |\xi - k|$$

be the positive distance between ξ and the nearest integer to ξ . The first theorem is concerned with

$$\min_{1 \leq k \leq m} \langle k\xi \rangle.$$

By the methods used for Theorem 1 we also solve some special cases of a problem raised in [2] concerning the existence of an integer k for which $m \leq k \leq mc$ ($c > 1$) and $\langle k\xi \rangle \leq \alpha/k$ (Theorem 2).

While studying $\min_{1 \leq k \leq n} \langle k\xi \rangle$, one is naturally led to consider the number of integers k for which $1 \leq k \leq m$ and $\langle k\xi \rangle \leq \gamma$. The third theorem deals with this quantity in p dimensions i.e. it considers $N(m, \gamma, p)$, the number of integers k , for which $1 \leq k \leq m$ and *simultaneously*

$$(1.1) \quad \langle k\xi_1 \rangle \leq \gamma, \langle k\xi_2 \rangle \leq \gamma, \dots, \langle k\xi_p \rangle \leq \gamma.$$

Theorem 4 gives some easy generalizations of the third theorem.

Our approach is probabilistic in the sense that we do not take $\xi, \xi_1, \dots, \xi_p$ fixed, but choose them randomly, according to a uniform distribution on $[0, 1]$. This makes $\min_{1 \leq k \leq m} \langle k\xi \rangle$ and $N(m, \gamma, p)$ random variables. Accordingly, Theorem 1 gives an asymptotic expression for the Lebesgue measure of the set

$$\left\{ \xi: 0 \leq \xi \leq 1, m \cdot \min_{1 \leq k \leq m} \langle k\xi \rangle \leq \alpha \right\}.$$

Theorem 3, which states that $N(m, \gamma, p)$ has asymptotically ($p \rightarrow \infty, m \cdot (2\gamma)^p \rightarrow \lambda$) a Poisson distribution with mean λ , can be paraphrased similarly. It then gives asymptotic expressions for the p dimensional Lebesgue measure of the sets

$$\{ \xi_1, \dots, \xi_p: 0 \leq \xi_j \leq p, N(m, \gamma, p) = k \}, \quad k = 0, 1, \dots.$$

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We shall stick further to the probabilistic language. As far as used here it is of such a simple kind that it should not cause any difficulties. All the required definitions can be found in [9].

Theorems 1, 2 are immediate extensions of the results of Friedman and Niven [3] and of Erdős, Szűsz, and Turan [2]. Theorem 3 is proved by the method of moments. We are forced, however, to prove the convergence of the moments in a rather roundabout way (cf. also the remarks after Lemma 1). One should compare Theorem 3 with the well-known result of Dirichlet, which states that if γ^{-1} is an integer, then there exists for *every* $(\xi_1, \dots, \xi_p)$ at least one $k \leq \gamma^{-p}$ satisfying (1.1). Our results show that the Lebesgue measure of the set of $(\xi_1, \dots, \xi_p)$'s, for which there exists such a $k \leq \lambda(2\gamma)^{-p}$, is approximately $1 - e^{-\lambda}$. Related results were also obtained by Erdős and Rényi [1] who studied

$$\max_{1 \leq k \leq m} \left| \sum_{j=1}^p e^{2\pi i k \xi_j} \right|.$$

2. The minimum of $\langle kx \rangle$, $k=1, \dots, m$. Let x have a uniform distribution⁽²⁾ on $[0, 1]$ and put

$$(2.1) \quad Z(m) = \min_{1 \leq k \leq m} \langle kx \rangle$$

$$(2.2) \quad N(m, \gamma) = \text{number of integers } k \text{ for which } 1 \leq k \leq m \text{ and } \langle kx \rangle \leq \gamma,$$

and

$$(2.3) \quad t(\xi, \gamma) = \text{smallest integer } k \geq 1 \text{ for which } \langle k\xi \rangle \leq \gamma.$$

Clearly

$$Z(m) \leq \alpha/m$$

is equivalent to

$$(2.4) \quad N(m, \alpha/m) \geq 1$$

which in turn is equivalent to

$$(2.5) \quad t(x, \alpha/m) \leq m.$$

Hence, if the limits exist⁽³⁾,

$$(2.6) \quad \begin{aligned} \lim_{m \rightarrow \infty} P\{Z(m) \leq \alpha/m\} &= \lim_{m \rightarrow \infty} P\{N(m, \alpha/m) \geq 1\} \\ &= \lim_{m \rightarrow \infty} P\{t(x, \alpha/m) \leq m\}. \end{aligned}$$

⁽²⁾ We shall always use x or x_i for random variables, whereas fixed numbers from $[0, 1]$ will be denoted by ξ or ξ_i .

⁽³⁾ $P\{A\}$ = probability of the event A , $P\{A|B\}$ = conditional probability of the event A , given B , $E(X)$ = expectation of the random variable X , $E(X|B)$ = conditional expectation of the random variable X given B , $\text{var}(X) = E(X^2) - (EX)^2$ = variance of the random variable X , $\text{var}(X|B) = E(X^2|B) - (E(X|B))^2$ = conditional variance of the random variable X given B .

The last limit in (2.6) can be found by the methods of Friedman and Niven [3] and of Erdős, Szűs and Turan [2]. In [3] the first two moments of $t(x, \alpha/m)$ were computed. Let F_k denote the Farey series of order k , that is the series of rational numbers⁽⁴⁾ a/b , $0 \leq a \leq b \leq k$, $(a, b) = 1$, in ascending order (cf. [5, Chapter III for more details]). It was proved in [3, p. 27] that for every ξ

$$t(\xi, 1/n) \leq n - 1.$$

Writing further, for some fixed n ,

$$I\left(\frac{a}{b}\right) = \left[\frac{a}{b} - \frac{1}{nb}, \frac{a}{b} + \frac{1}{nb}\right]$$

it was also shown in [3, p. 27] that

$$(2.7) \quad t(\xi, 1/n) \leq k$$

if and only if one has

$$(2.8) \quad \xi \in I\left(\frac{a}{b}\right) \quad \text{for some } \frac{a}{b} \in F_k.$$

Moreover, there are at most two fractions $a/b \in F_k$ ($k \leq n-1$) for which $\xi \in I(a/b)$ [2; 3]. Let

$$S_k = \left\{ \xi: 0 \leq \xi \leq 1; \text{there exist } \frac{a_1}{b_1}, \frac{a_2}{b_2} \in F_k, \frac{a_1}{b_1} \neq \frac{a_2}{b_2} \right. \\ \left. \text{such that } \xi \in I\left(\frac{a_1}{b_1}\right) \text{ and } \xi \in I\left(\frac{a_2}{b_2}\right) \right\}.$$

Thus S_k is the subset of $[0, 1]$ covered by two intervals I from F_k . If $\mu\{\cdot\}$ denotes Lebesgue measure, then it follows from the above remarks that (cf. [2, formula (12)])

$$(2.9) \quad P\left\{t\left(x, \frac{1}{n}\right) \leq k\right\} = \sum_{a/b \in F_k} \mu\left\{I\left(\frac{a}{b}\right)\right\} - \mu\{S_k\} = \sum_{a/b \in F_k} \frac{2}{nb} - \mu\{S_k\} \\ = \frac{2}{n} \sum_{b=1}^k \frac{\Phi(b)}{b} - \mu\{S_k\}$$

where $\Phi(\cdot)$ is Euler's function [5, p. 52].

It remains to find an expression for $\mu\{S_k\}$. If ξ is contained in two intervals $I(a_1/b_1)$, $I(a_2/b_2)$ with $a_i/b_i \in F_k$, $k \leq n-1$, then [3, p. 27] a_1/b_1 and a_2/b_2 must be consecutive elements of F_k . It is known [5, Theorem 28, p. 23] that in this case

⁽⁴⁾ $[a]$ = largest integer which does not exceed a . Although we also use square brackets with other meaning, confusion seems unlikely. (a, b) = greatest common divisor of a and b .

$$(2.10) \quad \left| \frac{a_1}{b_1} - \frac{a_2}{b_2} \right| = \frac{|a_1 b_2 - a_2 b_1|}{b_1 b_2} = \frac{1}{b_1 b_2}$$

and hence

$$(2.11) \quad \mu \left\{ I \left(\frac{a_1}{b_1} \right) \cap I \left(\frac{a_2}{b_2} \right) \right\} = \begin{cases} \frac{b_1 + b_2 - n}{n b_1 b_2} & \text{if } b_1 + b_2 \geq n, \\ 0 & \text{if } b_1 + b_2 < n. \end{cases}$$

One has therefore

$$(2.12) \quad \mu \{S_k\} = \sum_k' \frac{b_1 + b_2 - n}{n b_1 b_2}$$

where $\sum_k'$ ranges over all pairs $a_1/b_1, a_2/b_2$ which are consecutive elements of F_k and satisfy $b_1 < b_2$ and $b_1 + b_2 \geq n$. ($b_1 = b_2$ is impossible for successive fractions by Theorem 28, p. 23 in [5] and since we want to take every pair into account *only once* we may take $b_1 < b_2$.) It was proved in Lemma 1 of [3] that for fixed $b_1 < b_2 \leq k$, $(b_1, b_2) = 1$, $b_1 + b_2 \geq n$ there are exactly two choices for a_1, a_2 such that $a_1/b_1, a_2/b_2$ belong to $\sum_k'$ while for $(b_1, b_2) > 1$ there are no choices possible for a_1, a_2 (again by Theorem 28, p. 23 of [5]). Consequently

$$(2.13) \quad \mu \{S_k\} = \sum_k' \frac{b_1 + b_2 - n}{n b_1 b_2} = \frac{2}{n} \sum_{b_1=1}^k \sum_{\substack{b_2=\max(n-b_1, b_1+1) \\ (b_1, b_2)=1}}^k \frac{b_1 + b_2 - n}{b_1 b_2}.$$

With these preliminaries it is easy to prove

THEOREM 1.

