ON THE DIFFERENTIABILITY OF THE SOLUTIONS OF QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS(1)

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Introduction. In our thesis [2, Appendix, pp. 106–118] (cf. also [3]) we proved the infinite differentiability of the solutions of quasilinear partial differential equations of the form:

$$(1)\qquad \sum_{j=1}^{\mu} a_j(x, N_1(D)f, \cdots, N_\nu(D)f)M_j(D)f = g(x, N_1(D)f, \cdots, N_\nu(D)f).$$

Our hypotheses were: (1) The differential operator

$$\sum_{j=1}^{\mu} a_j(x_0, N_1(D)f(x_0), \cdots, N_\nu(D)f(x_0))M_j(D)$$

is, for every x_0, hypoelliptic and stronger than each $M_j(D)$. (2) Each $N_k(D)$ is strictly weaker than all $M_j(D)$ together (cf. [2, p. 116, d]). (3) The functions $a_j(x, t_1, \cdots, t_\nu)$ and $g(x, t_1, \cdots, t_\nu)$ are infinitely differentiable. In this paper we want to outline a different proof which is more elementary than the original one, in the sense that it utilizes neither the Sobolev nor the Gagliardo-Nirenberg estimates but only rather straightforward estimates obtained from the Fourier transformation in L^2. A severe disadvantage, on the other hand, is the fact that it requires rather strong a priori differentiability of the solutions.

1. **The Schauder algebra.** Let s be any real number. Put

$$\|f\|_s = \left(\int (1 + |\xi|)^{2s} |\hat{f}(\xi)|^2 d\xi\right)^{1/2}, \qquad f \in C_c^\infty$$

($\hat{f}(\xi)$ is the Fourier transform of $f=f(x)$, C_c^∞ is the set of infinitely differentiable functions with compact supports) and let H^s be the completion of C_c^∞ in this norm.

LEMMA 1. *If $s>n/2$, then there exists a constant K such that*

$$(2)\qquad \|fg\|_s \leq K\|f\|_s\|g\|_s, \qquad f \in C_c^\infty,\ g \in C_c^\infty.$$

Proof. Put $h=fg$. Now $\hat{h}(\xi)=\int\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta$. Hence

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$$\|h\|_s^2 = \int\int (1 + |\xi|)^{2s} \hat{f}(\xi - \eta)\hat{g}(\eta)(\hat{h}(\xi))^* d\xi d\eta. \tag{3}$$

By Schwarz's inequality, we then obtain

$$\begin{aligned}\|h\|_s^2 \leqq &\left(\int (1 + |\xi - \eta|)^{2s} |\hat{f}(\xi - \eta)|^2 d\xi\right)^{1/2} \\ &\times \left(\int (1 + |\eta|)^{2s} |\hat{g}(\eta)|^2 d\eta\right)^{1/2} \\ &\times \left(\int\int \left(\frac{1 + |\xi|}{(1 + |\xi - \eta|)(1 + |\eta|)}\right)^{2s} (1 + |\xi|)^{2s} |\hat{h}(\xi)|^2 d\xi d\eta\right)^{1/2}.\end{aligned} \tag{4}$$

The first two factors on the right hand side of (4) are $\|f\|_s$ and $\|g\|_s$ respectively. Now

$$\begin{aligned}\left(\frac{1 + |\xi|}{(1 + |\xi - \eta|)(1 + |\eta|)}\right)^{2s} &\leqq \left(\frac{(1 + |\xi - \eta|) + (1 + |\eta|)}{(1 + |\xi - \eta|)(1 + |\eta|)}\right)^{2s} \\ &= \left(\frac{1}{1 + |\xi - \eta|} + \frac{1}{1 + |\eta|}\right)^{2s} \\ &\leqq 2^{2s-1}\left(\frac{1}{(1 + |\xi - \eta|)^{2s}} + \frac{1}{(1 + |\eta|)^{2s}}\right).\end{aligned}$$

It follows that the square of the third factor is

$$\leqq 2^{2s-1} \int \left(\int \frac{d\eta}{(1 + |\xi - \eta|)^{2s}} + \int \frac{d\eta}{(1 + |\eta|)^{2s}}\right)(1 + |\xi|)^{2s} |\hat{h}(\xi)|^2 d\xi$$

$$= 2^{2s} \int \frac{d\eta}{(1 + |\eta|)^{2s}} \|h\|_s^2 = K^2 \|h\|_s^2$$

with $K = 2^s(\int (1 + |\eta|)^{-2s} d\eta)^{1/2}$, which is finite in view of the assumption that $s > n/2$. Hence $\|h\|_s^2 \leqq \|f\|_s \|g\|_s K \|h\|_s$, and (2) follows.

