THE ACTION OF AN ALGEBRAIC TORUS ON THE AFFINE PLANE

BY

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1. **Introduction.** Let G be a connected algebraic group, V a variety. G is said to operate regularly on V if we are given an everywhere defined rational map $g \times v \to g(v)$ of $G \times V \to V$ such that

$$(1) \qquad g_1(g_2(v)) = g_1 g_2(v) \quad \text{for any} \quad g_1, g_2 \in G,\ v \in V,$$

$$(2) \qquad e(v) = v \qquad \text{for any} \qquad v \in V.$$

Denote by k an algebraically closed field, by A^2 the affine plane over k, by G_m the multiplicative group of the universal domain.

Our purpose in this paper is to study by elementary means the regular operation of G_m on A^2. We shall denote by σ a regular operation of G_m on A^2 which is not the identity on A^2, σ: $G_m \times A^2 \to A^2$, and by $\sigma(t)$ the restriction of this map given by $\sigma(t)$: $t \times A^2 \to A^2$, where $t \in G_m$.

We recall that an algebraic torus is the direct product of a finite number of multiplicative groups.

2. **Change of coordinates in** A^2. Let (x,y) be a system of coordinates for A^2. Then if

$$(2.1) \qquad \begin{cases} x' = f(x,y), \\ y' = g(x,y), \end{cases} \quad \text{and} \quad \begin{cases} x = f'(x',y'), \\ y = g'(x',y'), \end{cases}$$

where $f, g \in k[x,y]$, $f', g' \in k[x',y']$, the system (x',y') will also be an allowable system of coordinates for A^2. (Considered in one coordinate system the map $(x,y) \to (x',y')$ is an entire Cremona-transformation.)

3. **Semi-invariant polynomials.** Let a regular operation of G_m on A^2 be given by $\sigma(t)$: $(t,x,y) \to (x^*,y^*)$ where $t \in G_m$, (x,y) is a coordinate system for A^2, and $x^*, y^* \in k[x,y,t,t^{-1}]$.

If $f \in k[x,y]$; we define $\lambda_t f$, the t-translate of f, by

$$(3.1) \qquad \lambda_t f(x,y) = f(x^*,y^*).$$

We say that f is a semi-invariant polynomial (abbreviated in the sequel as s.i.p.) of weight s if $\lambda_t f = t^s f$, where s is an integer. If $s = 0$, f is said to be an invariant polynomial.

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LEMMA 1. *Let $f \in k[x,y]$ and suppose that $\lambda_t f = pf$, with $p \in k[x,y,t,t^{-1}]$. Then $p = t^s$, s an integer.*

Proof. Indeed $\lambda_{t^{-1}}(\lambda_t f) = f$. Therefore $p(x,y,t^{-1},t)\lambda_{t^{-1}}p(x,y,t,t^{-1}) = 1$, so that p is independent of x and y, and the result follows.

LEMMA 2. (a) *Any factor of an s.i.p. is an s.i.p.*

(b) *Every semi-invariant rational function is the quotient of two s.i.p.*

Proof. (a) Let $f_1, f_2, \cdots, f_n$ be the irreducible factors of a semi-invariant polynomial; then $\lambda_t(f_1 \cdots f_n) = \lambda_t f_1 \lambda_t f_2 \cdots \lambda_t f_n = t^s f_1 \cdots f_n$.

Therefore

$$\lambda_t f_i = p(t,t^{-1}) \prod_{j \in I} f_j \tag{3.2}$$

where I is a subset of $1,2,\cdots,n$, $p(t,t^{-1}) \in k[t,t^{-1}]$. Taking $t = 1$ in (3.2) it follows that $\lambda_t f_i = p(t,t^{-1})f_i$.

(b) Let $\lambda_t(f/g) = t^\alpha f/g$; then $(\lambda_t f)g = t^\alpha f(\lambda_t g)$. Since f and g are relatively prime, the result follows by a standard divisibility argument.

COROLLARY A. *All polynomials of $k[x,y]$ which are semi-invariant under σ are products of irreducible s.i.p.*

PROPOSITION 1. *Every polynomial of $k[x,y]$ is a sum of s.i.p.*

Proof. Let $f \in k[x,y]$, since $\lambda_t f \in k[x,y,t,t^{-1}]$ we have

$$\lambda_t f = t^\alpha(f_0 + tf_1 + \dots + t^n f_n) = t^\alpha \sum_{i=0}^{n} t^i f_i \tag{3.3}$$

where, for all i, $f_i \in k[x,y]$ and $n \geqq 0$.

