CLASSIFICATIONS OF RECURSIVE FUNCTIONS BY MEANS OF HIERARCHIES(1)

BY

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1. **Introduction.** The motivations for attempting to find a satisfactory classification of recursive functions by ordinals are rather well known; cf., for example [6, pp. 67–68]. Among other things, such a classification should give insight into how the nonconstructively defined class of arbitrary recursive functions can be successively approximated by classes of functions whose members can be constructively recognized to be everywhere defined and computable. It should also provide a framework for the (partial) characterization of the strength of various formalized theories, through the classification of the provably recursive functions of those theories. Finally, it might be hoped that such a classification would provide a new tool for obtaining results of purely mathematical interest about recursive functions.

It has been pointed out by Myhill [11] and independently by Routledge [13] that the most obvious attempt to define such a classification, namely in terms of recursions over previously constructed recursive well-orderings of the natural numbers, already gives all recursive functions by suitable choice of primitive recursive well-orderings of order type ω. This is quite naturally considered a "breakdown," since none of the ends desired from such a classification are at all realized.

Another approach to the classification problem has been suggested by Kleene in [6]. This harks back to the idea that from any constructively generated class of recursive functions we are able to obtain new functions by diagonalization or, more generally, by enumeration. Transfinite iteration of this procedure leads to a hierarchy of recursive functions, most conveniently described with respect to some class of notations for recursive well-orderings. However, in order that such a classification not be trivialized at level ω, the set O of notations used should be restricted to those built up only by means of primitive recursive fundamental sequences [6, pp. 72–73]. We shall follow this restriction throughout this paper.

For functions ϕ, ψ on natural numbers put (for the moment) $\phi \ll \psi$ if ϕ is primitive recursive in ψ, but not conversely. Kleene's hierarchy of functions ρ_d (denoted by h_d in [6]) has the property

$$c <_O d \rightarrow \rho_c \ll \rho_d. \tag{1.1}$$

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Another hierarchy of recursive functions ρ_d which has this property when $\phi \ll \psi$ is interpreted to mean that ϕ is majorized by ψ, i.e., that $(Em)(n)(n > m \rightarrow \phi(n) < \psi(n))$, has been suggested to the author by Hartley Rogers, Jr. The possibility of obtaining such a hierarchy is easily seen from the fact that from any effectively enumerated class of recursive functions we can construct a recursive function which majorizes all elements of that class.

In this paper we consider a quite general class of hierarchies of recursive functions associated with given relations $\ll$, of which the above-mentioned are examples. Since, generally, uniqueness results fail for such hierarchies (cf. Axt [1], Kreisel [9], and 3.8 of this paper), it is natural to put questions of completeness in two ways. First, given $\kappa \leqq \omega_1$, what can be said about the set of functions ρ_d for $d \in O$, $|d| < \kappa$? Second, what can be said about the set of functions ρ_d for $d \in P$, where P is a path (set of notations well-ordered by $<_O$, and closed under predecessor) in O, $|P| = \kappa$? (We understand by $|d|$ the ordinal denoted by d, by $|P|$ the order type of P under $<_O$, and by ω_1 the least ordinal not denoted by any $d \in O$.)

The answers we obtain here to these questions are (under suitable conditions governing the "rate of growth" of the functions ρ_d) the following:

(1.2) *For any recursive function ϕ we can find $d \in O$ with $|d| = \omega^2$ and $\phi \ll \rho_d$.*

(1.3) *We can find paths $P \subseteq O$ such that $|P| = \omega^3$ and such that for any recursive ϕ there is a $d \in P$ with $\phi \ll \rho_d$. Moreover, given any ordinal $\kappa \leqq \omega_1$ we can find such paths with $|P| = \kappa + \omega^3$ if $\kappa < \omega_1$, $|P| = \omega_1$ otherwise.*

(1.4) *We can find incomplete paths P through O, in the sense that $|P| = \omega_1$ and there exist recursive functions ϕ such that $\rho_d \ll \phi$ (hence ϕ not $\ll \rho_d$) for all $d \in P$.*

The results (1.2), (1.3) answer Kleene's question P **236** [6, p. 77] for his hierarchy. An immediate corollary to (1.2), is the nonuniqueness of that classification for all $|d| \geqq \omega^2$, thus completing the answer given by Axt in [1] to the question P **238**. We also answer the question P **237** in part by means of the following result, which complements a converse result by Axt in [1]. (The present result has been obtained in collaboration with W. W. Tait.)

(1.5) *All the functions ρ_d of the Kleene sub-recursive hierarchy for which $|d| < \omega^2$ are ordinal recursive with respect to the "natural" well-ordering of the natural numbers in type ω^{ω^3}.*

A related result which we obtain is the following.

(1.6) *All the functions ρ_d of the majorizing hierarchy for which $|d| < \omega^2$ are primitive recursive.*

These results (1.5), (1.6) reinforce an opinion, which might already be taken on the basis of (1.2), that such hierarchies do not, when used with all notations, provide a satisfactory classification of recursive functions. Again, the reason for this breakdown can be localized in the liberality with which we have provided ourselves notations for well-orderings.

However, it turns out that these hierarchies can still be used to obtain some new information about recursive functions, thus providing something in one direction demanded from a suitable classification. This is the following (hierarchy-free) result:

(1.7) *There exists a set* Δ *of recursive functions densely ordered by* $\ll$; *hence for any denumerable ordinal* κ *there exists a sequence of recursive functions* ϕ_ι, $\iota<\kappa$, such that $\iota<\iota'\rightarrow\phi_\iota\ll\phi_{\iota'}$.

But here hierarchies are used in an unexpected way, namely through certain "nonstandard" extensions of them.

Some of the results described above and the methods used to obtain them are closely related to (and were suggested by) certain results concerning recursive progressions of theories. In particular, (1.2) is related to a completeness result for arithmetical sentences of the form $(\forall x)(\exists y)\gamma(x, y)=0$, γ primitive recursive, in suitable progressions of theories, obtained by us in [2, Theorem 5.2]. (1.3) is related to [2, Theorem 5.14], but the proof here is simpler since there are no problems of arithmetization involved. (1.4) is closely related to the incompleteness result of our paper with Spector [3, cf., Theorems 2.5, 4.4]. Finally, the methods used to obtain (1.7) exploit certain ideas incipient in [3].

It is perhaps accidental that these metamathematical results preceded the corresponding purely function-theoretic results. However, we believe that further work on the classification problem should involve metamathematical notions in an essential way. For this problem is intertwined with the question as to how we can generate recursive well-orderings which we can, in some sense, constructively verify on the basis of previously constructed functions and orderings to be well-orderings. An important step along these lines has already been taken in the work of Kreisel [10] on the question of classifying the class of finitistically acceptable recursive functions.

2. **Hierarchies of functions.** All lower case italics range as variables over the set 0, 1, 2, $\cdots$ of natural numbers. All lower case Greek letters (with minor exceptions) and certain italic capitals range as variables over the class of total functions (and, on occasion, also partial functions) of one or more arguments from the set of natural numbers into itself. We use the notation $a_{(i)}$ instead of the more usual $(a)_i$; thus $0_{(i)}=0$, and for $a\neq 0$ and $p_0, \cdots, p_n, \cdots$ the primes in increasing order, $a=\prod_{i=0}^{\infty} p_i^{a(i)}$.

The primitive recursive predicate $In^m(b)$ $(m>0)$ is taken as defined in [6, p. 70]. When it holds we say that b is an (n-) index for defining a function ϕ of $n=b_{(1)}$ arguments from any function θ of m arguments by adjoining instances of primitive recursive schemata to the true numerical equations for θ. We shall only need this notion for the case $m=1$. The n-ary function defined in this case from a given unary function θ by b is denoted by $[b]_n^\theta$. If b is not an n-index we take $[b]_n^\theta(x_1, \cdots, x_n)=0$. We set $[b]_n=[b]_n^{\lambda x(0)}$. We shall write $[b]^\theta$, or $[b]$, when $n=1$ and also, where there is no ambiguity, for other values of n. Thus $[b]([b]^\theta)$ for $b=0, 1, 2, \cdots$ provides an enumeration of all functions of one argument which are primitive recursive (in θ) [6, p. 71]. We write $\phi \subseteq_b \theta$ if $\phi = [b]^\theta$, $\phi \subseteq \theta$ if $(Eb)\phi \subseteq_b \theta$, and $\phi \subset \theta$ if $\phi \subseteq \theta$ but $\theta \nsubseteq \phi$. We use the notation $\{e\}$ for the partial recursive function with Gödel-number e, $\{e\}(x) \simeq U(\mu y T_1(e, x, y))$.

The following adaptation of the recursion theorem to primitive recursive functions, proved by Kleene in [6, p. 75], is of great usefulness. We add to it, in the second part of the statement, a corollary needed for simultaneous recursions.

2.1. LEMMA. (i) *Given any primitive recursive function* $\psi(z, x_1, \cdots, x_n)$ *we can find an* e *such that* $[e] = \lambda x_1 \cdots x_n \psi(e, x_1, \cdots, x_n)$.

(ii) *Given any primitive recursive functions* $\psi_i(z_0, z_1, x_1, \cdots, x_n)$ $(i = 0, 1)$ *we can find* e_0, e_1 *with* $[e_i] = \lambda x_1 \cdots x_n \psi_i(e_0, e_1, x_1, \cdots, x_n)$.

To prove (ii) from (i), first find primitive recursive θ_0, θ_1 such that for any $y, x_1, \cdots, x_n$, $[\theta_i(y)](x_1, \cdots, x_n) = ([y](x_1, \cdots, x_n))_{(i)}$. We then determine f by (i) so that $[f] = \lambda x_1 \cdots x_n 2^{\psi_0(\theta_0(f), \theta_1(f), x_1, \cdots, x_n)} \cdot 3^{\psi_1(\theta_0(f), \theta_1(f), x_1, \cdots, x_n)}$. We then take $e_i = \theta_i(f)$. (2.1 (ii) can obviously be generalized to obtain e_i, $i = 0, \cdots, m$.)