COROLLARY (SCHAUDER). *If $s > n/2$, then H^s is a Banach algebra (the Schauder algebra).*

REMARK. In the above reasoning one can as well replace $1 + |\xi|$ by an arbitrary "weight function" $E(\xi)$ such that

$$\frac{E(\xi)}{E(\xi - \eta)E(\eta)} \leqq c\left(\frac{1}{(1 + |\xi - \eta|)^\delta} + \frac{1}{(1 + |\eta|)^\delta}\right) \tag{5}$$

for some $\delta > 0$. In particular, (5) is fulfilled (Malgrange) if $E(\xi) = 1 + |P(\xi)|$ where $P(\xi)$ is a hypoelliptic polynomial. This can easily be seen by making use of Leibniz's formula in its "tensorial" form [1, p. 292].

LEMMA 2. *If $s>n/2+1$, then there exists a constant K' such that*

$$\|fg\|_s \leqq \sup |g| \, \|f\|_s + K'\|g\|_s\|f\|_{s-1}, \qquad f \in C_c^\infty,\ g \in C_c^\infty. \tag{6}$$

Proof. Write (3) as

$$\|h\|_s^2 = \iint (1 + |\xi|)^s(1 + |\xi - \eta|)^s \hat{f}(\xi - \eta)\hat{g}(\eta)(\hat{h}(\xi))^* d\xi d\eta$$

$$+ \iint (1 + |\xi|)^s((1 + |\xi|)^s - (1 + |\xi - \eta|)^s)\hat{f}(\xi - \eta)\hat{g}(\eta)(\hat{h}(\xi))^* d\xi d\eta$$

$$= \mathrm{I} + \mathrm{II}.$$

Since $\mathrm{I} = (g(1+|D|)^s f,\ (1+|D|)^s h)$, we obtain

$$|\mathrm{I}| \leqq \|g(1 + |D|)^s f\|_0 \|(1 + |D|)^s h\|_0 \leqq \sup |g| \, \|f\|_s \|h\|_s,$$

where $|D|$ stands for multiplication of the Fourier transform by $|\xi|$. It remains to estimate II. We observe that

$$|(1 + |\xi|)^s - (1 + |\xi - \eta|)^s| \leqq s((1 + |\xi|)^{s-1} + (1 + |\xi - \eta|)^{s-1})|\eta|.$$

Hence we have to consider:

$$\mathrm{II}_a = s\iint (1+|\xi|)^s(1+|\xi - \eta|)^{s-1}(1+|\eta|)|\hat{f}(\xi - \eta)| \, |\hat{g}(\eta)| \, |\hat{h}(\xi)| \, d\xi d\eta$$

and

$$\mathrm{II}_b = s\iint (1 + |\xi|)^s(1 + |\xi|)^{s-1}(1 + |\eta|)|\hat{f}(\xi - \eta)| \, |\hat{g}(\eta)| \, |\hat{h}(\xi)| \, d\xi d\eta$$

where

$$|\mathrm{II}| \leqq |\mathrm{II}_a| + |\mathrm{II}_b|.$$

As in the proof of Lemma 1, we obtain by Schwarz's inequality:

$$|\mathrm{II}_a| \leqq s\left(\int (1 + |\xi - \eta|)^{2(s-1)}|\hat{f}(\xi - \eta)|^2 d\xi\right)^{1/2}$$

$$\times \left(\int (1 + |\eta|)^{2s}|\hat{g}(\eta)|^2 d\eta\right)^{1/2}$$

$$\times \left(\int \frac{d\eta}{(1 + |\eta|)^{2(s-1)}}\right)^{1/2} \times \left(\int (1 + |\xi|)^{2s}|\hat{h}(\xi)|^2 d\xi\right)^{1/2}$$

$$= s\left(\int \frac{d\eta}{(1 + |\eta|)^{2(s-1)}}\right)^{1/2} \|f\|_{s-1}\|g\|_s\|h\|_s$$

and

$$
\begin{aligned}
|\,\mathrm{II}_b\,| &\leqq s\left(\int (1+|\xi-\eta|)^{2(s-1)}|\hat{f}(\xi-\eta)|^2 d\xi\right)^{1/2} \\
&\times\left(\int (1+|\eta|)^{2s}|\hat{g}(\eta)|^2 d\eta\right)^{1/2} \\
&\times\left(\iint\left(\frac{1+|\xi|}{(1+|\xi-\eta|)(1+|\eta|)}\right)^{2(s-1)}(1+|\xi|)^{2s}|\hat{h}(\xi)|^2 d\xi d\eta\right)^{1/2} \\
&\leqq s2^{s-1}\left(\int \frac{d\eta}{(1+|\eta|)^{2(s-1)}}\right)^{1/2}\|f\|_{s-1}\|g\|_s\|h\|_s.
\end{aligned}
$$