Since σ is a regular operation of G_m on A^2, the following conditions are satisfied:

$$\begin{aligned} &(1) \quad \sigma(1) \text{ is the identity on } A^2, \\ &(2) \quad \sigma(tt_1) = \sigma(t)\sigma(t_1). \end{aligned} \tag{3.4}$$

From (3.4) it follows that

$$(tt_1)^\alpha \sum_{i=0}^{n} t^i t_1^i f_i = t_1^\alpha \sum_{i=0}^{n} t_1^i(\lambda_t f_i). \tag{3.5}$$

Identifying corresponding coefficients of $t_1^{i+\alpha}$ in the two members of (3.5) we get

$$\lambda_t f_i = t^{\alpha+i} f_i. \tag{3.6}$$

Therefore all the f_i, $i = 1,\cdots,n$, in (3.5) are s.i.p. To finish the proof, take $t = 1$ in (3.3).

COROLLARY B. *Among the irreducible s.i.p. there are certainly two polynomials with independent linear terms.*

Proof. Apply Proposition 1 and Corollary A to $x,y \in k[x,y]$.

4. **Invariant functions.** Among the rational functions on A^2 some are invariant under σ. In the next few lemmas we investigate the connexions between those functions and the s.i.p.

LEMMA 3. *The field of rational functions on A^2 which are invariant under σ is a simple extension of k.*

Proof. Consider the variety W of orbits (see [3]) corresponding to the operation of G_m on A^2. W is of dimension one. Its function-field $k(W)$ is by Lüroth's theorem (see [4]) a simple extension of k. On the other hand $k(W)$ is exactly the subfield of $k(x,y)$ containing those functions of $k(x,y)$ which are invariant under σ. Q.E.D.

If σ is such that all s.i.p. are of positive weight we have $k(W) = k(f/g)$, where f and g are s.i.p. of the same weight (see [2, p. 84]). In the sequel $k(W)$ will denote the subfield of $k(x,y)$ formed by the functions on A^2 invariant under σ.

LEMMA 4. *Let σ be such that all s.i.p. are of positive weight and let f/g, (f and g relatively prime) be a generator for $k(W)$. Consider an element $p=f+\mu g$, $\mu \in k$. Then all the irreducible factors of p are equal.*

Proof. If f/g is a generator for $k(W)$, then so is $f + \mu g/g$. Therefore it suffices to prove that all the irreducible factors of f are equal. Suppose that $f = f_1^\alpha f_2$, where f_1 is irreducible and f_2 contains no factors equal to f_1. Since the weights of f, g, f_1 are all positive $f_1^a g^{-b}$ will be an invariant function for some well chosen a and b. Then $f_1^a g^{-b} \in k(f_1^\alpha f_2 g^{-1})$ i.e., since k is algebraically closed

$$\frac{f_1^a}{g^b} = \frac{c\prod_{i=1}^{n} ((f_1^\alpha f_2/g) + a_i)}{\prod_{j=1}^{m} ((f_1^\alpha f_2/g) + b_j)} \text{ with } c,\ a_i,\ b_j \in k,$$

or

$$f_1^a \prod_{j=1}^{m} (f_1^\alpha f_2 + b_j g) = c g^\varepsilon \prod_{i=1}^{n} (f_1^\alpha f_2 + a_i g), \tag{4.1}$$

(if ε is negative, we make the obvious change needed in order to get a polynomial identity).

Now f_1^a divides the right-hand member of (4.1). Since $f_1 \nmid g$ and $f_1 \nmid f_2$, it follows by standard divisibility arguments that $f_2 = 1$.

LEMMA 5. *Let a regular operation of G_m on A^2 be such that all s.i.p. are of positive weight. Let p_1, p_2 and p_3 be three irreducible s.i.p. of weights ω_1, ω_2 and ω_3 respectively, such that $p_1, p_2, p_3 \notin k$.*

Then they satisfy a relation of the type

(4.2) $$c_1 p_1^{\alpha} + c_2 p_2^{\beta} = p_3^{\gamma}$$

where $\alpha\omega_1 = \beta\omega_2$, $(\alpha,\beta) = 1$; $c_1, c_2 \in k$.

