FIXED POINT FREE INVOLUTIONS AND EQUIVARIANT MAPS. II(1)

BY

P. E. CONNER AND E. E. FLOYD

1. **Introduction.** We shall continue our discussion of fixed point free involutions which was begun in [2]. We denote by S^n the antipodal involution on the n-sphere. For any fixed point free involution on a space X the co-index was defined to be the least integer n for which there is an equivariant map $X \to S^n$. We abbreviate this invariant to co-ind X. In this terminology the classical Borsuk theorem states that co-ind $S^n = n$. There are also numerous results (for references, see [2]) which among other things relate co-index to the homology of the quotient space X/T. The main purpose of the present note is the computation of the co-index in several examples in which homotopy, rather than homology, considerations are of primary importance. It should be mentioned that A. S. Svarc has also recently studied the application of homotopy theory to equivariant maps [5]; there is a considerable overlap between his work and our previous paper [2].

We consider as in our previous paper the space $P(S^n)$ of paths on S^n which join a given point x to its antipode $A(x) = -x$ together with the natural involution of $P(S^n)$. It is shown that co-ind $P(S^n) = n$ for $n \neq 1, 2, 4$ or 8. Next we consider the space $V(S^n)$ of unit tangent vectors to S^n, with its involution (the antipodal map on each fibre), and show that co-ind $V(S^n) = n$ for $n \neq 1, 3$, or 7 and co-ind $V(S^n) = n - 1$ for $n = 1, 3$ or 7. We also compute the co-index of involutions on low dimensional projective spaces. The arguments rely on suspension and Hopf invariant theorems, using particularly the results of J. F. Adams [1] on maps of Hopf invariant one.

2. **The space of paths** $P(S^n)$. We choose a base point $x \in S^n$ and we let $P(S^n)$ denote the space of all paths in S^n which join x to its antipode $-x$. A fixed point free involution on $P(S^n)$ is given by $\Gamma(p)(t) = -p(1-t)$, where $p(t)$ is a point in $P(S^n)$. In this section we show

(2.1) THEOREM. *For* $n \neq 1, 2, 4$ *or* 8, co-ind $P(S^n) = n$.

We showed this for $n > 1$ and odd in [2, p. 425] and we conjectured this result as the general case. We see first that co-ind $P(S^n) \leqq n$ by defining an equivariant map $m: P(S^n) \to S^n$ as $m(p(t)) = p(1/2) \in S^n$.

Now we suppose there is an equivariant map m_1: $P(S^n) \to S^{n-1}$. We define an

Received by the editors October 18, 1961.

(1) The authors are Sloan Fellows; this work has also been supported in part by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under contract AF 49(638)-72 at the University of Virginia.

equivarient imbedding map $e: S^{n-1} \to P(S^n)$ as follows. We regard S^n as the suspension of S^{n-1} with vertices x and $-x$. Let $f: S^{n-1} \times I \to S^n$ be the natural map with $f(S^{n-1} \times 0) = x, f(S^{n-1} \times 1) = -x$. To each $y \in S^{n-1}$ we assign the path $p(t) = f(y,t)$. This is seen to be equivariant if we note that antipodal involution on S^n is induced by f from the involution $(y,t) \to (-y, 1-t)$.

The composite map $m_1 e: S^{n-1} \to S^{n-1}$ is an equivariant map of the antipodal involution into itself, thus it has odd degree. In particular $(m_1 e)_*: H_*(S^{n-1}; Z_2) \simeq H_*(S^{n-1}; Z_2)$. Now we have assumed $n \geqq 3$, thus we may apply the Whitehead theorem to conclude $(m_1 e)_* : \pi_k(S^{n-1}) \simeq \pi_k(S^{n-1}) \bmod C$, the class of finite groups with odd order. In particular then the kernel of $e_* : \pi_k(S^{n-1}) \to \pi_k(P(S^n))$ is a finite subgroup of odd order.