Putting together the estimates obtained for I, II_a, II_b, clearly (6) follows with

$$K' = s(2^{s-1}+1)\left(\int \frac{d\eta}{(1+|\eta|)^{2(s-1)}}\right)^{1/2}.$$

2. **The linear case.** We consider now (cf. [4]) differential operators of the form:

$$P = P(x, D) = \sum_{j=1}^{\mu} a_j(x) M_j(D) \tag{7}$$

where we assume that (1) $M = M(D) = P(0, D)$ is stronger than each M_j, (2) $a_j \in H^s$ for some $s > n/2+1$ and the quantity

$$\zeta = \sum_{j=1}^{\mu} \sup |a_j(x) - a_j(0)|$$

is less than a positive constant ζ_0 to be determined, depending on M and M_j only. We claim that the following inequality holds:

$$\|Mf\|_s \leqq C(\|Pf\|_s + \|Mf\|_{s-1}), \qquad f \in C_U^\infty, \tag{8}$$

where C_U^∞ is the subset of C^∞ of functions f in C^∞ such that the support of f is contained in the unit sphere U. Put

$$b_j(x) = w(x)(a_j(x) - a_j(0))$$

where w is in C_c^∞ and equals 1 in a neighborhood of U. By the triangle inequality we then obtain

$$\|Mf\|_s \leqq \|Pf\|_s + \sum \|b_j M_j f\|_s$$

and, by Lemma 2,

$$\leqq \|Pf\|_s + \sum \sup |b_j|\,\|M_j f\|_s + K'\|b_j\|_s\|M_j f\|_{s-1}.$$

Now there exist (cf. [2, Proposition 2, p. 43]) constants γ, γ', γ'', of which γ does not depend on s, such that

$$\|M_j f\|_s \leqq \gamma \|Mf\|_s + \gamma' \|Mf\|_{s-1},$$
$$\|M_j f\|_{s-1} \leqq \gamma'' \|Mf\|_{s-1}, \qquad f \in C_U^\infty.$$

Hence

$$\|Mf\|_s \leqq \|Pf\|_s + \zeta\gamma\|Mf\|_s + \zeta\gamma'\|Mf\|_{s-1} + Z\gamma''K'\|Mf\|_{s-1},$$

with $Z = \sum \|b_j\|_s$, so that (8) follows if we choose $\zeta_0 = 1/\gamma$.

From (8) we now obtain, as usual, the following

THEOREM 1. *Suppose that* (1) *and* (2) *hold and that moreover* $a_j \in H^{s+1}$ *with* $s > n/2+1$. *Suppose that the support of* f *is contained in* U *and that* $Mf \in H^s$ *and* $Pf \in H^{s+1}$. *Then* $Mf \in H^{s+1}$.

Proof. Apply (8) to $(f_h - f)/|h|$ where $f_h = f_h(x) = f(x+h)$. In view of Lemma 1 the right hand side will remain bounded as h tends to 0 and hence also the left hand side. A familiar weak compactness argument then gives $Mf \in H^{s+1}$.

Assume now that M is hypoelliptic. Then there is a number $d > 0$ such that

$$|M_j^\alpha(\xi)| \leqq C(1 + |\xi|)^{-d}(1 + |M(\xi)|), \qquad \alpha \neq 0,$$

so that $Mf \in H^s$ implies $M_j^\alpha f \in H^{s+d}$, $\alpha \neq 0$, provided f has compact support. Theorem 1 can now be "localized":

THEOREM 2. *Suppose that the hypotheses of Theorem* 1 *and the foregoing assumptions are fulfilled, with* $s > n/2 + 2 - d$. *If* $Mf \in H^s_{\text{loc}}(U)$ *and* $Pf \in H^{s+1}_{\text{loc}}(U)$, *then* $Mf \in H^{s+1}_{\text{loc}}(U)$.

(Here $H^s_{\text{loc}}(U)$ is the set of functions f in U such that $\phi f \in H^s$ for every $\phi \in C_U^\infty$.)

Proof. Cf. e.g. [4] for details.

REMARK. Suppose next that P is of the form (7) and such that (1') $P(x_0, D)$ is stronger than each $M_j(D)$ for every fixed x_0, (2') $a_j \in H^{s+1}$ for some $s > n/2 + 2 - d$. Let M be some fixed $P(x_0, D)$. Then we may conclude that $Mf \in H^s_{\text{loc}}(\mathcal{O})$ and $Pf \in H^{s+1}_{\text{loc}}(\mathcal{O})$ imply $Mf \in H^{s+1}_{\text{loc}}(\mathcal{O})$, for every open set $\mathcal{O}$ in R^n. In fact it is apparently sufficient to establish this fact when $\mathcal{O}$ is a small neighborhood of $x_0 = 0$, and in this case everything follows from Theorem 2, for P equals in the vicinity of 0 to some operator of the form (7) satisfying (1), (2).