Proof. Let d be the g.c.d. of ω_1 and ω_2. Then $\omega_1 = \beta d$ and $\omega_2 = \alpha d$. Consider $\phi = p_1^{\alpha} p_2^{(-\beta)} \cdot \phi$ is invariant under σ, so that, if fg^{-1} is a generator of $k(W)$ we have $p_1^{\alpha} p_2^{(-\beta)} \in k(fg^{-1})$. Therefore

$$p_1^{\alpha} p_2^{(-\beta)} = \frac{c\prod_{i=1}^{m} (f/g - a_i)}{\prod_{j=1}^{n} (f/g - b_j)} \text{ with } c,\ a_i,\ b_j \in k,$$

or

(4.3) $$p_1^{\alpha} \prod_{j=1}^{n} (f - b_j g) = c p_2^{\beta} g^{\varepsilon} \prod_{i=1}^{m} (f - a_i g),$$

(if ε is negative, we multiply both members of (4.3) by $g^{-\varepsilon}$, in order to get a polynomial identity).

Using Lemma 4 we then have, by standard divisibility arguments,

(4.4) $$p_1^{\alpha} = d_1 f + d_2 g,\ p_2^{\beta} = d_3 f + d_4 g, \qquad d_1, d_2, d_3, d_4 \in k,$$

whence we can choose as generator for $k(W)$ the element $p_1^{\alpha} p_2^{(-\beta)}$. We now repeat this argument with the invariant function $p_2^{\omega_3} p_3^{(-\omega_2)}$, taking $p_1^{\alpha} p_2^{(-\beta)}$ as a generator for $k(W)$. The relation which we then get instead of (4.4) is

$$p_3^{\gamma} = c_1 p_1^{\alpha} + c_2 p_2^{\beta}, \qquad c_1, c_2 \in k,$$

where γ is another positive number.

5. **Invariant polynomials.** Among the functions invariant under σ there may, or may not, be polynomials. The two following lemmas treat both possibilities, and lead up to the main theorem.

LEMMA 6. *If there exist nonconstant polynomials invariant under σ there is at most one irreducible s.i.p. of positive weight and similarly at most one irreducible s.i.p. of negative weight.*

Proof. Let f be an invariant polynomial of the lowest degree, $f \notin k$, and let $k(W) = k(g/h)$, where g and h are relatively prime polynomials. Then

$$f = \frac{c\prod_{i=1}^{n} (g/h - a_i)}{\prod_{j=1}^{m} (g/h - b_j)} = \frac{ch^{\varepsilon} \prod_{i=1}^{n} (g - a_i h)}{\prod_{j=1}^{m} (g - b_j h)} \text{ with } c, a_i, b_j \in k$$

for some suitable ε. It follows that some $g - b_j h$ is a constant, and therefore that $k(W) = k(h) = k(f)$.

Suppose now that p_1 and p_2 are different irreducible s.i.p. of positive weight. Then for an appropriate choice of a and b, $a > 0$, $b > 0$, $\phi = p_1^a p_2^{(-b)}$ is invariant under σ. Therefore $p_1^a p_2^{-b} \in k(f_i)$. We then get a polynomial identity

$$p_1^a \prod_{j=1}^{m} (f - c_j) = d p_2^b \prod_{i=1}^{m} (f - d_i) \quad \text{with } d, c_j, d_i \in k.$$

As p_1 and p_2 are both irreducible it then follows that $p_1^a \in k(f)$, $p_2^{-b} \in k(f)$, which is a contradiction. The proof is similar for p_1 and p_2 both of negative weight.

Lemma 7. *Let σ be such that there are no nonconstant invariant polynomials. Then after a suitable change of coordinates (a translation) all s.i.p. will be polynomials without constant terms.*

Proof. If there are no invariant polynomials the s.i.p. are either all of positive or all of negative weight. It is no loss of generality to take all weights positive.

Let $\sigma(t)$ be given by

$$\begin{aligned} x^* &= t^{\alpha}[f_0 + tf_1 + \cdots + t^n f_n], \\ y^* &= t^{\beta}[g_0 + tg_1 + \cdots + t^m g_m]. \end{aligned} \tag{5.1}$$

By (3.6) f_i is an s.i.p. of weight $(\alpha + i)$. Since, by assumption, all s.i.p. are of positive weight we may suppose $\alpha = \beta = 0$ in (5.1). On the other hand f_0 and g_0 are then reduced to constants, since there are no invariant polynomials. Hence (5.1) becomes

$$\begin{aligned} x^* &= a + tf_1(x,y) + \cdots + t^n f_n(x,y), \\ y^* &= b + tg_1(x,y) + \cdots + t^m g_m(x,y), \end{aligned} \tag{5.2}$$

where $m > 0$, $n > 0$, a, $b \in k$.

