ON THE RESIDUAL FINITENESS OF GENERALISED FREE PRODUCTS OF NILPOTENT GROUPS

BY

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1. Introduction.

1.1. NOTATION. Let $\mathscr{C}$ be a class of groups. Then $R\mathscr{C}$ denotes the class of those groups which are residually in $\mathscr{C}$[2], i.e., $G \in R\mathscr{C}$ if, and only if, for each $x \in G$ $(x \neq 1)$ there is an epimorph of G in $\mathscr{C}$ such that the element corresponding to x is not the identity. If $\mathscr{D}$ is another class of groups, then we denote by $\mathscr{C} \cdot \mathscr{D}$ the class of those groups G which possess a normal subgroup N in $\mathscr{C}$ such that $G/N \in \mathscr{D}$[3]. For convenience we call a group G a *Schreier product* if G is a generalised free product with one amalgamated subgroup. Let us denote by

$$\sigma(A, B)$$

the class of all Schreier products of A and B. It is useful to single out certain subclasses of $\sigma(A, B)$ by specifying that the amalgamated subgroup satisfies some condition Γ, say; we denote this subclass by

$$\sigma(A, B;\ \Gamma).$$

Thus $\sigma(A, B; \Gamma)$ consists of all those Schreier products of A and B in which *the amalgamated subgroup satisfies the condition* Γ.

We shall use the letters $\mathscr{F}$, $\mathscr{N}$, and Φ to denote, respectively, the class of finite groups, the class of finitely generated nilpotent groups *without elements of finite order*, and the class of free groups.

1.2. A NEGATIVE THEOREM. The simplest residually finite[4] groups are the finitely generated nilpotent groups (K. A. Hirsch [2]). In particular, then,

$$\mathscr{N} \subset R\mathscr{F}. \tag{1.21}$$

Since the free product of residually finite groups is residually finite (K. W. Gruenberg [3]), the free product of any given pair of finitely generated nilpotent groups is residually finite. Hence, if $A, B \in \mathscr{N}$,

$$\sigma(A, B;\ \text{trivial}) \subset R\mathscr{F}. \tag{1.22}$$

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(2) This term is due to P. Hall [1].

(3) We have here adopted the notation of Hanna Neumann [19].

(4) I.e., groups in $R\mathscr{F}$. Similarly G is residually a $\mathscr{C}$-group if $G \in R\mathscr{C}$.

This consequence of Gruenberg's theorem may be compared with the following theorem for Schreier products.

THEOREM 1. *Suppose $A, B \in \mathscr{N}$. If both A and B are nonabelian, then*

$$\sigma(A, B) \not\subset R\mathscr{F}.$$

Theorem 1 compares unfavourably with (1.22). It serves to show, once again, the complication of the Schreier product. Nevertheless, despite this complication, we shall arrive at a fairly satisfactory description of $\sigma(A, B)$ for any given pair of groups $A, B \in \mathscr{N}$.

It is convenient, at this point, to recall that a group G is hopfian if every epiendomorphism of G is an automorphism. Hopfian groups are connected to residually finite groups by virtue of the fact that finitely generated residually finite groups are hopfian (A. I. Mal′cev [4]). So a finitely generated group which is nonhopfian is certainly not residually finite.

These remarks facilitate the proof of Theorem 1. We first notice that as A and B are torsion-free and nonabelian they both contain copies C and D, respectively, of the free nilpotent group of class two on two generators. Now $\sigma(C, D)$ contains a nonhopfian group, P, say (G. Baumslag [5]). Therefore $\sigma(A, B)$ contains a group G which contains P. Since $P \notin R\mathscr{F}$, $G \notin R\mathscr{F}$. So we have proved Theorem 1.

It seems worthwhile, at this point, to place on record the following

CONJECTURE. Let $A, B \in \mathscr{N}$. If both A and B are nonabelian, then $\sigma(A, B)$ contains a nonhopfian group.

1.3. STATEMENT OF RESULTS. The following theorem, which answers affirmatively a question of Graham Higman[5], plays an important role in this investigation.

THEOREM 2. *If $A, B \in \mathscr{F}$, then*

$$\sigma(A, B) \subset R\mathscr{F}.$$

As an easy consequence of Theorem 2 we find

THEOREM 3. *If $A, B \in R\mathscr{F}$, then*

$$\sigma(A, B; \textit{finite}) \subset R\mathscr{F}.$$

At this point let us recall that Φ is the class of free groups. By a theorem of Levi [6]

$$\Phi \subset R\mathscr{F}. \tag{1.31}$$

This fact, repeated application of Theorem 2 and Hirsch's Theorem (1.21) may be used to good effect. Indeed we shall prove that if $A, B \in \mathscr{N}$, then

$$\sigma(A, B) \subset R\mathscr{F} \cdot R\mathscr{F}. \tag{1.32}$$

[5] In a letter.

Thus, despite the existence of Theorem 1, we have found, in (1.32), a pleasing generalisation of (1.22); by Theorem 1 it is, in a sense, a "best possible" generalisation of (1.22).