Now consider a map $\phi: P(S^n) \to \Omega(S^n)$ given as follows. Choose a great circle arc g which joins $-x$ to x and define $\phi(p)$ to be the composite path $g \cdot p$, which is a loop based at x. The composition

$$\pi_k(S^{n-1}) \overset{e_*}{\to} \pi_k(P(S^n)) \underset{\simeq}{\overset{\varphi_*}{\to}} \pi_k(\Omega(S^n)) \simeq \pi_{k+1}(S^n)$$

is exactly the Freudenthal suspension homomorphism $\Sigma : \pi_k(S^{n-1}) \to \pi_{k+1}(S^n)$. The kernel of e_* is just the kernel of Σ. Now G. W. Whitehead has characterized the kernel of $\Sigma : \pi_{2n-3}(S^{n-1}) \to \pi_{2n-2}(S^n)$ [6]. Namely Ker Σ is generated by the bracket $[i,i]$, where i is the identity map of S^{n-1}. For n odd, $[i,i]$ has Hopf invariant 2, and is thus of infinite order. This contradicts the assertion that $\ker(e_*)$ is finite. For n even, $2[i,i] = 0$, and $[i,i] = 0$ if and only if $\pi_{2n-1}(S^n)$ contains an element of Hopf invariant 1. Thus by Adams' theorem [1], $\ker(e_*) \simeq Z_2$, n even and $n \neq 2$, 4 or 8. This contradicts the assertion that $\ker(e_*)$ has odd order. The proof of (2.1) is now complete.

We have already shown results of the following type for n odd [2, p. 434]. It was also shown that $2n-2$ is the lowest dimension for which the following examples exist.

(2.2) *For $n \neq 1, 2, 4$ or 8 there is a fixed point free involution X on a compact space and an inessential equivariant map $m: X \to S^n$, $\dim X = 2n-2$ and co-ind $X = n$.*

We simply make a few revisions in the proof of (2.1). There is the imbedding $e: S^{n-1} \to P(S^n)$ and the element $[i,i] \in \pi_{2n-3}(S^{n-1})$ with $e_*([i,i]) = 0$. There is a compact $X \subset P(S^n)$ with $S^{n-1} \subset X$ such that under the homomorphism induced by $j: S^{n-1} \subset X$, $j_*([i,i]) = 0$. Such an X can be made invariant. Suppose that such an X could be equivariantly mapped into S^{n-1}. The proof of (2.1) reveals that $\ker j_* \in C$, and $\ker j_* \subset \ker e_*$. A contradiction results, and co-ind $X = n$. Now X can be mapped by an inessential equivariant map into S^n, since $X \subset P(S^n)$ The argument that X can be selected to satisfy the above, and also be of dimension $2n-2$ is tedious and we shall omit it.

The cases $n = 1, 2, 4, 8$ remain open. For $n = 1$ there is an equivariant map

$P(S^1) \to S^0$. For $n = 2$ it is at least true that every compact invariant subset of $P(S^2)$ can be equivariantly mapped into S^1 [2, p. 424]. We conjecture a like result for $n = 4, 8$.

3. **The tangent bundle** $V(S^n)$. Let $V(S^n)$ denote the unit tangent sphere bundle to S^n. Let $(A,V(S^n))$ be the fibre preserving fixed point free involution which, on each fibre, reduces to the antipodal involution. Clearly $(A,V(S^n)) \simeq (T,V_{n+1,2})$, where $V_{n+1,2}$ is the Stiefel manifold of orthonormal 2-frames in R^{n+1} and $T(v_1,v_2) = (v_2,v_1)$.

(3.1) THEOREM. *If* $n \neq 1, 3, 7$, co-ind $V(S^n) = n$, *while if* $n = 1, 3$ *or* 7, co-nd $V(S^n) = n - 1$.