Throughout the remainder of this paper O, $<_O$, and allied notions will be restricted to the case of primitive recursive fundamental sequences. (The notations O_p or O' are avoided for simplicity.) Much of the theory for these notions can be adapted from [8] as described in [6]. Briefly, we obtain first a transitive recursively enumerable relation $\prec$ such that (when we take $x \preceq y \leftrightarrow x \prec y \vee x = y$)

$$
(2.2) \qquad \begin{aligned} a \prec b \leftrightarrow b &= 2^{b(0)} \,\&\, b_{(0)} \neq 0 \,\&\, a \preceq b_{(0)}, \\ \vee\ b &= 3 \cdot 5^{b(2)} \,\&\, (En) a \preceq [b_{(2)}](n). \end{aligned}
$$

Then O is the smallest set which contains 1 and which, whenever it contains c, contains 2^c and, whenever it contains $[d](n)$ for all n, where $(n)\{[d](n) \prec [d](n+1)\}$, contains $3 \cdot 5^d$. We put $a <_O b$ if $a, b \in O$ and $a \prec b$. We denote by $C(b)$ the set of $x \prec b$ and by $C'(b)$ the set $C(b) \cup \{b\}$. We take $|b|$ to be the order type of $C(b)$ when $b \in O$. ω_1 is the least ordinal not thus represented. Kleene has shown in [6, p. 75] that this is the same as the ordinal obtained when O is defined with respect to arbitrary recursive fundamental sequences.

We put $0_O = 1$, $(n+1)_O = 2^{n_O}$ a suitable definition of $+_O$ has been given by Kleene in [6, p. 75]. We want, more generally, the following.

2.3. LEMMA. *Given primitive recursive functions* ψ_0, ψ_1, ψ_2 (*of* 1, 3, 3 *arguments*) *we can construct primitive recursive functions* ϕ, γ *with* (*for all* a, d)

(i) $\phi(a, 1) = \psi_0(a)$,

(ii) $\phi(a, 2^d) = \psi_1(a, d, \phi(a, d))$ *for* $d \neq 0$,

(iii) $\phi(a, 3 \cdot 5^d) = \psi_2(a, d, \gamma(a, d))$ *where* $[\gamma(a, d)](n) = \phi(a, [d](n))$ *for all* n.

The proof is similar to that for $+_O$ in [6]. Briefly, we can find primitive recursive θ with $[\theta(z, a, d)](n) = [z](a, [d](n))$ for all z, a, d, n. By course-of-values recursion we obtain primitive recursive $\phi^*(z, a, d)$ satisfying $\phi^*(z, a, 1) = \psi_0(a)$, $\phi^*(z, a, 2^d) = \psi_1(a, d, \phi^*(z, a, d))$ for $d \neq 0$, $\phi^*(z, a, 3\cdot 5^d) = \psi_2(a, d, \theta(z, a, d))$, and $\phi^*(z, a, b) = 0$ for all other b. We then choose e by 2.1(i) so that $[e](a, d) = \phi^*(e, a, d)$, and take $\phi = [e]$ and $\gamma(a, d) = \theta(e, a, d)$.

We shall write $a \oplus b$ instead of $a +_O b$. This is a primitive recursive function of a, b, satisfying $a \oplus 1 = a$, $a \oplus 2^d = 2^{a \oplus d}$ $(d \neq 0)$, and $a \oplus 3\cdot 5^d = 3\cdot 5^{\gamma(a,d)}$ where γ is primitive recursive and $[\gamma(a, d)](n) = a \oplus [d](n)$. For any a, b, n, $(a \oplus b) \oplus n_O = a \oplus (b \oplus n_O)$, but $\oplus$ is in general not associative.

Hierarchies of recursive functions corresponding to certain generation principles can be constructed as in [6, p. 74] by defining certain partial recursive functions $\sigma(z, a)$ such that for each $d \in O$, $\rho_d = \lambda x \sigma(d, x)$ is a recursive function, and such that the ρ_d's are related in the desired fashion for different d's. Alternatively, it is possible to generate the Gödel-numbers r_d of these functions. This is more convenient for us here, and the general possibility of doing this follows directly from 2.3, if we omit the parameter "a" there.

2.4. Corollary. *Suppose q_1 is the Gödel number of a recursive function and that ψ_1, ψ_2 are primitive recursive functions such that:*

(i) *if f is a Gödel-number of a recursive function, then so also is $\psi_1(f)$;*

(ii) *if for each n, $[e](n)$ is a Gödel-number of a recursive function then so also is $\psi_2(e)$.*

Then we can find primitive recursive ϕ, γ such that $\phi(1) = q_1$, $\phi(2^d) = \psi_1(\phi(d))$ for $d \neq 0$, $\phi(3\cdot 5^d) = \psi_2(\gamma(d))$ where for all n, $[\gamma(d)](n) = \phi([d](n))$. It follows from these conditions that for each $d \in O$, $\rho_d = \lambda x\{\phi(d)\}(x)$ is a recursive function.

We apply this to the construction of two hierarchies. First we have a slight variant of the hierarchy introduced by Kleene in [6, pp. 73–74].

2.5. Lemma. *We can find primitive recursive ψ_1, ψ_2 so that:*

(i) *if f is a Gödel-number of a recursive function $\theta(a)$ then $\psi_1(f)$ is a Gödel-number of the function $\theta'(a) = [a_{(0)}]^\theta(a_{(1)})$;*

(ii) *if for each n, $[e](n)$ is a Gödel-number of a recursive function $\theta_n(a)$ then $\psi_2(e)$ is a Gödel-number of the recursive function $\theta(a) = \theta_{a_{(0)}}(a_{(1)}) = \{[e](a_{(0)})\}(a_{(1)})$.*

We shall refer to the resulting ρ_d's associated by 2.4 with these ψ_1, ψ_2 and a Gödel-number q_1 of the constant function $\lambda x(0)$ as constituting the *Kleene sub-recursive hierarchy*. These have the property [6, p. 73] that if $c <_O d$ then $\rho_c \subset \rho_d$; this is generalized in 5.3 below. Further properties needed for the results of this paper will be established in the next section.

As a second example, we construct a class of *majorizing hierarchies* each of which is associated with a given recursive function $\chi(a, b)$. Suppose given a Gödel number of χ.

2.6. LEMMA. *We can find primitive recursive functions* ψ_1, ψ_2 *so that*:

(i) *if* f *is a Gödel-number of a recursive function* $\theta(a)$ *then* $\psi_1(f)$ *is a Gödel-number of the function* $\theta'(a)=\chi(a, \theta(a))+1$;

(ii) *if for each* n, $[e](n)$ *is a Gödel-number of a recursive function* $\theta_n(a)$ *then* $\psi_2(e)$ *is a Gödel-number of the recursive function*

$$\theta(a) = \max_{0\leq i\leq a} \theta_i(a) + 1 = \max_{0\leq i\leq a} \{[e](i)\}(a) + 1.$$

Put $\theta \prec \zeta$ if $(Em)(n)(n\geqq m\rightarrow\theta(n)<\zeta(n))$. Consider the functions ρ_d associated with the ψ_1, ψ_2 of 2.6 by 2.4, with q_1 the Gödel number of the function $\lambda x(0)$. These have the property that if $c<_O d$ then $\rho_c \prec \rho_d$ (cf. 5.3 below). We thus refer to these ρ_d's as constituting *the majorizing hierarchy associated with* χ. Other results in this paper will depend on taking χ to be a function which satisfies $b\leqq\chi(a, b)$ for all a, b. Some choices of such χ would be $\chi(a, b) =b$, $\chi(a, b)=(a+1)\cdot b$, $\chi(a, b)=(a+2)^b$, etc.; these lead to familiar number-theoretic majorizing relationships.

These hierarchies can be modified by taking ρ_1 to be any given recursive function. The results of this paper will still continue to hold for such modifications. We can further take ρ_1 to be an arbitrary function if the notion of recursiveness is replaced by that of recursiveness relative to ρ_1.

3. Completeness of primitive recursively expanding hierarchies.

3.1. DEFINITION. *By a hierarchy which is p.r.* (*primitive recursively*) *expanding with respect to certain relations* $\leqq_e$ $(e=0, 1, 2, \cdots)$ *between unary functions, we understand an assignment of unary functions* ρ_d *to each* $d\in O$ *for which there are primitive recursive functions* Tr, C, S, L, M *satisfying the following conditions*:

(i) *for any* ϕ, θ, ζ, *if* $\phi\leqq_e\theta$, $\theta\leqq_f\zeta$ *then* $\phi\leqq_{Tr(e,f)}\zeta$;

(ii) *for any* $d\in O$ *and any* k,

$$\lambda x(k) \leqq_{C(k,d)} \rho_{d\oplus(k+1)_0}$$

(iii) *for any* $d\in O$, $\rho_d\leqq_{S(d)}\rho_{2^d}$;

(iv) *for any* d, e, *and binary* ϕ *such that* $3\cdot5^d\in O$ *and* $\lambda x\phi(n, x)\leqq_{[e](n)}\rho_{[d](n)}$ *for all* n, *we have*

$$3\cdot5^{M(e,d)}\in O, \quad |3\cdot5^{M(e,d)}| = |3\cdot5^d|, \quad [d](n) <_O 3\cdot5^{M(e,d)}$$

for all n, *and* $\lambda x\phi(x, x)\leqq_{L(e,d)}\rho_{3\cdot5^{M(e,d)}}$.

We say that the hierarchy is strictly expanding if for all $d\in O$ *and all* e, $\rho_{2^d}\not\leqq_e\rho_d$.

3.2. THEOREM. *The Kleene sub-recursive hierarchy is strictly p.r. expanding with respect to the relations* $\subseteq_e$.

Proof. (i) A suitable function Tr has been defined by Kleene in [6, pp. 70–71].

(ii) Here we can take C to be a simple function $C(k)$ of k only, satisfying $\lambda x(k) \subseteq_{C(k)} \theta$ for any function θ.

(iii) Here we can take $S(d)$ to be a certain constant s where $\theta \subseteq_s \lambda x[x_{(0)}]^{\theta}(x_{(1)})$ for any function θ.

