# ON COMMUTATIVE ALGEBRAS OF DEGREE TWO(1)

BY

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Let $\mathfrak{A}$ be a simple, commutative, power-associative algebra of degree 2 over an algebraically closed field $\mathfrak{F}$ of characteristic not equal to 2, 3 or 5. The degree of $\mathfrak{A}$ is defined to be the number of elements in the maximal set of pairwise orthogonal idempotents in $\mathfrak{A}$. This algebra has a unit element 1 [1, Theorem 3]. The algebras $\mathfrak{A}$ of characteristic zero were considered by Kokoris [8] and found to be Jordan algebras. Kokoris also gave examples of algebras $\mathfrak{A}$ that were not Jordan [6]. This left the problem of determining those algebras $\mathfrak{A}$ that are not Jordan algebras.

Since $1 = e + f$ where $e$ and $f$ are primitive orthogonal idempotents, we have a decomposition $\mathfrak{A} = \mathfrak{A}_e(1) + \mathfrak{A}_e(1/2) + \mathfrak{A}_e(0)$ where $x \in \mathfrak{A}_e(\lambda)$ if and only if $ex = \lambda x$. We have $\mathfrak{A}_e(\lambda) = \mathfrak{A}_f(1-\lambda)$; $\mathfrak{A}_e(\lambda)\mathfrak{A}_e(1/2) \subseteq \mathfrak{A}_e(1-\lambda) + \mathfrak{A}_e(1/2)$ for $\lambda = 1,0$; and $\mathfrak{A}_e(1) = e\mathfrak{F} + \mathfrak{N}_1$, $\mathfrak{A}_e(0) = f\mathfrak{F} + \mathfrak{N}_0$ where $\mathfrak{N}_1$ and $\mathfrak{N}_0$ are nilideals of $\mathfrak{A}_e(1)$ and $\mathfrak{A}_e(0)$ respectively. If $\mathfrak{A}_e(\lambda)\mathfrak{A}_e(1/2) \subseteq \mathfrak{A}_e(1/2)$ for $\lambda = 1,0$ we say that $e$ is a stable idempotent. If $\mathfrak{A}_e(\lambda)\mathfrak{A}_e(1/2) \subseteq \mathfrak{A}_e(1/2) + \mathfrak{N}_{1-\lambda}$ for $\lambda = 1,0$ we say that $e$ is a nilstable idempotent.

The results of Albert extend the characteristic zero case to include algebras of characteristic $p \neq 2,3,5$ for which every idempotent is stable [2]. He also characterized those algebras of characteristic $p \neq 2,3,5$ that have at least one stable idempotent [3; 4]. Recently Kokoris announced [9] that every simple, flexible, power-associative algebra over an algebraically closed field of characteristic $\neq 2,3$ that is of degree two and in which every idempotent is nilstable is a $J$-simple algebra.

It is the purpose of this paper to fill in the remaining gap by giving a characterization of those algebras $\mathfrak{A}$ that have an idempotent that is not nilstable. An example is also given of an algebra $\mathfrak{A}$ that does not have a stable idempotent.

1. Let $\mathfrak{A}$ be an algebra that is simple, commutative, power-associative, of degree two and whose base field $\mathfrak{F}$ is an algebraically closed field of characteristic $p \neq 2,3,5$. Let $e$ be a primitive idempotent of $\mathfrak{A}$ that is not nilstable. Since $\mathfrak{A}$ is power-associative we have $x^2x^2 = x^4$ for all $x \in \mathfrak{A}$ and the linearization of this identity

$$\begin{aligned} P(x,y,s,t) \;=\; & 4(xy)(st) + 4(xs)(yt) + 4(xt)(ys) \\ & - x[y(st) + s(yt) + t(ys)] - y[x(ts) + t(xs) + s(xt)] \\ & - s[x(yt) + y(xt) + t(xy)] - t[x(ys) + y(xs) + s(xy)] = 0. \end{aligned} \tag{1}$$

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We will use $\mathfrak{C}$ to represent the space $\mathfrak{A}_e(1)+\mathfrak{A}_e(0)$, $a_\lambda$ to represent the $\mathfrak{A}_e(\lambda)$-component of $a$, $a_{10}$ to represent the $\mathfrak{C}$-component of $a$, and $z$ to represent $e$–$f$. We will make frequent use of some of the results of Albert on commutative power-associative algebras; namely, results (5), (6), (7), (8) of [1]. We state them as

$$[g(xy)_\lambda]_{1/2} = [(gx_\lambda)_{1/2}y_\lambda]_{1/2} + [(gy_\lambda)_{1/2}x_\lambda]_{1/2}, \tag{2}$$

$$[g(xy)_\lambda]_{1-\lambda} = 2[(gx_\lambda)_{1/2}y]_{1-\lambda} + 2[(gy_\lambda)_{1/2}x]_{1-\lambda}, \tag{3}$$

$$[(gx_\lambda)_{1/2}y_{1-\lambda}]_{1/2} = [(gy_{1-\lambda})_{1/2}x_\lambda]_{1/2}, \tag{4}$$

$$(gx_\lambda)_{1-\lambda}y_{1-\lambda} = 2[(gy_{1-\lambda})_{1/2}x_\lambda]_{1-2}, \tag{5}$$

where $\lambda=1,0; g\in\mathfrak{A}_e(1/2)$ and $x$ and $y$ are in $\mathfrak{C}$.

Two other relations

$$2[(x_\lambda g)_{1/2}g]_\lambda + [(x_\lambda g)_{1-\lambda}\, g]_\lambda = x_\lambda g^2, \tag{6}$$

$$(x_1 g)_{1/2} = (x_0 g)_{1/2} \quad \text{implies} \quad (x_1^2 g)_{1/2} = (x_0^2 g)_{1/2} \tag{7}$$

for $x$ and $g$ as above will be useful. The first of these is obtained from $P(x,e,g,g)=0$ while the second can be derived from (2) and (4).

THEOREM 1. *$\mathfrak{C}$ is an associative subalgebra of $\mathfrak{A}$ with an element $c\in\mathfrak{C}$ such that there is a $w\in\mathfrak{A}_e(1/2)$ with $z(cw)=1$, $(c_1w)_{1/2}=(c_0w)_{1/2}$ and $(c_1^2w)_0=-2c_0$.*

**Proof.** It is easily seen that the subset $\mathfrak{J}$ of $\mathfrak{A}_e(1)$ consisting of all elements of the form $(a_0g)_1$ is an ideal of $\mathfrak{A}_e(1)$ where $g\in\mathfrak{A}_e(1/2)$ and $a_0$ is a fixed element of $\mathfrak{A}_e(0)$ because by (5) we have $b_1(a_0g)_1=2[a_0(b_1g)_{1/2}]_1$. The additive property of an ideal is immediate.

We now let $b_1$, $d_1$ be elements of $\mathfrak{A}_e(1)$, $g\in\mathfrak{A}_e(1/2)$ and $a_0\in\mathfrak{A}_e(0)$ with $(a_0g)_1=a_1$. If we consider only the $\mathfrak{A}_e(1)$-components of each of the terms in $P(b_1,d_1,g,a_0)=0$ we get $2(b_1d_1)a_1 = b_1(d_1a_1)+d_1(b_1a_1)$. If $b_1$ is also in $\mathfrak{J}$ we can interchange $a_1$ and $b_1$ to get $a_1(d_1b_1)=2b_1(d_1a_1)-d_1(b_1a_1)$. Therefore $a_1(d_1b_1)=(a_1d_1)b_1$. Hence $\mathfrak{J}$ is associative.

It has been shown [1, Lemma 11] that if $(a_0g)_1\in\mathfrak{N}_1$ for all $a_0\in\mathfrak{A}_e(0)$ and $g\in\mathfrak{A}_e(1/2)$ then $(a_1g)_0\in\mathfrak{N}_0$ for all $a_1\in\mathfrak{A}_e(1)$ and $g\in\mathfrak{A}_e(1/2)$. From this result and the assumption that $e$ is not nilstable we can conclude that there is an element $c_0\in\mathfrak{A}_e(0)$ and an element $g$ in $\mathfrak{A}_e(1/2)$ such that $(c_0g)_1$ is nonsingular. If $b_1$ is the inverse of $(c_0g)_1$ in $\mathfrak{A}_e(1)$ then $[c_0(2b_1g)_{1/2}]_1 = b_1(c_0g)_1=e$. We may also conclude that $\mathfrak{A}_e(1)=\mathfrak{J}$ is associative. In a similar manner we obtain the result that $\mathfrak{A}_e(0)$ is associative.

If we take $c_0\in\mathfrak{A}_e(0)$ and $w\in\mathfrak{A}_e(1/2)$ such that $(c_0w)_1=e$ and let $2c_1 = (c_0^2w)_1 = 4[c_0(c_0w)_{1/2}]_1$ then we can quote the results of Kokoris [7, Lemma 4 and Identity 29] that $(c_1w)_0=-f$ or 0 and $(c_1w)_{1/2}=(c_0w)_{1/2}$. No generality will be lost if we also assume that $c_0$ is nilpotent because $\mathfrak{A}_e(0)=f\mathfrak{F}+\mathfrak{N}_0$ and

$(c_0w)_1 = [(\alpha f + c_0)w]_1$ for any $\alpha \in \mathfrak{F}$. To complete the proof of the theorem it remains only to show that $(c_1w)_0 \neq 0$. We assume that $(c_1w)_0 = 0$. If we examine the $\mathfrak{A}_e(1)$-components of the terms of the relation $P(c_0, c_0, w, w) = 0$ we get $8(c_0w)_1^2 + 8[(c_0w)_{1/2}^2]_1 = 4[c_0[w(wc_0)_1]_{1/2}]_1 + 2[w(c_0^2w)_{1/2}]_1 + 4[w[c_0(c_0w)_{1/2}]_{1/2}]_1$. Using this relation together with (2), (6), (7) and $(c_0w)_1 = e$, we get $6e + 8[(c_1w)_{1/2}^2]_1 = 2w^2c_1^2 - 2[w(c_1^2w)_0]_1$. But $(c_1^2w)_0 = 4[c_1(c_1w)_{1/2}]_0 = 4[c_1(c_0w)_{1/2}]_0 = 2c_0(c_1w)_0 = 0$. Therefore either $[(c_1w)_{1/2}^2]_1$ or $w^2c_1^2$ must be nonsingular. If we again use (1) with $P(c_1, c_1, w, w) = 0$ and examine the $\mathfrak{A}_e(0)$-components of the resulting terms we get $8[(c_0w)_{1/2}^2]_0 = 2c_0^2w^2$. But then $[(c_0w)_{1/2}^2]_0$ is nilpotent. Since $(c_0w)_{1/2}^2 = \alpha 1 + n$ where $n \in \mathfrak{N}_1 + \mathfrak{N}_0$ [1, Lemma 10] we must also have $[(c_1w)_{1/2}^2]_1$ nilpotent. Now by (6) we have $2[(c_0w)_{1/2}w]_1 = 2[(c_1w)_{1/2}w]_1 = -[(c_1w)_0w]_1 + c_1w^2 = c_1w^2$. But $2[(c_0w)_{1/2}w]_0 = -[(c_0w)_1w]_0 + c_0w^2 = c_0w^2$ is nilpotent. Therefore $c_1w^2$ and $c_1^2w^2$ are nilpotent. We have arrived at a contradiction. Hence $(c_1w)_0 = -f$ and the theorem is proved.