We see that S^{n-1} is equivariantly imbedded in $V(S^n)$ as a fibre. It follows that $n - 1 \leqq$ co-ind $V(S^n)$. There is also an equivariant map $m : V(S^n) \to S^n$. If we take $V_{n+1,2} = V(S^n)$, then there is the map m given by $m(V_1,V_2) = (V_1 - V_2)/\|V_1 - V_2\|$. Furthermore, for $n = 1, 3, 7$ there is an equivariant map $V(S^n) \to S^{n-1}$ since in these cases S^n is parallelizable. It remains to show that if $n \neq 1$, 3 or 7 then there is no equivariant map $V(S^n) \to S^{n-1}$.

Suppose there is such a map $g : V(S^n) \to S^{n-1}$. Let $i : S^{n-1} \to V(S^n)$ be the inclusion map of a fibre. By the Borsuk theorem, the map $gi : S^{n-1} \to S^{n-1}$ has odd degree. Putting it loosely, g is of odd degree on each fibre of $V(S^n)$. We shall now use a construction of Milnor and Spanier [3] to show the following and hence yield (3.1).

(3.2) *Suppose there exists a map* $g : V(S^n) \to S^{n-1}$, *n odd, which is of odd degree on each fibre; then* $n = 1$, 3 *or* 7.

We shall give in some detail the argument outlined for a similar purpose by Milnor and Spanier [3]. In $S^n \times S^n$ we consider a closed tubular neighborhood N of the diagonal $\Delta \subset S^n \times S^n$, with boundary $\dot{N}$ and interior N^0. Now $\dot{N}$ is an $(n - 1)$-sphere bundle over the diagonal Δ which is equivalent to the tangent sphere bundle $V(S^n)$. By hypothesis there is a map $g : \dot{N} \to S^{n-1}$ which is of odd degree on each fibre F_x of $\dot{N}$. It is well known that the inclusion $F_x \subset \dot{N}$ induces an isomorphism $H^{n-1}(\dot{N};Z_2) \simeq H^{n-1}(F_x;Z_2)$. We see thus that $g^* : H^{n-1}(S^{n-1};Z_2) \simeq H^{n-1}(\dot{N};Z_2)$. We now consider the closed n-cell D^n with boundary S^{n-1}. On each of the n-cell fibres F'_x of $N \to \Delta$ we may extend $g : F_x \to S^{n-1}$ to a mapping $F'_x \to D^n$. In this manner we obtain an extension $G : N \to D^n$ of $g : \dot{N} \to S^{n-1}$. The commutative diagram

$$\begin{array}{ccc} H^{n-1}(S^{n-1};Z_2) & \overset{\approx}{\to} & H^n(D^n,S^{n-1};Z_2) \\ \approx \downarrow g^* & & \downarrow G^* \\ H^{n-1}(\dot{N};Z_2) & \overset{\approx}{\to} & H^n(N,\dot{N};Z_2) \end{array}$$

shows that $G^* : H^n(N,\dot{N};Z_2) \simeq H^n(D^n,S^{n-1};Z_2)$.

We next consider a map $S^n \times S^n \to S^n$ given as follows. We think of S^n as obtained from D^n by collapsing S^{n-1} to a point. There is a composite map $N \to D^n \to S^n$ which maps $\dot{N}$ onto the point $a \in S$. We simply extend to a map $H : S^n \times S^n \to S^n$ which maps $S^n \times S^n - N$ onto a. Now we have the diagram

$$\begin{array}{ccccc} H^n(S^n,a;Z_2) & \overset{H_1^*}{\to} & H^n(N,\dot{N};Z_2) & \simeq & H^n(S^n \times S^n, S^n \times S^n - N^0;Z_2) \\ \downarrow\approx & & \downarrow & & \\ H^n(S^n;Z_2) & \overset{H^*}{\to} & H^n(S^n \times S^n;Z_2). & & \end{array}$$

We have seen that H_1^*, which is essentially G^*, is an isomorphism. If $\gamma \in H^n(S^n;Z_2)$ is the generator, then $H^*(\gamma)$ is the nonzero element of $\mathrm{Im}(H^n(N,\dot{N};Z_2) \to H^n(S^n \times S^n;Z_2))$. That is $H^*(\gamma) = \gamma \otimes 1 + 1 \otimes \gamma$.