(iv) In this case we can find M as a function $M(d)$ of d only and L as a function $L(e)$ of e only. M is chosen to satisfy $[M(d)](n) = 2^{[d](n)}$ $(= [d](n) \oplus 1_O)$, for all n, and a preliminary (primitive recursive) L_1 is chosen to satisfy

$$[L_1(e)](n, a) = 2^n \cdot 3^{2^{[e](n)} \cdot 3^a}$$

for all n, a. To see how L should be defined, suppose we had d, e, ϕ satisfying the hypothesis of 3.1 (iv) with respect to the relations $\subseteq_f$. Let $\phi_n = \lambda x \phi(n, x)$ for all n. Then for each n, a,

$$\begin{aligned} \phi_n(a) &= [[e](n)]^{\rho_{[d](n)}}(a) = \rho_{2^{[d](n)}}(2^{[e](n)} \cdot 3^a) \\ &= \rho_{[M(d)](n)}(2^{[e](n)} \cdot 3^a) = \rho_{3 \cdot 5^{M(d)}}(2^n \cdot 3^{2^{[e](n)} \cdot 3^a}) \\ &= \rho_{3 \cdot 5^{M(d)}}([L_1(e)](n, a)). \end{aligned}$$

Hence $\phi(a, a) = \rho_{3 \cdot 5^{M(d)}}([L_1(e)](a, a))$ for all a. Choose primitive recursive $L(e)$ to satisfy $[L(e)]^{\theta}(a) = \theta([L_1(e)](a, a))$ for all a, e, any θ. We see by the preceding argument that for such L, $\lambda x \phi(x, x) \subseteq_{L(e)} \rho_{3 \cdot 5^{M(d)}}$. That the expansion is strict is shown by Kleene in [6, p. 73].

3.3. Theorem. *Let $\chi(a, b)$ be recursive and $b \leqq \chi(a, b)$ for all a, b. Then the majorizing hierarchy associated with χ is strictly p.r. expanding with respect to the relations $\leqq_e$, when these are all taken to be the same relation $<$, where $\phi < \theta \leftrightarrow (n)(\phi(n) < \theta(n))$.*

Proof. The condition (i) is obviously fulfilled.

(ii) Using $\rho_{2^c}(a) = \psi(a, \rho_c(a)) + 1 \geqq \rho_c(a) + 1$ for any $c \in O$, and any a, we easily prove by induction on k that $i \leqq \rho_{d \oplus i_O}(a)$ for any $d \in O$, a and i. The first inequality also establishes (iii).

(iv) Here we can take $M(e, d) = d$. For suppose $\lambda x \phi(n, x) < \rho_{[d](n)}$ for all n. Then for any a,

$$\phi(a, a) < \rho_{[d](a)}(a) < \max_{0 \leqq i \leqq a} \rho_{[d](i)}(a) + 1 = \rho_{3 \cdot 5^d}(a).$$

That the expansion is strict is obvious by the inequality used to prove (ii).

We assume throughout the remainder of this section that we are dealing with any one hierarchy of functions ρ_d (not necessarily recursive) p.r. expanding (not necessarily strictly) with respect to certain relations $\leqq_e$. We assume that Tr, C, S, L, M are some fixed primitive recursive functions satisfying 3.1 (i)–(iv) *with these ρ_d, $\leqq_e$.*

The details of the proof of the completeness results which we shall give in this section would not be essentially simplified by restricting attention to the sub-recursive hierarchy, and would be simplified little more in the case of majorizing hierarchies. On the other hand, the theorems hold even for further (slight) generalizations of 3.1. For example, we could weaken 3.1 (ii) so that $k+1$ is replaced by some primitive recursive $D(k, d)$.

3.4. Lemma. *There is a primitive recursive function $N(d, e, i)$ such that for any $d\in O$ and any ϕ, if $\phi \leqq_e \rho_d$ then $\phi \leqq_{N(d,e,i)} \rho_{d\oplus i_O}$.*

Proof. Take $N(d, e, 0)=e$, $N(d, e, i+1)=Tr(N(d, e, i), S(d\oplus i_O))$, for any d, e, i.

Consider now any recursive function $\phi(a)=U(\mu y T_1(q, a, y))$, and let for each n ϕ_n be the constant function $\lambda x\phi(n)$. By 3.1 (ii), each $\phi_n \leqq_f \rho_h$ for certain f, h, namely $f=C(\phi(n), 1)$ and $h=1\oplus(\phi(n)+1)_O=(\phi(n)+1)_O$. However, in general these f, h are not chosen as primitive recursive functions of n. Our main argument towards completeness, which now follows, shows that, nevertheless, certain other f, h can be chosen primitive recursively to satisfy $\phi_n \leqq_f \rho_h$. This, when combined with the limit condition 3.1 (iv), will allow us to obtain a similar result for ϕ itself.

3.5. Theorem. *There are primitive recursive functions $f=F(q, b, n)$, $h=H(q, b, n)$ such that whenever $(Ey)T_1(q, n, y)$, $\phi=\lambda x U(\mu y T_1(q, x, y))$, $\phi_n=\lambda x\phi(n)$ and $b\in O$, then*

(i) *$h\in O$, $b<_O h$, $|h|=|b|+\omega\cdot m$ for some $m>0$, and*

(ii) *$\phi_n\leqq_f \rho_h$.*

Proof. We shall construct certain primitive recursive functions of q, b, n, k and i. Considering q, b, n as parameters, we concentrate on the definition of these as functions of k, i. Using the primitive recursive functions Sb_n^m of [6, p. 75], we have $[Sb_1^1(z, y)](x)=[z](y, x)$ for any x, y, z. We shall write z_y for $Sb_1^1(z, y)$. (This notation will be used only in this proof.) We consider three primitive recursive predicates of k (and, implicitly, also of q, n):

$$\begin{aligned} &\mathrm{Sec}^{(0)}(k) \leftrightarrow (Ey)_{y<k}T_1(q, n, y),\\ (1)\qquad &\mathrm{Sec}^{(1)}(k) \leftrightarrow T_1(q, n, k)\ \&\ (y)_{y<k} \rightharpoondown T_1(q, n, y),\\ &\mathrm{Sec}^{(2)}(k) \leftrightarrow (y)_{y\leq k} \rightharpoondown T_1(q, n, y).\end{aligned}$$

(Adapting the terminology of [8], we might say of these three cases, successively, that k is past secured, k is just secured, and k is unsecured.) Using 2.1(ii), we now find d, e, satisfying the following two conditions for all k, i:

$$(2)\qquad [d](k,i) = \begin{cases} 0 & \text{if } \mathrm{Sec}^{(0)}(k),\\ b\oplus(U(k)+1)_O\oplus i_O & \text{if } \mathrm{Sec}^{(1)}(k),\\ 3\cdot 5^{M(e_{k+1},d_{k+1})}\oplus i_O & \text{if } \mathrm{Sec}^{(2)}(k);\end{cases}$$

$$(3)\qquad [e](k,i)=\begin{cases}0 & \textit{if } \mathrm{Sec}^{(0)}(k),\\ N(b\oplus (U(k)+1)_O,\ C(U(k),b),\ i) & \textit{if } \mathrm{Sec}^{(1)}(k),\\ N(3\cdot 5^{M(e_{k+1},d_{k+1})},\ L(e_{k+1},d_{k+1}),\ i) & \textit{if } \mathrm{Sec}^{(2)}(k).\end{cases}$$

Now suppose $(Ey)T_1(q, n, y)$ and $b\in O$. Define ϕ and ϕ_n as in the statement of the theorem. Let $k_0=\mu y T_1(q, n, y)$. We shall prove by induction on j that

(4) *if* $j\leqq k_0$, $k=k_0-j$, *and* i *is arbitrary, then*

(a) $[d](k, i)\in O,\quad b\leqq_O [d](k, i)<_O [d](k, i+1)$,

$$|\,[d](k,i)\,| = |b| + \omega\cdot j + \begin{cases}U(k_0)+1+i & \textit{if } j=0,\\ i & \textit{if } j\neq 0,\end{cases}$$

and (b) $\phi_n \leqq_{[e](k,i)} \rho_{[d](k,i)}$

For $j=0$, we have $k=k_0$, so $\mathrm{Sec}^{(1)}(k)$. 4(a) is clearly true in this case. By 3.1(ii), $\phi_n=\lambda x U(k_0)\leqq_{C(U(k_0),b)}\rho_{b\oplus U(k_0)+1)_O}$, hence $\phi_n\leqq_{[e](k_0,i)}\rho_{[d](k_0,i)}$ by (2), (3) and 3.4.

Suppose (4) true for j; we shall now show it true for $j+1$. Suppose $j+1\leqq k_0$ and let $k=k_0-(j+1)$. We shall thus use (4)(a), (b) applied to j, $k+1$. We have from these $3\cdot 5^{d_{k+1}}\in O$, $b<_O 3\cdot 5^{d_{k+1}}$, $|3\cdot 5^{d_{k+1}}|=|b|+\omega\cdot(j+1)$, and $\phi_n\leqq_{[e_{k+1}](i)}\rho_{[d_{k+1}](i)}$ for all i. Let $\psi(i, x)=\phi_n(x)=\phi(n)$ for all i, x. Hence also $\lambda x\psi(i, x)\leqq_{[e_{k+1}](i)}\rho_{[d_{k+1}](i)}$ for all i. But then by 3.1(iv),

$$3\cdot 5^{M(e_{k+1},d_{k+1})}\in O,\qquad |3\cdot 5^{M(e_{k+1},d_{k+1})}| = |3\cdot 5^{d_{k+1}}|,$$
$$[d_{k+1}](i) <_O 3\cdot 5^{M(e_{k+1},d_{k+1})}\qquad \text{for all } i,$$

and

$$\lambda x\psi(x, x)\leqq_{L(e_{k+1},d_{k+1})}\rho_{3\cdot 5^{M(e_{k+1},d_{k+1})}}.$$

Since in this case $k<k_0$, and hence $\mathrm{Sec}^{(2)}(k)$, and since also $\phi_n=\lambda x\psi(x, x)$, we see by (2), (3) and 3.4 that (4) is also true for the case $j+1$, k.

As we remarked at the beginning of the proof, q, b, and n were taken as parameters in the definition of $[d](k, i)$, $[e](k, i)$. Hence $f=[e](0, 0)$ and $h=[d](0, 0)$ determine f and h as primitive recursive functions of q, b, n. When the hypotheses of our theorem concerning q, b, n, ϕ are met, we have $T_1(q, n, k_0)$, hence k_0 is the Gödel number of a deduction from the system of equations with number q, and thus $k_0\neq 0$. Taking $j=k_0$ in (4) thus gives us the desired result.