$$\begin{aligned} \lim_{m \rightarrow \infty} P \left\{ m \cdot \min_{1 \leq k \leq m} \langle kx \rangle \leq \alpha \right\} &= \lim_{m \rightarrow \infty} P \left\{ N \left(m, \frac{\alpha}{m} \right) \geq 1 \right\} \\ &= \lim_{m \rightarrow \infty} P \left\{ t \left(x, \frac{\alpha}{m} \right) \leq m \right\} = F(\alpha) \end{aligned}$$

where

$$(2.14) \quad F(\alpha) = \begin{cases} 0 & \text{if } \alpha < 0, \\ \frac{12\alpha}{\pi^2} & \text{if } 0 < \alpha \leq 1/2, \\ \frac{12\alpha}{\pi^2} - \frac{12}{\pi^2} \int_{1/2}^{\alpha} \left(2 - \frac{1}{y} - \frac{1-y}{y} \log \frac{y}{1-y} \right) dy & \text{if } 1/2 < \alpha \leq 1, \\ 1 & \text{if } 1 < \alpha. \end{cases}$$

Proof. Firstly,

$$(2.15) \quad \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{b=1}^k \frac{\Phi(b)}{b} = \frac{6}{\pi^2} \quad [3, \text{p. 29}].$$

Moreover, from (2.11) or (2.13) $\mu\{(S_k)\} = 0$ for $k \leq n/2$ because $b_1 < b_2 \leq k$ implies $b_1 + b_2 < n$. For $k > n/2$ we have, following [3](4),

$$(2.16) \quad \begin{aligned} & \frac{2}{n} \sum_{b_1=1}^k \sum_{\substack{b_2=\max(n-b_1, b_1+1) \\ (b_1, b_2)=1}}^k \frac{b_1 + b_2 - n}{b_1 b_2} \\ &= \frac{2}{n} \left(\sum_{b_1=n-k}^{[(n-1)/2]} \sum_{b_2=n-b_1}^k + \sum_{b_1=[(n+1)/2]}^k \sum_{b_2=b_1+1}^k \right) \frac{b_1 + b_2 - n}{b_1 b_2} \sum_{d|(b_1, b_2)} \mu(d) \\ &= T_1 + T_2, \text{ say,} \end{aligned}$$

where $\mu(\cdot)$ is the Möbius function [5, p. 234].

As an example we shall compute the asymptotic behaviour of T_1 , the computation for T_2 being very similar. Changing the order of summation and putting $b_2 = sd$ and

$$\{a\} = \text{smallest integer greater or equal to } a^{(5)}$$

one has

$$T_1 = \frac{2}{n} \sum_{b=n-k}^{[(n-1)/2]} \frac{1}{b} \sum_{d|b} \mu(d) \sum_{s=\{(n-b)/d\}}^{[k/d]} \frac{b + sd - n}{sd}.$$

However,

$$\sum_{s=\{(n-b)/d\}}^{[k/d]} \frac{b + sd - n}{sd} = \frac{b - n}{d} \log \frac{k}{n - b} + \frac{k - n + b}{d} + O(1).$$

Since [5, p. 235 and p. 260]

$$\sum_{d|b} \frac{\mu(d)}{d} = \frac{\Phi(b)}{b}, \quad \sum_{d|b} 1 = O(b^{1/2}),$$

$$\begin{aligned} T_1 &= 2 \sum_{b=n-k}^{[(n-1)/2]} \frac{\Phi(b)}{b^2} \left(1 - \frac{b}{n}\right) \log \left(1 - \frac{b}{n}\right) + \frac{2k}{n} \sum_{b=n-k}^{[(n-1)/2]} \frac{\Phi(b)}{b^2} \\ &+ 2 \left(1 + \log \frac{k}{n}\right) \frac{1}{n} \sum_{b=n-k}^{[(n-1)/2]} \frac{\Phi(b)}{b} - 2 \left(1 + \log \frac{k}{n}\right) \sum_{b=n-k}^{[(n-1)/2]} \frac{\Phi(b)}{b^2} \\ &+ O\left(\frac{1}{n} + \frac{1}{k}\right). \end{aligned}$$

Using the fact that [5, p. 268]

(5) This meaning of $\{a\}$ is used only in the next two formulae.

$$\sum_{b=1}^n \Phi(b) = \frac{3n^2}{\pi^2} + O(n \log n)$$

one obtains by partial summation as $n \rightarrow \infty$, $k \rightarrow \infty$, $k/n \rightarrow \alpha$, $1/2 \leq \alpha < 1$,

$$(2.17) \quad \lim_{k/n \rightarrow \alpha} T_1 = \frac{12}{\pi^2} \left[\int_{1-\alpha}^{1/2} \frac{1-y}{y} \log(1-y) dy - \alpha \log 2(1-\alpha) \right. \\ \left. + (1 + \log \alpha)(\alpha - 1/2 + \log 2(1-\alpha)) \right].$$

In an entirely similar manner one obtains

$$(2.18) \quad \lim_{k/n \rightarrow \alpha} T_2 = \frac{12}{\pi^2} \left[\alpha \log 2\alpha - (\alpha - 1/2) - \int_{1/2}^{\alpha} \frac{y-1}{y} \log y dy \right. \\ \left. + \log \alpha \int_{1/2}^{\alpha} \frac{y-1}{y} dy \right].$$

(2.17) and (2.18) imply

$$(2.19) \quad \lim_{k/n \rightarrow \alpha} (T_1 + T_2) = \frac{12}{\pi^2} \int_{1/2}^{\alpha} \left(2 - \frac{1}{y} - \frac{1-y}{y} \log \frac{y}{1-y} \right) dy$$

as one easily checks by comparing the derivatives of the right-hand sides of (2.17)–(2.19) as well as the values at $\alpha = 1/2$.

From (2.9), (2.13), (2.15), (2.16), (2.19) one has

$$(2.20) \quad \lim_{k/n \rightarrow \alpha} P \left\{ t \left(x, \frac{1}{n} \right) \leq k \right\} = F(\alpha),$$

where $F(\alpha)$ is defined in (2.14). To prove the general relation

$$(2.21) \quad \lim_{m \rightarrow \infty} P \left\{ t \left(x, \frac{\alpha}{m} \right) \leq m \right\} = F(\alpha)$$

we notice, that if

$$(2.22) \quad \frac{1}{n+1} \leq \frac{\alpha}{m} \leq \frac{1}{n}$$

then for every ξ

$$t \left(\xi, \frac{1}{n} \right) \leq t \left(\xi, \frac{\alpha}{m} \right) \leq t \left(\xi, \frac{1}{n+1} \right).$$

Consequently also

$$P\left\{t\left(x, \frac{1}{n}\right) \leq m\right\} \geq P\left\{t\left(x, \frac{\alpha}{m}\right) \leq m\right\} \geq P\left\{t\left(x, \frac{1}{n+1}\right) \leq m\right\}$$

which, together with (2.20), (2.22) completes the proof of (2.21).

Till now we considered integers $k \leq m$ for which $\langle k\xi \rangle \leq \alpha/m$. In [2] one considered integers k for which

$$\langle k\xi \rangle \leq \frac{\alpha}{k}.$$

In particular, putting $(\alpha > 0, c > 1)$

$S(m, \alpha, c) = \{\xi: 0 \leq \xi \leq 1, \text{ there exist integers } a, b \text{ for which}$

$$m \leq b \leq mc, (a, b) = 1, |b\xi - a| \leq \alpha/b\}.$$

Erdős, Szűs and Turán [2] raised the question of finding

$$\lim_{m \rightarrow \infty} \mu\{S(m, \alpha, c)\}$$

if it exists at all. They found this limit for $\alpha \leq c/(1+c^2)$ and gave bounds for $\mu\{S(m, \alpha, c)\}$ in several other cases. One has of course

$$\left\{\xi: 0 \leq \xi \leq 1, m \leq t\left(\xi, \frac{\alpha}{mc}\right) \leq mc\right\} \subseteq S(m, \alpha, c)$$

because

$$(2.23) \quad \left|t\left(\xi, \frac{\alpha}{mc}\right)\xi - r\right| \leq \frac{\alpha}{mc} \leq \frac{\alpha}{t(\xi, \alpha/mc)}$$

if $t(\xi, \alpha/mc) \leq mc$, and at the same time (2.23) implies $(t(\xi, \alpha/mc), r) = 1$ (cf. [3, p. 27]).

Consequently one has

$$(2.24) \quad \liminf_{m \rightarrow \infty} \mu\{S(m, \alpha, c)\} \geq F(\alpha) - F\left(\frac{\alpha}{c}\right).$$

Entirely obvious is the inclusion

$$S(m, \alpha, c) \subseteq \left\{\xi: 0 \leq \xi \leq 1, t\left(\xi, \frac{\alpha}{m}\right) \leq mc\right\},$$

and therefore

$$(2.25) \quad \limsup_{m \rightarrow \infty} \mu\{S(m, \alpha, c)\} \leq F(\alpha c).$$

(2.24) and (2.25) are improvements on the results of [2] for certain combinations of α and c . (2.25) however is only useful if $\alpha c < 1$ (compare (2.14)).

In this case, however, we can compute the limit of $\mu\{S(m, \alpha, c)\}$ exactly. The result is given by the next theorem.

THEOREM 2. *If $\alpha \leq c/(1+c^2)$, then*

$$(2.26) \quad \lim_{m \rightarrow \infty} \mu\{S(m, \alpha, c)\} = \frac{12\alpha}{\pi^2} \log c \quad [2].$$

If $c/(1+c^2) \leq \alpha \leq \min(1/2, 1/c)$, then

$$(2.27) \quad \begin{aligned} \lim_{m \rightarrow \infty} \mu\{S(m, \alpha, c)\} = & \frac{12\alpha}{\pi^2} \log c - \frac{12}{\pi^2} \left(\alpha c + \frac{\alpha}{c} - \alpha\beta - \frac{\alpha}{\beta} \right. \\ & \left. + \alpha \left(\frac{1}{\beta} - \beta \right) \log \frac{c}{\beta} - \frac{1}{2} \left(\log \frac{c}{\beta} \right)^2 \right), \end{aligned}$$

where

$$\beta = \frac{1 + (1 - 4\alpha^2)^{1/2}}{2\alpha}.$$

If $1/2 \leq \alpha \leq 1/c$, then

$$(2.28) \quad \lim_{m \rightarrow \infty} \mu\{S(m, \alpha, c)\} = \frac{12\alpha}{\pi^2} \log c - \frac{12}{\pi^2} \left(\alpha c - 2\alpha + \frac{\alpha}{c} - \frac{1}{2} (\log c)^2 \right).$$

Added in proof. Since this paper was written the following two references containing results related to this theorem have come to the author's attention: P. Erdős, *Some results on diophantine approximation*, Acta Arithmetica **5** (1959), 359–369 and Richard P. Gosselin, *On diophantine approximation and trigonometric polynomials*, Pacific J. Math. **9** (1959), 1071–1081.

