OVERRINGS OF COMMUTATIVE RINGS
I. NOETHERIAN OVERRINGS

BY

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1. **Introduction.** In this paper: "ring" means commutative ring with unit; a subring must always have the same identity element as the ring of which it is a subring; "dimension" (of a ring) means Krull dimension (the supremum of the lengths of all chains of prime ideals in the ring); ring T is an *overring* of ring R if R is a subring of T, and T is a sub-ring of the total quotient ring of R.

Cohen [2], generalizing work of Krull [4] and Akizuki [1], proved that every overring of a 1-dimensional Noetherian domain is Noetherian. Kaplansky suggested the question of whether the converse of this theorem holds. Precisely: If every overring of a domain is Noetherian, is the domain then either a field or of dimension 1? This question can be rephrased and successfully treated in the context of rings with divisors of zero. Call a ring a *C-ring* ("*C*" for Cohen) if all of its overrings are Noetherian. In §3 we give a rather precise characterization of C-rings.

In considering the question of C-rings we came upon a condition necessarily satisfied by every nondomain which has a Noetherian integral closure. (The condition is trivial for domains.) In §4 we address ourselves to the problem of determining to what extent this necessary condition is sufficient. Among other results, we obtain a modest generalization of the theorem of Mori and Nagata [7] which asserts that the integral closure of a 2-dimensional Noetherian domain is Noetherian.

2. **Preliminary notions.** A *Z-ring* (ring of zero-divisors) is a ring in which every nonunit is a zero-divisor. A *Z-ideal* is an ideal consisting entirely of zero-divisors. An *NZ-ideal* is an ideal containing a non-zero-divisor. A ring is said to be of *restricted dimension* n (n a positive integer) if it possesses an NZ-prime of rank n, but no NZ-prime of greater rank. We shall adopt the convention that Z-rings are of restricted dimension 0. The following technical lemma contains several completely elementary facts which the reader can easily establish for himself. We shall use these points repeatedly, both explicitly and implicitly.

LEMMA. *Let K be an overring of R, and let S be a multiplicatively closed subset of $R-\{0\}$. Then:*

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(1) *K_S is an overring of R_S. In particular, if K_S is a Z-ring, then it is the total quotient ring of R_S.*

(2) *Every overring of R_S contained in K_S is of the form T_S, where T is an over-ring of R contained in K.*

(3) *If R is integrally closed in K, then R_S is integrally closed in K_S.*

3. ***C*-rings.** Cohen's Theorem (1-dimensional Noetherian domains are C-rings) provides us with a class of nondomains which are trivially C-rings: the class of those Noetherian rings which are direct sums of Z-rings and 1-dimensional domains. A ring R of this class has the property that for every maximal ideal M of R, R_M is either a local Z-ring or a 1-dimensional local domain; that is, each R_M is a local C-ring. This local property is indeed characteristic of C-rings. We shall first prove this fact. Afterwards, in order to characterize C-rings, it will be enough to describe, as we shall, all local C-rings. We assure the reader that the class of C-rings is larger than the class of trivial examples described above(2).

PROPOSITION 1. *C-rings are of restricted dimension not greater than* 1.

Proof. We shall show that every ring of restricted dimension exceeding 1 must necessarily possess a non-Noetherian overring. Call the ring in question R. Let M be an NZ-prime in R of rank greater than 1, x a non-zero-divisor in M and P a prime contained in M and minimal over the ideal xR. Let y be a non-zero-divisor in $M-P$. We claim that if S is the set $\{x/y^n$, n positive$\}$, then the ideal generated by S in the overring $R[S]$ can have no finite basis. Otherwise $SR[S]$ is principal, generated by, say, x/y^n. It follows that for sufficiently large integers m and t and appropriately selected elements r_i of R, we have:

$$x/y^{n+1} = [r_0 + r_1(x/y^m) + \cdots + r_t(x/y^m)^t]x/y^n.$$

Multiplication of this equation by y^{mt+n+1}/x shows that $(1-yr_0)y^{mt}$ lies in xR. Since y was chosen outside of P, it follows that $1-yr_0$ lies in P. The resulting contradiction that 1 lies in M completes the proof.