3.6. THEOREM. *There are primitive recursive functions* $e=E(q, b)$ *and* $d=D(q, b)$ *such that whenever* $(x)(Ey)T_1(q, x, y)$, $\phi=\lambda x U(\mu y T_1(q, x, y))$, *and* $b\in O$, *then* $d\in O$, $b<_O d$, $|d|=|b|+\omega^2$ *and* $\phi\leqq_e\rho_d$.

Proof. Let F, H be chosen to satisfy the conditions of 3.5. Define primitive recursive H_1 by $H_1(q, b, 0)=b$, $H_1(q, b, n+1)=H(q, H_1(q, b, n), n)$. Treating

q, b as parameters, choose h with $[h] = \lambda x H_1(q, b, x+1)$. Also choose f with $[f] = \lambda x F(q, H_1(q, b, x), x)$. Thus h, f are primitive recursively chosen as functions of q, b. Let $d = 3 \cdot 5^{M(f,h)}$, $e = L(f, h)$. To see that these satisfy the required conditions, suppose $(x)(Ey)T_1(q, x, y)$ and that $b \in O$. By 3.5, each $[h](n) \in O$, $[h](n) <_O [h](n+1)$, and $|[h](n+1)| = |[h](n)| + \omega \cdot m_n$ for some $m_n > 0$; furthermore, $b <_O [h](0)$ and $|[h](0)| = |b| + \omega \cdot m$ for some m. Thus $3 \cdot 5^h \in O$, $b <_O 3 \cdot 5^h$ and $|3 \cdot 5^h| = |b| + \omega^2$. Also by 3.5, $\phi_n = \lambda x \phi(n) \leqq_{[f](n)} \rho_{[h](n)}$ for each n. Put $\psi(n, x) = \phi(n)$ for all n, x. Then also $\lambda x \psi(n, x) \leqq_{[f](n)} \rho_{[h](n)}$. Hence by 3.1(iv), $d \in O$, $|d| = |3.5^h|$, $b <_O [h](0) <_O d$, and $\phi = \lambda x \psi(x, x) \leqq_e \rho_d$.

By a path P within O we understand a subset of O simply (and hence well-) ordered by $<_O$, and containing with any d all predecessors of d. $|P|$ denotes the order type of this path.

3.7. THEOREM. *Suppose κ is any ordinal with $\kappa \leqq \omega_1$. Then there exist* ($\aleph_0$) *paths P_κ within O, $|P_\kappa| = \kappa + \omega^3$ for $\kappa < \omega_1$, $|P_{\omega_1}| = \omega_1$, such that for any recursive function ϕ there exists a $d \in P_\kappa$ and an e with $\phi \leqq_e \rho_d$. P_κ can be chosen to be arithmetically definable* (*in fact, in a* 4-*quantifier form*) *for $\kappa < \omega_1$, and recursive in O for $\kappa = \omega_1$.*

Proof. Let $q_0, \cdots, q_n, \cdots$ be an enumeration of all q such that $(x)(Ey)T_1(q, x, y)$; specifically each q_n is the least $q > q_{n-1}$ ($q > 0$, for $n = 0$) such that $(x)(Ey)T_1(q, x, y)$. The predicate $Q(n, a)$ which holds if and only if $a = q_n$ is arithmetically definable in the four alternating quantifier forms beginning with the existential quantifier.

To prove the theorem, consider first the case $\kappa < \omega_1$. Choose $b \in O$, $|b| = \kappa$. Let D, E be the primitive recursive functions of 3.6. Define $d_0 = b$, $d_{n+1} = D(q_n, d_n)$, and define $e_n = E(q_n, d_n)$. Thus $d_n <_O d_{n+1}$ and $|d_{n+1}| = |d_n| + \omega^2 = |b| + \omega^2 \cdot (n+1)$ for each n, and $\{q_n\} \leqq_{e_n} \rho_{d_{n+1}}$ for each n. Then the path P_κ can be chosen in this case to be the set of x such that $(En)(x \prec d_n)$. Using the evaluation of the predicate $Q(n, a)$ it is seen that P_κ can also be defined in the same form. There is no generality lost by taking $\kappa \geqq \omega$. By choosing $\aleph_0$ b's with $|b| = \kappa$, we obtain $\aleph_0$ distinct P_κ's.

The proof for the case $\kappa = \omega_1$ is obtained by a slight modification. Let $b_0, \cdots, b_n, \cdots$ be an enumeration of O; the predicate $B(n, a)$ which holds when $a = b_n$, is recursive in O. Define $d_0 = b_0$, $d_{n+1} = D(q_n, d_n \oplus b_n)$, and $e_n = E(q_n, d_n \oplus b_n)$. Thus $d_n \leqq_O d_n \oplus b_n <_O d_{n+1}$ and $|d_{n+1}| = |d_n| + |b_n| + \omega^2$ for each n, and $\{q_n\} \leqq_{e_n} \rho_{d_{n+1}}$ for each n. The path P_{ω_1} is again defined as the set of $x \prec d_n$ for some n. Since $|b_n| \leqq |d_{n+1}|$ for all n, $|P_{\omega_1}| = \omega_1$. Without loss of generality we can take $|b_0| = \omega$; by altering the enumeration of O for different choices of b_0 we obtain $\aleph_0$ distinct P_{ω_1}'s.

Uniqueness in hierarchies ρ_d with respect to relations $\leqq_e$ can be said to hold at level $\kappa < \omega_1$ if for all $d, d' \in O$ with $|d| = |d'| = \kappa$ there exist e, f with $\rho_d \leqq_e \rho_{d'}$, $\rho_{d'} \leqq_f \rho_d$. The following nonuniqueness result is obtained directly from 3.6 (cf. also [1; 9]).

3.8. Corollary. *Suppose the give hierarchy is strictly p.r. expanding and that for any c, $d \in O$, $c \leqq_O d \rightarrow (Ee)\rho_c \leqq_e \rho_d$. Then for any $d \in O$, $|d| \geqq \omega^2$, there exists $d' \in O$ with $|d'| = |d|$ and $(e)\{\rho_{d'} \nleqq_e \rho_d\}$.*

Proof. Let $|d| = \omega^2 + \kappa$, and let $b \in O$, $|b| = \kappa$. By 3.6 we can find $h \in O$, $|h| = \omega^2$, and f_1 such that $\rho_{2^d} \leqq_{f_1} \rho_h$. Let $d' = h \oplus b$; thus $h \leqq_O d'$ and $|d'| = |d|$. By hypothesis there is f_2 with $\rho_h \leqq_{f_2} \rho_{d'}$, hence by 3.1(i) there is f with $\rho_{2^d} \leqq_f \rho_{d'}$. If for any e, $\rho_{d'} \leqq_e \rho_d$, we would have $\rho_{2^d} \leqq_{e_1} \rho_d$ for certain e_1, contradicting the strictness of the expansion.

As we remarked in §2, the condition $c \leqq_O d \rightarrow (Ee)\rho_c \leqq_e \rho_d$ is met by the Kleene sub-recursive hierarchy with respect to the relations $\subseteq_e$. Axt showed in [1, pp. 87–91] that uniqueness holds in this hierarchy for $|d| < \omega^2$ and fails at $|d| = \omega^2$.

To apply 3.8 to the majorizing hierarchies, we consider the relation $\phi \leqq \theta \leftrightarrow (Em)(n)(n \geqq m \rightarrow \phi(n) \leqq \theta(n))$ instead of the relation $\phi < \theta$ of complete majorizing used to prove 3.6. This still has the property of strictness, $\rho_{2^d} \nleqq \rho_d$ and, as pointed out in §2, it satisfies $c \leqq_O d \rightarrow \rho_c \leqq \rho_d . \phi < \theta \rightarrow \phi \leqq \theta$, so we can use 3.6 for this relation too. The only other property needed in the proof of 3.8 was transitivity of $\leqq$, which certainly holds. Hence uniqueness with respect to the relation $\leqq$ also fails in the majorizing hierarchies for $|d| \geqq \omega^2$. It can be seen by special arguments for the case of the function $\chi(a, b) = b$ that it also fails for $|d| \geqq \omega$.

4. **Classification of certain hierarchies below ω^2.** In the first part of this section we give a classification, in terms of the notion of ordinal recursion, of the functions ρ_d in the Kleene sub-recursive hierarchy for $|d| < \omega^2$. This part of our work has been carried out in collaboration with W. W. Tait.

Consider a primitive recursive well-ordering $\angle$ of the natural numbers in which 0 is the first element. Put

$$x \mathbin{\overline{\wedge}} a = \begin{cases} x & \text{if } x \angle a, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

A function $\phi(x)$ is said to be defined by *nested $\angle$-recursion* from $\zeta_1, \cdots, \zeta_m$ if it satisfies

$$\begin{aligned} \phi(0) &= k, \\ \phi(a+1) &= \gamma(a), \end{aligned} \tag{4.2}$$

where $\gamma(a)$ is built up by composition from the functions $\zeta_1, \cdots, \zeta_m$ and the function ϕ, but where every application of ϕ has the form $\phi(s \mathbin{\overline{\wedge}} (a+1))$. This is said to be an *ordinary*, or *unnested*, recursion if $\gamma(a)$ has the form $\tau(a, \phi(\sigma(a) \mathbin{\overline{\wedge}} (a+1)))$, where τ, σ are built up from $\zeta_1, \cdots, \zeta_m$ alone. Generalization of this notion to functions of several variables is fairly direct. (The notion of nesting corresponds to Péter's "eingeschachtelte" recursions of [12, §10].) A function is said to be definable by ordinary $\angle$-recursion (by nested $\angle$-recursion) if it is the end term of a sequence of functions each of which is obtained from preceding functions in the sequence either by one of the usual schemas for primitive recursion or by the schema (4.2), in one or more variables, for ordinary $\angle$-recursion (for nested $\angle$-recursion).