After the change of coordinates

$$X = x - a, \qquad Y = y - b, \tag{5.3}$$

we get

$$\begin{aligned} X^* &= tf_1^*(X,Y) + \cdots + t^n f_n^*(X,Y), \\ Y^* &= tg_1^*(X,Y) + \cdots + t^m g_m^*(X,Y), \end{aligned} \tag{5.4}$$

where $f_i^*(X,Y) = f_i(X + a, Y + b)$, $g_j^*(X,Y) = g_j(X + a, Y + b)$. In this new system of coordinates no s.i.p. of positive weight can have a constant term, since all powers of t in the right-hand members of (5.4) are positive.

Proposition 2. *Let σ be a regular operation of G_m on A^2. Then the set of all polynomials of $k[x,y]$ semi-invariant under σ is generated over k by two of its elements.*

Proof. We distinguish two cases:

(A) *There are no invariant polynomials.* Then all weights may be taken positive. By Lemma 7 we may suppose that all s.i.p. have no constant terms. We know (by Corollary B) that among the s.i.p. there are at least two p_1 and p_2, with independent linear terms. Since all s.i.p. have no constant term, p_1 and p_2 are irreducible. Denote by ω_1, and ω_2 the weights of p_1 and p_2 respectively. Let p_3 of weight ω_3 be any other irreducible s.i.p. By Lemma 5 there exists a relation of the type:

$$p_3^\gamma = c_1 p_1^\alpha + c_2 p_2^\beta, \qquad c_1, c_2, \in k, \tag{5.5}$$

where $\alpha d = \omega_2$, $\beta d = \omega_1$, $(\alpha,\beta) = 1$. Then $\gamma\omega_3 = \alpha\beta d$. Let $\alpha \leqq \beta$. The polynomials p_1 and p_2 have independent linear terms—so that p_3^γ will have among its terms of lowest degree a term of the type $c_3 x^\delta y^\varepsilon$ with $\delta + \varepsilon = \alpha$, $c_3 \in k$. Therefore $\alpha = \mu\gamma$. Hence $\omega_3 = \mu\beta d = \mu\omega_1$. Applying Lemma 5 again, this time to p_3, p_1 and p_2, we get

$$c_5 p_3 + c_6 p_1^\mu = p_2^\varepsilon, \qquad c_5, c_6 \in k. \tag{5.6}$$

So that p_3 is a polynomial in p_1 and p_2.

(B) *There exist invariant polynomials.* By Lemma 6 there are at most two irreducible s.i.p., one of negative and one of positive weight. To conclude the proof it remains to show that they generate over k all the invariant polynomials. Let p_1 and p_2 be these two irreducible polynomials of weights ω_1 and ω_2 respectively, $\omega_1 > 0$, $\omega_2 < 0$. Let f be a generator for $k(W)$. Then $p_1^{(-\omega_2)} p_2^{\omega_1} \in k(f)$. The desired result follows by a divisibility argument, using the same methods as in Lemmas 4 and 5.

6. **Main theorems.**

Proposition 3. *Any regular operation of G_m on A^2 can, after a suitable change of coordinates, be reduced to the form*

$$(t,x,y,) \rightarrow (t^\mu x, t^\nu y) \tag{6.1}$$

with μ and ν integers.

Proof. Let σ be given by $(t,x,y) \rightarrow (x^*,y^*)$ where

$$\begin{aligned} x^* &= t^\alpha \sum_{i=0}^{n} f_i t^i, \\ y^* &= t^\beta \sum_{j=0}^{m} g_j t^j, \end{aligned} \tag{6.2}$$

with $m \geqq 0$, $n \geqq 0$, f_i, $g_j \in k[x,y]$.

By Proposition 1 the f_i and g_j are s.i.p. By Proposition 2 all s.i.p. are polynomials in two irreducible s.i.p., say X and Y. Then f_i, $g_j \in k[X,Y]$ and $X, Y \in k[x,y]$.

On the other hand $x = \sum f_i$, $y = \sum g_j$, so that x and y are polynomials in X and Y, say $x = P(X,Y)$, $y = Q(X,Y)$. Therefore we have the relations

$$\begin{cases} x = P(X,Y), \\ y = Q(X,Y), \end{cases} \qquad \begin{cases} X = X(x,y), \\ Y = Y(x,y), \end{cases} \tag{6.3}$$

which define an allowable change of coordinates in A^2. In terms of the (X, Y)-coordinate system σ is then given by

$$(t,X,Y) \to (t^{\mu}X,\ t^{\nu}Y) \tag{6.4}$$

since X and Y are s.i.p.