Nevertheless (1.32) is not the precise result we shall obtain. For we prove

THEOREM 4. *Suppose* $A, B \in \mathcal{N}$. *Then*

$$\sigma(A,B) \subset \Phi \cdot R\mathcal{F}. \tag{1.33}$$

We recall that if G is a group, then a subgroup H of G is *closed* (in G) if $a \in H$ whenever some nontrivial power of a lies in $H (a \in G)$. It turns out that it is precisely the freedom which enables us to form Schreier products in which the amalgamation is not closed in both A and B that makes the presence of Φ in (1.33) essential. Indeed we shall prove, again employing Theorem 2, the following theorem.

THEOREM 5. *If* $A, B \in \mathcal{N}$, *then*

$$\sigma(A,B; \textit{ closed in } A \textit{ and } B) \subset R\mathcal{F}.$$

This theorem may be compared with

THEOREM 6. *If* $A, B \in \mathcal{N}$, *then*

$$\sigma(A,B; \textit{ cyclic}) \subset R\mathcal{F}.$$

One might conjecture, on the basis of Theorem 6, that if we require only that A, $B \in R\mathcal{F}$, then

$$\sigma(A,B; \text{ cyclic}) \subset R\mathcal{F}.$$

It is easy to make counter-examples which refute this possible conjecture; in fact Graham Higman [7] has even constructed a nonhopfian group as a Schreier product of two finitely generated residually finite metabelian groups, amalgamating a cyclic subgroup.

Theorem 6 has some connection with an earlier theorem of G. Baumslag [8]. To explain how this comes about, suppose now that A, $B \in \Phi$. Furthermore suppose A and B are isomorphic. Let θ be an isomorphic mapping of A onto B:

$$\theta : A \twoheadrightarrow B.$$

If a is an element of A which generates its centraliser in A, then (see [8])

$$(A * B; \ a = a\theta) \in R\Phi. \tag{1.34}$$

Let us now suppose only of A and B that they are free and let $a \in A$, $b \in B$ ($a \neq 1 \neq b$). Suppose a and b generate their centralisers, in A and in B, respectively. Then Graham Higman has proved the following partial generalisation of (1.34):

(1.35) $(A * B;\ a = b)$ is residually a finite p-group.

Theorem 6 enables us to prove a generalisation (in a sense) of both these results (1.34) and (1.35).

THEOREM 7. *If $A, B \in \Phi$ then*

$$\sigma(A, B;\ cyclic) \subset R\mathscr{F}.$$

Theorem (1.35) is, as yet, unpublished. It is a consequence of another interesting unpublished theorem of Higman, viz: If A and B are finite p-groups, if a in A and b in B are of order p, then

$$(A * B;\ a = b)$$

is residually a finite p-group.

Our collection of theorems concerning $\sigma(A, B)$ ends in §6 with some rather special results.

1.4. It is a pleasure to acknowledge that part of this work grew out of some stimulating correspondence with Graham Higman.

2. The situation when the amalgamated subgroup is finite.

2.1. THE PROOF OF THEOREM 2. Suppose that A and B are finite groups and that P is a Schreier product of A and B, with amalgamated subgroup H:

$$P = (A * B;\ H).$$

It is obvious that there is a finite group P_1 containing isomorphic copies A_1 and B_1 of A and B, respectively, with isomorphisms

$$\theta : A \twoheadrightarrow A_1, \quad \phi : B \twoheadrightarrow B_1.$$

In fact P_1 can so be chosen that the isomorphisms θ and ϕ coincide on H (cf. e.g. B. H. Neumann [9, p. 532], B. H. Neumann [10]). Since P is a Schreier product of A and B amalgamating H, it follows that θ and ϕ can be simultaneously extended to a homomorphism μ of P into P_1. Let K be the kernel of μ. Since P_1 is finite it follows that

$$|P/K| < \infty.$$

Since μ is one-to-one when restricted to either A or B it follows also that

$$K \cap A = 1 = K \cap B.$$

So, by a well-known theorem of Hanna Neumann [11, p. 540] K is free. Since $K \in R\mathscr{F}$ (Levi [6]), $P \in R\mathscr{F}$ for a finite extension of a residually finite group is obviously residually finite. This then completes the proof of Theorem 2.

2.2. AN IMPORTANT PROPOSITION. Let A be a residually finite group. Then there exists a family $\{A_\lambda\}_{\lambda \in \Lambda}$ of normal subgroups of A, each of finite index (in A), with a trivial intersection. We shall call such a family of groups a *filtration*(6) of A. So $A \in R\mathscr{F}$ if and only if A possesses a filtration. Let B be a second group with a distinguished subgroup K and let H be a fixed subgroup of A which is isomorphic to K under a given isomorphism ϕ. Now let

$$\{A_\lambda\}_{\lambda \in \Lambda}, \quad \{B_\lambda\}_{\lambda \in \Lambda}$$

be equally indexed filtrations of A and B respectively. Then we say $\{A_\lambda\}_{\lambda \in \Lambda}$ and $\{B_\lambda\}_{\lambda \in \Lambda}$ are (H,K,ϕ)-*compatible* if ϕ induces an isomorphism between HA_λ/A_λ and KB_λ/B_λ for each λ in Λ; in other words, the mapping

$$\phi_\lambda : hA_\lambda \to (h\phi)B_\lambda$$

is to be an isomorphism between HA_λ/A_λ and KB_λ/B_λ for each $\lambda \in \Lambda$.