If now $\hat{\gamma} \in H^n(S^n;Z)$ is a generator of this integral cohomology group, then $H^*(\hat{\gamma}) = c(\hat{\gamma} \otimes 1) + d(1 \otimes \hat{\gamma})$, with c,d odd numbers. Thus $H: S^n \times S^n \to S^n$ is of type (c,d) in the sense of Hopf. In particular $\pi_{2n+1}(S^{n+1})$ contains an element of Hopf invariant 1, so by Adams' result $n=1$, 3 or 7. This completes the proof of (3.2) and of (3.1).

4. **Involutions on projective space.** We consider a fixed point free involution $\tilde{T} : P^n \to P^n$ on real projective n-space (n odd by necessity). Let $\pi : S^n \to P^n$ denote the 2-fold covering. Now $\tilde{T}$ is covered by a map $T : S^n \to S^n$ with $\pi T = \tilde{T}\pi$, thus T^2 covers $\tilde{T}^2 = 1$, hence $T^2 = \pm 1$. Note that T has no fixed points either. The case $T^2 = 1$ is impossible in fact, for if $T^2 = 1$, then (T,S^n) is an involution without fixed points and there must be a point $x \in S^n$ with $T(x) = -x$ (see (4.0) following), and $\pi(x)$ would then be a fixed point of $(\tilde{T},P^n)$. That is, every involution without fixed points on P^n is covered by a $T : S^n \to S^n$ with $T^2 = -1$. We restrict our attention to fixed point free involutions on P^n which are covered by an orthogonal map $T : S^n \to S^n$ with period 4. We shall call such involutions on P^n orthogonal.

Any two fixed point free orthogonal involutions $\tilde{T}_i : P^n \to P^n$ are equivalent in the sense that there is a homeomorphism $h : P^n \to P^n$ with $h\tilde{T}_1 = \tilde{T}_2 h$. It is in fact seen by the representation theory for the group Z_4 that the T_i which cover $\tilde{T}_i$ are equivalent via an orthogonal map $h : S^n \to S^n$ which must carry antipodal pairs into antipodal pairs, and thus inducing the equivalence h between $\tilde{T}_1$ and $\tilde{T}_2$.

(4.0) *If $T : S^n \to S^n$ is a fixed point free involution, there exists $x \in S^n$ with $T(x) = -x$.*

Proof. Suppose the conclusion is false. For every $x \in S^n$, let $g(x)$ be the midpoint of the shortest geodesic arc joining x to Tx; that is, $g(x) = (x + T(x))/\|x + Tx\|$. Then $g(x) = g(Tx)$, and also g is homotopic to the identity. Since $g(x) = g(Tx)$ we may factor g as $S^n \xrightarrow{p} S^n/T \xrightarrow{f} S^n$, where p is the orbit map. Now $p_* : H_n(S^n;Z_2) \to H_n(S^n/T;Z_2)$ is trivial. Hence $g = fp$ is of even degree, and we have a contradiction. That is, $Tx = -x$ for some x.

(4.1) *For every fixed point free orthogonal involution* P^3, co-ind $P^3 = 2$. *For* *every equivariant map $m:P^3\to S^2$ the composite $m\pi:S^3\to S^2$ has odd Hopf invariant.*

To display an equivariant map of P^3 into S^2 we consider the particular map of period 4, $T:S^3\to S^3$ given in complex co-ordinates by $T(Z_1,Z_2) = (-\bar{Z}_2,\bar{Z}_1)$. We consider S^2 as the Gauss sphere and we think of the antipodal map on S^2 as $A(Z) = -1/\bar{Z}$. There is the usual Hopf map $f:S^3\to S^2$ given by $f(Z_1,Z_2) = Z_1/Z_2$ and $fT=Af$, hence $f(-x) = f(x)$ on S^3 and f induces an equivariant map $m:P^3\to S^2$. Thus co-ind $P^3\leqq 2$. Now P^3 is connected and $H^1(P^3;Z)=0$, thus co-ind$_Z$ $P^3\geqq 2$. This refers to the integral co-index of [2]. Since co-ind$_Z$ $\leqq$ co-ind, it follows that co-ind $P^3=2$.