Proof. (2.26) is Theorem III of [2]. Instead of $I(a/b)$ we now define

$$J\left(\frac{a}{b}\right) = \left[\frac{a}{b} - \frac{\alpha}{b^2}, \frac{a}{b} + \frac{\alpha}{b^2} \right] \quad \text{for } \frac{a}{b} \in F_{[mc]}.$$

Then

$$(2.29) \quad S(m, \alpha, c) = \bigcup_{a/b \in F_{[mc]}; b \geq m} J\left(\frac{a}{b}\right).$$

$$(2.30) \quad \sum_{a/b \in F_{[mc]}; b \geq m} \mu\left\{J\left(\frac{a}{b}\right)\right\} = \sum_m^{[mc]} \frac{2\alpha\Phi(b)}{b^2} \rightarrow \frac{12\alpha}{\pi^2} \log c \quad (m \rightarrow \infty).$$

In addition, if a_1/b_1 and a_2/b_2 are two consecutive elements of $F_{[mc]}$ with $b_1, b_2 \geq m$, then

$$\left| \frac{a_2}{b_2} - \frac{a_1}{b_1} \right| = \frac{1}{b_1 b_2} \geq \max \left(\frac{\alpha}{b_1^2}, \frac{\alpha}{b_2^2} \right)$$

by [5, Theorem 28, p. 23] and the fact that $b_1/b_2 \leq c$, $b_2/b_1 \leq c$, $\alpha c \leq 1$. Consequently $a_1/b_1 \notin J(a_2/b_2)$, $a_2/b_2 \notin J(a_1/b_1)$ and therefore no ξ can be in more than two intervals $J(a/b)$, $b \geq m$, $a/b \in F_{[mc]}$. If ξ is in $J(a_1/b_1) \cap J(a_2/b_2)$ then a_1/b_1 and a_2/b_2 must be consecutive elements of $F_{[mc]}$. In this case

$$\mu \left\{ J \left(\frac{a_1}{b_1} \right) \cap J \left(\frac{a_2}{b_2} \right) \right\} = \max \left(0, \frac{\alpha}{b_1^2} + \frac{\alpha}{b_2^2} - \frac{1}{b_1 b_2} \right).$$

Notice that for $\alpha \geq 1/2$, always

$$\frac{\alpha}{b_1^2} + \frac{\alpha}{b_2^2} - \frac{1}{b_1 b_2} \geq 0$$

where as for $\alpha \leq 1/2$, $b_2 > b_1$

$$\frac{\alpha}{b_1^2} + \frac{\alpha}{b_2^2} - \frac{1}{b_1 b_2} \geq 0$$

only if

$$\frac{b_2}{b_1} \geq \beta = \frac{1 + (1 - 4\alpha^2)^{1/2}}{2\alpha}.$$

Thus, by (2.29) and (2.30)

$$\mu \{ S(m, \alpha, c) \} = \frac{12\alpha}{\pi^2} \log c - \sum'' \left(\frac{\alpha}{b_1^2} + \frac{\alpha}{b_2^2} - \frac{1}{b_1 b_2} \right) + o(1)$$

where $\sum''$ ranges over all pairs a_1/b_1 , a_2/b_2 of consecutive elements of $F_{[mc]}$ with

$$(2.31) \quad m \leq b_1 < b_2 \leq mc \quad \text{if } \alpha \geq 1/2$$

and with

$$(2.32) \quad m \leq b_1, \quad \beta b_1 \leq b_2 \leq mc \quad \text{if } \alpha \leq 1/2.$$

It follows again from Lemma 1 in [3], that for given b_1 , b_2 satisfying (2.31) if $\alpha \geq 1/2$ or (2.32) if $\alpha \leq 1/2$, there are exactly two pairs (a_1, a_2) such that $a_1/b_1, a_2/b_2$ belong to $\sum''$ if $(b_1, b_2) = 1$ and no such pairs if $(b_1, b_2) > 1$. $\sum''$ can now be computed exactly as in Theorem 1.

3. The distribution of $N(m, \gamma)$ in more dimensions. As we have seen in the last section, the study of the distribution of $\min_{1 \leq k \leq m} \langle kx \rangle$ was equivalent to finding $P \{ N(m, \gamma) = 0 \}$ for appropriate γ . This raises the question of finding the complete distribution of $N(m, \gamma)$. Even though the methods of §2 probably allow us to find the asymptotic distribution of $N(m, \alpha/m)$, it seems

very hard to find the asymptotic distribution of $N(m, \gamma) - EN(m, \gamma) = N(m, \gamma) - 2m\gamma$ after proper normalization, for fixed γ . The difficulty seems to be the "strong" dependence between the random variables $\langle kx \rangle$, $k = 1, 2, \dots$. This shows a.o. in the fact that the distribution of $m^{-1/2}(N(m, \gamma) - 2m\gamma)$ does *not* approach a normal distribution, as it would if the random variables $\langle kx \rangle$ were strictly independent. In fact $m^{-1/2}$ is not at all the correct normalization factor [7]. It was suggested by M. Kac in a discussion with the author that independence would approximately hold again for analogous random variables in high dimensions. This will be shown to be correct in a certain sense in the next theorem. The limiting distribution obtained in Theorem 3 is precisely the limiting distribution which would pertain if the Y 's were strictly independent.

Let $x_1, x_2, \dots$ be independent random variables, each with a uniform distribution on $[0, 1]$. Define

$$Y_k^j(\gamma) = \begin{cases} 1 & \text{if } \langle kx_1 \rangle \leq \gamma, \langle kx_2 \rangle \leq \gamma, \dots, \langle kx_j \rangle \leq \gamma, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$N(m, \gamma, j) = \sum_{k=1}^m Y_k^j(\gamma).$$

$N(m, \gamma, j)$ is the number of indices $k \leq m$ for which *simultaneously* $kx_1, \dots, kx_j$ are closer than γ to an integer.

THEOREM 3. *If $0 < \gamma < 1/2$ is fixed and $p \rightarrow \infty, m \rightarrow \infty$ such that*

$$(3.1) \quad m(2\gamma)^p \rightarrow \lambda > 0,$$

then

$$\lim P\{N(m, \gamma, p) = k\} = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, \dots,$$

that is, $N(m, \gamma, p)$ has asymptotically a Poisson distribution with mean λ .

Proof. Put for $k_1, \dots, k_r$ pairwise different

$$(3.2) \quad A_\gamma(k_1, \dots, k_r) = P\{Y_{k_1}^1(\gamma) = \dots = Y_{k_r}^1(\gamma) = 1\}.$$

γ will be fixed throughout ($0 < \gamma < 1/2$) and the index or subscript γ will only be written explicitly when necessary to avoid confusion. Since the $x_1, x_2, \dots$ are independent one has clearly

$$(3.3) \quad \begin{aligned} P\{Y_{k_1}^j = \dots = Y_{k_r}^j = 1\} &= (P\{Y_{k_1}^1 = \dots = Y_{k_r}^1 = 1\})^j \\ &= A^j(k_1, \dots, k_r). \end{aligned}$$

Moreover,

$$(3.4) \quad \begin{aligned} E(N(m, \gamma, j)(N(m, \gamma, j) - 1)(N(m, \gamma, j) - 2) \cdots (N(m, \gamma, j) - r + 1)) \\ = E \sum_{k(r)} Y_{k_1}^j \cdots Y_{k_r}^j = \sum_{k(r)} A^j(k(r)). \end{aligned}$$

We adopted here the convention to write $k(r)$ for a generic r -tuple $(k_1, \dots, k_r)$ of different integers k_i , $1 \leq k_i \leq m$, and to include in $\sum_{k(r)}$ all such r -tuples in the $r!$ orders in which they can appear. This convention will be used through the remainder of this section.

One easily checks that if N has a Poisson distribution with mean λ , then its r th factorial moment,

$$E(N(N-1)(N-1) \cdots (N-r+1)) = \lambda^r, \quad r = 1, 2, \dots.$$

Conversely, if $\{N(p)\}$ is a sequence of random variables for which

$$\lim_{p \rightarrow \infty} E(N(p)(N(p)-1)(N(p)-2) \cdots (N(p)-r+1)) = \lambda^r, \quad r = 1, 2, \dots$$

then the limiting distribution ($p \rightarrow \infty$) of $N(p)$ is a Poisson distribution with mean λ (by [9, p. 185 C; 6, p. 115 4.30, p. 109 4.21]).

Theorem 3 will therefore be proved, if we can show

$$\lim \sum_{k(r)} A^p(k(r)) = \lambda^r, \quad r = 1, 2, \dots,$$

or equivalently, by (3.1)

$$(3.5) \quad \lim \frac{\sum_{k(r)} \left(\frac{A_\gamma(k(r))}{(2\gamma)^r} \right)^p}{m(m-1) \cdots (m-r+1)} = 1, \quad r = 1, 2, \dots.$$

We shall prove (3.5) by induction on r .

Let us put

$$(3.6) \quad \nu_r^{(j)}(\gamma) = \frac{\sum_{k(r)} \left(\frac{A_\gamma(k(r))}{(2\gamma)^r} \right)^j}{m(m-1) \cdots (m-r+1)}.$$

Then

$$\nu_1^{(p)}(\gamma) = \frac{\sum_{k=1}^m A_\gamma^p(k)}{m(2\gamma)^p} = \frac{\sum_{k=1}^m (P\{Y_k^1(\gamma) = 1\})^p}{m \cdot (2\gamma)^p} = 1$$

since

$$P\{Y_k^1(\gamma) = 1\} = P\{kx_1 \leq \gamma\} = 2\gamma.$$

Thus (3.5) holds for $r=1$. The remaining details will be split up in a number of lemmas. C_i , $i=1, 2, \dots$, will always stand for some finite, positive constant which depends on γ, r, λ only.