It is useful to single out certain special filtrations of A which depend for their definition on some given subgroup H, say, of A. Thus we call a filtration $\{A_\lambda\}_{\lambda \in \Lambda}$ of A an H-*filtration* if

$$\bigcap_{\lambda \in \Lambda} (HA_\lambda) = H.$$

The following proposition indicates the practicality of these notions for our purposes.

PROPOSITION 1. *Let A and B be a given pair of residually finite groups, let H be a subgroup of A, K a subgroup of B and let ϕ be an isomorphism between H and K:*

$$\phi : H \twoheadrightarrow K.$$

Furthermore, let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a filtration of A and let $\{B_\lambda\}_{\lambda \in \Lambda}$ be a filtration of B. Now suppose that $\{A_\lambda\}_{\lambda \in \Lambda}$ and $\{B_\lambda\}_{\lambda \in \Lambda}$ are (H,K,ϕ)-compatible. Then

$$(A * B;\ H \overset{\phi}{=} K)(^7) \in \Phi \cdot R\mathscr{F}.$$

Proof. We put

$$P = (A * B;\ H \overset{\phi}{=} K).$$

Let, further, $a \in A (a \neq 1)$. Now $\{A_\lambda\}_{\lambda \in \Lambda}$ is a filtration of A. So there exists $\mu \in \Lambda$ such that $a \notin A_\mu$; observe that

$$|A/A_\mu| < \infty$$

since $\{A_\lambda\}_{\lambda \in \Lambda}$ is a filtration of A. But $\{B_\lambda\}_{\lambda \in \Lambda}$ is a filtration of B; so

$$|B/B_\mu| < \infty.$$

(6) I.e. an $\mathscr{F}$-filter, in the sense of Gruenberg [3].

(7) Here $(A * B;\ H \overset{\phi}{=} K)$ stands for that generalised free product of A and B where H is identified with K according to the isomorphism ϕ.

Now $\{A_\lambda\}_{\lambda \in \Lambda}$ is (H,K,ϕ)-compatible with $\{B_\lambda\}_{\lambda \in \Lambda}$. In particular, therefore,

$$\phi_\mu : hA_\mu \to (h\phi)B_\mu \quad (h \in H)$$

is an isomorphism between HA_μ/A_μ and KB_μ/B_μ.

We form now the Schreier product P_1 of the finite groups A/A_μ and B/B_μ, amalgamating HA_μ/A_μ with KB_μ/B_μ according to the isomorphism ϕ_μ:

$$P_1 = (A/A_\mu * B/B_\mu; \quad HA_\mu/A_\mu \overset{\phi_\mu}{=} KB_\mu/B_\mu).$$

Let v be the natural homomorphism of A onto A/A_μ and let η be the natural homomorphism of B onto B/B_μ. Since the restriction of $v\phi_\mu$ to H coincides with the restriction of $\phi\eta$ to H, the mappings v and η can simultaneously be extended to a homomorphism θ, say, of P onto P_1 (cf. B. H. Neumann [9, p. 505]).

We observe that

$$a\theta = av = aA_\mu \neq A_\mu.$$

But $P_1 \in R\mathscr{F}$ by Theorem 2. So there is a normal subgroup N_1 of P_1 of finite index which does not contain $a\theta$. Now P_1 is an epimorph of P; so we can also find a normal subgroup N of P of finite index which does not contain a.

It follows from the above argument that if N is the intersection of the normal subgroups of finite index in P, then

$$N \cap A = 1.$$

This argument is clearly symmetrical in A and B; so

$$N \cap B = 1.$$

These two restrictions of N ensure that it is free (Hanna Neumann [11, p. 540]). So

$$P \in \Phi \cdot R\mathscr{F}.$$

2.3. A SECOND PROPOSITION. We assume here he notation and hypotheses of Proposition 1. In addition let us assume that the filtration $\{A_\lambda\}_{\lambda \in \Lambda}$ is an H-filtration of A and that the filtration $\{B_\lambda\}_{\lambda \in \Lambda}$ is a K-filtration of B. Then the argument used to prove Proposition 1 can be utilised to prove a little more.