THEOREM 2. *There is an isomorphism $T$ between $\mathfrak{A}_e(1)$ and $\mathfrak{A}_e(0)$ such that for $b_1 \in \mathfrak{A}_e(1)$, $T(b_1)$ is the unique element of $\mathfrak{A}_e(0)$ satisfying $(b_1w)_{1/2} = [T(b_1)w]_{1/2}$. The subset $\mathfrak{B}$ of $\mathfrak{C}$ of all elements of the form $b_1 + T(b_1)$ is an associative subalgebra of $\mathfrak{C}$ isomorphic to both $\mathfrak{A}_e(0)$ and $\mathfrak{A}_e(1)$.*

**Proof.** We use $c_1$, $c_0$ and $w$ as in Theorem 1. If we consider only the $\mathfrak{A}_e(1/2)$-components of the terms in $P(c_0, b_1, w, w) = 0$ we get $8[(c_0w)_{1/2}(b_1w)_0]_{1/2} + 4(b_1w)_{1/2} = 2[w\{b_1 + [(b_1w)_{1/2}c_0]_1\}]_{1/2} + 2\{w[(c_0w)_{1/2}b_1 + (b_1w)_0c_0]_0\}_{1/2} + 2\{c_0[w(wb_1)_0]_{1/2}\}_{1/2} + (b_1w)_{1/2}$. Using (5) and (2) on the terms $[(c_0w)_{1/2}b_1]_0$, $[(b_1w)_{1/2}c_0]_1$ and $\{c_0[w(wb_1)_0]_{1/2}\}_{1/2}$ this relation reduces to $[(c_0w)_{1/2}(b_1w)_0]_{1/2} = \{w[(c_0w)_{1/2}b_1]_0\}_{1/2}$. We now consider the $\mathfrak{A}_e(1/2)$-component of each term in $P(c_1, b_1, w, w) = 0$. We have

$$\begin{aligned} -4(b_1w)_{1/2} &+ 8[(c_1w)_{1/2}(b_1w)_0]_{1/2} \\ &= 2\{w[(c_1b_1)w + c_1(b_1w)_{1/2} + b_1(c_1w)_{1/2}]_0\}_{1/2} \\ &\quad - (b_1w)_{1/2} + 2\{c_1[w(wb_1)_0]_{1/2}\}_{1/2}. \end{aligned}$$

This relation together with (2) and (4) gives us $2[(c_1w)_{1/2}(b_1w)_0]_{1/2} = (b_1w)_{1/2} + \{w[(c_1b_1)w]_0\}_{1/2}$. But $[(c_0w)_{1/2}(b_1w)_0]_{1/2} = \{w[(c_0w)_{1/2}b_1]_0\}_{1/2}$ and $(c_0w)_{1/2} = (c_1w)_{1/2}$. Therefore $(b_1w)_{1/2} = (\{2[(c_1w)_{1/2}b_1]_0 - [(c_1b_1)w]_0\}w)_{1/2} = -2\{[(b_1w)_{1/2}c_1]_0\,w\}_{1/2}$. We can now define $T(b_1) = -2[(b_1w)_{1/2}c_1]_0$ to be the element $b_0$ in $\mathfrak{A}_e(0)$ such that $(b_1w)_{1/2} = (b_0w)_{1/2}$. To show that $T$ is well-defined we assume $(a_0w)_{1/2} = 0$. We have $a_0 = -a_0(c_1w)_0 = -2[c_1(a_0w)_{1/2}]_0 = 0$ by (5). Therefore $(b_0w)_{1/2} = (b_0'w)_{1/2}$ implies $b_0 = b_0'$. Simply by changing the signs of $c_1$ and $c_0$ and interchanging 1 and 0 we can get a similar result for $\mathfrak{A}_e(0)$; i.e., for every $b_0 \in \mathfrak{A}_e(0)$ there is a unique $b_1 = 2\{[(b_0w)_{1/2}c_0]_1w\}_{1/2}$ such that $(b_0w)_{1/2} = (b_1w)_{1/2}$. Therefore $T$ is onto $\mathfrak{A}_e(0)$ and is a 1–1 correspondence between $\mathfrak{A}_e(1)$ and $\mathfrak{A}_e(0)$.

Now if $a$ and $b$ are elements of $\mathfrak{B}$ as defined in the theorem we have, with the help of (2) and (4), that

$$\begin{aligned}[w(a_1b_1)]_{1/2} &= [(wb_1)a_1 + (wa_1)b_1]_{1/2} \\ &= [(wb_0)a_1 + (wa_0)b_1]_{1/2} = [(wa_1)b_0 + (wb_1)a_0]_{1/2} \\ &= [(wa_0)b_0 + (wb_0)a_0]_{1/2} = [w(a_0b_0)]_{1/2}.\end{aligned}$$

Therefore $T(a_1b_1) = a_0b_0$ and $ab = a_1b_1 + a_0b_0 \in \mathfrak{B}$. Clearly $\mathfrak{B}$ is closed under addition and scalar multiplication.

Define $S(b) = be$ for every $b \in \mathfrak{B}$. It follows immediately from the above results that $S$ is a 1–1 correspondence of $\mathfrak{B}$ onto $\mathfrak{A}_e(1)$. From the definition we have $S(ab) = (ab)e = (ae)(be) = S(a)S(b)$ and $S(a + b) = S(a) + S(b)$ for all $a$ and $b$ in $\mathfrak{B}$. Therefore $\mathfrak{B}$ and $\mathfrak{A}_e(1)$ are isomorphic as rings and hence as algebras. In the same manner we show that $\mathfrak{B}$ is isomorphic to $\mathfrak{A}_e(0)$. We have shown also that $T$ is an isomorphism. The associativity of $\mathfrak{B}$ follows from that of $\mathfrak{C}$.

From the definition of $\mathfrak{B}$ it is clear that $c = c_1 + c_0$ is in $\mathfrak{B}$. From $P(w,w,w,z)=0$ it follows that $w^2$ is in $\mathfrak{B}$. Theorem 2 also implies that $\mathfrak{C} = \mathfrak{B} + \mathfrak{B}z$.

THEOREM 3. *The mapping* $b \to D(b) = (bw)z$ *is a derivation of* $\mathfrak{B}$ *into* $\mathfrak{B}$ *such that* $D(c) = 1$.

**Proof.** Let $a$ and $b$ be arbitrary elements of $\mathfrak{B}$. Then $[(ab)w]_{10} = [(ab)_1w]_0 + [(ab)_0w]_1 = [(a_1b_1)w]_0 + [(a_0b_0)w]_1 = 2[a_1(b_1w)_{1/2}]_0 + 2[b_1(a_1w)_{1/2}]_0 + 2[a_0(b_0w)_{1/2}]_1 + 2[b_0(a_0w)_{1/2}]_1 = 2[a_1(b_0w)_{1/2}]_0 + 2[b_1(a_0w)_{1/2}]_0 + 2[a_0(b_1w)_{1/2}]_1 + 2[b_0(a_1w)_{1/2}]_1 = b_0(a_1w)_0 + a_0(b_1w)_0 + b_1(a_0w)_1 + a_1(b_0w)_1 = b(aw)_{10} + a(bw)_{10}$ by (3), (5) and the definition of $\mathfrak{B}$. If this relation is multiplied by $z$ we have $D(ab) = aD(b) + bD(a)$ and $D$ is a derivation on $\mathfrak{B}$ into $\mathfrak{C}$.

To show that $D(b)$ lies in $\mathfrak{B}$ for $b = b_1 + b_0$, an element of $\mathfrak{B}$, we need several de)ntities; the first of which is obtained from $P(b_0,w,w,c_1) = 0$. We get $8[(b_0w)_1(wc_1)_{1/2}]_0 + 8[(b_0w)_{1/2}\,(wc_1)_{1/2}]_0 = 2(b_0c_0)w^2 + 2\{[c_1(wb_0)_1]w\}_0$ after the usual simplifications using (2), (5), (6) and $(c_1w)_{1/2} = (c_0w)_{1/2}$. $8[b_0w)_{1/2}(wc_1)_{1/2}]_0 = 2(b_0c_0)w^2 + 2\{[c_1(wb_0)_1]w\}_0$ after the usual simplifications. We consider $P(b_0,w,w,c_0) = 0$ next to get

$$\begin{aligned}-3(b_0c_0)w^2 + 8[(b_0w)_{1/2}(wc)_{1/2}]_0 + 6[(b_0w)_1(c_0w)_{1/2}]_0 \\ = -2\{[(b_0c_0)w]_{1/2}w\}_0.\end{aligned}$$

Finally we obtain $3(b_1w)_0 = 4[(b_1w)_{1/2}(c_0w)_{1/2}]_0 - 2\{w[(c_0b_0)w]_{1/2}\}_0$ from $P(b_1,w,w,c_1) = 0$. Now, from the proof of Theorem 2 and from (3) we have

$$\begin{aligned}6T[b_0w)_1] &= -12\{[(b_0w)_1w]_{1/2}c_1\}_0 \\ &= 12[(c_1w)_{1/2}(b_0w)_1]_0 - 6\{[(b_0w)_1c_1]w\}_0.\end{aligned}$$

By successively applying to this relation the three identities above in the order we obtained them we get $6T[(b_0w)_1] = -12[(b_0w)_1(c_1w)_{1/2}]_0 - 24[(b_0w)_{1/2}(c_1w)_{1/2}]_0 + 6b_0c_0w^2 = -12[(b_0w)_1(c_0w)_{1/2}]_0 - 24[(b_0w)_{1/2}(c_0w)_{1/2}]_0 + 6c_0b_0w^2 = 4\{[(b_0c_0)w]_{1/2}w\}_0 - 8[(b_0w)_{1/2}(c_0w)_{1/2}]_0 = -6(b_1w)_0$. Therefore we have $D(b)z = (bw)_{10}z = (b_0w)_1 - (b_1w)_0 \in \mathfrak{B}$. The fact that $D(c) = 1$ follows immediately from the definition of $c$.

THEOREM 4. *If $a$ and $b$ are elements of $\mathfrak{B}$ then $[(wa)_{1/2}b]_{1/2} = [w(ab)]_{1/2}$, $[(wa)_{1/2}b]_{10} = (wb)_{10}a$ and $(wa)_{1/2}(wb)_{1/2} \in \mathfrak{B}$.*

**Proof.** By (2) and (4) and the definition of $\mathfrak{B}$ we have $[w(ab)]_{1/2} = 2[w(ab)_1]_{1/2} = 2[(wa_1)_{1/2}b_1]_{1/2} + 2[(wb_1)_{1/2}a_1]_{1/2} = [(wa)_{1/2}b_1]_{1/2} + 2[(wb_0)_{1/2}a_1]_{1/2} = [(wa)_{1/2}b_1]_{1/2} + 2[(wa_1)_{1/2}b_0]_{1/2} = [(wa)_{1/2}b_1]_{1/2} + [(wa)_{1/2}b_0]_{1/2} = [(wa)_{1/2}b]_{1/2}$. By (5) we have $[(wa)_{1/2}b]_{10} = 2[(wa_0)_{1/2}b_1]_0 + 2[(wa_1)_{1/2}b_0]_1 = (wb_1)_0a_0 + (wb_0)_1a_1 = (wb)_{10}a$. Now use $P(w,w,a,b)$ to get $4w^2ab + 8(wa)(wb) = 2w[(ab)w + (aw)b + (bw)a] + a[w^2b + 2w(wb)] + b[w^2a + 2w(wa)]$. If we consider only the $\mathfrak{C}$-components of each of the terms and if we use the facts that $(wa)_{10} \in \mathfrak{B}z$ for all $a \in \mathfrak{B}$ and $[w(az)]_{10} \in \mathfrak{B}$ then $8(wa)_{1/2}(wb)_{1/2} - 4w[(ab)w]_{1/2} - 2a[w(wb)_{1/2}] - 2b[w(wa)_{1/2}]$ is in $\mathfrak{B}$. Now $P(w,w,az,z) = 0$ implies $2w^2a = -2D^2(a) + 2w(wa)_{1/2}$. Since $a, w^2$ and $D(a)$ are in $\mathfrak{B}$, so also is $w(wa)_{1/2}$. Hence $8(wa)_{1/2}(wb)_{1/2}$ is in $\mathfrak{B}$.