LEMMA 1. *There exists a C_1 such that for any sequence of s positive integers $k_1 < k_2 < \dots < k_s$*

$$E\left(\sum_{j=1}^s Y_{k_j}^1(\gamma) - 2s\gamma\right)^2 \leq \frac{s}{2} (C_1 + (\log s)^2).$$

C_1 does not depend on s .

Proof⁽³⁾.

$$\begin{aligned} E\left(\sum_{j=1}^s Y_{k_j}^1 - 2s\gamma\right)^2 &= \text{var}\left(\sum_{j=1}^s Y_{k_j}^1\right) \\ (3.7) \qquad &= s2\gamma(1-2\gamma) + 2 \sum_{1 \leq i < j \leq s} (A(k_i, k_j) - 4\gamma^2). \end{aligned}$$

It was proved in [8, p. 217 line 4] that

$$(3.8) \qquad |A_\gamma(k_i, k_j) - 4\gamma^2| \leq \frac{(k_i, k_j)^2}{k_i \cdot k_j} = \frac{1}{k'_i \cdot k'_j}$$

where (k_i, k_j) = greatest common divisor of k_i and k_j and

$$k'_i = \frac{k_i}{(k_i, k_j)}, \qquad k'_j = \frac{k_j}{(k_i, k_j)}.$$

Notice that for any pair of integers $a < b$ there are at most s possible pairs $k_i < k_j$ such that $k'_i = a$, $k'_j = b$ and hence, for any positive integer c there are at most $sd(c)/2$ pairs $k_i < k_j$ with $k'_i \cdot k'_j = c$ ($d(c)$ = number of divisors of c). Therefore,

$$(3.9) \qquad \sum_{1 \leq i < j \leq s} |A_j(k_i, k_j) - 4\gamma^2| \leq \frac{s}{2} \sum_{c=1}^{c_0} \frac{d(c)}{c}$$

where c_0 is the smallest integer for which

$$\frac{s}{2} \sum_{c=1}^{c_0} d(c) \geq \frac{s(s-1)}{2}.$$

In fact there are $s(s-1)/2$ pairs $k_i < k_j$ and we have replaced each $1/k'_i k'_j$ by some $1/c$ with $1/k'_i k'_j \leq 1/c$. Since

$$\sum_{c=1}^n d(c) = n \log n + O(n)$$

[5, Theorem 320, p. 264],

$$\frac{s}{2} \sum_{c=1}^{c_0} \frac{d(c)}{c} \leq \frac{s}{2} \sum_{c=1}^{2s/\log s} \frac{d(c)}{c} \leq \frac{s}{4} (\log s)^2$$

for sufficiently large s . This together with (3.7) and (3.8) implies the lemma.

The estimate of [8] for $|A(k_i, k_j) - 4\gamma^2|$ would suffice to prove directly $\nu_2^{(p)}(\gamma) \rightarrow 1$, and one might want to follow LeVeque's method [8] of estimating $A_\gamma(k_1, k_j)$ also for $A_\gamma(k_1, \dots, k_r)$ in order to prove (3.5) directly for all r . This would require an estimate of

$$A_\gamma(k_1, \dots, k_r) = \int_0^1 d\xi \prod_{i=1}^r \left(2\gamma + \frac{2}{\pi} \sum_{n_i=1}^{\infty} \frac{\sin 2\pi n_i \gamma \cos 2\pi n_i k_i \xi}{n_i} \right).$$

We have been unable to follow this direct procedure and are forced to prove (3.5) by a detour.

LEMMA 2. *If X is any random variable satisfying*

$$(3.10) \quad E(X - a)^2 \leq c^2$$

and

$$(3.11) \quad 0 \leq X \leq b \quad \text{with probability one,}$$

then, there exists constants C_2 , independent of the distribution of X such that

$$(3.12) \quad |EX - a| \leq c,$$

$$(3.13) \quad \int_{2a}^{\infty} y^r dP\{X \leq y\} \leq C_2(r) b^{r-2} c^2, \quad r = 3, 4, \dots,$$

and

$$(3.14) \quad |EX^r - (EX)^r| \leq C_2(r) b^{r-2} c^2, \quad r = 2, 3, 4, \dots$$

Proof. Put $EX = \mu$. Then

$$(3.15) \quad \begin{aligned} c^2 &\geq E(X - a)^2 = E(X - \mu + \mu - a)^2 = E(X - \mu)^2 + (\mu - a)^2 \\ &\geq (EX - a)^2. \end{aligned}$$

This proves (3.12). As for (3.13) and (3.14) we use the one sided analogue of Tchebyshev's inequality. For $\lambda \geq 0$,

$$(3.16) \quad \begin{aligned} P\{X - a \geq \lambda\} &\leq \frac{1}{(\lambda^2 + c^2)^2} \int_a^{\infty} (\lambda(y - a) + c^2)^2 dP\{X \leq y\} \\ &\leq \frac{2}{(\lambda^2 + c^2)^2} \int_a^{\infty} (\lambda^2(y - a)^2 + c^4) dP\{X \leq y\} \\ &\leq \frac{2}{(\lambda^2 + c^2)^2} (\lambda^2 c^2 + c^4) = \frac{2c^2}{\lambda^2 + c^2}. \end{aligned}$$

Putting

$$G(\lambda) = \begin{cases} 0 & \text{if } \lambda \leq c, \\ 1 - \frac{2c^2}{\lambda^2 + c^2} & \text{if } c \leq \lambda < b - a, \\ 1 & \text{if } b - a \leq \lambda, \end{cases}$$

one has by (3.11) and (3.16)

$$1 - P\{X \leq a + \lambda\} \leq 1 - G(\lambda).$$

Consequently, for $r \geq 3$

$$\begin{aligned} \int_{2a}^{\infty} y^r dP\{X \leq y\} &= \int_{2a}^{b+} y^r dP\{X \leq y\} \\ &= -(1 - P\{X \leq y\})y^r \Big|_{2a}^b + r \int_{2a}^b y^{r-1}(1 - P\{X \leq y\})dy \\ &= (1 - P\{X \leq 2a\})(2a)^r + r \int_{2a}^b y^{r-1}(1 - P\{X \leq y\})dy \\ &\leq (2a)^r(1 - G(a)) + r \int_a^{b-a} (z+a)^{r-1}(1 - G(z))dz \\ &\leq (2a)^r \frac{2c^2}{a^2} + 2^{r-1}r \int_a^{b-a} z^{r-1} \cdot \frac{2c^2}{z^2 + c^2} dz. \end{aligned}$$

This implies (3.13) when $2a \leq b$. When $2a > b$ (3.13) becomes trivial by (3.11).

(3.14) is proved similar to (3.13). By (3.11), $0 \leq \mu \leq b$ and by (3.15) $E(X - \mu)^2 \leq c^2$. Thus, with a replaced by μ one has from (3.16)

$$(3.17) \quad 1 - P\{X \leq \mu + \lambda\} \leq 1 - G(\lambda)$$

and similarly with a replaced by $b - \mu$

$$(3.18) \quad P\{X \leq \mu - \lambda\} \leq 1 - G(\lambda).$$

Now, since $E(X - \mu)$ equals zero,

$$\begin{aligned} EX^r &= E(\mu + X - \mu)^r = \mu^r + \binom{r}{2} \mu^{r-2} E(X - \mu)^2 \\ (3.19) \quad &+ \sum_{j=3}^r \binom{r}{j} \mu^{r-j} E(X - \mu)^j. \end{aligned}$$

By (3.17), for $j \geq 3$

$$(3.20) \quad \int_{y \geq \mu} (y - \mu)^j dP\{X \leq y\} \leq \int_0^{b-\mu} z^j dG(z) \leq C_3(j) b^{j-2} c^2$$

and by (3.18)

$$(3.21) \quad \int_{y \leq \mu} |y - \mu| dP\{X \leq y\} \leq C_3(j) b^{i-2} c^2.$$

(3.14) follows from (3.19)–(3.21).

An important consequence of Lemma 1 and Lemma 2 is the following:

LEMMA 3. *If $p \rightarrow \infty$, $m \rightarrow \infty$ such that $m(2\gamma)^p \rightarrow \lambda$, then for some C_4 , depending on γ , r , λ only, and $j \leq p$*

$$(3.22) \quad \exp(-C_4(2\gamma)^{p-j}(1 + (p-j)^2)) \leq \nu_r^{(j)}(\gamma) \leq \exp(C_4(2\gamma)^{p-j}(1 + (p-j)^2)).$$

Proof. As we remarked already, Lemma 3 is obvious for $r = 1$, and we may assume $r \geq 2$.

$$\begin{aligned} \nu_r^{(j)}(\gamma) &= \frac{E(N(m, \gamma, j) \cdots (N(m, \gamma, j) - r + 1))}{m(m-1) \cdots (m-r+1)(2\gamma)^{rj}} \\ &\leq \frac{EN^r(m, \gamma, j)}{m(m-1) \cdots (m-r+1)(2\gamma)^{rj}}. \end{aligned}$$

Denote by $F(k_1, \dots, k_s; u)$ the event

$$Y_{k_1}^u = Y_{k_2}^u = \cdots = Y_{k_s}^u = 1 \quad \text{whereas} \quad Y_k^u = 0 \quad \text{if } k \neq k_i, \quad i = 1, 2, \dots, s.$$

Then, dropping the indices m and γ for this proof,

$$EN^r(j) = \sum_{1 \leq k_1 < k_2 < \cdots < k_s \leq m} P\{F(k_1, \dots, k_s; j-1)\} E(N^r(j) | F(k_1, \dots, k_s; j-1)).$$