For suppose $u \in P$ $(u \neq 1)$. If $u \in A$ or if $u \in B$, then there is a normal subgroup N of P of finite index which does not contain u. We shall show that this is also the case when neither $u \in A$ nor $u \in B$. For then we can write u in the form

$$u = u_1 u_2 \cdots u_n \ (n > 1),$$

where the u_i come alternately out of A and B but not out of both (cf. B. H. Neumann [9, p. 511]). Since $\{A_\lambda\}_{\lambda \in \Lambda}$, $\{B_\lambda\}_{\lambda \in \Lambda}$ are not only filtrations of A and B respectively, but H- and K-filtrations (respectively), we can find $\mu \in \Lambda$ such that

$$u_i \notin HA_\mu, \quad u_i \notin KB_\mu \quad (i = 1, 2, \cdots, n). \tag{2.31}$$

Now either $u_1 \in A$ or $u_1 \in B$; for definiteness let us assume $u_1 \in A$. Then (cf. the proof of Proposition 1)

$$u\theta = (u_1 A_\mu)(u_2 B_\mu) \cdots . \tag{2.32}$$

It follows from (2.31) and (2.32) that $u\theta$ is a product of elements coming alternately from A/A_μ and B/B_μ but not both. Hence (cf. B. H. Neumann [9, p. 511])

$$u\theta \neq 1.$$

But, just as before, $P_1 \in R\mathscr{F}$, by Theorem 2. Hence we can find a normal subgroup N_1 of P_1 of finite index which does not contain $u\theta$. Since P_1 is an epimorph of P we can also find a normal subgroup $\bar{N}$ of finite index in P which does not contain u. This means that the normal subgroups of finite index in P have a trivial intersection, i.e., $P \in R\mathscr{F}$. Thus we have proved

PROPOSITION 2. *Let A and B be given residually finite groups, let H be a subgroup of A, and K a subgroup of B which is isomorphic to H:*

$$\phi : H \twoheadrightarrow K.$$

Let $\{A_\lambda\}_{\lambda \in \Lambda}$, $\{B_\lambda\}_{\lambda \in \Lambda}$ be (H,K,ϕ)-compatible H- and K-filtrations of A and B, respectively. Then

$$(A * B;\ H \overset{\phi}{=} K) \in R\mathscr{F}.$$

2.4. THE PROOF OF THEOREM 3. The situation here is that A, $B \in R\mathscr{F}$, H is a subgroup of A, K is a subgroup of B and ϕ is an isomorphism:

$$\phi : H \twoheadrightarrow K;$$

in addition H (and therefore K) is finite. It is an easy matter now to find filtrations $\{A_\lambda\}_{\lambda \in \Lambda}$ of A and $\{B_\lambda\}_{\lambda \in \Lambda}$ of B such that

$$A_\lambda \cap H = 1 = B_\lambda \cap K \ \ (\lambda \in \Lambda).$$

Hence the filtrations $\{A_\lambda\}_{\lambda \in \Lambda}$, $\{B_\lambda\}_{\lambda \in \Lambda}$ are not only H- and K-filtrations, respectively, but they are also (H,K,ϕ)-compatible. So we can apply Proposition 2 and the theorem follows.

It is useful to state a corollary of Theorem 3. The proof of this corollary is similar to the proof of Proposition 2; however, it relies on Theorem 3, whereas Proposition 2 relies on Theorem 2. The argument thus follows closely the proof of Proposition 2 and is left to the reader.

COROLLARY 2.41. *Let A and B be given groups, let H be a subgroup of A and K a subgroup of B which is isomorphic to H under the isomorphism ϕ:*

$$\phi : H \twoheadrightarrow K.$$

Let $\{A_\lambda\}_{\lambda\in\Lambda}$, $\{B_\lambda\}_{\lambda\in\Lambda}$ be families of normal subgroups of A and B respectively satisfying the conditions

(i) $A/A_\lambda \in R\mathscr{F}$, $B/B_\lambda \in R\mathscr{F}$,

(ii) $|HA_\lambda/A_\lambda| < \infty$, $|KB_\lambda/B_\lambda| < \infty$,

(iii) $\bigcap_{\lambda\in\Lambda} HA_\lambda = H$, $\bigcap_{\lambda\in\Lambda} KB_\lambda = K$.

Suppose that the families $\{A_\lambda\}_{\lambda\in\Lambda}$, $\{B_\lambda\}_{\lambda\in\Lambda}$ are (H,K,ϕ)-compatible, i.e., for each $\lambda\in\Lambda$ the mapping

$$\phi_\lambda : hA_\lambda \to (h\phi)B_\lambda$$

is an isomorphism. If $\bigcap_{\lambda\in\Lambda} A_\lambda = \bigcap_{\lambda\in\Lambda} B_\lambda = 1$, then

$$(A*B;\ H \overset{\phi}{=} K) \in R\mathscr{F}.$$

3. **The proof of Theorem 4.**

3.1. Preliminaries. Let p be a prime. Then a subgroup Y of a group X is called *p-closed* (in X) if $u \in Y$ whenever $u^p \in Y$ $(u \in X)$. An obvious necessary and sufficient condition for a subgroup Y, normal in X, to be p-closed (in X) is that the factor group X/Y is p-free, i.e., there are no elements of order p in X/Y. If X is locally nilpotent then the elements of finite order form a subgroup $f(X)$ (cf. e.g. Kurosh [12, vol. 2, p. 229]) of X. If A is any finitely generated nilpotent group, then every factor group and every subgroup of A is a finitely generated nilpotent group (K. A. Hirsch [13]); hence $f(A)$ is finite.