Since co-ind$_Z$ $P^3=$ co-ind$_Z S^2=2$ it follows that for every equivariant map $m:P^3\to S^2$, $m^*:H^2(S^2;Z)\to H^2(P^3;Z)$ is nontrivial [2], that is an epimorphism. It follows that $m^*:H^2(S^2;Z_2)\simeq H^2(P^3;Z_2)$.

We now consider the composite map $m\pi:S^3\to S^2$, and we shall apply Steenrod's characterization of the Hopf invariant [4]. Let M denote the mapping cylinder of $m\pi$, and let M/S^3 be the space obtained from M by shrinking its face S^3 to a point. Now let γ be the generator of $H^2(M/S^3;Z_2)\simeq Z_2$, and note that $H^4(M/S^3;Z_2)\simeq Z_2$ also. As Steenrod points out, $m\pi:S^3\to S^2$ has odd Hopf invariant if and only if $Sq^2(\gamma)=\gamma\cup\gamma\neq 0$. Now we shall compute.

Let M_1,M_2 respectively denote the mapping cylinders of $\pi:S^3\to P^3$ and $m:P^3\to S^2$ respectively. Let $\bar{M}=M_1\cup M_2$ denote their union identified along their common face P^3. It is readily seen that $\bar{M}/S^3$ and M/S^3 have the same homotopy type, so that we may consider $\bar{M}/S^3$. Moreover

$$\bar{M}/S^3 = M_1/S^3 \cup M_2,\ (M_1/S^3)\cap M_2 = P^3.$$

Note that $M_1/S^3=P^4$ and $P^3\subset M_1/S^3$ as P^3 is usually contained in P^4. We now take the Mayer-Vietoris sequence of $\bar{M}/S^3$

$$\begin{array}{ccccc} \dots & H^2(\bar{M}/S^3;Z_2) & \to & H^2(P^4;Z_2)+H^2(S^2;Z_2) & \overset{i^*+m^*}{\to} & H^2(P^3) \\ & \downarrow Sq^2 & & \downarrow Sq^2 & & \\ & H^4(\bar{M}/S^3;Z_2) & \overset{h}{\to} & H^4(P^4;Z_2)+H^4(S^2;Z_2) & \to & H^4(P^3;Z) \end{array}$$

where i^* is induced by inclusion. Both i^* and m^* are isomorphisms. Let α_1, α_2 respectively be the nonzero elements of $H^2(P^4;Z_2)$ and $H^2(S^2;Z_2)$ so that $i^*(\alpha_1)+m^*(\alpha_2)=0$. Hence $h(\gamma)=\alpha_1+\alpha_2$ and

$$hSq^2(\gamma) = Sq^2\alpha_1 + Sq^2\alpha_2 = Sq^2\alpha_1 \neq 0.$$

Thus $Sq^2(\gamma)\neq 0$ and $m\pi:S^3\to S^2$ has odd Hopf invariant.

(4.2) *For every orthogonal fixed point free involution* P^5, co-ind $P^5=4$, *and every equivariant map $m:P^5\to S^4$ is essential.*

First there is no equivariant map $m:P^5\to S^2$, for if there were we would consider $m\pi:S^5\to S^2$ and note that $m:P^3\to S^2$ is equivariant so by (4.1) $m\pi$: $S^3\to S^2$ has odd Hopf invariant. This is impossible since $m\pi : S^3 \to S^2$ is inessential. Hence co-ind $P^5 \geqq 3$.

Next we assert that there is an equivariant map $m : P^5 \to S^4$. It is sufficient to find $f : S^5 \to S^4$ with $fT = Af$. We consider S^5 as the join $S^3 \circ S^1$ and (T,S^5) as the join of (T,S^3) with $z \to iz$. We take (A,S^4) as the join of the antipodal involutions (A,S^2) and (A,S^1). There is already a map $f_1 : S^3 \to S^2$ with $f_1T = Af_1$ and there is the map $f_2 : S^1 \to S^1$ given by $f_2(z) = z^2$. The join $f_1 \circ f_2 : S^5 \to S^4$ is the desired map.