We deal here only with "standard" or "natural" well-orderings $\angle$, which notion is well understood at the very least for ordinals $\leqq \epsilon_0$. We might specifically take for these the orderings defined in [4, p. 361], or consider orderings satisfying certain minimal conditions, as in [15, 1.2]. The classification of (part of) the class of recursive functions by such orderings does not collapse at low ordinals, in contrast to [11; 13]. If α is the order type of $\angle$, we shall speak of (ordinary or nested) α-recursion. The ω^k-recursions thus correspond to the "k-fache" recursions of [12, §§11–12]; the nested ω^k-recursions correspond to what Axt calls k-recursive functions in [1, p. 93]. The main fact that we shall use for such standard α-recursions is the following proved by Tait in [15, Theorem 2], for $\alpha \geqq \omega$: if a function ϕ is definable by nested α-recursion then it is also definable by ordinary ω^α-recursion. (He has also shown in [15] that, when $\omega \cdot \alpha = \alpha$, the converse is also true.)

4.3. Theorem. *The functions ρ_d of the Kleene sub-recursive hierarchy are all nested ω^3-recursive, hence all ordinary ω^{ω^3}-recursive for $d \in O$, $|d| < \omega^2$.*

Proof. It is more convenient here to return to Kleene's definition of his hierarchy in [6] as consisting of functions $h_d(b, a)$ satisfying

$$\text{(1)}\qquad \begin{aligned} h_1(b, a) &= 0, \\ h_{2^d}(b, a) &= [b]^{h_d}(a) \qquad \text{for } d \neq 0, \\ h_{3\cdot 5^d}(b, a) &= h_{[d](b_{(1)})}(b_{(0)}, a). \end{aligned}$$

By the uniqueness result of Axt [1], it is sufficient to classify the functions h_d corresponding to the "natural" notations $|d| < \omega^2$, all others at these levels being primitive recursive in these particular functions. Thus to each m, k we associate $d_{m,k}$ with $|d_{m,k}| = \omega \cdot m + k$, $d_{m,k} <_O d_{m_1,k_1}$ if and only if $|d_{m,k}| < |d_{m_1,k_1}|$. We shall now define a sequence of functions $H_m(k, b, a)$ such that $h_{d_{m,k}}(b, a) = H_m(k, b, a)$ for all m, k.

$$\text{(2)}\qquad \begin{aligned} &\text{(i)} & H_0(0, b, a) &= 0, \\ &\text{(ii)} & H_0(k+1, b, a) &= [b]^{\lambda yx H_0(k,y,x)}(a), \end{aligned}$$

$$\text{(3)}\qquad \begin{aligned} &\text{(i)} & H_{m+1}(0, b, a) &= H_m(b_{(1)}, b_{(0)}, a), \\ &\text{(ii)} & H_{m+1}(k+1, b, a) &= [b]^{\lambda yx H_{m+1}(k,y,x)}(a). \end{aligned}$$

We next analyze the form of (2)(ii), (3)(ii). In general, the definition of $\phi(b, a) = [b]^\theta(a)$ from given binary θ can be put in the form

$$\text{(4)}\qquad \begin{aligned} \phi(0, a) &= \theta(a_{(0)}, a_{(1)}), \\ \phi(b+1, a) &= \chi\Big(b, a, \phi\Big(\tau_0(b), \tau_1\Big(a, b, \prod_{i \leq b} p_i^{\phi(i,a)}, \phi(b+1, [a/2])\Big)\Big)\Big), \end{aligned}$$

where $\tau_0(b) < b+1$, *and* τ_1 *is a function of* b *only when* $a=0$. This can be obtained from [6, p. 74]. Here χ, τ_0, τ_1 are certain primitive recursive functions. Assigning the ordinal $\omega \cdot b + a$ to (b, a) shows that this is a definition of $[b]^\theta(a)$ by nested ω^2-recursion from θ. Assigning the ordinal $\omega^2 \cdot k + \omega \cdot b + a$ to (k, b, a) in (2), and replacing the equation (2)(ii) by two equations obtained from (4) by substituting $\lambda yx H_0(k, y, x)$ for θ and $\lambda yx H_0(k+1, y, x)$ for ϕ shows that H_0 is nested ω^3-recursive. Similarly, H_{m+1} is nested ω^3-recursive in H_m. Hence induction and Tait's result shows that each H_m is ordinary ω^{ω^3}-recursive. It follows that the same is true of each function $d_{d_{m,k}} = \lambda yx H_m(k, y, x)$.

Axt has shown in [1, p. 99] that the converse to 4.3 is true, i.e., that every nested ω^3-recursive function is primitive recursive in one of the ρ_d for $|d| < \omega^2$, and in fact more generally for nested ω^{k+1}-recursions and $|d| < \omega^k$. It is clear how 4.3 can be extended to the classification of ρ_d for "natural" d with $|d| < \omega^k$. These various results constitute a partial answer to Kleene's problem P **237** in [6, p. 77].

4.4. THEOREM. *The functions* ρ_d *of the majorizing hierarchy associated with a function* $\chi(a, b)$ *are all primitive recursive in* χ *for* $d \in O$, $|d| < \omega^2$.

Proof. For any $d \in O$, $|d| < \omega^2$, there are only finitely many limit notations $3 \cdot 5^c \leqq_O d$. It suffices then to consider any sequence $3 \cdot 5^{c_1}, \cdots, 3 \cdot 5^{c_n}, \cdots$ (not necessarily primitive recursive) with $|3 \cdot 5^{c_n}| = \omega \cdot n$ and $3 \cdot 5^{c_n} <_O 3 \cdot 5^{c_{n+1}}$ and to show that for any $d <_O 3 \cdot 5^{c_n}$ for some n, we have ρ_d primitive recursive in χ. We regard this sequence as fixed throughout the following.

Define the following primitive recursive functions M, E (no relation to functions used in §3) by course-of-values recursion.

$$M(2^d) = M(d), \quad M(a) = a \quad otherwise;$$

$$E(2^d) = E(d) + 1, \quad E(a) = 0 \quad otherwise. \tag{1}$$

Thus if $d \in O$, $|d| < \omega^2$, $M(d)$ is the maximum limit notation $3 \cdot 5^c$ which is $\leqq_O d$, or 1 if there is no such. Further $d = M(d) \oplus (E(d))_O$, so $E(d)$ measures the excess of d over this notation. We shall also need an iteration of the function $\chi(a, b) + 1$,

$$\chi^*(0, a, b) = b, \qquad \chi^*(n+1, a, b) = \chi(a, \chi^*(n, a, b)) + 1. \tag{2}$$

χ^* is primitive recursive in χ.

Let $L_n = \{d : d <_O 3 \cdot 5^{c_n}\}$. We shall prove by induction on n that

(3) *There exists a function* ϕ_n, *primitive recursive in* χ, *such that for every* $d \in L_n$ *and every* a,

$$\rho_d(a) = \phi_n\left(a, d, \prod_{i \leqq a} p_i^{[c_1](i)}, \cdots, \prod_{i \leqq a} p_i^{[c_{n-1}](i)}\right).$$

The defining conditions for the majorizing hierarchy are $\rho_1(a) = 0$,

$$\rho_{2^d}(a) = \chi(a, \rho_d(a)) + 1, \qquad \rho_{3 \cdot 5^d}(a) = \max_{0 \leqq j \leqq a} \rho_{[d](j)}(a) + 1.$$

Using (1) and (2) we have for any $d\in O$, $|d| < \omega^2$,

$$\rho_d(a) = \chi^*(E(d), a, \rho_{M(d)}(a)). \tag{4}$$

Thus for the case $n=1$, we can take $\phi_1(a, d) = \chi^*(E(d), a, 0)$. Suppose now that (3) is true for n, and let us prove it for $n+1$. If $d\in L_{n+1}$, we have $d\in L_n \leftrightarrow M(d) \neq 3\cdot 5^{c_n}$. We can define the desired function ϕ_{n+1} by separating the cases $d\in L_n$, $d\in L_{n+1}-L_n$. Set

$$\psi_n(a, x_1, \cdots, x_{n-1}, y) = \max_{0\leqq j\leqq a} \phi_n(a, y_{(j)}, x_1, \cdots, x_{n-1}) + 1. \tag{5}$$

Then we define

$$\phi_{n+1}(a, d, x_1, \cdots, x_n) = \begin{cases} \phi_n(a, d, x_1, \cdots, x_{n-1}) & \text{if } M(d) \neq 3\cdot 5^{c_n}, \\ \chi^*(E(d), a, \psi_n(a, x_1, \cdots, x_{n-1}, x_n)) & \text{if } M(d) = 3\cdot 5^{c_n}. \end{cases} \tag{6}$$

Then by induction hypothesis ϕ_n and hence ψ_n and ϕ_{n+1} are primitive recursive in χ. To see that (3) continues to hold true for $n+1$, we need only consider the case $M(d) = 3\cdot 5^{c_n}$. By (4), it is sufficient to see that

$$\rho_{3\cdot 5^{c_n}}(a) = \psi_n\left(a, \prod_{i\leq a} p_i^{[c_1](i)}, \cdots, \prod_{i\leq a} p_i^{[c_{n-1}](i)}, \prod_{i\leq a} p_i^{[c_n](i)}\right). \tag{7}$$

But the right side here is just

$$\max_{0\leqq j\leqq a} \phi_n\left(a, [c_n](j), \prod_{i\leq a} p_i^{[c_1](i)}, \cdots, \prod_{i\leq a} p_i^{[c_{n-1}](i)}\right) + 1,$$

which, since each $[c_n](j) <_O 3\cdot 5^{c_n}$, i.e., $[c_n](j)\in L_n$, is by (3) for n the same as $\max_{0\leq j\leq a} \rho_{[c_n](j)}(a)+1$.

Thus (7) is proved and the induction is complete. Now for any particular $d\in O$, $|d| <\omega^2$, (3) gives the value of $\rho_d(a)$ as a function, primitive recursive in χ, of a, d and the values of the primitive recursive functions $[c_1], \cdots, [c_{n-1}]$ obtained from all limit notations which are $\leqq_O d$. This proves the theorem.

5. **Nonstandard extensions of hierarchies.** In this section we use the nonstandard extension O^* of O, defined in [3] (restricted here to primitive recursive fundamental sequences) to obtain an incompleteness result for hierarchies and to give some information on the structure of the set of recursive functions with respect to certain partial orderings.