Thus, the converse of Cohen's Theorem:

COROLLARY 1. *A C-domain is a field or of dimension* 1.

Moreover:

COROLLARY 2. *If R is a C-ring, then for every maximal ideal M of R, R_M is a C-ring.*

Proof. If M is a Z-ideal, then R_M is a Noetherian Z-ring, and therefore

(2) For an explicit nontrivial example consider the ring $R=Z[X]/(X^2,2X)$, where Z is the ring of rational integers, and X is an indeterminate over Z. Every NZ-ideal of R is principal. From this fact it follows that every overring of R is of the form R_S, where S is a multiplicatively closed set of non-zero-divisors of R. Thus every overring of R is Noetherian.

trivially a C-ring. If M is an NZ-ideal (and therefore of rank 1), then since $R-M$ meets every nonminimal prime of K, the total quotient ring of R, K_{R-M} is a Z-ring; that is, K_{R-M} is the total quotient ring of R_M. Since every overring of R_M is then of the form T_{R-M}, where T is an over-ring of R, it follows that every overring of R_M is Noetherian.

Proposition 2, our first step towards establishing the converse of Corollary 2, is of some independent interest; we shall consider it further in §4.

PROPOSITION 2. *If the ring R has a Noetherian integral closure, then for every maximal ideal M of R, R_M is a Z-ring or a ring without nilpotent elements.*

Proof. Let $\bar{R}$ and K be respectively the integral closure and total quotient ring of R. Suppose that x is a non-zero-divisor in the maximal ideal M of R. If y is a nilpotent element of $\bar{R}_{R-M}$, then since $\bar{R}_{R-M}$ is integrally closed in K_{R-M}, it follows that y/x lies in $\bar{R}_{R-M}$. Thus, if N is the ideal of nilpotent elements of $\bar{R}_{R-M}$, then $xN=N$. Since x clearly lies in every maximal ideal of $\bar{R}_{R-M}$, by Nakayama's Lemma, $N=(0)$. Since $\bar{R}_{R-M}$ is an overring of R_M, it follows that R_M has no nilpotent elements.

DEFINITION. A ring R is said to be *locally nilpotentfree* if for every maximal ideal M of R, R_M is a Z-ring or a ring without nilpotent elements. In this terminology:

COROLLARY 3. *C-rings are locally nilpotentfree.*

The proof of the converse of Corollary 2 requires two technical lemmas; we state them here in the generality demanded by several later applications.

LEMMA 1. *Let I and J be submodules of some R-module. If for every maximal ideal M of R, $I_M=J_M$, then $I=J$. If every I_M has a finite basis, and there exists a finite set of elements of I which is a basis of I_M for all but a finite number of M, then I has a finite basis.*

Proof. To prove the first statement it suffices, by symmetry, to show that I contains J. Let x be an arbitrary element of J. For every maximal ideal M_α of R there is an s_α in $R-M_\alpha$ such that xs_α lies in I. Since no maximal ideal contains all the s_α, for appropriate r_i in R, $1=s_1r_1+\cdots+s_tr_t$. It follows that $x=x(s_1r_1+\cdots+s_tr_t)=(xs_1)r_1+\cdots+(xs_t)r_t$, an element of I. To prove the second statement let $\{x_i\}$ be a finite set of elements of I which generates all but a finite number of I_M. Since every I_M has a finite basis, there is a finite set $\{y_j\}$ of elements of I which is a basis for each of the exceptional I_M. $\{x_i, y_j\}$ is a basis for every I_M. The second statement now follows from the first.

LEMMA 2. *Let R be a ring with only finitely many minimal primes P. If each R/P is Noetherian, then R can have only finitely many rank 1 primes which contain more than one minimal prime. In particular, a Noetherian ring can have only finitely many rank 1 primes which contain more than one minimal prime.*

Proof. If the lemma is false, then we may assume that infinitely many rank 1 primes contain at least two minimal primes, of which one is the minimal prime P. Let x belong to every minimal prime different from P, but not to P. It follows that the Noetherian ring $R/(P, x)$ has infinitely many minimal primes—impossible.