COROLLARY. *If $a \in \mathfrak{C}$ and $b \in \mathfrak{B}$ then $[(wa)_{1/2}b]_{1/2} = [w(ab)]_{1/2}$.*

**Proof.** We can write $a = a' + a''z$ where $a'$ and $a''$ are in $\mathfrak{B}$. Since $[(a''z)w]_{1/2} = [(a''bz)w]_{1/2} = 0$ we have $[(wa)_{1/2}b]_{1/2} = [(wa')_{1/2}b]_{1/2} = [w(a'b)]_{1/2} = [w(ab)]_{1/2}$.

We now define $\mathfrak{G}$ to be the set of all $g \in \mathfrak{A}_e(1/2)$ such that $(gc)_{10}$ is in $\mathfrak{B}$.

THEOREM 5. *$\mathfrak{A}_e(1/2)$ is the direct sum of the two subspaces $(w\mathfrak{B})_{1/2}$ and $\mathfrak{G}$. Moreover $(\mathfrak{G}a)_{1/2} \subseteq \mathfrak{G}$, $[\mathfrak{G}(az)]_{1/2} \subseteq (w\mathfrak{B})_{1/2}$, and $[(w\mathfrak{B})_{1/2}(az)]_{1/2} \subseteq \mathfrak{G}$, for all $a \in \mathfrak{B}$.*

**Proof.** If $g$ is any element of $\mathfrak{A}_e(1/2)$, let $(gc)_{10} = a + a'z$ where $a$ and $a'$ are in $\mathfrak{B}$. Since $[(a'w)_{1/2}c]_{10} = a'z$ we have $\{[g-(a'w)_{1/2}]c\}_{10} = a, [g-(a'w)_{1/2}] \in \mathfrak{G}$ and $g$ is equal to the sum of an element of $\mathfrak{G}$ and an element of $(w\mathfrak{B})_{1/2}$. If $h$ lies in both $(w\mathfrak{B})_{1/2}$ and $\mathfrak{G}$ then $(hc)_{10}$ lies in $\mathfrak{B}z$ and $\mathfrak{B}$. Hence $(hc)_{10} = 0$. But $[(wa)_{1/2}c]_{10} = az$. Therefore if $h = (wa)_{1/2}$ then $a = (wa)_{1/2} = 0$ and $h = 0$. Hence $\mathfrak{A}_e(1/2)$ is the direct sum of $\mathfrak{G}$ and $(w\mathfrak{B})_{1/2}$.

Since $D(c^2) = 2c$, the $\mathfrak{A}_e(1/2)$-components of the terms obtained from $P(c,c,w,g) = 0$ with $g \in \mathfrak{G}$ yield the relation

$$8[(cw)_{1/2}(cg)_{10}]_{1/2} = \{2c[w(cg)_{10}]_{1/2} + w(c^2g)_{10} + 2w[c(cg)_{10}]\}_{1/2} + \{2w[c(cg)_{1/2}]_{10} + 6g(cz)\}_{1/2}.$$

Using this relation, Theorem 4 and the property that $(cg)_{10} \in \mathfrak{B}$ it is easily seen that $[g(cz)]_{1/2}$ is in $(w\mathfrak{B})_{1/2}$. Therefore $[g(cz)_{1/2}c]_{10} \in z\mathfrak{B}$. But

$$\begin{aligned}[g(cz)_{1/2}c]_{10} &= [(gc_1)_{1/2}c_1 - (gc_0)_{1/2}c_1 - (gc_0)_{1/2}c_0 + (gc_1)_{1/2}c_0]_{10} \\ &= [(1/4)(c_1^2g) - (1/4)(c_0^2g) - (1/2)(gc_1)_0c_0 + (1/2)(gc_0)_1c_1]_1 \\ &= -(1/4)(c^2g)_{10}\, z + (1/2)(cz)(cg)_{10}.\end{aligned}$$

Therefore since $(cg)_{10}$ is an element of $\mathfrak{B}$ we also have $(c^2g)_{10}$ is an element of $\mathfrak{B}$. Similarly $[(cg)_{1/2}c]_{10} = (1/4)(c^2g)_{10} + (1/2)c(cg)_{10}$ is in $\mathfrak{B}$. Therefore $(cg)_{1/2}$ is in $\mathfrak{G}$. We now examine the $\mathfrak{A}_e(1/2)$-components of the terms resulting from $P(a_1,c_1,w,g) = 0$. With the help of (3) and (4) we get

$$\begin{aligned}&[2(a_1w)_0(c_1g)_{1/2} + 2(a_1w)_{1/2}(c_1g)_0 + 2(c_1w)_{1/2}(a_1g)_0 + 2(c_1w)_0(a_1g)_{1/2}]_{1/2} \\ &\qquad = \{w[(a_1c_1)g]_0 + g[(a_1c_1)w]_0\}_{1/2}.\end{aligned} \tag{8}$$

Interchanging the subscripts 1 and 0 we obtain

$$\begin{aligned}&[2(a_0w)_1(c_0g)_{1/2} + 2(a_0w)_{1/2}(c_0g)_1 + 2(c_0w)_{1/2}(a_0g)_1 + 2(c_0w)_1(a_0g)_{1/2}]_{1/2} \\ &\qquad = \{w[(a_0c_0)g]_1 + g[(a_0c_0)w]_1\}_{1/2}.\end{aligned}$$

But

$$\begin{aligned}\{g[(a_0c_0)w]_1\}_{1/2} &= \{2g[a_0(c_0w)_{1/2} + c_0(a_0w)_{1/2}]_1\}_{1/2} \\ &= \{2g[a_0(c_1w)_{1/2}]_1 + g[a_1(c_0w)_1]_1\}_{1/2} \\ &= \{g[c_1(a_0w)_1] + ga_1\}_{1/2}.\end{aligned}$$

Therefore

$$\begin{aligned}&[2(a_0w)_1(c_0g)_{1/2} + 2(a_0w)_{1/2}(c_0g)_1 + 2(c_0w)_{1/2}(a_0g)_1 + 2(c_0w)_1(a_0g)_{1/2}]_{1/2} \\ &\qquad = \{g[c_1(a_0w)_1] + w[(a_0c_0)g]_1 + [a_1g]\}_{1/2}.\end{aligned} \tag{9}$$

Again consider only the $\mathfrak{A}_e(1/2)$-components of the terms of $P(a_0, c_1, w, g) = 0$. This relation together with (2), (3) and (4) gives us

$$\begin{aligned}&[2(a_0w)_{1/2}(c_1g)_0 + 2(a_0w)_1(c_1g)_{1/2} + 2(a_0g)_1(c_1w)_{1/2} + 2(c_1w)_0(a_0g)_{1/2}]_{1/2} \\ &\quad = \{[a_0(c_1w)_0]g + g[c_1(a_0w)_1] + w[a_0(c_1g)_0] + w[c_1(a_0g)_1]\}_{1/2}.\end{aligned} \tag{10}$$

Interchanging the subscripts 0 and 1 in (10) we obtain

$$\begin{aligned}&[2(a_1w)_{1/2}(c_0g)_1 + 2(a_1w)_0(c_0g)_{1/2} + 2(a_1g)_0(c_0w)_{1/2} + 2(c_0w)_1(a_1g)_{1/2}]_{1/2} \\ &\quad = \{[a_1(c_0w)_1]g + g[c_0(a_1w)_0] + w[a_1(c_1g)_1] + w[c_0(a_1g)_0]\}_{1/2}.\end{aligned} \tag{11}$$

We now subtract the sum of identities (10) and (11) from the sum of the identities (8) and (9) and use the facts that $(a_1w)_{1/2} = (a_0w)_{1/2}$, $(c_1w)_0 = -f$ and $(c_0w)_1 = e$. We have $\{2[(az)w]_{10}[(cz)g]_{1/2} + (a_0g) - 2(a_1g) - g[(a_1c_1)w]_0 + g[c_0(a_1w)_0]\}_{1/2}$ is in $(w\mathfrak{B})_{1/2}$. Therefore

$$\{-2D(a)[(cz)g]_{1/2} + (a_0g) - 2(a_1g) - 2g[(a_1c_1w)_{1/2}]_0$$
$$- 2g[c_1(a_1w)_{1/2}]_0 + 2g[a_1(c_0w)_{1/2}]_0\}_{1/2}$$
$$= \{-2D(a)[(cz)g]_{1/2} + (a_0g) - 2(a_1g) - 2g[a_1(c_1w)_{1/2}]_0$$
$$- g[a_0(c_1w)_0] + 2g[a_1(c_1w)_{1/2}]_0\}_{1/2}$$
$$= \{-2D(a)[(cz)g]_{1/2} + (a_0g) - 2(a_1g) + ga_0\}_{1/2}$$
$$= \{-2D(a)[(cz)g]_{1/2} - 2(az)g\}_{1/2}$$

is in $\mathfrak{B}$. Since $[(cz)g]_{1/2} \in (w\mathfrak{B})_{1/2}$ we have $[(az)g]_{1/2} \in (w\mathfrak{B})_{1/2}$. To show that $(ga)_{1/2} \in \mathfrak{G}$ for $a \in \mathfrak{B}$ we consider

$$[(ga)_{1/2}c]_{10} = [(ga_1)_{1/2}c_1 + (ga_0)_{1/2}c_1 + (ga_1)_{1/2}c_0 + (ga_0)_{1/2}c_0]_{10}$$
$$= [2(ga_0)_{1/2}c_1 + [g(az)]_{1/2}c_1 + 2(ga_1)_{1/2}c_0 - [g(az)]_{1/2}c_0]_{10}$$
$$= [(gc_{10}a_0 + (gc_0)_1a_1 + g(az)_{1/2}(cz)]_{10}$$
$$=(gc)_{10}a + \{[g(az)]_{1/2}(cz)\}_{10}.$$

Since $(gc)_{10} \in \mathfrak{B}$ so is $(gc)_{10}a$. Also since $[g(az)]_{1/2} \in (w\mathfrak{B})_{1/2}$ we have

$$\{[g(az)]_{1/2}(cz)\}_{10} \in \mathfrak{B}.$$

Hence $[(ga)_{1/2}c]_{10} \in \mathfrak{B}$ and $(ga)_{1/2} \in \mathfrak{G}$. Finally if we take $a$, $b$ and $h$ in $\mathfrak{B}$ we have $\{[(wa_0)_{1/2}(bz)]_{1/2}h_1\}_0 = \{[(wa_0)_{1/2}b_1]_{1/2}h_1 - [(wa_0)_{1/2}b_0]_{1/2}h_1\}_0$ $= \{[(wb_1)a_0]_{1/2}h_1 - (1/4)(wh_1)_0a_0b_0\}_0 = \{[(wb_0)a_0]_{1/2}h_1 - (1/4)(wh_1)_0a_0b_0\}_0$ $=(1/4)(wh_1)_0b_0a_0 - (1/4)(wh_1)_0b_0a_0 = 0$. Similarly $\{[(wa_0)(bz)]_{1/2}h_0\}_1 = 0$. By taking $h = c$ we can see that the $(w\mathfrak{B})_{1/2}$ component of $[(wa)_{1/2}(bz)]_{1/2}$ is 0. Hence $[(wa)_{1/2}(bz)]_{1/2}$ is in $\mathfrak{G}$.