However,

$$(3.23) \quad E(N^r(j) | F(k_1, \dots, k_s; j-1)) = E\left(\sum_{t=1}^s Y_{k_t}^1\right)^r$$

because if $Y_{k_1}^{j-1} = 1$, then $Y_{k_1}^j = 1$ if and only if $\langle k_1 x_j \rangle \leq \gamma$ and, if $Y_{k_1}^{j-1} = 0$ then $Y_{k_1}^j$ always equals zero. Hence,

$$(3.24) \quad \begin{aligned} E(N(j) | F(k_1, \dots, k_s; j-1)) &\leq 2\gamma s, \\ \text{var}(N(j) | F(k_1, \dots, k_s; j-1)) &\leq \frac{s}{2} (C_1 + (\log s)^2) \quad (\text{Lemma 1}) \end{aligned}$$

and since $0 \leq N(j) \leq s$ if $F(k_1, \dots, k_s; j-1)$ occurs,

$$E(N^r(j) | F(k_1, \dots, k_s; j-1)) \leq (2\gamma)^r s^r + C_2(r) \frac{s^{r-1}}{2} (C_1 + (\log s)^2) \quad (\text{by (3.14)}).$$

If we take into account that

$$N(j-1) = s \quad \text{if } F(k_1, \dots, k_s; j-1)$$

occurs one has

$$\begin{aligned}
 EN^r(j) &\leq \sum_{1 \leq k_1 < \dots < k_s \leq m} P\{F(k_1, \dots, k_s; j-1)\} \\
 (3.25) \quad &\cdot \left[(2\gamma)^r N^r(j-1) + \frac{C_2}{2} N^{r-1}(j-1) (C_1 + (\log N(j-1))^2) \right] \\
 &= (2\gamma)^r EN^r(j-1) + \frac{C_2 C_1}{2} EN^{r-1}(j-1) \\
 &\quad + \frac{C_2}{2} EN^{r-1}(j-1) (\log N(j-1))^2.
 \end{aligned}$$

Applying Jensen's inequality [9, p. 156, c,] twice gives

$$(3.26) \quad EN^{r-1}(j-1) \leq (EN^r(j-1))^{(r-1)/r} \leq \frac{EN^r(j-1)}{EN(j-1)} = \frac{EN^r(j-1)}{m \cdot (2\gamma)^{i-1}}.$$

The last term of (3.25) is estimated first by using Hölder's inequality [9, p. 156]

$$\begin{aligned}
 &EN^{r-1}(j-1) (\log N(j-1))^2 \\
 &\leq (2r-1)^2 EN^{r-1}(j-1) + \int_{\exp(2r-1)}^{\infty} y^{r-1} (\log y)^2 dP\{N(j-1) \leq y\} \\
 &\leq (2r-1)^2 EN^{r-1}(j-1) + \left(\int_{\exp(2r-1)}^{\infty} y^r dP\{N(j-1) \leq y\} \right)^{(r-1)/r} \\
 &\quad \cdot \left(\int_{\exp(2r-1)}^{\infty} (\log y)^{2r} dP\{N(j-1) \leq y\} \right)^{1/r}.
 \end{aligned}$$

Since $d^2(\log y)^{2r}/dy^2 \leq 0$ for $y \geq \exp(2r-1)$, Jensen's inequality [4, Theorem 95, p. 77; 9, p. 159, e] implies

$$\begin{aligned}
 &\int_{\exp(2r-1)}^{\infty} (\log y)^{2r} dP\{N(j-1) \leq y\} \\
 (3.27) \quad &\leq P\{N(j-1) \geq \exp(2r-1)\} \left\{ \log \frac{\int_{\exp(2r-1)}^{\infty} y dP\{N(j-1) \leq y\}}{\int_{\exp(2r-1)}^{\infty} dP\{N(j-1) \leq y\}} \right\}^{2r} \\
 &= O(\log EN(j-1))^{2r} \\
 &= O(\log m(2\gamma)^i)^{2r}.
 \end{aligned}$$

Applying again (3.26) to $EN^{r-1}(j-1)$ and combining (3.25)–(3.27) one obtains for some $C_5(\gamma, r)$

$$\begin{aligned} EN^r(j) &\leq (2\gamma)^r EN^r(j-1) \left(1 + C_5 \frac{1 + (\log m(2\gamma)^j)^2}{m \cdot (2\gamma)^j} \right) \\ &\leq (2\gamma)^r EN^r(j-1) \exp \left(C_5 \frac{1 + (\log m(2\gamma)^j)^2}{m(2\gamma)^j} \right). \end{aligned}$$

This also holds for $j=1$ if we put $N(0)=m$ with probability one. Thus by induction on j

$$EN^r(j) \leq (2\gamma)^r m^r \exp \left(C_5 \sum_{n=1}^j \frac{1 + (\log m(2\gamma)^n)^2}{m(2\gamma)^n} \right),$$

which immediately gives the right-hand inequality of (3.22) if we take into account $m(2\gamma)^p \rightarrow \lambda$.

In the same way one shows^(*)

$$EN^r(j) \geq (2\gamma)^r m^r \exp \left(-C_5 \sum_{n=1}^j \frac{1 + (\log m(2\gamma)^n)^2}{m(2\gamma)^n} \right)$$

which implies the left-hand inequality of (3.22) because

$$EN(N-1) \cdots (N-r+1) = EN^r + O \left(E \sum_{k=0}^{r-1} N^k \right).$$

LEMMA 4. *There exists a constant $C_6(r, \gamma) < \infty$ such that*

$$(3.28) \quad A_\gamma^{-1}(k(r)) \leq C_6(r, \gamma).$$

Proof.

$$2\gamma + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2\pi n\gamma \cos 2\pi n\xi}{n} = \begin{cases} 1 & \text{if } \langle \xi \rangle < \gamma, \\ 0 & \text{if } \langle \xi \rangle > \gamma \end{cases}$$

(cf. [8]). Hence

$$A_\gamma(k_1, \dots, k_r) = \int_0^1 d\xi \prod_{t=1}^r \left(2\gamma + \frac{2}{\pi} \sum_{n_t=1}^{\infty} \frac{\sin 2\pi n_t \gamma \cos 2\pi n_t k_t \xi}{n_t} \right).$$

Since

(*) We shall use the left-hand inequality of (3.22) only for $j \leq (2p+6)/3$. For this range the argument in the next few lines suffices but not for all $j \leq p$. However, the left-hand inequality of (3.22) is true for all $j \leq p$. The same remark applies to the left-hand inequality of (3.45).

$$\sum_{n=M+1}^{\infty} \frac{\sin 2\pi n\gamma \cos 2\pi n k \xi}{n}$$

is bounded uniformly in M, k, ξ (being the tail of the Fourier series of a function of bounded variation [10, p. 408]) and since

$$\int_0^1 \left| \sum_{n=M+1}^{\infty} \frac{\sin 2\pi n\gamma \cos 2\pi n k \xi}{n} \right| d\xi = \int_0^1 \left| \sum_{n=M+1}^{\infty} \frac{\sin 2\pi n\gamma \cos 2\pi n \xi}{n} \right| d\xi$$

tends to zero as $M \rightarrow \infty$ (uniformly in k), it is possible to choose $M = M(\gamma, \iota)$ such that

$$\begin{aligned} & A(k_1, \dots, k_r) \\ & \geq \int_0^1 d\xi \prod_{i=1}^r \left(2\gamma + \frac{2}{\pi} \sum_{n_i=1}^M \frac{\sin 2\pi n_i \gamma \cos 2\pi n_i k_i \xi}{n_i} \right) - \frac{(2\gamma)^r}{2} \\ (3.29) \quad & = \frac{(2\gamma)^r}{2} + \sum_{j=1}^r (2\gamma)^{r-j} \left(\frac{2}{\pi} \right)^j \sum_{1 \leq i_1 \leq \dots \leq i_j \leq r} \int_0^1 d\xi \prod_{i=1}^j \frac{\sin 2\pi n_{i_i} \gamma \cos 2\pi n_{i_i} k_{i_i} \xi}{n_{i_i}}. \end{aligned}$$

The last integrals can only be different from zero if some relation

$$\sum_{i=1}^r \epsilon_i n_i k_i = 0, \quad \epsilon_i = 0, +1, -1$$

holds, with at least two ϵ 's different from zero. We can now prove the lemma by induction on r . $A_\gamma(k_1) = 2\gamma$ i.e. $C_0(1, \gamma) = (2\gamma)^{-1}$ satisfies (3.28) for $r=1$. Let (3.28) be proved already for $r-1$ and assume there exists a sequence $(k_1^{(n)}, \dots, k_r^{(n)})$ $n=1, 2, \dots$ of r -tuples $k^{(n)}(r)$ such that

$$(3.30) \quad \lim_{n \rightarrow \infty} A(k^{(n)}(r)) = 0.$$

By virtue of (3.29) and (3.30) we may assume, if necessary by selecting a subsequence and rearranging the indices, that for some fixed $\epsilon_1, \dots, \epsilon_r$ and $n_1, \dots, n_r \leq M(\gamma, r)$ and all n

$$\sum_{i=1}^r \epsilon_i n_i k_i^{(n)} = 0, \quad \epsilon_{r-1} \epsilon_r \neq 0.$$

In this case, however, one has for any ξ satisfying

$$\langle k_j^{(n)} \xi \rangle \leq \frac{\gamma}{rM}, \quad j = 1, \dots, r-1,$$

also

$$\langle n_r k_j^{(n)} \xi \rangle \leq \frac{\gamma}{r} \leq \gamma, \quad j = 1, \dots, r-1$$

and

$$\langle n_r k_r^{(n)} \xi \rangle = \left\langle \sum_{i=1}^{r-1} \epsilon_i n_i k_i^{(n)} \xi \right\rangle \leq \frac{(r-1)M}{rM} \gamma \leq \gamma.$$

Consequently,

$$(3.31) \quad \begin{aligned} A_\gamma(n_r k_1^{(n)}, n_r k_2^{(n)}, \dots, n_r k_r^{(n)}) &\geq A_{\gamma/rM}(k_1^{(n)}, \dots, k_{r-1}^{(n)}) \\ &\geq C_6 \left(r-1, \frac{\gamma}{rM} \right)^{-1}. \end{aligned}$$

But $A(k_1^{(n)}, \dots, k_r^{(n)}) = A(n_r k_1^{(n)}, \dots, n_r k_r^{(n)})$ because $n_r x$ has a uniform distribution modulo one if x has. (3.31) contradicts (3.30) which proves the lemma.