PROPOSITION 3. *If R is a Noetherian ring such that for every maximal ideal M of R, R_M is a C-ring, then R is a C-ring.*

Proof. We must show that in any overring T of R, every ideal I has a finite basis. It follows from the fact that every R_M is a C-ring (and therefore it and all of its overrings are Noetherian and of restricted dimension not more than 1) that there are but finitely many maximal ideals of T lying over any given maximal ideal of R. Further, for every maximal ideal M of T, T_M is Noetherian. We intend to exploit these facts for the purpose of applying Lemma 1. Let $J = I \cap R$.

Case I. J is contained in no minimal prime. Let x belong to J, but to no minimal prime. Any NZ-prime of R containing x is minimal over xR; there are but finitely many of these, and each is, by Proposition 1, a maximal ideal. It follows that x is contained in only finitely many maximal ideals of T. Thus, for all but a finite number of maximal ideals M of T, $I_M = xT_M = T_M$. By Lemma 1, I has a finite basis.

Case II. J is contained in a minimal prime. Let x belong to J, but to no minimal prime that does not contain J. Any NZ-prime which contains x, but does not contain J, is minimal over xR; there are but finitely many of these. Therefore all but a finite number of maximal ideals of R, and so of T, either contain J or do not contain x. It follows from Lemma 2 and Corollary 3 that for all but a finite number of maximal ideals M of T, $T_{R-M\cap R}$ is a domain. Thus, for all but a finite number of maximal ideals M of T, either $I_M = xT_M = T_M$ or $I_M = (0)$. An application of Lemma 1 finishes this case and thereby completes the proof of Proposition 3.

REMARK. Both Cohen [2] and MacLane and Schilling [5] have shown that any overring of a Dedekind ring is also a Dedekind ring. Let us note that a much simplified version of the preceding argument yields this result. The crucial point is that the local ring associated with a maximal ideal of a Dedekind ring is a very simple sort of C-ring: a discrete valuation ring. Let T be an overring of the Dedekind ring R. That for every maximal ideal M of T, $T_M = R_{M\cap R}$, a discrete valuation ring, is a triviality; we must show that T is Noetherian. If I is a nonzero ideal of T, let x be a nonzero element of $I \cap R$. Only finitely many maximal ideals of R, and so only finitely many maximal ideals of T, contain x. Thus, for all but a finite number of maximal ideals M of T, $I_M = xT_M = T_M$. By Lemma 1, I has a finite basis.

It now remains to find all local C-rings. We already know that a local C-ring which is not a Z-ring must be 1-dimensional and free of nilpotent elements. It is a theorem that these two conditions guarantee that a local ring is a C-ring. The proof of this fact rests heavily on Cohen's arguments, of which, for the convenience of the reader, we present here enough to satisfy our immediate needs.

Lemma 3 (Cohen's Theorem). *If T is an overring of the 1-dimensional local ring R, and x is a non-zero-divisor in the maximal ideal of R, then T/xT is a finite R-module. The NZ-ideals of T are finitely generated.*

Proof. Let $A_i = R \cap x^i T$. Since the ring R/xR has the minimum condition, there exists an n such that for $m \geqq n$, $(A_m, x) = (A_n, x)$. We claim that under the natural homomorphism of T onto T/xT, A_n/x^n maps onto T/xT. (It follows from this fact that T/xT is a finite R-module.) Let t be an arbitrary element of T; we must show that t has a representative (mod xT) of the form a/x^n, with a in A_n. Let $t = u/v$ be a representation of t as a quotient of elements of R[3]. There is an $m \geqq n$ such that x^m lies in vR. Therefore $t = ux^m/vx^m = w/x^m$, with w in $R \cap x^m T = A_m$. Now suppose that $m \geqq n$ is minimal such that t has a representative (mod xT) of the form a/x^m, with a in A_m. We claim that $m = n$. Otherwise, since $(A_{m+1}, x) = (A_m, x)$, $a = bx + c$, with b in R and c in A_{m+1}. Let $c = x^{m+1}t'$, with t' in T. Then $t \equiv a/x^m \equiv (bx + x^{m+1}t')/x^m \equiv b/x^{m-1} + xt'$ (mod xT). Since b clearly lies in A_{m-1}, this congruence contradicts the minimality of m. The first statement is thereby established. The second statement follows from the first.