THEOREM 6. $[(w\mathfrak{B})_{1/2}(\mathfrak{B}z)]_{1/2} = 0$.

**Proof.** Let $a$ be a nilpotent element of $\mathfrak{A}_e(1)$. There exists a $\lambda \in \mathfrak{F}$ such that $d = a + \lambda c$ has the property that $(d_0w)_1$ is a nonsingular element $b_1$ of $\mathfrak{A}_e(1)$. Then $[d(2b_1^{-1}w)_{1/2}]_1 = b_1^{-1}(dw)_1 = e$. If we let $b$ be the unique element of $\mathfrak{B}$ whose $\mathfrak{A}_e(1)$-component is $b_1$ we have by the isomorphism established in Theorem 2 that $[d(b^{-1}w)_{1/2}]_{10} = b^{-1}D(d)z = z$. For these elements $d \in \mathfrak{C}$ and $(wb^{-1})_{1/2} \in \mathfrak{A}_e(1/2)$ we get a $\tilde{\mathfrak{B}} \subseteq \mathfrak{C}$ such that $\tilde{\mathfrak{B}} + \tilde{\mathfrak{B}}z = \mathfrak{C}$ and where $\tilde{\mathfrak{B}}$ has the properties described for $\mathfrak{B}$ in Theorems 2–5. Let $t + sz \in \tilde{\mathfrak{B}}z$ where $t$ and $s \in \mathfrak{B}$. We have $[(wb^{-1})_{1/2}(t + sz)]_{1/2} = 0$. Therefore $(wb^{-1}t)_{1/2} + [(wb^{-1})_{1/2}(sz)]_{1/2} = 0$. Since $[(wb^{-1})_{1/2}(sz)]_{1/2} \in \mathfrak{G}$ we must have $(wb^{-1}t)_{1/2} = 0$ and $b^{-1}t = 0$. Therefore $t = 0$ and $\tilde{\mathfrak{B}}z \subseteq \mathfrak{B}z$. If $\tilde{\mathfrak{B}}z$ is a proper subset of $\mathfrak{B}z$ then $\tilde{\mathfrak{B}}$ is a proper subset of $\mathfrak{B}$. But this would imply that $\tilde{\mathfrak{B}} + \tilde{\mathfrak{B}}z$ is a proper subset of $\mathfrak{C}$ which is a contradiction. Therefore we must have $\tilde{\mathfrak{B}}z = \mathfrak{B}z$ and $[(wb^{-1})_{1/2}(\mathfrak{B}z)]_{1/2} = 0$. Now let $\mathfrak{S}$ be the subset of $\mathfrak{B}$ of all elements $s$ such that $[(ws)_{1/2}(\mathfrak{B}z)]_{1/2} = 0$. Let $x, y \in \mathfrak{B}$. The relation $P(y,x,w,z)=0$ yields $[(wx)(yz)]_{1/2}+[(wy)(xz)]_{1/2}=0$. Let $t\in\mathfrak{B}$, $s$ and $s'\in\mathfrak{S}$. Then we get $\{[w(ss')]_{1/2}(tz)\}_{1/2}=-\{(wt)_{1/2}(ss'z)\}=0$ from $P(tw,s,s'z)=0$. Hence $\mathfrak{S}$ is a subalgebra of $\mathfrak{B}$. If we let $b_{-1}=\alpha+n$ where $b$ is as described above and $n$ is a nilpotent element of $\mathfrak{B}$ and $\alpha\in\mathfrak{F}$, then $n\in\mathfrak{S}$ and hence every power of $n$ is in $\mathfrak{S}$. But $b$ is the sum of a multiple of the identity and a linear combination of powers of $n$. Hence $b=\lambda+D(a)\in\mathfrak{S}$ and the derivative of every element of $\mathfrak{B}$ is in $\mathfrak{S}$. Now $a\in\mathfrak{B}$ implies $a=D(ca)-cD(a)$. Since $D(ca)$, $c=(1/2)D(c^2)$ and $D(a)$ are in $\mathfrak{S}$ we have $\mathfrak{B}\subseteq\mathfrak{S}$ and $[(w\mathfrak{B})_{1/2}(\mathfrak{B}z)]_{1/2}=0$.

At this point we have obtained partial results on the multiplications of $\mathfrak{A}$. However, the chief remaining gap in the characterization of $\mathfrak{A}$ lies with the products involving elements of $\mathfrak{G}$. To facilitate the determination of these products we shall introduce some symbols $Q_g$, $\phi_g$, $k_g$, $f_g$, and $h_g$ on $\mathfrak{B}$ into $\mathfrak{B}$ for every $g\in\mathfrak{G}$ by letting

$$[g(bz)]_{1/2}=[wQ_g(b)]_{1/2}, \tag{12}$$

$$(gb)_{10}=h_g(b)+k_g(b)z, \tag{13}$$

$$[g(wb)_{1/2}]_{10}=f_g(b)+\phi_g(b)z \tag{14}$$

for every $b\in\mathfrak{B}$. In our subscripts we abbreviate $(ga)_{1/2}$ to $ga$.

From (2) and (3) and the definition of $\mathfrak{G}$ we have

$$\begin{aligned}\{[(ga)_{1/2}(bz)]_{1/2}c\}_1 &= \{[(ga)_{1/2}b_1]_{1/2}c_0\}_1-\{[(ga)_{1/2}b_0]_{1/2}c_0\}_1\\ &=\{2[(ga_1)_{1/2}b_1]_{1/2}c_0+[(wQ_g(a))_{1/2}b_1]_{1/2}c_0\\ &\quad -2[(ga_1)_{1/2}b_0]_{1/2}c_0-[(wQ_g(a))_{1/2}b_1]_{1/2}c_0\}_1\\ &=(1/2)(gc_0)_1a_1b_1+(1/2)b_1Q_g(a)-(1/2)b_1Q_g(a)\\ &\quad -2\{[(gb_0)_{1/2}a_1]_{1/2}c_0\}_1\\ &=(1/2)(gc_0)_1a_1b_1-2\{[(gb_1)_{1/2}a_1]_{1/2}c_0\}_1\\ &\quad +2\{[(wQ_g(b))_{1/2}a_1]_{1/2}c_0\}_1\\ &=a_1Q_g(b).\end{aligned}$$

Now $[(ga)_{1/2}(bz)]_{1/2}=[wQ_{ga}(b)]_{1/2}$ and therefore $[(wQ_{ga}(b))_{1/2}c]_{10}=Q_{ga}(b)z$. Hence

$$Q_{ga}(b)=aQ_g(b). \tag{15}$$

Consider $h_{gb}(a)+k_{gb}(a)z=[(gb)_{1/2}a]_{10}=[(gb)_{1/2}a_1]_0+(gb)_{1/2}a_0]_1$ $=2[(gb_0)_{1/2}a_1]_0+[(wQ_g(b))_{1/2}a_1]_0+2[(gb_1)_{1/2}a_0]_1-[(wQ_g(b))_{1/2}a_0]_1$ $=b_0(ga_1)_0+b_1(ga_0)_1+Q_g(b)[(az)w]_{10}=bh_g(a)+bzk_g(a)-Q_g(b)D(a)$. From this relation we obtain

$$h_{gb}(a)=bh_g(a)-Q_g(b)D(a), \tag{16}$$

$$k_{gb}(a)=bk_g(a). \tag{17}$$

We now consider the $\mathfrak{C}$-components of the terms of $P(a,a,g,z)=0$. We have $3ah_g(a)z + 3ak_g(a) - 5h_{ga}(a)z - 5k_{ga}(a) = Q_g(a)D(a)z - h_g(a^2)z - k_g(a^2)$. If we equate $\mathfrak{B}$-components and $\mathfrak{B}z$-components we have

$$k_g(a^2) = 2ak_g(a), \tag{18}$$

$$h_g(a^2) = 2ah_g(a) - 4Q_g(a)D(a) \tag{19}$$

by using (16) and (17).

We have proved that $k_g$ is a derivation for every $g \in \mathfrak{G}$. We shall now prove that $Q_g$ is a derivation for every $g \in \mathfrak{G}$. We have

$$\begin{aligned}[wQ_g(ab)]_{1/2} &= [g(abz)]_{1/2} = [g(ab)_1]_{1/2} - [g(ab)_0]_{1/2} \\ &= [(ga_1)_{1/2}b_1 + (gb_1)_{1/2}a_1 - (ga_0)_{1/2}b_0 - (gb_0)_{1/2}a_0]_{1/2} \\ &= [(ga_1)_{1/2}b_1 + (gb_0)_{1/2}a_1 + (wQ_g(b))_{1/2}a_1 - (ga_0)_{1/2}b_0 \\ &\qquad - (gb_1)_{1/2}a_0 + (wQ_g(b))_{1/2}a_0]_{1/2} \\ &= [(ga_0)_{1/2}b_1 + (wQ_g(a))_{1/2}b_1 + (gb_0)_{1/2}a_1 + (wQ_g(b))_{1/2}a_1 \\ &\quad -(ga_1)_{1/2}b_0 + (wQ_g(a))_{1/2}b_0 - (gb_1)_{1/2}a_0 + (wQ_g(b)_{1/2}a_0]_{1/2} \\ &= [(ga_0)_{1/2}b_1 - (gb_1)_{1/2}a_0 + (gb_0)_{1/2}a_1 - (ga_1)_{1/2}b_0 \\ &\qquad + w(Q_g(a)b) + w(Q_g(b)a)]_{1/2}.\end{aligned}$$

By (4) we have $(wQ_g(ab))_{1/2} = [w(Q_g(a)b + Q_g(b)a)]_{1/2}$. Therefore

$$Q_g(ab) = Q_g(a)b + Q_g(b)a\,. \tag{20}$$

Next, we consider the $\mathfrak{G}$-components of the terms of $P(g,a,bz,z)=0$ to get $4[(ga)b]_{1/2} = [3g(ab) + (gb)a]_{1/2}$. However

$$\begin{aligned}[(ga)b]_{1/2} &= [2(ga_0)b_1 + (wQ_g(a))b_1 + 2(ga_1)b_0 - (wQ_g(a))b_0]_{1/2} \\ &= 2[(gb_1)a_0 + (gb_0)a_1]_{1/2} \\ &= [(gb)a_0 + (wQ_g(b))a_0 + (gb)a_1 - (wQ_g(b))a_1]_{1/2} \\ &= [(gb)a]_{1/2}.\end{aligned}$$