LEMMA 5. *There exists a positive constant $C_7(r, \gamma)$ for every $r \geq 2$, such that for every $k(r) = (k_1, \dots, k_r)$*

$$\frac{A_\gamma(k(r))}{A_\gamma(k_1, \dots, k_{j-1}, k_{j+1}, \dots, k_r)} \leq 1 - C_7(r, \gamma) < 1$$

for some $1 \leq j \leq r$.

Proof. For shortness put $k(r, j) = (k_1, \dots, k_{j-1}, k_{j+1}, \dots, k_r)$. By the definition of A

$$(3.32) \quad \begin{aligned} \frac{A(k(r))}{A(k(r, j))} &= P\{Y_{k_j}^1 = 1 \mid Y_{k_i}^1 = 1, 1 \leq i \leq r, i \neq j\} \\ &= 1 - A^{-1}(k(r, j))P\{Y_{k_j}^1 = 0, Y_{k_i}^1 = 1, 1 \leq i \leq r, i \neq j\}. \end{aligned}$$

It therefore suffices to prove

$$(3.33) \quad \begin{aligned} B(k(r, j)) &= P\{Y_{k_j}^1 = 0, Y_{k_i}^1 = 1, 1 \leq i \leq r, i \neq j\} \\ &\geq C_8(r, j) > 0 \end{aligned} \quad \text{for some } 1 \leq j \leq r$$

where $C_8(r, j)$ does not depend on $k(r)$.

Assume (3.33) does not hold and that $k^{(n)}(r)$, $n = 1, 2, \dots$, is a sequence of r -tuples for which

$$(3.34) \quad \lim_{n \rightarrow \infty} B(k^{(n)}(r, j)) = 0, \quad j = 1, \dots, r.$$

Without loss of generality we may rearrange the indices and select a subsequence such that

$$(3.35) \quad k_1^{(n)} < k_2^{(n)} < \dots < k_r^{(n)} \quad \text{for all } n$$

and

$$(3.36) \quad \lim_{n \rightarrow \infty} \frac{k_r^{(n)}}{k_i^{(n)}} \geq 1 \quad \text{exists for } i = 1, \dots, r$$

(infinity is allowed as a limit in (3.36)). Let

$$(3.37) \quad \lim_{n \rightarrow \infty} \frac{k_r^{(n)}}{k_i^{(n)}} = 1 \quad \text{for } i = s+1, \dots, r$$

while

$$\lim_{n \rightarrow \infty} \frac{k_r^{(n)}}{k_i^{(n)}} > 1 \quad \text{for } i = 1, \dots, s.$$

By deleting some n 's we can assume

$$(3.38) \quad \frac{k_j^{(n)}}{k_i^{(n)}} \geq 1 + \delta,$$

$i = 1, \dots, s, \quad j = s+1, \dots, r$ for all n and some $0 < \delta \leq 1$.

Let $g(\xi)$ be the fractional part of ξ minus the integer closest to ξ . More precisely

$$g(\xi) = \begin{cases} \xi & \text{if } 0 \leq \xi \leq 1/2, \\ \xi - 1 & \text{if } 1/2 < \xi \leq 1, \end{cases}$$

$$g(\xi + 1) = g(\xi).$$

One has

$$P\{|g(k_i x) - g(k_j x)| \leq \eta\} \leq P\{|g((k_i - k_j)x)| \leq \eta\} \leq 2\eta$$

so that for

$$(3.39) \quad \eta = \min \left[\left(4 \binom{r}{2} C_\delta \left(r, \frac{\gamma\delta}{4(1+\delta)} \right) \right)^{-1}, \frac{1}{\gamma} - 2, \frac{\delta}{(2+\delta)} \right],$$

$$(3.40) \quad \begin{aligned} & P \left\{ |g(k_i x)| \leq \frac{\gamma\delta}{4(1+\delta)}, |g(k_i x) - g(k_j x)| \geq \eta, 1 \leq i, j \leq r, i \neq j \right\} \\ & \geq A_{\gamma\delta(4+\delta)^{-1}}(k(r)) - \sum_{1 \leq i < j \leq r} P\{|g(k_i x) - g(k_j x)| \leq \eta\} \\ & \geq C_\delta^{-1} \left(r, \frac{\gamma\delta}{4(1+\delta)} \right) - \binom{r}{2} \cdot 2\eta \\ & \geq \frac{1}{2} C_\delta^{-1} \left(r, \frac{\gamma\delta}{4(1+\delta)} \right) > 0. \end{aligned}$$

There exists therefore a ξ , depending on n and satisfying

$$(3.41) \quad |g(k_i^{(n)} \xi)| \leq \frac{\gamma \delta}{4(1+\delta)}, \quad |g(k_i^{(n)} \xi) - g(k_j^{(n)} \xi)| \geq \eta,$$

$1 \leq i, j \leq r$, $i \neq j$. Let j_1 be determined by

$$g(k_{j_1}^{(n)} \xi) = \max_{s+1 \leq j \leq r} g(k_j^{(n)} \xi) \quad (\text{thus } j_1 \geq s+1)$$

and put

$$(3.42) \quad \bar{\xi} = \frac{\gamma(1+\eta/2) - g(k_{j_1}^{(n)} \xi)}{k_{j_1}^{(n)}} \leq \frac{\gamma(1+\eta/2 + \delta/4(1+\delta))}{k_{j_1}^{(n)}}.$$

Then

$$(3.43) \quad g(k_{j_1}^{(n)} \xi) + k_{j_1}^{(n)} \bar{\xi} = \gamma \left(1 + \frac{\eta}{2}\right) \leq \frac{1}{2}.$$

On the other hand, by (3.38), (3.41) and (3.42), for any $i \leq s$

$$g(k_i^{(n)} \xi) + k_i^{(n)} \bar{\xi} \leq \frac{\gamma \delta}{4(1+\delta)} + \frac{\gamma(1+\eta/2 + \delta/4(1+\delta))}{1+\delta} \leq \gamma \left(1 - \frac{\eta}{2}\right),$$

and for any $i \geq s+1$, $i \neq j_1$

$$(3.44) \quad \begin{aligned} g(k_i^{(n)} \xi) + k_i^{(n)} \bar{\xi} &\leq g(k_{j_1}^{(n)} \xi) - \eta + k_{j_1}^{(n)} \bar{\xi} + (k_i^{(n)} - k_{j_1}^{(n)}) \bar{\xi} \\ &\leq \gamma \left(1 + \frac{\eta}{2}\right) - \eta + \frac{|k_i^{(n)} - k_{j_1}^{(n)}|}{k_{j_1}^{(n)}} \gamma \left(1 + \frac{\eta}{2} + \frac{\delta}{4(1+\delta)}\right). \end{aligned}$$

Since, for $i, j_1 \geq s+1$

$$\frac{k_i^{(n)}}{k_{j_1}^{(n)}} \rightarrow 1$$

the last member of (3.44) will eventually also be less than $\gamma(1-\eta/2)$. Thus for sufficiently large n

$$g(k_{j_1}^{(n)} \xi) + k_{j_1}^{(n)} \bar{\xi} = g(k_{j_1}^{(n)} (\xi + \bar{\xi})) = \gamma \left(1 + \frac{\eta}{2}\right)$$

and by (3.41)–(3.44)

$$-\frac{\gamma \delta}{4(1+\delta)} \leq g(k_i^{(n)} (\xi + \bar{\xi})) \leq \gamma \left(1 - \frac{\eta}{2}\right) \quad \text{if } i \neq j_1.$$

If now

$$\langle k_i^{(n)} t \rangle < \gamma \frac{\eta}{2}, \quad i = 1, \dots, r,$$

then

$$\langle k_i^{(n)} (\xi + \bar{\xi} + t) \rangle \leq \gamma \left(1 - \frac{\eta}{2} + \frac{\eta}{2} \right) = \gamma, \quad i \neq j_1,$$

whereas for j_1

$$\langle k_{j_1}^{(n)} (\xi + \bar{\xi} + t) \rangle \geq |g(k_{j_1}^{(n)} (\xi + \bar{\xi}))| - \langle k_{j_1}^{(n)} t \rangle > \gamma.$$

Thus, for sufficiently large n ,

$$\begin{aligned} B(k^{(n)}(r, j_1)) &\geq \mu \left\{ v: v = \xi + \bar{\xi} + t, 0 \leq t \leq 1, \langle k_i^{(n)} t \rangle < \gamma \frac{\eta}{2}, i = 1, \dots, r \right\} \\ &= P \left\{ \langle k_i^{(n)} x \rangle < \gamma \frac{\eta}{2}, i = 1, \dots, r \right\} = A_{\gamma\eta/2}(k^{(n)}(r)) \geq C_6^{-1} \left(r, \frac{\gamma\eta}{2} \right). \end{aligned}$$

This contradicts (3.34) and therefore proves the lemma.