If n is an integer and X a group, then we define

$$X^n = gp(x^n;\ x \in X).$$

If X is nilpotent, then X^p is the *set* of pth powers of elements of X, for almost all primes p:

$$X^p = \{x^p \mid x \in X\};$$

this fact follows readily from the commutator collecting process of P. Hall [14] (cf. e.g. G. Higman [15]). Hence

$$X^{p^i} = \{x^{p^i} \mid x \in X\} \qquad (i = 1, 2, \cdots) \tag{3.11}$$

for almost all primes p.

3.2. The proof of the following simple lemma will make use of some of the above results, sometimes without explicit mention.

Lemma 3.21. *Let A be a finitely generated nilpotent group and let H be a given subgroup of A. Then H is p-closed in A for almost all primes p.*

Proof. Suppose first that A is abelian. Then H is a normal subgroup of A. Hence we can form the factor group A/H. Now $f(A/H)$ is finite. Therefore A/H is p-free for almost all primes p. So H is p-closed for almost all primes p.

If A is nonabelian, say of class $c > 1$, then inductively we may assume the truth of the lemma for all finitely generated nilpotent groups of class at most $c-1$. Now let Z be the centre of A. Inductively, HZ/Z is p-closed in A/Z for almost all primes p; say for all primes p in a set Ω consisting of almost all the primes. It follows then that if $p \in \Omega$, if $u \in A$ and if $u^p \in HZ$, then $u \in HZ$.

Now H is clearly a normal subgroup of HZ. So by Hirsch's theorem [13] $f(HZ/H)$ is finite. Hence HZ/H is p-free for every $p \in \mathrm{P}$, a set consisting of almost all the primes.

Put

$$\Pi = \Omega \cap \mathrm{P}.$$

Then all but a finite number of primes belong to Π.

Suppose $p \in \Pi$, $u \in A$ and

$$u^p \in H.$$

Then

$$u^p \in HZ.$$

Since $p \in \Omega$, it follows that

$$u \in HZ.$$

Now

$$(uH)^p = H.$$

But $p \in \mathrm{P}$ and HZ/H is p-free. Since $uH \in HZ/H$, we are forced to the conclusion $uH = H$; more conveniently,

$$u \in H.$$

So H is p-free for almost all primes p, as claimed.

3.3 It is easy to check that if n is an integer and A a group, then A^n is a fully invariant subgroup of A. In particular A^n is always normal in A. So if p is a prime, then the family

$$\{A^{p^i}\}_{i=1}^{\infty} \tag{3.31}$$

of subgroups of A is in fact a normal family, i.e., A^{p^i} is normal in A for every i. Suppose now that $A \in \mathcal{N}$. Then, by a theorem of K. W. Gruenberg [3],

$$\bigcap_{i=1}^{\infty} A^{p^i} = 1 \tag{3.32}$$

for every prime p. Therefore the family (3.31) of subgroups of A is a filtration of A, since in this situation A/A^{p^i} is finite.

Now suppose that H is a given subgroup of A. Consider

$$H \cap A^{p^i} \quad (i = 1, 2, \cdots).$$

Let us suppose that p is suitably large; then A^{p^i} *consists* of the p^ith powers of the elements of A (cf. (3.11)). Moreover H is p-closed for almost all primes p (Lemma 3.21). It follows that for almost all primes p,

$$H \cap A^{p^i} = H^{p^i} \qquad (i = 1, 2, \cdots). \tag{3.33}$$

Let us suppose next that $B \in \mathcal{N}$ and, furthermore, that K is a subgroup of B which is isomorphic to H under ϕ:

$$\phi : H \twoheadrightarrow K.$$

Now the subgroups

$$\{B^{p^i}\}_{i=1}^{\infty} \tag{3.34}$$

are a filtration of B for all choices of p (cf. (3.31), (3.32)). Moreover (cf. (3.33))

$$K \cap B^{p^i} = K^{p^i} \qquad (i = 1, 2, \cdots), \tag{3.35}$$

for almost all primes p. It follows that there is a prime p, say, such that (3.33) and (3.35) are simultaneously satisfied. Clearly, then, for such a prime p (3.31) and (3.35) are in fact compatible (H,K,ϕ)-filtrations.

The upshot of the above considerations is that we have proved the following lemma.

LEMMA 3.36. *Let $A, B \in \mathcal{N}$, let H be a subgroup of A and K a subgroup of B isomorphic to H under ϕ:*

$$\phi : H \twoheadrightarrow K.$$

Then

$$\{A^{p^i}\}_{i=1}^{\infty}, \qquad \{B^{p^i}\}_{i=1}^{\infty}$$

are (H,K,ϕ)-compatible filtrations of A and B respectively, for almost all primes p.

The proof of Theorem 4 is now a triviality; we simply apply Proposition 1 (this is in order, since we can invoke Lemma 3.36).

4. The proof of Theorem 5.

4.1. A CRUCIAL PROPOSITION. The proof of Theorem 4 required no more than a verification that Proposition 1 was applicable, and Lemma 3.36 was provided for just that purpose. Here we want to justify the use of Proposition 2; but Lemma 3.36 is no longer of any use. We are therefore compelled to prove a somewhat stronger form of Lemma 3.36. To do this we first prove a partial generalisation of Gruenberg's theorem (Theorem 2.2 in [3]).