PROPOSITION 4. *If G_m operates regularly on A^2 there is always a fixed point.*

Proof. This is an obvious corollary of the previous proposition. Note that G_m can operate on A^2 in such a way that there exists a curve of fixed points, but that in this case there is a fixed point adherent to every orbit.

7. **Algebraic torus.** The next two propositions concern the action of an algebraic torus on A^2.

PROPOSITION 5. *If an algebraic torus operates regularly on A^2 there always exists a fixed point.*

Proof. Consider the algebraic torus $G_m^{(1)} \times G_m^{(2)}$ and let σ_1 and σ_2 respectively be the regular operations of $G_m^{(1)}$ and $G_m^{(2)}$ on A^2. Since $G_m^{(1)}$ and $G_m^{(2)}$ commute, the set of orbits corresponding to σ_1 is globally invariant under σ_2, and so is the set of fixed points of σ_1. Therefore, if σ_i has only one fixed point, it is also a fixed point for σ_j, $i \neq j$. If on the other hand both σ_1 and σ_2 have curves of fixed points, the curve of fixed points of σ_1 is either a curve of fixed points for σ_2, or an orbit of σ_2. In either case it has at least one fixed point. This proves that σ_1 and σ_2 have a common fixed point. The proposition then follows by induction.

PROPOSITION 6. *Let an algebraic torus $G_m^{(1)} \times G_m^{(2)} \times \cdots \times G_m^{(r)}$ operate regularly on A^2. Then after an appropriate change of coordinates this operation can be described by*

$$(s_1,s_2,\cdots,s_r,x,y) \to \left(x\prod_{i=1}^{r} s_i^{\alpha_i},\ y\prod_{j=1}^{r} s_j^{\beta_j} \right), \tag{7.1}$$

where $s_i \in G_m^{(i)}$, $i = 1,2,\cdots,l$, and the α_i and β_j are integers.

Proof. Let a regular operation of $G_m \times G_m$ on A^2 be given by

$$(s,t,x,y) \to (x^*,y^*) \tag{7.2}$$

with $x^*,y^* \in k[x,y,s,s^{-1},t,t^{-1}]$.

In particular (7.2) defines by restriction two regular operations of G_m on A^2 which we denote by

$$\sigma_1\colon (s,x,y) \to (x_1,y_1), \tag{7.3}$$

$$\sigma_2 \colon (t,x,y) \to (x_2,y_2). \tag{7.4}$$

Suppose that the coordinate system is chosen so that σ_2 is of the form

$$\sigma_2 \colon (t,x,y) \to (t^\alpha x, t^\beta y). \tag{7.5}$$

Obviously in some allowable coordinate system σ_1 is of the form

$$(s,x',y') \to (s^\gamma x', s^\delta y') \tag{7.6}$$

and in this coordinate system the orbits have as equation

$$x'^\delta + \mu y'^\gamma = 0 \tag{7.7}$$

if we suppose that $\gamma\delta > 0$. Let the change of coordinates from (x',y') to (x,y) be given by

$$x' = f(x,y), \ y' = g(x,y), \qquad f,g \in k[x,y]. \tag{7.8}$$

Then in the (x,y)-coordinate system the equation of the orbits (7.7) is

$$f^\delta + \mu g^\gamma = 0. \tag{7.9}$$

These orbits have to be globally invariant under σ_2, so that f^δ/g^γ is a semi-invariant function for σ_2. By Lemma 2, f^δ and g^γ are then s.i.p., which means that both f and g are s.i.p. for σ_2, that is, in the (x',y')-coordinate system x' and y' are s.i.p. for σ_2. This proves that in the (x',y')-coordinate system both σ_1 and σ_2 are reduced to the canonical form of Proposition 3. The proof is similar if $\gamma\delta \leqq 0$. Our proposition follows by induction.

References

1. A. Borel, *Seminar on transformation groups*, Annals of Mathematics Studies, No. 46, Princeton Univ. Press, Princeton, N.J., 1960.

2. C. Chevalley, *Théorie des groupes de Lie*. II, Hermann, Paris, 1951.

3. M. Rosenlicht, *Some basic theorems on algebraic groups*, Amer. J. Math. **78** (1956), 401–443.

4. P. Samuel, *Some remarks on Lüroth's theorem*, Mem. Coll. Sci. Univ. Kyoto Ser. A **27** (1953), 223–224.

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