LEMMA 6. If $p \rightarrow \infty$, $m \rightarrow \infty$ such that $m(2\gamma)^p \rightarrow \lambda$ then for some C_9 , depending on γ, r, λ only, and $j \leq p$

$$\begin{aligned} &\exp(-C_9(2\gamma)^{(p-j)/2}(1+p-j)) \\ (3.45) \quad &\leq \frac{1}{m-r+1} \sum_{\substack{k \neq k_1, \dots, k_{r-1} \\ 1 \leq k \leq m}} \left(\frac{A_\gamma(k_1, \dots, k_{r-1}, k)}{A_\gamma(k_1, \dots, k_{r-1})2\gamma} \right)^j \\ &\leq \exp(C_9(2\gamma)^{(p-j)/2}(1+p-j)). \end{aligned}$$

Proof. Put

$$(3.46) \quad \nu^{(j)}(\gamma, k(r-1)) = \frac{1}{m-r+1} \sum_{\substack{k \neq k_1, \dots, k_{r-1} \\ 1 \leq k \leq m}} \left(\frac{A_\gamma(k_1, \dots, k_{r-1}, k)}{A_\gamma(k_1, \dots, k_{r-1})2\gamma} \right)^j.$$

In this proof we assume $k(r-1)$ and γ fixed and shall abbreviate $\nu^{(j)}(\gamma, k(r-1))$ by $\nu^{(j)}$. For the same fixed $k(r-1)$ and γ we put

$$N'(j) = \sum_{\substack{k \neq k_1, \dots, k_r \\ 1 \leq k \leq m}} Y_k^j(\gamma).$$

Then (cf. (3.32)),

$$\nu^{(j)} = \frac{E(N'(j) \mid Y_{k_1}^j = \dots = Y_{k_{r-1}}^j = 1)}{(m-r+1) \cdot (2\gamma)^j}.$$

The proof will very much resemble the proof of Lemma 3.

$$\begin{aligned}
 & E(N'(j) \mid Y_{k_1}^j = \dots = Y_{k_{r-1}}^j = 1) \\
 (3.47) \quad & = \sum_{\substack{1 \leq t_1 < \dots < t_s \leq m \\ t_u \neq k_v}} P\{F(k_1, \dots, k_{r-1}, t_1, \dots, t_s; j-1) \mid Y_{k_1}^{j-1} = \dots \\
 & \qquad \qquad \qquad = Y_{k_{r-1}}^{j-1} = 1\} \\
 & \cdot E(N'(j) \mid Y_{k_1}^j = \dots = Y_{k_{r-1}}^j = 1, F(k_1, \dots, k_{r-1}, t_1, \dots, t_s; j-1)).
 \end{aligned}$$

However,

$$E(N'(j) \mid F(k_1, \dots, k_{r-1}, t_1, \dots, t_s; j-1)) = 2\gamma s$$

and thus by Lemma 1 (compare (3.23), (3.24)),

$$E((N'(j) - 2\gamma s)^2 \mid F(k_1, \dots, k_{r-1}, t_1, \dots, t_s; j-1)) \leq \frac{s}{2} (C_1 + (\log s)^2).$$

Hence, by Lemma 4

$$\begin{aligned}
 & E((N'(j) - 2\gamma s)^2 \mid Y_{k_1}^{(j)} = \dots = Y_{k_{r-1}}^{(j)} = 1, \\
 & \qquad \qquad \qquad F(k_1, \dots, k_{r-1}, t_1, \dots, t_s; j-1)) \\
 (3.48) \quad & \leq \frac{s(C_1 + (\log s)^2)}{2P\{Y_{k_1}^{(j)} = \dots = Y_{k_{r-1}}^{(j)} = 1 \mid F(k_1, \dots, k_{r-1}, t_1, \dots, t_s; j-1)\}} \\
 & = \frac{s(C_1 + (\log s)^2)}{2A(k_1, \dots, k_{r-1})} \leq C_6(r-1, \gamma) \frac{s}{2} (C_1 + (\log s)^2).
 \end{aligned}$$

By (3.12) and (3.48) one has

$$\begin{aligned}
 & E(N'(j) \mid Y_{k_1}^j = \dots = Y_{k_{r-1}}^j = 1, \\
 (3.49) \quad & \qquad \qquad \qquad F(k_1, \dots, k_{r-1}, t_1, \dots, t_s; j-1)) - 2\gamma s \\
 & \qquad \qquad \qquad \leq C_{10}(r, \gamma) s^{1/2} (1 + \log s)
 \end{aligned}$$

for appropriate C_{10} .

Since, under the condition $F(k_1, \dots, k_{r-1}, t_1, \dots, t_s; j-1)$

$$N'(j-1) = s,$$

(3.47) and (3.49) give immediately

$$\begin{aligned}
 & | E(N'(j) \mid Y_{k_1}^j = \dots \\
 & \qquad \qquad \qquad = Y_{k_{r-1}}^j = 1) - 2\gamma E(N'(j-1) \mid Y_{k_1}^{j-1} = \dots = Y_{k_{r-1}}^{j-1} = 1) | \\
 (3.50) \quad & \leq C_{10} E(N'^{1/2}(j-1)(1 + \log N'(j-1)) \mid Y_{k_1}^{j-1} = \dots = Y_{k_{r-1}}^{j-1} = 1) \\
 & \leq 5C_{10} E(N'^{3/4}(j-1) \mid Y_{k_1}^{j-1} = \dots = Y_{k_{r-1}}^{j-1} = 1).
 \end{aligned}$$

Thus

$$(3.51) \quad \nu^{(j)} \leq \nu^{(j-1)} + \frac{5C_{10}(\nu^{(j-1)})^{3/4}}{(m-r+1)^{1/4}(2\gamma)^{j/4+1}}, \quad \nu^{(0)} = 1.$$

First we get from (3.51) by induction,

$$\nu^{(j)} \leq \exp \left(5C_{10} \sum_{u=0}^{j-1} (m-r+1)^{-1/4} (2\gamma)^{-u/4-1} \right)$$

which is bounded for $j \leq p$. Then we complete the proof from (3.50) as in Lemma 3 by applying Hölder's and Jensen's inequalities and induction on $j^{(6)}$.

An immediate corollary of (3.45) is

$$(3.52) \quad \frac{1}{m-r+1} \sum_{\substack{k \neq k_1, \dots, k_{r-1} \\ 1 \leq k \leq m}} \left(\frac{A_\gamma(k_1, \dots, k_{r-1}, k)}{A_\gamma(k_1, \dots, k_{r-1}) 2\gamma} \right)^{3[p/3]+3} \leq (2\gamma)^{-3} \exp C_9 = C_{11}, \text{ say.}$$

We are now in the position to prove (3.5). Put

$$U_r(k(r)) = \left(\frac{A(k(r))}{(2\gamma)^r} \right)^{[p/3]+1}.$$

If we consider U_r as a random variable taking each of the values $U_r(k(r))$ with probability $1/m(m-1) \cdots (m-r+1)$ then (3.6) and Lemma 3 state

$$\begin{aligned} EU_r &= \frac{1}{m(m-1) \cdots (m-r+1)} \sum_{k(r)} \left(\frac{A(k(r))}{(2\gamma)^r} \right)^{[p/3]+1} \\ &\geq \exp(-C_4(2\gamma)^{(2p-3)/3} p^2) \quad \text{for sufficiently large } p. \end{aligned}$$

Similarly

$$EU_r^2 \leq \exp C_4(2\gamma)^{(p-6)/3} p^2$$

and consequently

$$\text{var}(U_r) = EU_r^2 - (EU_r)^2 \leq C_{12}(\gamma, r, \lambda) p^2 (2\gamma)^{p/3}$$

for appropriate C_{12} . In other words

$$U_r \rightarrow 1 \quad \text{in probability} \quad (p \rightarrow \infty)$$

and

$$\frac{1}{m(m-1) \cdots (m-r+1)} \sum_{k(r)} \left(\frac{A(k(r))}{(2\gamma)^r} \right)^p = EU_r^{p/([p/3]+1)}$$

will indeed tend to one if we can show

$$(3.53) \quad \lim \frac{1}{m(m-1) \cdots (m-r+1)} \sum_{k(r); U_r^3 \geq \epsilon} U_r^3(k(r)) = 0$$

[9, p. 184 B].

Unfortunately, Lemma 3 alone does not seem to be strong enough to prove (3.53), and we have to proceed by induction. (3.53) certainly holds for $r=1$ and let us assume it has already been proved with r replaced by $r-1$. We shall then prove that it also holds for r . For this purpose, we define

$$(3.54) \quad V_r(k(r)) = \begin{cases} \left(\frac{A_\gamma(k_1, \dots, k_r)}{A_\gamma(k_1, \dots, k_{r-1}) 2\gamma} \right)^{[p/3]+1} & \text{if } \frac{A_\gamma(k_1, \dots, k_r)}{A_\gamma(k_1, \dots, k_{r-1})} \leq C_{13}(\gamma, r), \\ \frac{1}{m-r+1} \sum_{k \neq k_1, \dots, k_{r-1}} \left(\frac{A_\gamma(k_1, \dots, k_{r-1}, k)}{A_\gamma(k_1, \dots, k_{r-1}) 2\gamma} \right)^{[p/3]+1} & \text{otherwise} \end{cases}$$

where

$$(3.55) \quad C_{13}(\gamma, r) = \max(1 - C_7(\gamma, r), (2\gamma)) < 1 \quad (\text{cf. Lemma 5}).$$

Any set of r different integers $k_1, \dots, k_r \leq m$ will appear in $r!$ orders as a $k(r)$. For some j

$$\frac{A_\gamma(k(r))}{A_\gamma(k(r, j))} \leq C_{13}(\gamma, r) \quad \text{by Lemma 5.}$$

This j will appear in $(r-1)!$ permutations of $1, \dots, r$ at the end. Thus taking into account

$$(3.56) \quad \left(\frac{A(k(r))}{(2\gamma)^r} \right)^{3[p/3]+3} = \left(\frac{A(k(r-1))}{(2\gamma)^{r-1}} \right)^{3[p/3]+3} \left(\frac{A(k(r))}{A(k(r-1)) 2\gamma} \right)^{3[p/3]+3},$$

$$\sum_{\substack{k(r) \\ U_r \geq 3^r}} U_r^3(k(r)) \leq r \sum_{k(r-1)} U_{r-1}^3(k(r-1)) \sum_{\substack{k_r \neq k_1, \dots, k_{r-1} \\ U_{r-1} V_r \geq 3^r}} V_r^3(k(r)).$$

Since $U^3 V^3 \geq 3^r$ implies $U^3 \geq 3^{r-1}$ or $V \geq 3$ one obtains from (3.56)

$$(3.57) \quad \frac{1}{m(m-1) \cdots (m-r+1)} \sum_{\substack{k(r) \\ U_r \geq 3^r}} U_r^3(k(r))$$

$$\leq \frac{r}{m(m-1) \cdots (m-r+2)} \sum_{\substack{k(r) \\ U_{r-1} \geq 3^{r-1}}} U_{r-1}^3(k(r-1)) \frac{1}{m-r+1}$$

$$\cdot \sum_{k_r \neq k_1, \dots, k_{r-1}} V_r^3(k(r))$$

$$+ \frac{r}{m(m-1) \cdots (m-r+2)} \sum_{k(r-1)} U_{r-1}^3(k(r-1)) \frac{1}{m-r+1}$$

$$\cdot \sum_{\substack{k_r \neq k_1, \dots, k_{r-1} \\ V_r \geq 3}} V_r^3(k(r)).$$