Finally we must show there is no equivariant map $m : P^5 \to S^3$. Suppose there is a map $f : S^5 \to S^3$ with $fT = Af$. We shall first show that we may as well suppose $f(S^3) \subset S^2$. The map $f : S^3 \to S^3$ is certainly inessential, so by the Hopf classification theorem $m : P^3 \to S^3$ is also inessential since $H^3(P^3;Z) \to H^3(S^3;Z)$ is a monomorphism (of degree 2). Hence both $m : P^3 \to S^3$ and the equivariant map $m_1 : P^3 \to S^2 \subset S^3$ constructed in (4.1) are both inessential in S^3 and are homotopic. But just as in [2] it follows that $m,m_1 : P^3 \to S^3$ are equivariantly homotopic. Since m can be equivariantly extended over P^5 it follows that m_1 also admits an equivariant extension. Thus we may as well say $m(P^3) \subset S^2$ and $f(S^3) \subset S^2$.

By (4.1) the map $f : S^3 \to S^2$ has odd Hopf invariant. We now use the particular model $T : S^5 \to S^5$ with $T^2 = -1$ which is given by $T(Z_1,Z_2,Z_3) = (iZ_1,iZ_2, iZ_3)$. Let D_1^4 be the closed 4-cell $\{(Z_1,Z_2,r) : r \geqq 0\}$, and let D_2^4 be $\{(Z_1,Z_2,ir) : r \geqq 0\}$. Let S^4 denote the union $D_1^4 \cup D_2^4$. We claim that $f : S^4 \to S^3$ represents in $\pi_4(S^3)$ the suspension of the element $\pi_3(S^2)$ given by $f : S^3 \to S^2$. Since $f : S^3 \to S^2$ has odd Hopf invariant it follows that $f : S^4 \to S^3$ is essential. On the other hand $f | S^4$ can be extended to $f : D^5 \to S^3$, where D^5 is the 5-cell $\{(Z_1,Z_2,Z_3), \mathrm{Re}(Z_3) \geqq 0, \mathrm{Im}\, Z_3 \geqq 0\}$. From this contradiction (4.2) follows.

We need the following consequence of $\pi_4(S^3) \simeq Z_2$, which we leave to the reader. The consequence is that $f : D_1^4 \to S^3$ is homotopic via h_t, relative to $\partial D_1^4 = S^3$, to a map h_1 of D_1^4 into the upper hemisphere of S^3 (or to a map of D_1^4 into the lower hemisphere). Next we note that $T(D_1^4) = D_2^4$, and recall that $fT = Af$. Then $Ah_tT^{-1} : D_2^4 \to S^3$ is a homotopy, relative to the boundary $\partial D_2^4 = S^3$, with $Ah_0T^{-1} = h_0 = f | D_2^4$ and with Ah_1T^{-1} having values in the lower hemisphere of S^3 (or in the upper hemisphere). The homotopy, h_t on D_1^4, Ah_tT^{-1} on D_2^4, gives a homotopy of $f : S^4 \to S^3$ to the suspension of $f : S^3 \to S^2$. The theorem follows. That every equivariant map $P^5 \to S^4$ is essential follows from the stability theorem of [2, p. 424].

We are unable to compute co-ind P^{2n+1} for $n \geqq 3$. In particular, it is an open question as to whether co-ind $P^7 = 4$ or co-ind $P^7 = 5$. In commenting on our faulty argument in the original manuscript that co-ind $P^7 = 4$, the referee has pointed out that if there is an equivariant map $S^7 \to S^4$ then it represents an element of order two in $\pi_7(S^4)$. He conjectures co-ind $P^7 = 5$.

5. **An addendum.** In this section we consider in more detail the technique in (4.1) involving Steenrod squares. The aim is to obtain information concerning which maps $S^n \to S^k$ can be factored through P^n.