PROPOSITION 3. *Le $A \in \mathcal{N}$ and let H be a closed subgroup of A. Then*

$$\bigcap_{=1}^{\infty} HA^{p^i} = H \tag{4.11}$$

for almost all primes p.

So when H is the trivial group we have the theorem of Gruenberg quoted above.

The proof of Proposition 3 is somewhat roundabout. Our argument will proceed via an induction. To start out with we consider the case where A is abelian. It follows easily from the basis theorem for finitely generated abelian group (cf. e.g. Kurosh [12, vol. 1, p. 149]) that a closed subgroup of a finitely generated abelian group is a direct factor. So

$$A = H \times J,$$

for a suitable choice of J. Therefore, for every prime p,

$$HA^{p^i} = H \times J^{p^i}.$$

So (4.11) follows immediately from the fact that

$$\bigcap_{i=1}^{\infty} J^{p^i} = 1.$$

We now assume that A is of class $c > 1$. Moreover we shall assume that the proposition holds for all groups of class not exceeding $c-1$ in $\mathcal{N}$. Under these assumptions we proceed to a proof of a lemma.

Lemma 4.12. *Let $B \in \mathcal{N}$ and let K be a closed subgroup of B. Let Z be the centre of B and suppose that KZ/Z is a closed subgroup of B/Z. Then*

$$\bigcap_{i=1}^{\infty} KB^{p^i} = K$$

for almost all primes p.

Proof. If B is abelian, then our introductory remarks (above) suffice. So we assume that B is of class $c > 1$. Let us put

$$L_p = \bigcap_{i=1}^{\infty} KB^{p^i}.$$

Then the induction hypothesis of Proposition 3 yields

$$L_p \leqq KZ$$

for almost all primes p. Hence

$$K \leqq L_p \leqq KZ \tag{4.121}$$

for almost all primes p. We can therefore find a set Ω, say, containing all but a finite number of primes, such that if $p \in \Omega$, then both (4.121) and

$$B^{p^i} = \{v^{p^i} \mid v \in B\} \tag{4.122}$$

hold simultaneously. Notice that KZ is closed in B since KZ/Z is closed in B/Z.

Now let $p \in \Omega$ and suppose $u \in L_p$. Then, noting (4.122), for each $i = 1, 2, \cdots$ there exists $b_i \in B$ such that

$$u = b_i^{p^i} k_i \quad (k_i \in K).$$

By (4.121),

$$u \in KZ.$$

But $k_i \in KZ$. So $b_i^{p^i} \in KZ$ and as KZ is closed in B,

$$b_i \in KZ \quad (i = 1, 2, \cdots). \tag{4.123}$$

Clearly K is a normal subgroup of KZ. Let us consider KZ/K; since K is a closed subgroup of B, KZ/K is a finitely generated torsion-free nilpotent group, i.e., $KZ/K \in \mathcal{N}$. But every finitely generated nilpotent group satisfies the maximum condition (Hirsch [13]). Consequently in a torsion-free nilpotent group on a finite number of generators only the unit element has p^ith roots for arbitrarily large values of i. Now (cf. (4.123)) for each $i = 1, 2, \cdots$

$$uK = b_i^{p^i} K = (b_i K)^{p^i}.$$

Therefore $uK = K$; i.e.,

$$u \in K.$$

So

$$L_p = K.$$

This is precisely what is required and so the proof of Lemma 4.12 is complete.

We come now to the proof of Proposition 3. We are interested in the case where A is a nonabelian group in $\mathcal{N}$ and H is a closed subgroup of A. Let Z now denote the centre of A and consider the smallest closed subgroup $\overline{HZ}$ (the so-called *closure* of HZ) of A containing HZ. Thus $\overline{HZ}$ is simply the intersection of the closed subgroups of A containing HZ. We consider

$$L_p = \bigcap_{i=1}^{\infty} HA^{p^i},$$

where p is a prime. By Lemma 4.12

$$H \leqq L_p \leqq \overline{HZ} \tag{4.13}$$

for almost all primes p. Now $\overline{HZ}$ consists of precisely those elements of A which have a positive power in HZ (cf. e.g. Kurosh [12, vol. 2, p. 249]). So if

$$u \in \overline{HZ},$$

then, for a suitable choice of $n > 0$,

$$u^n \in HZ.$$

Clearly then

$$u^n \text{ normalises } H.$$

But the normaliser of a closed subgroup of a group in $\mathcal{N}$ is itself closed (cf. e.g. Kurosh [12, vol. 2, p. 249]). So

$$u \text{ normalises } H.$$

In other words H is a normal subgroup of $\overline{HZ}$.

We now form

$$\overline{HZ}/H.$$

Since H is closed in A, $\overline{HZ}/H$ is torsion-free.