Remarks. 1. Observe that the preceding argument does not require the full force of R's being Noetherian; it is enough to assume that the maximal ideal (and so all NZ-ideals) of R is finitely generated. As there do exist examples of rings satisfying the weaker hypothesis, it is well to be aware of the stronger result. However, Lemma 3, as given above, will do for our purposes.

2. Note that Lemma 3 and the argument given in Case I of the proof of Proposition 3 show the following: If T is an overring of a Noetherian ring of restricted dimension 1, then the ideals in T of positive rank (ideals not contained in minimal primes) are finitely generated. Thus, should a ring such as T fail to be Noetherian, the offending (i.e., nonfinitely generated) ideals are all of rank 0.

Proposition 4. *A 1-dimensional local ring without nilpotent elements is a C-ring.*

Proof. Call the ring in question R. In view of the preceding lemma, it suffices to show that in any overring of R, the Z-ideals are finitely generated. Denote the minimal primes of R by $P_1, \cdots, P_n$. We proceed by induction on n. Since R is a domain when $n=1$, we may assume that $n > 1$. If I is a Z-ideal in some overring of R, then $I \cap R$ is contained in a minimal prime, say

(3) Here, as elsewhere in this paper, we assume that in the representation of an element of a total quotient ring as a fraction, the denominator is a non-zero-divisor.

P_1. Let Q be the intersection of $P_2, \cdots, P_n$, $\overline{R}=R/Q$, and denote by $\bar{x}$ (for x in R) the image of x under $R \rightarrow R/Q$. Let I also denote the set $\{x_\alpha/y_\alpha$, x_α and y_α in $R\}$ of elements of the ideal I; let $\bar{I}$ be $\{\bar{x}_\alpha/\bar{y}_\alpha\}$. By the induction hypothesis, $\overline{R}$ is a C-ring. Thus the ideal $\bar{I}\overline{R}[\bar{I}]$ has a finite basis $\{\bar{x}_1/\bar{y}_1, \cdots, \bar{x}_t/\bar{y}_t\}$ in the ring $\overline{R}[\bar{I}]$. If x/y is an arbitrary element of I, then it follows that $\bar{x}/\bar{y}=\bar{u}_1\bar{x}_1/\bar{v}_1\bar{y}_1+ \cdots +\bar{u}_t\bar{x}_t/\bar{v}_t\bar{y}_t$, with the u_i/v_i in $R[I]$. Then $x/y=u_1x_1/v_1y_1 + \cdots +u_tx_t/v_ty_t+u/v$, with u in Q. Now u is clearly in P_1; thus $u=0$. Then $\{x_i/y_i\}$ is a basis for I.

The promised characterization of C-rings:

THEOREM 1. *If R is a Noetherian ring, then the following statements are equivalent.*

(1) *R is a C-ring.*

(2) *For every maximal ideal M of R, R_M is a C-ring.*

(3) *R is of restricted dimension not greater than* 1 *and locally nilpotentfree.*

(4) *R is of restricted dimension not greater than* 1, *and the integral closure of R is Noetherian.*

REMARK. Call a Noetherian ring a *D-ring* if all of its overrings are integrally closed([4]). It is not hard to see that an integrally closed C-ring is a D-ring. In part II of this study [3] it is seen that the following statement can be added to the list given in Theorem 1.