By the induction hypothesis and (3.52) the first sum in the right-hand side of (3.57) tends to zero. As for the second term, we have

$$(3.58) \quad \frac{1}{m(m-1) \cdots (m-r+2)} \sum_{k(r-1)} U_{r-1}^2(k(r-1)) \frac{1}{m-r+1} \\ \cdot \sum_{k \neq k_1, \dots, k_{r-1}} \left(\frac{A(k_1, \dots, k_{r-1}, k)}{A(k(r-1))2\gamma} \right)^{2[p/3]+2} \leq \exp(C_4(2\gamma)^{(p-6)/3} p^2)$$

by (3.6) and Lemma 3. On the other hand by (3.6), Lemma 3 and Lemma 6

$$(3.59) \quad \frac{1}{m(m-1) \cdots (m-r+2)} \sum_{k(r-1)} U_{r-1}^2(k(r-1)) \\ \cdot \left(\frac{1}{m-r+1} \sum_{k \neq k_1, \dots, k_{r-1}} \left(\frac{A(k_1, \dots, k_{r-1}, k)}{A(k(r-1))2\gamma} \right)^{[p/3]+1} \right)^2 \\ \geq \exp(-C_4(2\gamma)^{(p-6)/3} p^2) \exp(-2C_9(2\gamma)^{(p-3)/3} p).$$

Putting

$$D(k(r-1)) = \frac{1}{m-r+1} \sum_{\substack{k \neq k_1, \dots, k_{r-1} \\ 1 \leq k \leq m}} \left(\frac{A(k_1, \dots, k_{r-1}, k)}{A(k(r-1))2\gamma} \right)^{[p/3]+1}$$

and subtracting (3.59) from (3.58) one sees

$$(3.60) \quad \frac{1}{m(m-1) \cdots (m-r+2)} \sum_{k(r-1)} U_{r-1}^2(k(r-1)) \\ \cdot \frac{1}{m-r+1} \sum_{k \neq k_1, \dots, k_{r-1}} \left(\left(\frac{A(k_1, \dots, k_{r-1}, k)}{A(k(r-1))2\gamma} \right)^{[p/3]+1} - D(k(r-1)) \right)^2 \\ \leq C_{14}(\gamma, r, \lambda)(2\gamma)^{p/3} p^2$$

for appropriate C_{14} . Thus

$$(3.61) \quad \frac{1}{m-r+1} \sum_{k \neq k_1, \dots, k_{r-1}} \left(\left(\frac{A(k_1, \dots, k_{r-1}, k)}{A(k(r-1))2\gamma} \right)^{[p/3]+1} - D(k(r-1)) \right)^2 \\ \leq C_{14}(2\gamma)^{p/3} p^2 C_{13}^{-p/6}$$

except for a set S of $(r-1)$ -tuples $k(r-1)$ for which

$$(3.62) \quad \frac{1}{m(m-1) \cdots (m-r+2)} \sum_{k(r-1) \in S} U_{r-1}^2(k(r-1)) \leq C_{13}^{p/6}.$$

However, by (3.52)

$$\begin{aligned}
& \frac{1}{m(m-1) \cdots (m-r+2)} \sum_{k(r-1) \in S} U_{r-1}^3(k(r-1)) \frac{1}{m-r+1} \sum_{k \neq k_1, \dots, k_{r-1}} V_r^3(k(r)) \\
& \leq \frac{2C_{11}}{m(m-1) \cdots (m-r+2)} \sum_{k(r-1) \in S} U_{r-1}^3(k(r-1)) \\
& \leq \frac{2C_{11}}{m(m-1) \cdots (m-r+2)} \sum_{\substack{k(r-1) \\ U_{r-1} \geq 3^{r-1}}} U_{r-1}^3(k(r-1)) \\
& \quad + \frac{2C_{11} \cdot 3^{r-1}}{m(m-1) \cdots (m-r+2)} \sum_{k(r-1) \in S} U_{r-1}^2(k(r-1)).
\end{aligned}$$

Both these terms tend to zero, the first by the induction hypothesis and the second by (3.62) and (3.55). In view of (3.57) it remains to prove

$$\begin{aligned}
(3.63) \quad & \frac{1}{m(m-1) \cdots (m-r+2)} \sum_{k(r-1) \notin S} U_{r-1}^3(k(r-1)) \\
& \cdot \frac{1}{m-r+1} \sum_{\substack{k \neq k_1, \dots, k_{r-1} \\ V_r \geq 3}} V_r^3(k(r)) \rightarrow 0.
\end{aligned}$$

Let $k(r-1) \notin S$ be fixed and look at V_r as a random variable which takes each of the values $V_r(k(r))$ with probability $1/(m-r+1)$. Then

$$\begin{aligned}
& E(V_r - D_r(k(r-1)))^2 \\
& \leq \frac{1}{m-r+1} \sum_{k \neq k_1, \dots, k_{r-1}} \left(\left(\frac{A(k_1, \dots, k_{r-1}, k)}{A(k(r-1))2\gamma} \right)^{[p/3]+1} - D(k(r-1)) \right)^2 \\
& \leq C_{14}(2\gamma)^{p/3} p^2 C_{13}^{-p/6} \quad \text{by (3.61).}
\end{aligned}$$

On the other hand

$$0 \leq V_r \leq \max \left(\left(\frac{C_{13}}{2\gamma} \right)^{[p/3]+1}, D(k(r-1)) \right)$$

while $D(k(r-1)) \leq 3/2$ for sufficiently large p by Lemma 6. Thus, by Lemma 2, (3.13)

$$\begin{aligned}
(3.64) \quad & \frac{1}{m-r+1} \sum_{\substack{k \neq k_1, \dots, k_r \\ V_r \geq 3}} V_r^3(k(r)) \\
& \leq C_2(3) \left(\frac{3}{2} + \left(\frac{C_{13}}{2\gamma} \right)^{[p/3]+1} \right) \cdot C_{14}(2\gamma)^{p/3} p^2 C_{13}^{-p/6} \rightarrow 0 \quad (p \rightarrow \infty).
\end{aligned}$$

Since, by Lemma 3

$$\frac{1}{m(m-1) \cdots (m-r+2)} \sum_{k(r-1)} U_{r-1}^3(k(r-1))$$

is uniformly bounded, (3.63) follows from (3.64) and this implies (3.53). By induction (3.53) and (3.5) follow now for all r and this proves the theorem.

COROLLARY. If $p \rightarrow \infty$, $m \rightarrow \infty$ such that

$$m(2\gamma)^p \rightarrow \infty \quad \text{ufficiently slow,}$$

then

$$P \left\{ \frac{N(m, \gamma, p) - m(2\gamma)^p}{(m(2\gamma)^p)^{1/2}} \leq \alpha \right\} \rightarrow \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\alpha} e^{-t^2/2} dt.$$

Proof. This follows from Theorem 3, since for random variables $X(\lambda)$, having a Poisson distribution with mean λ , one has [6, p. 116, Ex. 4.9]

$$\lim_{\lambda \rightarrow \infty} P \left\{ \frac{X(\lambda) - \lambda}{\lambda^{1/2}} \leq \alpha \right\} = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\alpha} e^{-t^2/2} dt.$$

Our estimates are not sharp enough, however, to tell how fast $m(2\gamma)^p$ may tend to infinity.

One might want to prove Theorem 3 for any interval of length 2γ . More precisely, put

$$Y_k^j(\delta, 2\gamma + \delta) = \begin{cases} 1 & \text{if there exist integers } n_1, \dots, n_j \text{ such that} \\ & \delta + n_i \leq kx_i \leq \delta + 2\gamma + n_i, \quad i = 1, \dots, j, \\ 0 & \text{otherwise.} \end{cases}$$

As long as

$$(3.65) \quad -\frac{1}{2} < \delta < 0 < \delta + 2\gamma < \frac{1}{2}$$

one can still follow the proof of Theorem 3. In fact, the only places in the above proof where a difference between $Y(\gamma)$ and $Y(\delta, 2\gamma + \delta)$ might come in are the Lemmas 1, 4 and 5. However, the estimate (3.8) remains valid (cf. [8]) so that the proof of Lemma 1 needs no change and one easily sees also that Lemmas 4 and 5 remain valid. This is not so when (3.65) is violated. The analogue of Lemma 4 is then false for certain choices of γ and δ . We formulate this with another generalization in

THEOREM 4. If γ, δ satisfy (3.65) and if $p \rightarrow \infty$, $m \rightarrow \infty$ such that $m(2\gamma)^p \rightarrow \lambda$ and if for every m , $k_1(m), \dots, k_m(m)$ are m different positive integers, then

$$\lim P \left\{ \sum_{i=1}^m Y_{k_i(m)}^p(\delta, 2\gamma + \delta) = k \right\} = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \dots$$

The transition to arbitrary sequences $\{k_i(m), i=1, \dots, m\}$ of integers requires no change of proof since our estimates are uniform in $\{k_i(m)\}$. Finally we mention the possibility of taking intervals of length 2γ at a random location. I.e. let $\delta_1, \delta_2, \dots$ also be random variables, independent of each other and of $x_1, x_2, \dots$, and each with a uniform distribution on $[0, 1]$. Put

$$\tilde{Y}_k^j(\gamma) = \begin{cases} 1 & \text{if there exist integers } n_1, \dots, n_j \text{ such that} \\ & \delta_i + n_i \leq kx_i \leq \delta_i + 2\gamma + n_i, \quad i = 1, \dots, j, \\ 0 & \text{otherwise.} \end{cases}$$

Then Theorem 4 holds with $Y(\delta, 2\gamma + \delta)$ replaced by $\tilde{Y}(\gamma)$. Again the proof is almost identical with that of Theorem 3.

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THE HEBREW UNIVERSITY,
JERUSALEM, ISRAEL