We shall briefly describe some of the notions and results of [3] as adapted to the present situation. We put a set $A\in\Pi$ if it can be defined in the form $n\in A\leftrightarrow(\alpha)(Ex)R(n, \bar{\alpha}(x))$ with primitive recursive R, $A\in\Sigma\leftrightarrow\overline{A}\in\Pi$, and $A\in$H.A.$\leftrightarrow A\in\Pi\cap\Sigma$ (H.A. = hyperarithmetical). We put $d\in M$ if $C'(d)$ is simply ordered by $<$, $1 \in C'(d)$, $C'(d)$ contains with each x also each $y < x$, and if for all $x \in C'(d)$, either $x = 1$, $x = 2^{x_{(0)}}$ where $x_{(0)} \neq 0$, or $x = 3 \cdot 5^{x_{(2)}}$ where $[x_{(2)}](y) < [x_{(2)}](y+1)$ for all y. We put $d \in O^*$ if $d \in M$ and for all $A \in$ H.A., $A \cap C'(d) \neq \Lambda$ implies $A \cap C'(d)$ has a least element under $<$. It is shown in [3, §3], that $O \subseteq O^*$, $O^* \in \Sigma$. On the other hand, $O \in \Pi - \Sigma$. In fact for any $A \in \Pi$ we can find primitive recursive ξ such that $(x)[x \in A \leftrightarrow \xi(x) \in O]$; the proof of this can be directly adapted from Kleene's proof of the corresponding theorem for the usual definition of O in [8]. Hence we also have here the result of [3, 3.6] that for any $a \in O$ we can find $d \in O^* - O$ such that $a < d$. The argument of [3, 3.7] also served to show that for any such d, $P = O \cap C'(d)$ is a path through O with $P \in \Pi$. The only thing to check in that proof for the present O is that for each $c \in O$, $\{x : x \in O \,\&\, |x| < |c|\} \in$ H.A. This is true for the full O by Spector [14, p. 158]. However, the present O is in 1-1 correspondence with the intersection of an arithmetically defined set (similar to M above) with the full O, and this correspondence is easily used to carry the result over. Hence we obtain directly the existence, as in [3, 4.4] of $\aleph_0$ paths P through O such that $P \in \Pi$. Moreover, it is useful to note, just as in [3, 3.8], that for any such P there is a $d \in O^* - O$ with $P = O \cap C'(d)$.

In any strictly expanding hierarchy (3.1) we have a relation $\phi \leqq \psi$ defined by $(Ee)\phi \leqq_e \psi$; this has the property $\rho_d \leqq \rho_{2^d}$ and $\rho_{2^d} \not\leqq \rho_d$ for each $d \in O$. We need rather less of the conditions on a hierarchy of 3.1 for the developments of this section, but a little more on the relation $\leqq$. *Throughout the remainder of this section, we assume q_1 is any fixed Gödel-number of a recursive function and that ψ_1, ψ_2 are any fixed primitive recursive functions satisfying the conditions* 2.4(i), (ii). *We take ϕ, γ to be primitive recursive functions satisfying the conclusion of* 2.4 *and $\rho_d = \lambda x\{\phi(d)\}(x)$ for any d.*

5.1. Definition. *A relation $\ll$ between functions is said to conform with ψ_1, ψ_2 if $\ll$ is transitive and irreflexive and if*

(i) *whenever f is the Gödel-number of a recursive function then $\{f\} \ll \{\psi_1(f)\}$,*

(ii) *whenever $[e](n)$ is the Gödel-number of a recursive function for each n and $(n)(\{[e](n)\} \ll \{[e](n+1)\})$ then $\{[e](n)\} \ll \{\psi_2(e)\}$, and*

(iii) *the relation $\{e\} \ll \{f\}$ is a hyperarithmetical relation between e, f.*

We note, for applications, the following easily derived result.

5.2. Lemma. (i) *The relation $\subset$ conforms with any functions ψ_1, ψ_2 satisfying* 2.5(i), (ii).

(ii) *The relation $<$ (of majorizing) conforms with any functions ψ_1, ψ_2 satisfying* 2.6(i), (ii) *with respect to any given recursive χ.*

We now assume throughout the following that $\ll$ is any relation which conforms with the general ψ_1, ψ_2 we are considering here.

Just as is shown in [3, 5.2] we see that O^* is the intersection of all $X \in$ H.A. satisfying the following closure conditions:

(i) $1 \in X$;

(ii) if $c\in X$ then $2^c\in X$;

(iii) if $(n)\{[d](n)\in X \;\&\; [d](n)\prec[d](n+1)\}$ then $3\cdot 5^d\in X$.

This characterization of O^* permits us to make inductive proofs, in the usual style, that various hyperarithmetical properties hold for all $d\in O^*$. In this way, we easily obtain the following from 2.4 and 5.1.

5.3. LEMMA. (i) *For any $d\in O^*$, ρ_d is a recursive function.*

(ii) *For any c, $d\in O^*$, $c\prec d\rightarrow\rho_c\ll\rho_d$.*

Here 5.3(ii) generalizes the statements of §2 that, for the sub-recursive hierarchy $c<_O d\rightarrow\rho_c\subset\rho_d$, and for the majorizing hierarchy $c<_O d\rightarrow\rho_c\prec\rho_d$. These results now lead us directly to the following incompleteness theorem for certain paths in hierarchies (cf. [3, 2.5], for a corresponding incompleteness theorem for progressions of theories).

5.4. THEOREM. *For any path P through O, $P\in\Pi$, we can find a recursive function θ such that $\rho_c\ll\theta$, and hence θ not $\ll\rho_c$, for all $c\in P$.*

Proof. As we noted earlier, $P=O\cap C'(d)$ for a certain $d\in O^*-O$. We take $\theta=\rho_d$ and apply 5.3 and the transitivity and irreflexivity of $\ll$.

We shall now devote the remainder of the paper to a proof of an essentially new result, namely that there is a subset of O^* densely ordered by $\prec$. This, via 5.3(ii), thus gives us some information regarding the structure of $\ll$ on the set of recursive functions. We first need an extension of ordinal notation arithmetic to O^*. The reason for this will be seen in connection with 5.16–5.18 below.

We wish to introduce operations corresponding to addition, multiplication, and exponentiation of ordinals. We already have a $\oplus$ operation and, for uniformity, repeat the definition of this in 5.5(i) next. In order to apply a certain general result below (5.14) insuring the proper growth of these functions, we modify slightly the usual definitions of the other operations at the initial values.

5.5. DEFINITION. *$\oplus$, $\circ$, and $^\circ$, and ν_1, ν_2, ν_3 are chosen by 2.3 to be binary primitive recursive functions satisfying the following conditions for all a, d:*

(i) $a\oplus 1=a$, $a\oplus 2^d=2^{a\oplus d}(d\neq 0)$, $a\oplus 3\cdot 5^d=3\cdot 5^{\nu_1(a,d)}$ *where*

$$(n)\{[\nu_1(a, d)](n) = a \oplus [d](n)\};$$

(ii) $a\circ 1 = a$, $a\circ 2^d = (a\circ d)\oplus a$ $(d\neq 0)$, $a\circ 3\cdot 5^d = 3\cdot 5^{\nu_2(a,d)}$ *where* $(n)\{[\nu_2(a, d)](n)=a\circ [d](n)\}$;

(iii) $a^{\circ 1} = a$, $a^{\circ(2^d)} = (a^{\circ d})\circ a$ $(d\neq 0)$, $a^{\circ 3\cdot 5^d} = 3\cdot 5^{\nu_3(a,d)}$ *where* $(n)\{[\nu_3(a, d)](n)=a^{\circ([d])(n)}\}$.

More generally:

5.6. DEFINITION. *Let $\theta(a, b)$ be any primitive recursive function. Choose primitive recursive $\sigma(a, d)$, $\nu(a, d)$ by 2.3 satisfying:*

(i) $\sigma(a, 1)=a$;

(ii) $\sigma(a, 2^d) = \theta(\sigma(a, d), a)$ *for* $d \neq 0$;

(iii) $\sigma(a, 3 \cdot 5^d) = 3 \cdot 5^{\nu(a,d)}$ *where* $(n)\{[\nu(a, d)](n) = \sigma(a, [d](n))\}$.

We shall refer to σ *as being the function determined by notation recursion from* θ *via* ν.

Thus we see that, for suitable ν_1, ν_2, ν_3, if we set $\sigma_0(a, b) = 2^a$, $\sigma_1(a, b) = a \oplus b$, $\sigma_2(a, b) = a \circ b$, $\sigma_3(a, b) = a^{\circ b}$, then each σ_{i+1} is determined by notation recursion from σ_i via ν_{i+1}.

It is seen that $a \circ b$, $a^{\circ b}$ correspond respectively to the operations $\alpha(1+\beta)$ and $\alpha \cdot (1+\alpha)^\beta$ on ordinals; these are strictly increasing functions of β for $\alpha \geqq 1$.

We wish to show that O^* is closed under the operations $\oplus, \circ, {}^\circ$ (for $a \succ 1$), and that these have various properties on O^*. We might expect an inductive proof on O^* of these properties. However, closure of O^* under such operations is not a hyperarithmetical property. It is necessary therefore to generally prove something stronger.

5.7. Definition. *Let* X *be any set,* $\theta(a, b)$ *any function. We write* $\mathrm{Cl}(X)$ *if the following conditions* (i)–(v) *hold*:

(i) $1 \in X$;

(ii) $d \in X \to 2^d \in X$;

(iii) $(n)\{[d](n) \in X \ \&\ [d](n) \prec [d](n+1)\} \to 3 \cdot 5^d \in X$;

(iv) $d \in X \ \&\ c \prec d \to c \in X$;

(v) $d \in X \to 1 \preceq d$.

If, in addition, the following conditions (vi), (vii) *or* (vi), (viii) *hold, we write* $\mathrm{Cl}^\theta(X)$ *or* $\mathrm{Cl}_1^\theta(X)$, *respectively*:

(vi) $a, b \in X \ \&\ 1 \prec a \to \theta(a, b) \in X$;

(vii) $a, c \in X \ \&\ 1 \prec a \ \&\ b \prec c \to \theta(a, b) \prec \theta(a, c)$;

(viii) $a, b \in X \ \&\ 1 \prec a \ \&\ 1 \prec b \to a \prec \theta(a, b)$.

As we remarked earlier, it has been proved in [3] that O^* is the intersection of all sets $X \in H.A.$ satisfying (i)–(iii). Since the set M is arithmetical, each $X \cap M \in H.A.$, and it is easily seen that $\mathrm{Cl}(X \cap M)$. Hence we have

$$O^* = \bigcap X[\mathrm{Cl}(X) \ \&\ X \in \mathrm{H.A.}]. \tag{5.8}$$

The following is easily obtained from 5.7.