(5) *The integral closure of R is a D-ring.*

We conclude this section with the observation that the only C-rings in which (0) is an unmixed ideal are the "obvious" examples. Specifically:

PROPOSITION 5. *Let R be a Noetherian ring in which* (0) *is an unmixed ideal. Then R is a C-ring if, and only if, it is a direct sum of primary rings and a* 1-*dimensional ring having no nilpotent elements.*

Proof. The sufficiency of the direct sum condition is clear; we turn to the question of its necessity. Let $Q_1, \cdots, Q_n$ be those primary components of (0) whose associated primes are maximal ideals, and let Q be the intersection of all the other primary components of (0). By the Chinese Remainder Theorem, R is the direct sum of the R/Q_i and R/Q. Each R/Q_i is a primary ring, and every maximal ideal of R/Q is an NZ-ideal. Since each of the summands (and in particular R/Q) must be a C-ring, R/Q is 1-dimensional, and for every maximal ideal M of R/Q, $(R/Q)_M$ has no nilpotent elements. By Lemma 1, R/Q has no nilpotent elements.

4. **Noetherian integral closures.** In this section we consider some of the implications of Proposition 2. First, a more or less standard fact for which we shall have considerable use throughout this section:

([4]) "D" because D-domains are Dedekind rings. The example described in the footnote on p. 3 is a D-ring.

LEMMA 4. *Let R be a ring having no nilpotent elements and only finitely many minimal primes, P_i. Then T, the direct sum of the R/P_i, may be regarded as an overring of R; T is a finite R-module. Further, the integral closure of R is the direct sum of the integral closures of the R/P_i.*

Proof. Since (0) is the intersection of the P_i, R may be imbedded, by means of the homomorphisms $R \rightarrow R/P_i$, as a subring of T. It is easily seen that each of the idempotents associated with this direct sum is a fraction (i.e., a quotient of elements of R). Consequently T is an overring of R and a finite R-module; thus T is integral over R. Clearly then, T and R have the same integral closure, the direct sum of the integral closures of the R/P_i.

We may now record a curious consequence of Proposition 2:

THEOREM 2. *An integrally closed local ring is a Z-ring or an integral domain.*

Proof. If the ring in question is not a Z-ring, then, by Proposition 2, it has no nilpotent elements. In this event, by the preceding lemma, the ring is a direct sum of as many domains as it has minimal primes. As it has but one maximal ideal, the ring must then be a domain.

By *affine ring* we shall mean a homomorphic image of a polynomial ring over a field. A classical theorem of F. K. Schmidt asserts that the integral closure of an affine domain R is a finite R-module. A standard argument (Lemma 4) instantly extends Schmidt's theorem to apply to affine rings without nilpotent elements. In view of Proposition 2, further extension of this result will require at least the additional hypothesis "locally nilpotentfree." It is a theorem that this condition is sufficient. We separate out from the proof that portion which does not deal specifically with the affine case.

LEMMA 5. *A locally nilpotentfree ring contains every nilpotent element of its total quotient ring.*

Proof. Let N and $\overline{N}$ be respectively the ideals of nilpotent elements of R and its total quotient ring. If the maximal ideal M of R is a Z-ideal, then every element of $\overline{N}_M$ has a representative of the form x/y, with x in N and y in $R-M$; thus, in this case, $\overline{N}_M = N_M$. If the maximal ideal M contains a non-zero-divisor, then $\overline{N}_M = N_M = (0)$. Therefore, by Lemma 1, $\overline{N} = N$.

THEOREM 3. *If R is an affine ring with integral closure $\overline{R}$, then the following statements are equivalent.*

(1) *R is locally nilpotentfree.*
(2) *$\overline{R}$ is a finite R-module.*
(3) *$\overline{R}$ is Noetherian.*

Proof. It suffices to show that (1) implies (2). Let N be the ideal of nilpotent elements of R (and also, by the preceding lemma, of $\overline{R}$). $\overline{R}/N$ is an (R/N)-submodule of the integral closure of R/N, an affine ring without nilpotent elements. By Schmidt's theorem, $\overline{R}/N$ is a finite (R/N)-module, and therefore a finite R-module. Since N is a finite R-module (even an ideal of R), it follows that $\bar{R}$ is a finite R-module.