(5.1) *If* $n \not\equiv 3 \bmod 4$, *then for every essential map* $f: S^n \to S^{n-1}$ *there is a pair of antipodal points on* S^n *which is mapped into a pair of antipodal points of* S^{n-1} *by* f.

Suppose on the contrary $f(x)$ and $f(-x)$ are never antipodal. Define $g: S^n \to S^{n-1}$ so that $g(x)$ is the midpoint of the minor arc of the great circle joining $f(x)$ and $f(-x)$. Then g is a homotopic to f, g is also essential and $g(x) = g(-x)$ for all $x \in S^n$ That is $g = \tilde{g}\pi$, where $\pi: S^n \to P^n$ is the covering map and $\tilde{g}: P^n \to S^{n-1}$.

As in (4.1), consider the mapping cylinder M of g, as well as the cylinder M_1 of π and M_2 of $\tilde{g}$. Let $\bar{M}$ denote the union $M_1 \cup M_2$ joined along their common face. We then have $\bar{M}/S^n = M_1/S^n \cup M_2$, $M_1/S^n = p^{n+1}$, and Mayer-Vietoris sequence

$$\begin{array}{ccccc} \ldots \to H^{n-1}(\bar{M}/S^n;Z_2) & \xrightarrow{h} & H^{n-1}(P^{n+1};Z_2) + H^{n-1}(S^{n-1};Z_2) & \xrightarrow{i^* + g^*} & H^{n-1}(P^n;Z_2) \\ \downarrow Sq^2 & & \downarrow Sq^2 & & \\ \ldots \to H^{n+1}(\bar{M}/S^n;Z_2) & \xrightarrow{h'} & H^{n+1}(P^{n+1};Z_2) + H^{n+1}(S^{n-1};Z_2) \ldots . & & \end{array}$$

Let $\gamma \in H^{n-1}(\bar{M}/S^n;Z_2)$ be the generator of this cohomology group. Since g is essential, $Sq^2(\gamma) \neq 0$. Since h' is a monomorphism, $h'(Sq^2(\gamma)) \neq 0$ and $Sq^2(h,(\gamma)) \neq 0$. In particular $h(\gamma)$ has a nonzero component in $H^{n-1}(P^{n+1};Z_2)$, but $h(\gamma)$ is in the kernel of $i^* + g^*$, so $h(\gamma)$ must also have a nonzero component in $H^{n-1}(S^{n-1};Z_2)$. In particular it follows that $g^*: H^{n-1}(S^{n-1},Z_2) \to H^{n-1}(P^n;Z_2)$ is nontrivial. Examination of Sq^1 shows n must then be odd. Since $Sq^2: H^{n-1}(P^{n+1};Z_2) \to H^{n+1}(P^{n+1};Z_2)$ is nontrivial, so $n = 3, 7, 11, \ldots$. The result (5.1) follows.

We note the following addition to (4.2). Let $\tilde{f}: P^5 \to S^4$ be an equivariant map. We have seen that f is always essential; however it follows from (5.1) that the composition $f = \tilde{f}\pi: S^5 \to S^4$ is always inessential.

REFERENCES

1. J. F. Adams, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. (2) **72** (1960), 20–104.

2. P. E. Conner and E. E. Floyd, *Fixed point free involutions and equivariant maps*, Bull. Amer. Math. Soc. **66** (1960), 416–441.

3. John Milnor and Edwin Spanier, *Two remarks on fiber homotopy type*, Pacific J. Math. **10** (1960), 585–590.

4. N. E. Steenrod, *Cohomology invariants of mappings*, Ann. of Math. (2) **50** (1949), 954–988.

5. A. S. Svarc, *Some notions connected with that of the genus of a fibre space*, Dokl. Akad. Nauk. SSSR **136** (1961), 72–74.

6. G. W. Whitehead, *A generalization of the Hopf invariant*, Ann. of Math. (2) **51** (1950), 192–237.

UNIVERSITY OF VIRGINIA,
CHARLOTTESVILLE, VIRGINIA