5.9. Lemma. *Let* Γ *be any collection of sets,* $\theta(a, b)$ *any function. If for each* $X \in \Gamma$ *we have* $\mathrm{Cl}(X)$ *then* $\mathrm{Cl}(\bigcap X[X \in \Gamma])$. *The same holds true for each of* "Cl^θ," "Cl_1^θ" *instead of* "Cl."

5.10. Definition. *Let* X *be any set,* $\sigma(a, b)$ *any function. We put*

$$X/\sigma = \{d : d \in X \ \&\ (a)(b)(c)[a \in X \ \&\ 1 \prec a \ \&\ c \preceq d \to \sigma(a, c) \in X \ \&\ (b \prec c \to \sigma(a, b) \prec \sigma(a, c))]\}.$$

It is seen that if $X \in$ H.A. then $X/\sigma \in$ H.A. Actually, we need a little more. A set $X \in$ H.A. if and only if there exist e_0, e_1 such that for all a

$$a \in X \leftrightarrow (\alpha)(Ey)T_1^{\alpha}(e_0, a, y) \leftrightarrow (E\alpha)(y)\overline{T}_1^{\alpha}(e_1, a, y).$$

For any number f, let $L_f = \{x: (\alpha)(Ey)T_1^{\alpha}(f, x, y)\}$. Put $e \in$ h.a. if $L_{e_{(0)}} = \overline{L}_{e_{(1)}}$. For each e, put $K_e = L_{e_{(0)}}$. Thus if $e \in$ h.a., $K_e \in$ H.A.; conversely, for any $X \in$ H.A. there exists an $e \in$ h.a. with $X = K_e$.

5.11. Lemma. *With each recursive function $\sigma(a, b)$ is associated a recursive function $\sigma^*(e)$ such that whenever $e \in$ h.a. then $\sigma^*(e) \in$ h.a. and $K_{\sigma^*(e)} = K_e/\sigma$.*

The proof of this is by standard methods of analytic hierarchy theory; cf., for example, Kleene's [5] or [7].

5.12. Lemma. *Suppose θ is primitive recursive and that σ is determined by notation recursion* (5.9) *from θ via ν. Then for any set X,* $\mathrm{Cl}_1^{\theta}(X)$ *implies* $\mathrm{Cl}(X/\sigma)$.

Proof. We must check conditions 5.7(i)–(v) for X/σ.

(i) $1 \in X/\sigma$ since for any $a \in X$, $\sigma(a, 1) = a \in X$.

(ii) Suppose $d \in X/\sigma$. To show $2^d \in X/\sigma$, consider any $a \in X$, $1 \prec a$ and any $c \preceq 2^d$. If $c \prec d$ then $\sigma(a, c) \in X$ by hypothesis. Otherwise $c = 2^d$. (We can assume $d \neq 0$.) But $\sigma(a, 2^d) = \theta(\sigma(a, d), a) \in X$, by 5.7(vi), since $\mathrm{Cl}^{\theta}(X)$ and $\sigma(a, d) \in X$ and $1 \prec a = \sigma(a, 1) \preceq \sigma(a, d)$. Note also $\sigma(a, d) \prec \sigma(a, 2^d)$ by 5.7(viii) for X, θ. Thus if we have $b \prec c \preceq 2^d$, either $b \prec c \preceq d$, in which case $\sigma(a, b) \prec \sigma(a, c)$ from $d \in X/\sigma$, or $b \preceq d$, $c = 2^d$, in which case $\sigma(a, b) \preceq \sigma(a, d) \prec \sigma(a, 2^d)$, hence $\sigma(a, b) \prec \sigma(a, 2^d)$.

(iii) Suppose that for each n, $[d](n) \in X/\sigma$ and $[d](n) \prec [d](n+1)$. To show $3 \cdot 5^d \in X/\sigma$, consider any $a \in X$, $1 \prec a$ and any $c \preceq 3 \cdot 5^d$. If $c \prec 3 \cdot 5^d$ then $c \preceq [d](n)$ for some n and $\sigma(a, c) \in X$ & $(b)(b \prec c \rightarrow \sigma(a, b) \prec \sigma(a, c))$ by $[d](n) \in X/\sigma$. Hence we may assume $c = 3 \cdot 5^d$. We have $\sigma(a, 3 \cdot 5^d) = 3 \cdot 5^{\nu(a,d)}$, where $[\nu(a, d)](n) = \sigma(a, [d](n))$ for every n. Since each $[d](n)$, $[d](n+1) \in X/\sigma$ it follows that $[\nu(a, d)](n) \in X$, $[\nu(a, d)](n) \prec [\nu(a, d)](n+1)$ for all n. But $\mathrm{Cl}(X)$, so $\sigma(a, 3 \cdot 5^d) = 3 \cdot 5^{\nu(a,d)} \in X$. Moreover each $\sigma(a, [d](n)) \prec \sigma(a, 3 \cdot 5^d)$. Thus if $b \prec 3 \cdot 5^d$, $b \preceq [d](n)$ for some n, and hence by $[d](n) \in X/\sigma$ and transitivity of $\prec$, $\sigma(a, b) \prec \sigma(a, 3 \cdot 5^d)$.

(iv) It is clear that if $d \in X/\sigma$ and $d_1 \prec d$ then $d_1 \in X/\sigma$, since $\mathrm{Cl}(X)$.

(v) is obvious from $X/\sigma \subseteq X$.

Since σ is not in general associative (not even in the simplest case, $\oplus$), we are not able to see that X/σ is closed under σ. However, we shall see in 5.14 below that by forming a suitable intersection, we can pass from any $X \in$ H.A. such that $\mathrm{Cl}_1^{\theta}(X)$ to a subset $Y \in$ H.A. such that $\mathrm{Cl}^{\sigma}(y)$. First we need the following effectiveness condition for such an intersection.

5.13. Lemma. *There is a primitive recursive function ι such that whenever*

$\{f\}$ is a recursive function with $\{f\}(n) \in h.a.$ for all n then $\iota(f) \in h.a.$ and

$$K_{\iota(f)} = \bigcap K_{\{f\}(n)} \ (n = 0, 1, 2, \cdots).$$

The proof of this is obtained by standard methods [5] for effectively converting the two equivalent conditions for a to be in the intersection, $(x)(\alpha)(Ey)T_1^\alpha((\{f\}(x))_{(0)}, a, y)$ and $(x)(E\alpha)(y)\overline{T}_1^\alpha((\{f\}(x))_{(1)}, a, y)$ to the forms $a \in L_{e_0}$, $a \in \overline{L}_{e_1}$ for suitable e_0, e_1.

5.14. LEMMA. *Suppose θ is primitive recursive and that σ is defined by notation recursion from θ via σ. Suppose we have a recursive function τ such that for any e, if $e \in h.a.$ and $\mathrm{Cl}(K_e)$ then $\tau(e) \in h.a.$, $\mathrm{Cl}_1^\theta(K_{\tau(e)})$ and $K_{\tau(e)} \subseteq K_e$. Then*:

(i) *we can find a recursive function π such that for any $e \in$ h.a. if $\mathrm{Cl}(K_e)$ then we have $\pi(e) \in h.a.$, $\mathrm{Cl}^\sigma(K_{\pi(e)})$ and $K_{\pi(e)} \subseteq K_e$;*

(ii) $\mathrm{Cl}^\sigma(O^*)$.

Proof. Define $\psi(e, 0) = e$, $\psi(e, n+1) = \sigma^*(\tau(\psi(e, n)))$, where σ^* is the function of 5.11. Let $\{g\}(e, n) = \psi(e, n)$ all n. We define $\pi(e) = \iota(S_1^1(g, e))$. To prove that this satisfies (i), consider any $e \in$ h.a. such that $\mathrm{Cl}(K_e)$. Let $f = S_1^1(g, e)$, $f_n = \{f\}(n)$ for all n. Then we have

$$K_{f_0} = K_e, \qquad K_{f_{n+1}} = K_{\sigma^*(\tau(f_n))} = K_{\tau(f_n)}/\sigma \qquad \text{for all } n.$$

It is seen by induction that $f_n \in$ h.a. for all n; hence $\pi(e) = \iota(f)$ is h.a. by 5.13. We prove by induction on n that $\mathrm{Cl}(K_{f_n})$. Suppose this for n. Then $\mathrm{Cl}_1^\theta(K_{\tau(f_n)})$, and hence $\mathrm{Cl}(K_{\tau(f_n)}/\sigma)$ by 5.12, thus proving it for $n+1$.

By 5.13, $K_{\pi(e)} = K_{\iota(f)} = \bigcap K_{f_n}$ $(n = 0, 1, 2, \cdots)$, so that $\mathrm{Cl}(K_{\iota(f)})$ by 5.8. It remains only to check 5.6(vi), (vii) for $K_{\iota(f)}$, σ. Let $a, b, c \in K_{\iota(f)}$, $1 \prec a$. Then for each n, we have $a, b \in K_{f_{n+1}}$, hence $\sigma(a, b) \in K_{\tau(f_n)} \subseteq K_{f_n}$; thus $\sigma(a, b) \in K_{\iota(f)}$. Already from $a, b \in K_{f_1}$ we can conclude $b \prec c \rightarrow \sigma(a, b) \prec \sigma(a, d)$. Thus $\mathrm{Cl}^\sigma(K_{\pi(e)})$ is proved.

To prove part (ii) of our theorem, we use part (i) and 5.9 to establish the series of inclusions

$$\begin{aligned} O^* = \bigcap K_e[e \in \text{h.a.} \ \& \ \mathrm{Cl}(K_e)] &\subseteq \bigcap K_f[f \in \text{h.a.} \ \& \ \mathrm{Cl}^\sigma(K_f)] \\ &\subseteq \bigcap K_{\pi(e)}[e \in \text{h.a.} \ \& \ \mathrm{Cl}(K_e)] \\ &\subseteq \bigcap K_e[e \in \text{h.a.} \ \& \ \mathrm{Cl}(K_e)] = O^*, \end{aligned}$$

to conclude by 5.9 that $\mathrm{Cl}^\sigma(O^*)$.