If we combine the above two relations we have

$$[(ga)b]_{1/2} = [g(ab)]_{1/2}\,. \tag{21}$$

A similar computation using $P(w,w,a,z)=0$ and $P((wa)_{1/2},w,a,z)=0$ gives us

$$w(wa)_{1/2} = w^2a + D^2(a) \tag{22}$$

$$(wa)_{1/2}^2 = w^2a^2 + 2aD^2(a) - D(a)D(a)\,. \tag{23}$$

If we consider the $(w\mathfrak{B})_{1/2}$-components of the terms of $P(z,(aw)_{1/2},w,g)=0$ we have $[wQ_g(w^2a) + wQ_g(D^2(a)) + w(a\phi_g(1)) + w\phi_g(a)]_{1/2} = 0$. By letting $a=1$ we get

$$\phi_g(1) = -\frac{1}{2}\,Q_g(w^2)\,. \tag{24}$$

Therefore

$$\phi_g(a) = \frac{1}{2} a\, Q_g(w^2) - Q_g(aw^2) - Q_g(D^2(a)). \tag{25}$$

From (15) and (25) we have

$$\phi_{ga}(b) = a\phi_g(b). \tag{26}$$

We now wish to express $h_g$ in terms of $Q_g$ and $D$. We examine the $\mathfrak{B}z$-components of $P(w,g,c,a)=0$ and use (21) and (26) to get $3\phi_g(c)a + 3\phi_g(a)c + 3h_g(a) + 3aD(h_g(c)) + 3D(a)h_g(c) + 3cD(h_g(a)) - 4h_{gc}(D(a)) = 3\phi_g(1)ca + D(h_g(ca) + h_{gc}(a) + h_g(c)a + h_{ga}(c) + h_g(a)c) + 3\phi_g(ca) - 3h_g(D(ca)) - ch_g(D(a)) + \phi_g(D(a))$. We simplify this relation using (25), (16) and the linearized form of (19) to get $-3Q_g(D^2(a))c + 3h_g(a) = -3Q_g(D^2(ca)) - Q_g(c)D^2(a) - 3D(Q_g(c))D(a) - 3D(Q_g(a)) - 3h_g(c)D(a) + 7Q_g(D(a))$. Since $Q_g$ and $D$ are derivations we have

$$3h_g(a) = -3D(Q_g(c))D(a) - 3D(Q_g(a)) - 3h_g(c)D(a) + Q_g(D(a)) - 4Q_g(c)D^2(a). \tag{27}$$

If we let $a = c$ in (27) we get $h_g(c) = -D(Q_g(c))$. Therefore (27) simplifies to

$$3h_g(a) = -3D(Q_g(a)) + Q_g(D(a)) - 4Q_g(c)D^2(a). \tag{28}$$

We substitute the values obtained from (28) in $h_g(ac) = ch_g(a) + ah_g(c) - 2Q_g(a) - 2Q_g(c)D(a)$, a linearized form of (19), to get

$$Q_g(a) = Q_g(c)D(a). \tag{29}$$

If we use this relation in (28) we obtain

$$h_g(a) = -D(Q_g(c))D(a) - 2Q_g(c)D^2(a). \tag{30}$$

We now investigate the behaviour of $f_g$. Consider the $\mathfrak{B}z$-components of the terms of $P((wb)_{1/2}, g, a, z) = 0$. We have

$$2f_g(b)a = f_{ga}(b) + f_g(ab) - bD(k_g(a)) - bk_g(D(a)) - D(a)k_g(b) \tag{31}$$

and when $b = 1$

$$2f_g(1)a = f_{ga}(1) + f_g(a) - D(k_g(a)) - k_g(D(a)). \tag{32}$$

We define a new mapping $T_g$ on $\mathfrak{B}$ into $\mathfrak{B}$ for each $g$ by

$$T_g(a) + f_g(1)a - f_{ga}(1) + D(k_g(a)). \tag{33}$$

This definition together with (32) gives us $f_g(a) = f_g(1)a + T_g(a) + k_g(D(a))$ and $f_{ga}(1) = f_g(1)a - T_g(a) + D(k_g(a))$. Now $f_{ga}(b) = -f_g(ab) + 2f_g(b)a + b(Dk_g + k_gD)(a) + k_g(b)D(a)$ and $f_{ga}(b) = -f_{gab}(1) + 2f_{ga}(1)b + a(Dk_g + k_gD)(b) + D(a)k_g(b)$ by (31) and (32). Substituting the values for $f_g(ab)$, $f_g(b)$, $f_{gab}(1)$ and $f_{ga}(1)$ expressed in terms of $T_g$ in these relations and simplifying we have

$$T_g(ab) = T_g(a)b + T_g(b)a \tag{34}$$

and

$$f_{ga}(b) = f_g(1)ab + T_g(b)a - bT_g(a) + ak_g(D(b)) + bD(k_g(a)) - k_g(a)D(b). \tag{35}$$

It follows readily that

$$T_{ga}(b) = aT_g(b) - D(b)k_g(a). \tag{36}$$

We have already shown that $\phi_g(a) = Q_g(c)[(1/2)aD(w^2) - D(w^2a) - D^3(a)]$. We also have that $P(g,g,(aw)_{1/2},z) = 0$ implies $[g\phi_g(a)]_{1/2} = 0$. If we let $a = c^3$ we have $\phi_g(c^3) = Q_g(c)[-(1/2)c^3D(w^2) - 3c^2D(w^2) - 6]$. Since the second factor on the right-hand side is nonsingular we have $[gQ_g(c)]_{1/2} = 0$. Multiplying by $cz$ and considering the $(w\mathfrak{B})_{1/2}$-component we get

$$Q_g(c)^2 = 0. \tag{37}$$

Similarly we have

$$Q_g(c)k_g(a) = 0. \tag{39}$$

Now consider the element $w' = [w - wD(Q_g(c))]_{1/2} + g$ of $\mathfrak{A}_e(1/2)$. We have $(c_2w')_0 = -f$. By Theorem 1 and its proof, $c_2 - (1/2)(c_2^2w')_0$ is an element $a$ in $\mathfrak{C}$ such that $(aw')z = 1$. Also $(c_2^2w')_0 = -2c_0 - 4(Q_g(c))_0$. Therefore $(aw')z = \{[c + Q_g(c) - Q_g(c)z]w'\}z = 1 - 2D(Q_g(c))^2 - 2D(Q_g(c))^2z - 2Q_g(c)D^2(Q_g(c))z + k_g(Q_g(c)) - 2Q_g(c)D^2(Q_g(c)) + k_g(Q_g(c))z$. Simple properties of derivations and the fact that $Q_g(c)^2 = 0$ gives us $(aw')z = 1 + k_g(Q_g(c)) + k_g(Q_g(c))z$. Therefore

$$k_g(Q_g(c)) = 0. \tag{40}$$

We also have from (35) and (36) that

$$T_g(Q_g(c)) = f_g(1)Q_g(c) \text{ and } T_g(b)Q_g(c) = 0 \tag{41}$$

for every $b \in \mathfrak{B}$.

For $w'$ and $c' = c + Q_g(c) - Q_g(c)z$ we have a corresponding $\mathfrak{B}'$ and $\mathfrak{B}'z$ as described in Theorem 2. To determine these two subspaces we let $a + bz$ be an element of $\mathfrak{C}$ with $a$, $b \in \mathfrak{B}$ and such that the 1/2-component of $w'(a + bz)$ is 0. We obtain $wa - wD(Q_g(c)a + ga + wQ_g(b))_{1/2} = 0$. Therefore $a[1 - D(Q_g(c))] = -Q_g(c)D(b)$. Solving for $a$ we have $a = -D(b)Q_g(c)$. Since $\mathfrak{B}' + \mathfrak{B}'z = \mathfrak{C}$, we can conclude from the above result that $\mathfrak{B}'$ consists of all elements of the form $a - Q_g(a)z$. We note that the $\mathfrak{C}$-component of the element $(a - Q_g(a)z)w'$ must be an element of $\mathfrak{B}'z$ by Theorem 3. If we calculate this element we obtain $D(a)z - D(a)D(Q_g(c))z + Q_g(c)D^2(a) + k_g(a)z - D(Q_g(a))D(Q_g(c)) + D(Q_g(c))^2 \cdot D(a)z$. In order for this element to be in $\mathfrak{B}'z$ we must have $Q_g(c)D^2(a) + D(Q_g(c))^2D(a) = Q_g(c)D[D(a) - D(a)D(Q_g(c)) + k_g(a) + D(Q_g(c))^2D(a)]$ by the definition of $\mathfrak{B}'z$. Therefore

(42) $$Q_g(c)D(k_g(a)) = k_g(a)D(Q_g(c)) = 0.$$

We also have

(43) $$Q_g(c)k_t(b) = 0$$

for any $t \in \mathfrak{G}$ and any $b \in \mathfrak{B}$ since $Q_g(c)k_t(b) = k_t(Q_g(c)b) - k_t(Q_g(c))b = k_t(Q_{gb}(c)) - k_t(Q_g(c))b = -k_{gb}(Q_t(c)) - k_t(Q_g(c))b = -bk_g(Q_t(c)) - k_t(Q_g(c))b = 0$.

We define $t'$ to be the 1/2-component of

$$w[-D(Q_t(c)D(Q_g(c)) + Q_t(c)D^2(Q_g(c)) - k_t(Q_g(c))] + t$$

for $t \in \mathfrak{G}$. Then the $\mathfrak{C}$-component of $(c + Q_g(c) - Q_g(c)z)t'$ is

(44) $$-D(Q_t(c)) - D(Q_t(c))D(Q_g(c)) - 2Q_t(c)D^2(Q_g(c) + k_t(Q_g(c)) + Q_g(c)D^2(Q_t(c))z$$

since $Q_t(c)D^2(Q_g(c)) + 2D(Q_g(c))D(Q_t(c)) + Q_g(c)D^2(Q_t(c)) = 0$ and

$2D(Q_t(c))D(Q_g(c))D(Q_g(c)) = -Q_t(c)D^2(Q_g(c))D(Q_g(c)) = 3Q_t(c)Q_g(c)D^3(Q_g(c)) = 0.$

Hence $t'$ is in $\mathfrak{G}'$. We now compute $D'$ and $Q'_{t'}$. We have simply that

(45) $$\begin{aligned} D' : a - Q_g(a)z \to D(a) - D(Q_g(c))D(a) + D(Q_g(c))^2D(a) + k_g(a) \\ - [Q_g(c)D^2(a) + D(Q_g(c))^2D(a)]\, z, \end{aligned}$$

(46) $$Q'_{t'} : c + Q_g(c) - Q_g(c)z \to Q_t(c) + Q_t(c)D(Q_g(c)) - Q_g(c)D(Q_t(c))z.$$

Therefore

$$\begin{aligned} D'Q'_{t'} : c + Q_g(c) - Q_g(c)z \to D(Q_t(c)) + D(Q_t(c))D(Q_g(c)) \\ + Q_t(c)D^2(Q_g(c)) - D(Q_g(c))D(Q_t(c)) + k_g(Q_t(c)) - Q_g(c)D^2(Q_t(c))z. \end{aligned}$$