Let us now choose a prime p so that (4.13) holds and so that the product of pth powers in A is again a pth power in A; notice that almost all primes satisfy the above two conditions. Consider L_p/H. Suppose $v \in L_p$. Then there are elements $a_i \in A$ such that

$$v = a_i^{p^i} h_i \quad (h_i \in H)$$

for $i = 1,2,\cdots$. But $v \in \overline{HZ}$ (by (4.13)), $h_i \in \overline{HZ}$ and so $a_i^{p^i} \in \overline{HZ}$. Since $\overline{HZ}$ is a closed subgroup of A it follows that

$$a_i \in \overline{HZ} \quad (i = 1,2,\cdots).$$

Hence

$$vH = (a\,H)^{p^i} \quad (i = 1,2,\cdots).$$

Now $\overline{HZ}/H \in \mathcal{N}$; so it follows from the fact that the groups in $\mathcal{N}$ satisfy the maximum condition for subgroups that $vH = H$. Therefore

$$v \in H.$$

So

$$L_p = H.$$

This completes the proof of Proposition 3.

The proof of Theorem 5 follows easily. For choose a prime p such that

$$\bigcap_{i=1}^{\infty} A^{p^i}H = H, \quad \bigcap_{i=1}^{\infty} B^{p^i}K = K.$$

Then $\{A^{p^i}\}_{i=1}^{\infty}$ and $\{B^{p^i}\}_{i=1}^{\infty}$ are, respectively, an H-filtration of A and a K-filtration of B; these filtrations are, moreover, (H,K,ϕ)-compatible. So Proposition 2 applies, and this completes the proof of Theorem 5.

5. **The proofs of Theorem 6 and Theorem 7.**

5.1. We follow a now familiar procedure. Thus we suppose $A \in \mathcal{N}$ and that H is a cyclic subgroup of A:

$$H = gp(u).$$

Now $\bar{H}$, the closure of H in A, is also cyclic. To verify that this is indeed the case we first observe that $\bar{H}$ is abelian (Baumslag [16]). Now $\bar{H}$ consists of all those elements of A which have a positive power in H (cf. e.g. Kurosh [12, vol. 2, p. 249]). So $\bar{H}$ is locally cyclic. But $\bar{H}$ satisfies the maximum condition for subgroups (Hirsch [13]) and therefore is necessarily cyclic. Suppose then that

$$\bar{H} = gp(a).$$

Consequently

$$u = a^m.$$

Let U_1 be the last term of the upper central series of A which does not contain a. Then the succeeding term U_2 of the upper central series of A does contain a. We observe that U_2/U_1 is a finitely generated torsion-free abelian group and that $U_2 \in \mathcal{N}$ (Hirsch [13]).

Now let n be any nonzero integer and let

$$X_1 = U_2^{m^2 n}$$

Then X_1 is a normal subgroup of A; moreover

$$aX_1 \text{ is of order } m^2 n$$

because even

$$a(X_1 U_1) \text{ is of order } m^2 n$$

by virtue of the fact that U_2/U_1 is a direct product of infinite cyclic groups. It follows that

$$uX_1 \text{ is of order } mn. \tag{5.11}$$

Let us recall that $H_1 = gp(u^{mn})$ is p-closed in A for almost all primes p, by Lemma 3.21. So we can choose a prime p such that the product of pth powers in A is again a pth power, such that H_1 is p-closed in A and such that

$$\bigcap_{i=1}^{\infty} \bar{H}A^{p^i} = \bar{H},$$

by Lemma 3.21 and Proposition 3. We then define

$$X_{i+1} = X_i^p \quad (i = 1, 2, \ldots).$$

Then the family $\{X_i\}_{i=1}^{\infty}$ of subgroups of A possesses the following properties:

(i) X_i is a normal subgroup of A for $i = 1, 2, \cdots$;

(ii) $\bigcap_{i=1}^{\infty} X_i = 1$;

(iii) uX_i is of order mnp^{i-1} for $i = 1, 2, \cdots$;

(iv) $\bigcap_{i=1}^{\infty} HX_i = H$.

To verify this claim, we observe, firstly, that (i) is trivially true, and that (ii) is a consequence of a theorem of Gruenberg [3]. Secondly, (iii) is a consequence of (5.11) and our choice of the prime p. The verification of (iv) is a little more involved. We have, by Proposition 3 and the choice of p,

$$K = \bigcap_{i=1}^{\infty} HX_i \leqq \bigcap_{i=1}^{\infty} \bar{H}X_i \leqq \bigcap_{i=2}^{\infty} \bar{H}X_i \leqq \bigcap_{i=1}^{\infty} \bar{H}A^{p^i} \leqq \bar{H}.$$

Now

$$|\bar{H}/H| = m. \tag{5.12}$$

Clearly K contains H. Let us consider $|\bar{H}/K|$. Since a is of order m modulo HX_1, it follows that

$$|\bar{H}/K| = m. \tag{5.13}$$

It follows immediately from (5.12) and (5.13) that $K = H$ and so we have established (iv).

The outcome of the above considerations is the following lemma.