5.15. THEOREM. *Let $\sigma_1(a, b) = a \oplus b$, $\sigma_2(a, b) = a \circ b$, $\sigma_3(a, b) = a^{\circ b}$. Then for $i = 1, 2, 3$, $\mathrm{Cl}^{\sigma_i}(O^*)$.*

Proof. σ_1 is defined by notation recursion from $\theta(a, b) = 2^a$ via ν_1. σ_2 is defined by notation recursion from σ_1 via ν_2, and σ_3 is defined by notation recursion from σ_2 via ν_3. Clearly for any X such that $\mathrm{Cl}(X)$ we have $\mathrm{Cl}_1^\theta(X)$. Hence if we take $\tau(e) = e$, we can find by 5.14 recursive π_1 such that whenever

$e \in$ h.a. and $\mathrm{Cl}(K_e)$ then $\pi_1(e) \in$ h.a., $\mathrm{Cl}^{\sigma_1}(K_{\pi_1(e)})$ and $K_{\pi_1(e)} \subseteq K_e$, and then also $\mathrm{Cl}^{\sigma_1}(O^*)$. Noting that $\mathrm{Cl}^{\sigma_1}(K_{\pi_1(e)})$, we see that for $a, b \in K_{\pi_1(e)}$, if $1 \prec a$ and $1 \prec b$ then $a = \sigma_1(a, 1) \prec \sigma_1(a, b)$, hence $\mathrm{Cl}_1^{\sigma_1}(K_{\pi_1(e)})$. Hence we can apply 5.14 to $\sigma_1, \pi_1, \sigma_2$ instead of θ, τ, σ, to obtain π_2 satisfying 5.14(i) for σ_2 and thence 5.14(ii). Repeating this argument gives us the desired result for σ_3 as well.

For any $d \in O^* - O$, $C'(d) - O$ has no least element; for any element of $C'(d)$ has one of the forms 1, 2^b where $b \in C'(d)$, or $3 \cdot 5^b$ where $(n)\{[b](n) \in C'(d) \;\&\; [b](n) \prec [b](n+1)\}$. Hence we can find an infinite sequence $c_0, c_1, \cdots, c_n, \cdots$ such that

$$(5.16) \qquad \begin{array}{l} \text{(i) } c_0, c_1, \cdots, c_n, \cdots \in O^*, \\ \text{(ii) } \textit{for each } n,\ c_n \succ c_{n+1} \oplus 1_O. \end{array}$$

In other words we have a subset of order type ω^* (under $\prec$) in O. Our construction of a densely ordered subset of O^* is based on finding a subset of O^* whose ordering is of order type 2^{ω^*}, which is dense. We assume the sequence of c_n's is fixed throughout the following.

5.17. Definition. *We denote by Sq the set of all infinite sequences ξ such that $\xi_k = 0$ or $\xi_k = 1$ for all $k = 0, 1, 2, \cdots$, and such that $\xi_k = 1$ for at least one but only finitely many k. For $\xi, \eta \in Sq$ we put $\xi \angle \eta$ if*

$$(En)((k)(k < n \rightarrow \xi_k = \eta_k) \;\&\; \xi_n < \eta_n).$$

For any a and any $\xi \in Sq$, if $k_0 < \cdots < k_n$ are all the values k such that $\xi_k = 1$ we put

$$\textstyle\sum_\xi (a) = a^{\circ c_{k_0}} \oplus (a^{\circ c_{k_1}} \oplus (\cdots \oplus (a^{\circ c_{k_{n-1}}} \oplus a^{\circ c_{k_n}}) \cdots))$$

(or simply $a^{\circ c_{k_0}}$ if $n = 0$). Let $\sum^*(a)$ be the set of all values $\sum_\xi(a)$ for $\xi \in Sq$.

It is clear that $\angle$ is a dense ordering of Sq with no first or last element.

5.18. Lemma. *For any $a \in O^*$, $1 \prec a$, we have $\sum^*(a) \subseteq O^*$. Moreover, for any $\xi, \eta \in Sq$, $\xi \angle \eta \leftrightarrow \sum_\xi(a) \prec \sum_\eta(a)$.*

Proof. The first part of the statement is immediate from 5.15. Consider any $\xi, \eta \in Sq$ with $\xi \angle \eta$. Let n be the least number with $\xi_n \neq \eta_n$, hence $\xi_n = 0$, $\eta_n = 1$. Let $k_0 < \cdots < k_{r-1}$ be all $k < n$ where $\xi_k = 1$, equivalently where $\eta_k = 1$. Let $l_0 < \cdots < l_s$ be all $l \geqq n$ where $\eta_l = 1$, starting with $l_0 = n$. Let $m_0 < \cdots < m_{t-1}$ be all $m \geqq n$ where $\xi_m = 1$; hence $m_0 > n$. We write $f_i = a^{\circ c_i}$. We shall write all sums of the form $\sum_\xi(a)$ without parentheses, with the understanding that association is always to the right. Hence we have

$$(1) \qquad \begin{array}{l} \sum_\eta (a) = f_{k_0} \oplus \cdots \oplus f_{k_{r-1}} \oplus f_{l_0} \oplus \cdots \oplus f_{l_s}, \\ \sum_\xi (a) = f_{k_0} \oplus \cdots \oplus f_{k_{r-1}} \oplus f_{m_0} \oplus \cdots \oplus f_{m_{t-1}}. \end{array}$$

If $r \neq 0$, $1 \prec a \prec a^{\circ c_{k_0}} \prec f_{k_0} \oplus \cdots \oplus f_{k_{r-1}}$, since for $1 \prec b$, $1 \prec d$, we have $b \prec b^{\circ d}$ and $b \prec b \oplus d$. Since also $1 \prec f_{l_0} \oplus \cdots \oplus f_{l_s}$ by the same argument, we see by

5.15 that $\sum_\xi(a) \prec \sum_\eta(a)$ when $t=0$. On the other hand, if $t\neq 0$, it suffices by this to prove

(2) $$f_{m_0} \oplus \cdots \oplus f_{m_{t-1}} \prec f_{l_0} \oplus \cdots \oplus f_{l_s}.$$

We first show

(3) $$f_{m_0} \oplus \cdots \oplus f_{m_{t-1}} \prec a^{\circ(c_{m_0}\oplus 1_O)}.$$

This is proved by induction on t, for $t\geqq 1$. For $t=1$, this simply says $a^{\circ c_{m_0}} \prec a^{\circ(c_{m_0}\oplus 1_O)}$, which is clear. Suppose for $t-1\geqq 1$. Then $f_{m_1}\oplus \cdots \oplus f_{m_{t-1}} \prec a^{\circ(c_{m_1}\oplus 1_O)} \prec a^{\circ c_{m_0}}$, since $c_{m_1}\oplus 1_O \prec c_{m_0}$ by 5.16(ii). Hence $f_{m_0}\oplus \cdots \oplus f_{m_{t-1}} \prec f_{m_0}\oplus a^{\circ c_{m_0}} = a^{\circ c_{m_0}}\oplus a^{\circ c_{m_0}} = a^{\circ c_{m_0}} \circ 1_O \prec a^{\circ c_{m_0}} \circ a = a^{\circ(c_{m_0}\oplus 1_O)}$. Thus (3) is proved. Now since $l_0 < m_0$, we have $c_{m_0}\oplus 1_O \prec c_{l_0}$ by 5.16(ii), hence $a^{\circ(c_{m_0}\oplus 1_O)} \prec a^{\circ c_{l_0}} = f_{l_0} \preceq f_{l_0}\oplus \cdots \oplus f_{l_s}$. Thus (2) is proved, and we have now the proof that $\xi \angle \eta \rightarrow \sum_\xi(a) \prec \sum_\eta(a)$. Since $\angle$ is a simple ordering and $\prec$ is a partial ordering on O^*, the equivalence follows immediately.

5.19. THEOREM. *There exists a set* Δ *of recursive functions densely ordered by* $\ll$.

Proof. Pick any $a\in O^*$, $1 \prec a$. By 5.18, $\sum^*(a)$ is densely ordered without first or last element. By 5.3(ii), $d_1, d_2 \in \sum^*(a)$ & $d_1 \prec d_2 \rightarrow \rho_{d_1} \ll \rho_{d_2}$. The theorem follows immediately from this.

This result can be obtained for special cases of the majorizing relation very simply. For example, for the function $\chi(a, b)=b$ we can take the set of functions θ_r for r rational, $0<r<1$, where $\theta_r(n) = [r\cdot n]$ (greatest integer function), so that $\theta_r \prec \theta_s$ whenever $r<s$; similarly for the function $\chi(a, b) = (a+1)\cdot b$ we take the functions $\theta_r(n) = [r^n]$, so that $n[r^n] < [s^n]$ for sufficiently large n. However, we have seen no way of obtaining the result directly for the case of arbitrary recursive χ with the majorizing relationship. Neither have we seen a way of obtaining 5.19 for the relation $\subset$ without an excursion through nonstandard extensions of hierarchies.

Actually, a somewhat stronger statement than 5.19 can be made, but it is one which is formulated in terms of hierarchies. Let $|a|$ be the order type of $C(a)$ for $a\in O^*$. One can prove by induction on $b\in O^*$ that (for any given $a\in O^*$), $(x)[x \prec a\oplus b \rightarrow x \prec a \vee (Ey)(y\prec b \;\&\; x = a\oplus y)]$. Thus it is seen that $|a\oplus b| = |a| \oplus |b|$. Now in 5.18, we do not have that $\sum_\xi(a)\oplus a \prec \sum_\eta(a)$ for any $\xi \angle \eta$, because of the problem of association. However, it is seen that (by first considering ζ with $\xi \angle \zeta \angle \eta$), we have $|\sum_\xi(a)| + |a| < |\sum_\eta(a)|$. Thus any two elements d_1, d_2 of $\sum^*(a)$ with $d_1 \prec d_2$ have a distance greater then $|a|$ between them. In particular, if we choose $a\in O^* - O$, we have $|a| > \omega_1$. Loosely stated, the functions ρ_d associated with $d\in O^*(a)$ are densely ordered by a relation $\ll$ which, when it holds between ρ_{d_1}, ρ_{d_2} implies that there is a sequence of functions ρ_c obtained from a path through O in the subrecursive hierarchy starting with the function $\theta = \rho_{d_1}$, all of which are $\subset \rho_{d_2}$.

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