3. **The quasilinear case.** We turn now to the quasilinear equation (1). Our hypotheses are the hypotheses (1), (2), (3) of the introduction. Let M be some fixed operator equally strong as all M_j together. Then we have the following

THEOREM 3. *There is a number* σ_0 *such that if* f *satisfies* (1) *and if* $Mf \in H^s_{\text{loc}}(\mathcal{O})$ *for some* $s > \sigma_0$, $\mathcal{O}$ *being an open set of* R^n, *then* $f \in C^\infty(\mathcal{O})$.

For the proof we need the following

LEMMA 3. *There exists a number σ with the following properties. Let $a(x, t_1, \cdots, t_\nu)$ be infinitely differentiable in $R^n \times C^\nu$ and suppose that all derivatives are bounded in every set of the form*

$$|t_1| \leqq W, \cdots, |t_\nu| \leqq W, \text{ } x \text{ arbitrary.}$$

Suppose further that $a(x, 0, \cdots, 0) \in H^s$ for some $s > \sigma$. Then $\phi_1 \in H^s, \cdots, \phi_\nu \in H^s$ imply $a(x, \phi_1, \cdots, \phi_\nu) \in H^s$.

Let us assume the lemma for a moment. Then Theorem 3 follows at once from Theorem 2, in the form given to it in the Remark following it. In fact, if $Mf \in H^s_{\text{loc}}(\mathcal{O})$, then, for some $\epsilon > 0$, $N_k f \in H^{s+\epsilon}_{\text{loc}}(\mathcal{O})$ and hence in virtue of Lemma 3, if $s > \sigma - \epsilon$, $a_j(x, N_1 f, \cdots, N_\nu f)$ and $g(x, N_1 f, \cdots, N_\nu f) \in H^{s+\epsilon}_{\text{loc}}(\mathcal{O})$, so that in view of Theorem 2, if further $s + \epsilon - 1 > n/2 + 2 - d$, $Mf \in H^{s+\epsilon}_{\text{loc}}(\mathcal{O})$. Repeating this argument l times, one obtains $Mf \in H^{s+\epsilon l}_{\text{loc}}(\mathcal{O})$ and, letting l tend to infinity, $Mf \in C^\infty(\mathcal{O})$ and hence $f \in C^\infty(\mathcal{O})$. This proves the theorem, with

$$\sigma_0 = \sup(\sigma - \epsilon, n/2 + 3 - d - \epsilon).$$

It remains to prove Lemma 3.

Proof of Lemma 3. Introducing Re t_k and Im t_k as new variables, we may as well assume that $t_1, \cdots, t_\nu$ are real variables. Also it is no restriction to assume that $a(x, t_1, \cdots, t_\nu)$ is periodic in each t_k with period, say, 2π. Expand $a(x, t_1, \cdots, t_\nu)$ in trigonometric series,

$$a(x, t_1, \cdots, t_\nu) = \sum a_{l_1 \cdots l_\nu}(x) e^{i(l_1 t_1 + \cdots + l_\nu t_\nu)}$$

where $l_1, \cdots, l_\nu$ are integers. We obtain

$$a(x, \phi_1, \cdots, \phi_\nu) = a(x, 0, \cdots, 0) + \sum a_{l_1 \cdots l_\nu}(x)(e^{i(l_1\phi_1 + \cdots + l_\nu\phi_\nu)} - 1).$$

The first term is in H^s by hypothesis. Also the coefficients $a_{l_1 \cdots l_\nu}(x)$ and all their derivatives tend to zero rapidly, i.e., faster than any expression of the form $l/(1 + |l_1| + \cdots + |l_\nu|)^A$. So everything will follow if we can show that

$$\psi = e^{i(l_1\phi_1 + \cdots + l_\nu\phi_\nu)} - 1$$

is in H^s and moreover subject to a majorization of the form

$$\|\psi\|_s \leqq C(1 + |l_1| + \cdots + |l_\nu|)^B. \tag{9}$$

Clearly $\psi \in H^t$ for some $t > n/2$, provided $s > o$, and o is chosen so that all derivatives of order $\leqq t$ will be continuous and bounded, which is possible in view of a (weak) form of Sobolev's lemma. Differentiating we get

$$D_\alpha \psi = i(\psi + 1) \sum l_k D_\alpha \phi_k, \qquad |\alpha| = 1,$$

so that, by the Corollary of Lemma 1, $D_\alpha \psi \in H^{\inf(t, s-1)}$. This improves the regularity of ψ from t to $\inf(t+1, s)$ so finally $\psi \in H^s$. The same argument (we omit the details!) proves also the inequality (9).

REMARK. Actually, as follows from [2, Proposition 1, p. 116], one can take $\sigma = n/2$. For our purpose the present weaker but more elementary statement is of course sufficient.

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