Mori and Nagata have proved [7] that the integral closure of a 2-dimensional Noetherian domain is Noetherian. (Nagata [6] has found examples to show that this theorem can not in general be extended to domains of dimension greater than 2.) A standard argument (Lemma 4) extends the result to 2-dimensional Noetherian rings without nilpotent elements. Hereafter we shall refer to this version of the theorem as *Mori-Nagata*. It is easy to see that Mori-Nagata and Proposition 2 yield:

PROPOSITION 6. *Let R be a 2-dimensional Noetherian ring in which* (0) *is an unmixed ideal* (*i.e., each of the primes belonging to* (0) *is of rank* 0). *Then the integral closure of R is Noetherian if, and only if, R is locally nilpotentfree.*

We know (Theorem 3) that if R is an affine ring, then the conclusion of Proposition 6 remains valid even without the restriction on the ranks of the primes belonging to (0). It is natural to ask whether this restriction can be relaxed in the general case. The remainder of this section is devoted to the consideration of this question. Although we do not reach the ultimate goal of determining whether the restriction can be removed entirely, we do, in Corollary 4, achieve a certain improvement of Proposition 6.

Mori-Nagata equips us with a trivially new class of rings with Noetherian integral closures: the class of those Noetherian rings which are direct sums of Z-rings and 2-dimensional rings without nilpotent elements. A ring of this class has the property that none of its NZ-primes contains a nonminimal Z-prime. It is a theorem that this property and "locally nilpotentfree" force the integral closure of a 2-dimensional Noetherian ring to be Noetherian(5). The proof of this fact will require the following technical lemmas.

LEMMA 6 (COHEN [2]). *A ring is Noetherian if its prime ideals are finitely generated.*

LEMMA 7 (NAGATA [7]). *Let P be a rank* 1 *prime in the integral closure of the Noetherian ring R. If x is a non-zero-divisor in $P\cap R$, then $P\cap R$ belongs*(6) *to the ideal xR.*

LEMMA 8. *If a ring has only finitely many minimal primes, and each of its nonminimal primes is finitely generated, then there are only finitely many primes minimal over any of its ideals.*

Proof. It suffices to show that any radical ideal (an ideal which is the intersection of prime ideals) I is the intersection of a finite number of prime ideals. If I is contained in no minimal prime, then the ring modulo I is, by

(5) Examples show that a ring satisfying these two conditions need not belong to the trivial class described above.

(6) "Belongs" in the sense of the primary decomposition.

Lemma 6, Noetherian, and there is nothing to prove. Assume then that I is contained in the minimal primes $P_1, \cdots, P_n$, but in no other. Then I is the intersection of the P_i and a radical ideal J, where J is the intersection of the prime ideals $\{Q_\alpha\}$, none of which contains any P_i. If J is contained in no minimal prime, we are finished. Otherwise, suppose that J is contained in the minimal prime P. Clearly P is one of the P_i. Now P contains the intersection of all minimal primes that are contained in some Q_α. It follows that P is contained in some Q_α—a contradiction.

THEOREM 4. *Let R be a locally nilpotentfree Noetherian ring of restricted dimension 2. If no NZ-prime of R contains a nonminimal Z-prime, then the integral closure of R is Noetherian.*

Proof. Let $\overline{R}$ be the integral closure of R. By Lemma 6, it suffices to show that the prime ideals of $\overline{R}$ have finite bases. It follows from the condition that no NZ-prime of R contain a nonminimal Z-prime that for every NZ-prime P of $\overline{R}$, $\overline{R}_{R-P\cap R}$ is the integral closure of $R_{P\cap R}$. Thus, by Mori-Nagata and Theorem 2, each $\overline{R}_P$ is an integrally closed Noetherian domain. As in the proof of Proposition 3, we intend to exploit this fact along with the fact that only finitely many maximal ideals of $\overline{R}$ lie over any given maximal ideal of R for the purpose of applying Lemma 1. Let P denote an arbitrary prime of $\overline{R}$.