By (30) and (44) we have

$$\begin{aligned} D(Q_t(c)) + D(Q_t(c))D(Q_g(c)) + Q^2(c)D^2(Q_g(c)) - D(Q_g(c))D(Q_t(c)) + k_g(Q_t(c)) \\ = D(Q_t(c)) + D(Q_t(c))D(Q_g(c)) + 2Q_t(c)D_2(Q_g(c)) - k_t(Q_g(c)). \end{aligned}$$

Therefore $Q_t(c)D^2(Q_g(c)) - D(Q_g(c))D(Q_t(c)) = 2Q_t(c)D^2(Q_g(c))$ and

(47) $$Q_t(c)D^2(Q_g(c)) = -D(Q_t(c))D(Q_g(c)).$$

Replacing $t$ by $(ct)_{1/2}$ we have $cQ_t(c)D^2(Q_g(c)) = -cD(Q_t(c))D(Q_g(c)) - Q_t(c)D(Q_g(c))$ and therefore

(48) $$Q_t(c)D(Q_g(c)) = 0.$$

We now examine the $\mathfrak{B}$-components of the terms of $P(g, t, a, z) = 0$ for $g, t \in \mathfrak{G}$ and $a \in \mathfrak{B}$. We have

(49) $$\begin{aligned} m(1, a) + m(a, 1) = 2m(1, 1)a + 2D(Q_t(c))D(D(Q_g(c))D(a)) \\ + 2D(Q_g(c))D(D(Q_t(c))D(a)) + (k_gk_t + k_tk_g)(a) \end{aligned}$$

where $m(a,b)$ denotes the $\mathfrak{B}$-component of $(ga)_{1/2}\cdot(tb)_{1/2}$. Since $m(a,b)$ does depend on $g$ and $t$ also, we will use $m_{g,t}(a,b)$ for $m(a,b)$ when there is any chance of confusion. Replacing $t$ by $(tb)_{1/2}$ in (49) we obtain

$$\begin{aligned} m(1,ab)+m(a,b) &= 2m(1,b)a+2bD(Q_t(c))D(D(Q_g(c))D(a)) \\ &\quad + 2bD(Q_g(c))D(D(Q_t(c))D(a))+2D(Q_t(c))D(b)D(a) \\ &\quad + k_g(b)k_t(a)+b(k_gk_t+k_tk_g)(a). \end{aligned} \tag{50}$$

Define

$$S_{g,t}(a)=m(1,a)-m(1,1)a-2D(Q_g(c))D(D(Q_t(c))D(a))-k_gk_t(a) \tag{51}$$

for all $a\in\mathfrak{B}$. If $g=t$ the right-hand side of (51) reduces to identity (49) with $g=t$. Therefore $S_{g,g}$ is identically zero. A simple linearization gives us

$$S_{g,t}=-S_{t,g}. \tag{52}$$

Substituting (51) into (50) and letting $a=b$ we have $S_{g,t}(a^2)+2L_gL_t(a^2)+m(a,a)+k_gk_t(a^2)=2S_{g,t}(a)a+m(1,1)a^2+4aL_gL_t(a)+2ak_gk_t(a)$ where $L_g=D(Q_g(c))D$ and $L_t=D(Q_t(c))D$ are derivations. Interchanging $g$ and $t$ in this result and subtracting gives us $2S_{g,t}(a^2)+2L_gL_t(a^2)-2L_tL_g(a^2)+(k_gk_t-k_tk_g)(a^2)=4S_{g,t}(a)a+4a(L_gL_t-L_tL_g)(a)+2a(k_gk_t-k_tk_g)(a)$. Since both $L_gL_t-L_tL_g$ and $k_gk_t-k_tk_g$ are derivations this relation reduces to $S_{g,g}(a^2)=2aS_{g,t}(a)$. Hence $S_{g,t}$ is a derivation of $\mathfrak{B}$ into $\mathfrak{B}$.

We can now replace (50) by

$$\begin{aligned} m(a,b) &= m(1,1)ab+aS_{g,t}(b)-bS_{g,t}(a)+2aL_gL_t(b)+2bL_tL_g(a) \\ &\quad -2L_g(a)L_t(b)+ak_gk_t(b)+bk_tk_g(a)-k_g(a)k_t(b). \end{aligned} \tag{53}$$

By setting $g=t$, $a=1$ and $b=Q_g(c)$ in (53) we have

$$m_{g,g}(1,1)Q_g(c)=0. \tag{54}$$

An examination of the $(w\mathfrak{B})_{1/2}$-components of the terms of $P(g,g,g,z)=0$ gives us

$$Q_g(c)D(m_{g,g}(1,1))=0. \tag{55}$$

Finally we compute $P((ga)_{1/2},(tb)_{1/2},w,z)=0$ to get

$$n_{g,t}(a,b)=-aQ_g(f_t(1)b-T_t(b)+D(k_t(b)-)bQ_t(f_g(1)a-T_g(a)+D(k_g(a)) \tag{56}$$

where $n_{g,t}(a,b)$ is the $\mathfrak{B}z$-component of $(ga)_{1/2}\cdot(tb)_{1/2}$. Now $P(g,g,(wa)_{1/2},z)=0$ Therefore $n_{g,g}(1,1)a+2Q_g(f_g(1)a)+2Q_g(T_g(a))=0$. From (56) with $g=t$ and $a=b=1$ we have

$$Q_g(T_g(a))=-Q_g(a)f_g(1). \tag{57}$$

2. In the previous section we expressed the multiplications of $\mathfrak{A}$ in terms of constants and derivations. In this section we use these multiplicative properties to construct a simple power-associative algebra of degree two from an associative algebra.

Let $\mathfrak{B}$ be an associative, commutative algebra over a field $\mathfrak{F}$ of characteristic $p > 5$. Also assume that $\mathfrak{B}$ has a single nonzero idempotent 1 that is a unity quantity.

Let $\mathfrak{B}_0, \dots, \mathfrak{B}_{n-1}$ be $n$ homomorphic images of the vector space $\mathfrak{B}$. We let $\mathfrak{L}$ be a sum of these $n$ vector spaces, but not necessarily the vector space direct sum. We let $z\mathfrak{B}$ be a one-dimensional module over $\mathfrak{B}$. Clearly $z\mathfrak{B}$ is a vector space over $\mathfrak{F}$ and we form the vector space direct sum $\mathfrak{A} = \mathfrak{B} + \mathfrak{L} + z\mathfrak{B}$. We now extend the multiplication of $\mathfrak{B}$ to $\mathfrak{A}$ in such a way that $\mathfrak{A}$ remains a commutative, power-associative algebra. First we define

$$(za)(zb) = (zb)(za) = ab, \tag{58}$$

$$1x = x, \tag{59}$$

$$zy = 0 \tag{60}$$

for every $a$ and $b$ in $\mathfrak{B}$, every $x$ in $\mathfrak{A}$ and every $y$ in $\mathfrak{L}$. The element $e = (1/2)(1+z)$ is an idempotent. We have already defined sufficient multiplicative properties to determine an idempotent decomposition of $\mathfrak{A}$. Clearly $\mathfrak{L} \subseteq \mathfrak{A}_e(1/2)$ and $\mathfrak{B} + \mathfrak{B}z \subseteq \mathfrak{A}_e(1) + \mathfrak{A}_e(0)$. The second part of this statement follows by consideration of $a + bz = (c + cz) + (d - dz)$ with $2c = a + b$ and $2d = a - b$. For each of the vector spaces $\mathfrak{B}_i$ and the corresponding homomorphism of $\mathfrak{B}$ onto $\mathfrak{B}_i$ we define $(g_i b)_{1/2}$ to be the image of $b$. Since this notation is consistent with that of the decomposition of $\mathfrak{A}$ with respect to $e$ we will allow the confusion of the two notations.

In order to complete our definitions of the multiplications of $\mathfrak{A}$ we choose elements $b_{ij}$ and $b_i$ of $\mathfrak{B}$ and derivations $D_{ij}$ and $D_i$ on $\mathfrak{B}$ into $\mathfrak{B}$ for $i,j = 0,1,\dots,n-1$ with the following restrictions:

$$D_{ij} = -D_{ji}, \qquad b_{ij} = b_{ji}, \qquad b_0 = 0 \tag{61}$$

for all values of $i$ and $j$ and

$$\begin{gathered} b_i b_j = (b_i + b_j)b_{ij} = 0, \\ b_i D_0(b_j) = (b_i + b_j)D_0(b_{ij}) = D_i(b_j b) + D_j(b_i b) = 0, \\ b_j D_0 D_i(b) + b_i D_0 D_j(b) = b_j D_i(b) = 0, \\ (b_i g_j + b_j g_i)_{1/2} = 0, \qquad b_i b_{0i} D_0 = -b_i D_0 D_{0i} \end{gathered} \tag{62}$$

for all $i$ and $j$ different from 0 and all $b \in \mathfrak{B}$. We now define

$$(g_i a)_{1/2} b = [g(ab)]_{1/2} - D_0(ab_i)D_0(b) - 2b_i a D_0^2(b) + aD_i(b)z, \tag{63}$$

$$(g_i a)_{1/2}(bz) = -[(g_i a)_{1/2} b]z + \{g_0[aD_0(b)b_i]\}_{1/2}, \tag{64}$$

$$\begin{aligned} (g_i a)_{1/2}(g_j b)_{1/2} = {} & abb_{ij} + aD_{ij}(b) - bD_{ij}(a) + aD_j D_i(b) + bD_i D_j(a) \\ & - D_j(b)D_i(a) + 2aL_i L_j(b) + 2bL_j L_i(a) - 2L_j(b)L_i(a) \\ & + ab_i\{D_0[D_{0j}(b) - b_{0j}b - D_0 D_j(b)]\}z \cdot bb_j D_0[D_{0i}(a) - b_{0i}a - D_0 D_i(a)]z \end{aligned} \tag{65}$$

where $L_i = D_0(b_i)D_0$, $i,j = 0,\ldots,n-1$, and $a$ and $b \in \mathfrak{B}$. Since we did not restrict $\mathfrak{L}$ to be a direct sum of subspaces it is necessary to assume that our multiplications in $\mathfrak{A}$, as defined above, are well-defined. We place two additional assumptions on $\mathfrak{A}$. If $\mathfrak{D}$ is the set of derivations consisting of $D_i$ and $D_{ij}$ for all $i$ and $j$ we assume, in the terminology of Albert [3], that $\mathfrak{B}$ is $\mathfrak{D}$-simple; i.e., there is no nontrivial ideal $\mathfrak{J}$ of $\mathfrak{B}$ such that $\mathfrak{J}$ is $\mathfrak{D}$-admissible. The second assumption is that for every element $g$ in $\mathfrak{L}$ there is a $t$ in $\mathfrak{L}$ such that $gt$ is not zero.