LEMMA 5.14. *Let $A \in \mathcal{N}$, let H be a cyclic subgroup of A and let H be of index m in its closure $\bar{H}$. Furthermore, let n be any positive integer. Then for almost all primes p there is a family $\{X_i\}_{i=1}^{\infty}$ of normal subgroups of A, with a trivial intersection, such that*

$$|HX_i/X_i| = mnp^{i-1} \quad (i = 1,2,\cdots) \quad \text{and} \quad \bigcap_{i=1}^{\infty} HX_i = H.$$

5.2. PROPOSITION 4. It is now an easy matter to deduce the following somewhat technical proposition from Lemma 5.14.

PROPOSITION 4. *Let $A, B \in \mathcal{N}$ and let H be a cyclic subgroup of A, say*

$$H = gp(u).$$

Furthermore, let K be a cyclic subgroup of B, say

$$K = gp(v),$$

and suppose

$$\phi : u \to v$$

is a given isomorphism between H and K. Then there exist families $\{X_i\}_{i=1}^{\infty}$, $\{Y_i\}_{i=1}^{\infty}$ of normal subgroups of A and B, respectively, satisfying the conditions

(a) $|HX_i/X_i| < \infty$, $|KY_i/Y_i| < \infty$;
(b) $\bigcap_{i=1}^{\infty} HX_i = H$, $\bigcap_{i=1}^{\infty} KY_i = K$;
(c) $|HX_i/X_i| = |KY_i/Y_i|$ $(i = 1,2,\cdots)$;
(d) $\bigcap_{i=1}^{\infty} X_i = 1 = \bigcap_{i=1}^{\infty} Y_i$.

Proof. Suppose $|\bar{H}/H| = m$ and $|\bar{K}/K| = n$. It follows from Lemma 5.14 that we can find a prime p such that

$$|HX_i/X_i| = mnp^i = nmp^i = |KY_i/Y_i| \quad (i = 1,2,\cdots),$$

where $\{X_i\}_{i=1}^{\infty}$ and $\{Y_i\}_{i=1}^{\infty}$ are families of normal subgroups of A and B respectively satisfying (b) and (d).

But now Proposition 4 entitles us to apply Corollary 2.31 and so the proof of Theorem 6 follows immediately.

5.3. THE PROOF OF THEOREM 7. The proof of Theorem 7 is an easy consequence of Theorem 6. We recall first the data of Theorem 7. We have two free groups A and B with subgroups $H = gp(u)$ $(u \neq 1)$ and $K = gp(v)$ $(v \neq 1)$ respectively, and an isomorphism ϕ of H onto K defined by

$$u\phi = v.$$

In addition

$$P = (A * B;\quad H \overset{\phi}{=} K).$$

It is well known that $\Phi \subset R\mathcal{N}$ (Magnus [18]); this helps us to establish that $P \in R\mathcal{F}$. Now suppose $w \in P$ $(w \neq 1)$. Then we can express w in the form

$$w = x_1 x_2 \cdots x_n \quad (n > 1),$$

where the x_i come alternately out of A and B, but not out of both. Now it follows from Magnus' theorem and the fact that the centraliser of any element different from 1 in a free group is cyclic, that we can find normal subgroups M and N, of A and B, respectively, such that

(i) $M \cap H = 1 = N \cap K$;

(ii) $A/M \in \mathcal{N}$, $B/N \in \mathcal{N}$;

(iii) If $x_i \in A - H$, then $x_i M \notin HM$ $(i = 1, 2, \cdots, n)$;

(iv) If $x_i \in B - K$, then $x_i N \notin KN$ $(i = 1, 2, \cdots, n)$.

Now let μ be the natural homomorphism of A onto A/M and let ν be the natural homomorphism of B onto B/N. It follows from (i) that μ and ν can be simultaneously extended to a homomorphism θ of P onto

$$P_1 = (A/M * B/N;\quad HM/M \overset{\bar{\phi}}{=} KN/N),$$

where $\bar{\phi}$ is the isomorphism defined by

$$(uM)\bar{\phi} = (u\phi)N.$$

It is important to observe that

$$w\theta = (x_1\theta)(x_2\theta)\cdots(x_n\theta)$$

is a product of elements coming alternatively out of A/M and B/N but not out of both. So

$$w\theta \neq 1.$$

But Theorem 6 applies and therefore $P_1 \in R\mathcal{F}$. Since $w\theta \neq 1$ the argument that we used before applies and so $P \in R\mathcal{F}$; this completes the proof of Theorem 7.

6. **Miscellaneous results.**

6.1. We end this discussion of Schreier products of nilpotent groups by stating some of the theorems that can be proved by using the same basic idea that we have employed throughout.

THEOREM 8. *If A and B are polycyclic, then*

$$\sigma(A,B\colon \textit{in the centre of } B) \subset R\mathcal{F}.$$

THEOREM 9. *If A and B are polycylic, then*

$$\sigma(A,B;\ \textit{normal in both } A \textit{ and } B) \subset R\mathcal{F}.$$

THEOREM 10. *If $A \in R\mathscr{F}$, if H is a subgroup of A and if ϕ is an isomorphic mapping of A onto a group B then*

$$(A * B;\ H \overset{\phi}{=} H\phi) \in \Phi \cdot R\mathscr{F}.$$

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