Case I. P is a maximal ideal or a nonminimal Z-prime. In this case, the generators of $P\cap R$ are all contained in only finitely many maximal ideals of R, and so in only finitely many maximal ideals of $\overline{R}$. Thus, for all but a finite number of maximal ideals M of $\overline{R}$, the finite set of generators of $P\cap R$ is a basis for $P_M=\overline{R}_M$. By Lemma 1, P has a finite basis.

Case II. P is a nonmaximal (and therefore rank 1) NZ-prime. Since $\overline{R}_P$ is a discrete valuation ring, there is an x in P such that $P_P=x\overline{R}_P$([7]). Without loss of generality, x may be taken to be a non-zero-divisor. Any prime minimal over $x\overline{R}$ is of rank 1; it follows from Lemma 7 that there are only finitely many of these. Let y belong to P, but to no other prime minimal over $x\overline{R}$. Any maximal ideal containing (x, y), but not P, is minimal over (x, y); by Lemma 8, there are only finitely many of these. Of those maximal ideals containing P, only finitely many can contain any other prime minimal over $x\overline{R}$ (Lemma 2). If the maximal ideal M contains P, but no other prime minimal over $x\overline{R}$, then since the principal ideals of $\overline{R}_M$ are unmixed, $x\overline{R}_M$ is primary for P_M; since $P_P=x\overline{R}_P$, it then follows that $P_M=x\overline{R}_M$. Thus, for all but a finite number of maximal ideals M of $\overline{R}$, either $P_M=(x, y)_M=\overline{R}_M$ or $P_M=x\overline{R}_M$. By Lemma 1, P has a finite basis.

Case III. P is a minimal prime. Let x belong to P, but to no other minimal prime. By Lemma 8, there are only finitely many primes minimal over $x\overline{R}$;

([7]) It is quite generally true that if P is a rank 1 NZ-prime in the integral closure of a Noetherian ring, then P_P is principal. Lemma 3 shows that P_P has a finite basis; easy arguments, familiar for domains, then show that P_P is invertible, and hence principal.

let y belong to P, but to none of the others. Any maximal NZ-prime which contains (x, y), but not P, is minimal over (x, y); there are only finitely many of these (again Lemma 8). Thus, all but a finite number of maximal ideals of $\bar{R}$ either contain P or do not contain (x, y). For every NZ-prime M of $\bar{R}$, $P_M=(0)$ or $\bar{R}_M$ according to whether or not M contains P. Thus, for all but a finite number of maximal ideals M of $\bar{R}$, either $P_M=(x, y)_M=\bar{R}_M$ or $P_M=(0)$. An application of Lemma 1 finishes the argument in this case and thereby establishes Theorem 4.

Corollary to Theorem 4 is the promised improvement of Proposition 6:

COROLLARY 4. *Let R be a 2-dimensional Noetherian ring having the property that none of its primes belonging to* (0) *is of rank greater than* 1. *Then the integral closure of R is Noetherian if, and only if, R is locally nilpotentfree.*

Proof. We need only establish the sufficiency of "locally nilpotentfree." In view of the preceding theorem, it suffices to show that no NZ-prime of R contains a nonminimal Z-prime. Suppose that the maximal NZ-ideal M contains the nonminimal Z-prime P. Because of the restriction on the ranks of primes belonging to (0), P belongs to (0). Thus (0) in the ring R_M has an imbedded primary component, namely, that corresponding to P_M. Then, contrary to the hypothesis, R_M has nontrivial nilpotent elements.

Whether this result remains valid without the restriction on the primes belonging to (0) is yet an open question. The major difficulty in treating the general case seems to be that encountered in dealing with precisely those primes ruled out by the hypothesis of Corollary 4: Z-primes which do not belong to (0). Such Z-primes can become NZ-primes on localization and in so doing, slip completely out of the control of the methods we have employed. It is possible that a study of specific examples of 2-dimensional affine rings will give some clue to the general behavior of these recalcitrant primes.

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