THEOREM 7. *Every commutative, power-associative, simple algebra of degree two over an algebraically closed field $\mathfrak{F}$ of characteristic $p \neq 2,3,5$ is an algebra of the type described above.*

**Proof.** We choose a set of elements $g_1,\ldots,g_{n-1}$ in $\mathfrak{G}$ such that every element of $\mathfrak{G}$ is expressible in the form $\sum(g_ia_i)_{1/2}$ where $a_i \in \mathfrak{B}$. We translate the notation of §1 to the notation of this section by letting $\mathfrak{L} = \mathfrak{A}_e(1/2)$, $g_0 = w$, $D_0 = D$, $b_{00} = w^2$, $b_{0i} = f_{g_i}(1)$, $D_{0i} = T_{g_i}$, $D_i = k_{g_i}$, $b_i = Q_{g_i}(c)$, $b_{ij} = m_{g_i,g_j}(1,1)$ and $D_{ij} = S_{g_i,g_j}$ where $i,j \neq 0$. Identities (25)–(57) give us the relations (61)–(65).

If $\mathfrak{J}$ is a nontrivial ideal of $\mathfrak{B}$ that is $\mathfrak{D}$-admissible then if $a \in \mathfrak{J}$ we have $Q_g(a)$, $f_{ga}(b)$, $\phi_g(a)$, $\phi_{ga}(b)$, $f_{gb}(a)m_{g,t}(a,b)$ and $n_{g,t}(a,b) \in \mathfrak{J}$. This is sufficient to guarantee that $\mathfrak{J} + \mathfrak{J}z + (w\mathfrak{J})_{1/2} + (\mathfrak{G}\mathfrak{J})_{1/2}$ is a proper ideal of $\mathfrak{A}$. Since this contradicts the simplicity of $\mathfrak{A}$ we have that $\mathfrak{B}$ is $\mathfrak{D}$-simple.

Let $(wa)_{1/2} + g$ be an element of $\mathfrak{A}_e(1/2)$ such that there is no element $t$ in $\mathfrak{A}_e(1/2)$ such that $(wa)_{1/2}t + gt \neq 0$. Choosing $t$ to be successively $w$, $(wc)_{1/2}$ and $(wc^2)_{1/2}$ and considering only the $\mathfrak{B}$-components of the resulting terms we have $w^2a + D^2(a) + f_g(1) = w^2ac + cD^2(a) - D(a) + f_g(1)c + T_g(c) = w^2ac^2 + c^2D^2(a) + 2a - 2cD(a) + f_g(1)c^2 + 2cT_g(c) = 0$. Eliminating $w^2$ from these equations we have $-D(a) + T_g(c) = 2a - cD(a) + cT_g(c) = 0$. Hence $a = 0$ and $f_g(1) = T_g(c) = 0$. If we multiply $g$ by $(wb)_{1/2}$ for $b \in \mathfrak{B}$ we have $f_g(b) = \phi_g(b) = 0$ by our assumption on $g$. By a previous result we had that $Q_g(c)$ was a multiple of $\phi_g(c^3)$. Hence $Q_g(c) = 0$. Now $f_g(b) = T_g(b) + k_g(D(b)) = 0$ for all $b \in \mathfrak{B}$. If we substitute $bc$ for $b$ we have $cT_g(b) + ck_g(D(b)) + k_g(b) = 0$. Therefore $k_g(b) = 0$. We now have that $\mathfrak{C}g = \{(ag)_{1/2} : a \in \mathfrak{B}\}$. With this choice of $g$ and for any $b \in \mathfrak{B}$ we have $f_{ga}(b) = 0$ by (35) and $\phi_{ga}(b) = 0$ since $Q_{ga}(c) = aQ_g(c)$. Also $m_{g,t}(a,b) = aS_{g,t}(b) - bS_{g,t}(a)$. But by the assumption on $g$ and (51) we have $S_{g,t} = 0$. Therefore $m_{g,t}(a,b) = 0$ for all $a$ and $b\mathfrak{B}$. Combining this result with (56) we have $(ga)_{1/2}t = 0$ for all $a \in \mathfrak{B}$ and all $t \in \mathfrak{A}_e(1/2)$. Therefore the ideal generated by $g$ is $\{(ag)_{1/2} : a \in \mathfrak{B}\}$. This contradicts the assumption of simplicity of $\mathfrak{A}$. Hence for each $x \in \mathfrak{A}_e(1/2)$ there is an element $t$ in $\mathfrak{A}_e(1/2)$ such that $xt \neq 0$.

THEOREM 8. *An algebra $\mathfrak{A}$ over a field $\mathfrak{F}$ of characteristic $p \neq 2,3,5$ as described in identities* (58)–(65) *is a commutative, power-associative, simple algebra.*

**Proof.** It follows readily from the definition of $\mathfrak{A}$ that $\mathfrak{B}+\mathfrak{B}z+(g_0\mathfrak{B})_{1/2}$ is a subalgebra of $\mathfrak{A}$. We shall show that this subalgebra is power-associative by examining $P(x,y,s,t)$ for various values in $\mathfrak{B}+\mathfrak{B}z+(g_0\mathfrak{B})_{1/2}$. If $P(x,y,s,t)=0$ for all possible choices of the variables $x$,$y$,$s$ and $t$ in $\mathfrak{B}$, $\mathfrak{B}z$ or $(g_0\mathfrak{B})_{1/2}$ we have $\mathfrak{B}+\mathfrak{B}z+(g_0\mathfrak{B})_{1/2}$ power-associative. We examine the powers of $x=a+g_0$ for $a\in\mathfrak{B}$. We have $x^2=a^2+b_{00}+(ag_0)_{1/2}+2D_0(a)z$, $x^3=a^3+2ab_{00}-D_0^2(a)+5aD_0(a)z+D_0(b_{00})z+[(2a^2+b_{00})g_0]_{1/2}$ and $x^2x^2=x^3x$. The proof of this result depends on the properties

$$
\begin{aligned}
a(bz) &= (ab)z,\\
(az)(bz) &= ab,\\
(bz)(g_0a)_{1/2} &= -aD_0(b), \qquad (66)\\
b(g_0a)_{1/2} &= [(ab)g_0]_{1/2}+aD_0(b)z,\\
(g_0a)_{1/2}(g_0b)_{1/2} &= abb_{00}+aD_0^2(b)+bD_0^2(a)-D_0(a)D_0(b).
\end{aligned}
$$

If $d\in\mathfrak{B}$ and if we replace $D_0$ by $dD_0$, $b_{00}$ by $b_{00}d^2+2dD_0^2(d)-D_0(d)^2$ and $g_0$ by $(g_0d)_{1/2}$ we see that relations similar to those expressed in (66) hold. Therefore we can conclude that $a+(g_0d)_{1/2}$ has a unique fourth power.

Next we investigate the fourth powers of $x=az+g_0$. We have $x^2=a^2+b_{00}-2D_0(a)$, $x^3=a^3z+b_{00}az+D_0(b_{00})z-2D_0^2(a)z+a^2+b_{00}-[2D_0(a)g_0]_{1/2}$ and $x^2x^2=x^3x$. Again the only multiplicative properties used were those expressed in (66). Therefore $az+(g_0b)_{1/2}$ has a unique fourth power for all $a$ and $b\in\mathfrak{B}$. It is easily seen that $\mathfrak{B}+\mathfrak{B}z$ is associative. Hence $a+bz$ has a unique fourth power. The assumption on the characteristic and simple linearizations of these three fourth powers we have obtained give us the result that $P(x,y,s,t)=0$ provided that in any evaluation the four values $x$, $y$, $s$, and $t$ are chosen from only two of the three subspaces $\mathfrak{B}$, $\mathfrak{B}z$ and $(g_0\mathfrak{B})_{1/2}$. This leaves us those choices of $x,y,s$ and $t$ for which $x\in\mathfrak{B}$, $y\in\mathfrak{B}z$, $s\in(g_0\mathfrak{B})_{1/2}$ and $t$ is arbitrary. Because of the linearization process we need only consider $P(a,bz,(g_0d)_{1/2},a)$, $P(a,bz,(g_0d)_{1/2},bz)$ and $P(a,bz,(g_0d)_{1/2},(g_0d)_{1/2})$. Straightforward computations, which we omit, show that each of these relations is zero. Therefore $\mathfrak{B}+\mathfrak{B}z+(g_0\mathfrak{B})_{1/2}$ is power-associative.

Now let $g=\sum(g_ia_i)_{1/2}$ where $a_i\in\mathfrak{B}$. The index $i$, or indices $i$ and $j$, of this summmation and all subsequent ones will run from 1 to $n-1$. Define

$$
\begin{aligned}
b_g &= \textstyle\sum a_ib_i,\\
D_g &= \textstyle\sum a_iD_i,\\
b_{0g} &= \textstyle\sum a_ib_{0i}-\sum D_{0i}(a_i)+\sum D_0D_i(a_i), \qquad (67)\\
D_{0g} &= \textstyle\sum a_iD_{0i}-\sum D_i(a_i)D_0,\\
b_{gg} &= \textstyle\sum b_{ij}a_ia_j+2\sum a_iD_{ij}(a_j)+4\sum a_jL_jL_i(a_i)\\
&\quad -\textstyle\sum D_i(a_i)D_j(a_j).
\end{aligned}
$$

From (62) and (67) we have

$$(68)\qquad \begin{aligned} b_g^2 &= b_g b_{gg} = b_g D_0(b_{gg}) = b_g D_g(b) = D_g(b_g) = b_g D_0 D_g(b) = 0,\\ b_g b_{0g} D_0(a) &= -b_g D_0 D_{0g}(a),\\ (gb_g)_{1/2} &= 0. \end{aligned}$$

From (65) we have $(ga)_{1/2}(ga)_{1/2} = b_{gg} + 2aD_g^2(a) - D_g(a)^2 + 4\sum aa_iL_ia_jL_j(a - 2\sum a_iL_i(a)a_jL_j(a)$. Now $\sum a_iL_i(a) = \sum a_iD_0(b_i)D_0(a) = D_0(b_g)D_0(a) - \sum b_iD_0(a_i)D_0(a)$. Therefore $\sum a_iL_i(a)a_jL_j(a) = L_g(a)^2$ where $L_g = D_0(b_g)D_0$. Also

$$\begin{aligned} \sum a_iL_ia_jL_j(a) &= \sum L_ga_jL_j(a) - \sum b_iD_0(a_i)D_0a_jL_j(a)\\ &= L_g^2(a) - \sum L_gb_jD_0(a_i)D_0(a) - \sum b_iD_0(a_i)D_0a_jL_j(a)\\ &= L_g^2(a) - \sum D_0(b_i)D_0(b_j)a_iD_0(a_j)D_0(a) - \sum b_iD_0^2(b_j)a_jD_0(a_i)D_0(a)\\ &= L_g^2(a). \end{aligned}$$

Therefore

$$(69)\qquad (ga)_{1/2}^2 = b_{gg} + 2aD_g^2(a) - D_g(a)^2 + 4aL_g^2(a) - 2L_g(a)^2.$$

We also have

$$(70)\qquad \begin{aligned} b(ga)_{1/2} &= g(ab)_{1/2} - D_0(ab_g)D_0(b) - 2b_gaD_0^2(b) + aD_g(b)z,\\ (bz)(ga)_{1/2} &= g_0(aD_0(b)b_g)_{1/2} - [(ga)_{1/2}b]z \end{aligned}$$

for all $a$ and $b$ in $\mathfrak{B}$.

We now let $g_0' = g_0 + g$ and $a' = a - b_gD_0(a)z$ for $a \in \mathfrak{B}$. We define a derivation $D_0'(a') = [D_0(a) + D_0(b_g)^2D_0(a) + D_g(a)]'$ and let $t = b_{00} + 2b_{0g} - b_gD_0(b_{00})z + b_{gg} - 2b_gD_0(b_{0g})z$. Now $(D_0 + D_0(b_g)^2D_0 + D_g)^2 = (D + D_g)^2 + 2L_g^2$. Therefore $a'D_0'^2(a') = a(D_0 + D_g)^2(a) + 2aL_g^2(a) - b_g[aD_0^3(a) + aD_0^2D_g(a) + D_0(a)D_0^2(a)]z$ since $3b_gD_0L_g^2(a) = 3b_gD_0^2b_gD_0^2(b_g)D_0(a) = -3D_0(b_g)D_0(b_g)D_0^2(b_g)D_0(a) = 2D_0(b_g)b_gD_0^3(b_g)D_0(a) = 0$. Also $[(D_0 + D_0(b_g)^2D_0 + D_g)(a)]^2 = [(D_0 + D_g)(a)]^2 + 2L_g(a)^2$. Therefore $[D_0'(a')]^2 = [(D_0 + D_g)(a)]^2 + 2L_g(a)^2 - 2b_gD_0^2(a)D_0(a)z$. We have, using these results, that $(g_0'a')_{1/2}^2 = (g_0'a)_{1/2}^2 = b_{00}a^2 + 2aD_0^2(a) - D_0(a)^2 + 2a^2b_{0g} + 2aD_gD_0(a) + 2aD_0D_g(a) - 2D_g(a)D_0(a) - ab_gaD_0(b_{00})z - 2b_{00}D_0(a)ab_gz - 2ab_gD_0^3(a)z + b_{gg}a^2 + 4aL_g^2(a) - 2L_g(a)^2 + 2aD_g^2(a) - D_g(a)^2 - 2b_ga^2D_0(b_{0g})z - 2ab_gb_{0g}D_0(a) + 2ab_gD_0D_{0g}(a)z - 2b_gaD_0^2D_g(a)z = t(a^2)' + 2a'D'^2(a') - D'(a')^2 + 2b_gb_{0g}aD_0(a) + 2b_gaD_0D_{0g}(a) = t(a^2)' + 2a'D'^2(a') - D'(a')^2$. Since $t = g_0'^2$ we have

$$(71)\qquad (g_0'a')_{1/2}^2 = g_0'^2 + 2a'D_0'^2(a) - D_0'(a')^2.$$

From (68) and (70) we have

$$
\begin{aligned}
a'(b'z) &= (a'b')z = (ab)'z, \\
(a'z)(b'z) &= a'b' = (ab)', \\
(b'z)(g_0a')_{1/2} &= -a'D_0'(b'), \\
b'(g_0a')_{1/2} &= [(a'b')g_0']_{1/2} + a'D_0'(b')z.
\end{aligned} \tag{72}
$$

If $\mathfrak{B}'$ is the set of all elements of the form $a'$ where $a \in \mathfrak{B}$ then $\mathfrak{B}' + \mathfrak{B}'z + (g_0'\mathfrak{B}')_{1/2}$ is a subalgebra with multiplications similar to those expressed in (66). Hence we can conclude that this subalgebra is power-associative and that $a' + b'z + (g_0'd')_{1/2}$ has a unique fourth power for every $a', b'$ and $d' \in \mathfrak{B}'$. But $\mathfrak{C} = \mathfrak{B}' + \mathfrak{B}'z$. Therefore $a + bz + (g_0d + gd)_{1/2}$ has a unique fourth power for every $a, b, d \in \mathfrak{B}$ and every $g$. If $d$ is nonsingular then $d$ can be absorbed in the coefficients $a_i$ of $g_i$ in the expression for $g$. Hence $a + bz + (g_0d)_{1/2} + g$ has a unique fourth power if $d$ is nonsingular. We can restate this as $x = g_0 + \alpha(a + bz) + \beta(g_0d)_{1/2} + \gamma g$ has a unique fourth power for $d$ a singular element of $\mathfrak{B}, a, b \in \mathfrak{B}$, $g = \sum(a_ig_i)_{1/2}$ and $\alpha, \beta \in \mathfrak{F}$. The characteristic is sufficiently high so that the attached polynomials of the expression $x^2x^2 - x^4$ are all zero [6]. The sum of those polynomials with a coefficient $\alpha^i\beta^j\gamma^k$ where $i + j + k = 4$ is of course also equal to zero. But by replacing $\alpha, \beta$ and $\gamma$ by 1 in this sum we get $y^2y^2 - y^4 = 0$ where $y = (a + bz + (g_0d)_{1/2} + g)$. Hence any element of $\mathfrak{A}$ has a unique fourth power and $\mathfrak{A}$ is power-associative.

To complete the proof it remains only to show the simplicity of $\mathfrak{A}$. Let $\mathfrak{I}$ be a proper ideal of $\mathfrak{A}$ with the nonzero element $a + bz + t$ where $a, b \in \mathfrak{B}$ and $t \in \mathfrak{L}$. Since $z\mathfrak{I} \subseteq \mathfrak{I}$ we have $az + b \in \mathfrak{I}$. Now multiply $az + b$ by $g_0$ to get $(ag_0)_{1/2} + D_0(a)z - D_0(b) \in \mathfrak{I}$. By the above $(ag_0)_{1/2} \in \mathfrak{I}$. Multiplying this element by $cz$ we get $a \in \mathfrak{I}$ and therefore $b, t$, $D(a)$ and $D(b) \in \mathfrak{I}$. Let $\mathfrak{P}$ be the set of all elements of $\mathfrak{B}$ that are in $\mathfrak{I}$. Clearly, $\mathfrak{P}$ is a proper ideal of $\mathfrak{B}$. Since $\mathfrak{P}\mathfrak{L} \subseteq \mathfrak{I}$ and $(\mathfrak{P}\mathfrak{L})_{1/2}\mathfrak{L} \subseteq \mathfrak{I}$ it can be easily shown that $\mathfrak{P}$ is $\mathfrak{D}$-admissible. Hence $\mathfrak{P} = 0$ and the only nonzero elements that could be in $\mathfrak{I}$ are of the form $t$ where $t \in \mathfrak{L}$. But by the assumption on $\mathfrak{A}$ there is an $x \in \mathfrak{L}$ such that $gx \neq 0$. Since $gx \in \mathfrak{B} + \mathfrak{B}z$ and $\mathfrak{I} \cap (\mathfrak{B} + \mathfrak{B}z) = 0$ we must have $\mathfrak{I} = 0$. Therefore $\mathfrak{A}$ is simple.

To further characterize the algebra $\mathfrak{A}$ and its subalgebra $\mathfrak{B}$ we quote a result of Harper [5, Theorem 1].

THEOREM 9. *Let $\mathfrak{B}$ be a commutative, associative algebra with unity* 1 *over an algebraically closed field $\mathfrak{F}$, and let $\mathfrak{B}$ be $\mathfrak{D}$-simple relative to a set of derivations of $\mathfrak{B}$ over $\mathfrak{F}$. Then $\mathfrak{B} = \mathfrak{F}[1, x_1, \ldots, x_n]$ is an algebra with generators $x_1, \ldots, x_n$ over $\mathfrak{F}$ which are independent except for the relations $x_1^p = \ldots = x_n^p = 0$ where $p$ is the characteristic of $\mathfrak{F}$.*

3. Let $p$ be a prime $\neq 2, 3, 5$ and let $\mathfrak{B}$ be the associative commutative algebra of all polynomials $\sum_{i=0}^{p-1}\alpha_ic^i$ in $c$ with $c^p = 0$ and $c^0 = 1$, the identity of $\mathfrak{B}$. Let $\mathfrak{L}$ be $\{(g_0a)_{1/2} : a \in \mathfrak{B}\}$. Then $\mathfrak{A} = \mathfrak{B} + \mathfrak{B}z + (g_0\mathfrak{B})_{1/2}$. Let $b_{00} = 0$ and $D_0$ be ordinary polynomial differentiation; i.e., $D_0(c) = 1$. Assume that $u = a + bz$

$+ (g_0 d)_{1/2}$, where $a, b, d \in \mathfrak{B}$, is an idempotent of $\mathfrak{A}$ that is not in $\mathfrak{C}$. Then $a^2 + b^2 + 2dD_0^2(d) - D_0(d)^2 - 2dD_0(b) + 2abz + 2dD_0(a)z + 2(g_0(da))_{1/2} = a + bz + (g_0 d)_{1/2}$. Therefore $d(2a-1) = 0$ and $2ab + 2dD_0(a) = b$. If $d = 0$ then $u \in \mathfrak{C}$. By our assumptions $d \neq 0$ and we must have $2a - 1$ is singular. Therefore we can write $a = 1/2 + c^t s$ where $s$ is a nonsingular element of $\mathfrak{B}$ and $t \geqq 1$. We have $dc^t = 0$ and $c^t b + tc^{t-1}d = 0$. Hence $c^{t+1}b = 0$. Since

$$a^2 + b^2 + 2dD_0^2(d) - D_0(d)^2 - 2dD_0(b) = a \tag{73}$$

it follows that $a^2 c^{t+1} = ac^{t+2}$. But this implies that $c^{t+1} = 2c^{t+1}$. Hence $t + 1 \geqq p$. Assume $t = p - 1$; then $c^{p-1}b = c^{p-2}d$. Now if $b = \sum_0^{p-1} \beta_i c^i$ and $d = \sum_0^{p-1} \alpha_i c^i$ then we must have $\alpha_0 = 0$ and $\beta_0 = \alpha_1$. From (73) we must also have $\beta_0^2 - \alpha_1^2 = 1/4$ which is a contradiction. Therefore $t + 1 > p$ and $a = 1/2$.

Let $x' = a' + b'z + (g_0 d')_{1/2}$ be an arbitrary element of $\mathfrak{A}$. By considering the product $x'u$ we see that a necessary and sufficient condition that $x' \in \mathfrak{A}_u(1)$ is that

$$\begin{aligned} 2a'd &= d', \\ 2ba' + 2D_0(a')d &= b'. \end{aligned} \tag{74}$$

The correspondence $a' \to a' + 2a'bz + 2D_0(a')dz + 2[g_0(a'd)]_{1/2}$ is clearly a 1-1 correspondence between $\mathfrak{B}$ and $\mathfrak{A}_u(1)$ preserving the vector space operations. Therefore $\mathfrak{A}_u(1)$ is of dimension $p$.

If $u$ is a stable idempotent then Albert has shown [3; 4] that $\mathfrak{A} = \mathfrak{A}_u(1) + \mathfrak{A}_u(0) + (w\mathfrak{C}') + \mathfrak{G}$ where $\mathfrak{C}' = \mathfrak{A}_u(1) + \mathfrak{A}_u(0)$ and $w\mathfrak{C}' + \mathfrak{G} = \mathfrak{A}_u(1/2)$. Albert also showed that the dimensions of $\mathfrak{A}_u(1)$, $\mathfrak{A}_u(0)$ and $w\mathfrak{C}'$ are all equal. Therefore $\mathfrak{G} = 0$. A further result of Albert's is that $\mathfrak{A}_u(1) + \mathfrak{A}_u(0) + w\mathfrak{C}'$ is associative. This implies that $\mathfrak{A}$ is a simple, associative algebra and hence we must have $c = 0$. We can conclude that our example contains no stable idempotents.

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