SOME HILBERT SPACES OF ANALYTIC FUNCTIONS. I

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The spaces of analytic functions studied arise from the problem of finding invariant subspaces for bounded linear transformations in Hilbert space. Two fundamental problems are (1) to determine the invariant subspaces of any bounded transformation, (2) to reconstruct a transformation from its invariant subspaces. Satisfactory answers are known for self-adjoint transformations, for unitary transformations, and more generally for normal transformations, but for all other kinds of transformations the known results are less complete. Beurling [2] illustrates how to find invariant subspaces for isometric transformations which are not unitary, and in so doing uncovers an important connection with analytic function theory. Aronszajn and Smith [1] are able to show the existence of invariant subspaces for completely continuous transformations, a result which they ascribe to von Neumann in the Hilbert space case with which we are now concerned. A general study of transformations T with $T - T^*$ completely continuous is started by Livšic [11] and continued by Brodskiĭ and Livšic [9]. The Livšic approach forms an interesting link between the methods of analytic function theory and those which depend on compactness in linear spaces. When the spectrum of the transformation is a point, Gohberg and Krein [10] give an integral representation of the transformation in terms of invariant subspaces. This construction makes an interesting contrast with the spectral representation of a self-adjoint operator. Relations between the existence of invariant subspaces and the factorization of related operator valued entire functions are obtained by Brodskiĭ [7; 8]. Our purpose now is to give an exposition of the function theoretic background to this interesting observation.

Recall that we have previously [3–6] made a study of Hilbert spaces, whose elements are entire functions and which have these properties:

(H1) Whenever $F(z)$ is in the space and has a nonreal zero w, the function $F(z)(z - \bar{w})/(z - w)$ is in the space and has the same norm as $F(z)$.

(H2) For every complex number w, the linear functional defined on the space by $F(z) \to F(w)$ is continuous.

(H3) Whenever $F(z)$ is in the space, the function $F^*(z) = \bar{F}(\bar{z})$ is in the space and has the same norm as $F(z)$. The axiom (H2) which appears here is conjectured to be a consequence of (H1). Several apparently weaker conditions, of various degrees of subtlety, are known to imply (H2), and one of these is quoted

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in previous work in place of the present axiom (continuity not assumed on the real axis). Present work requires a similar but more general class of Hilbert spaces, whose elements are vector valued analytic functions, and which satisfy a suitable interpretation of (H1) and (H2). Some notation is necessary to describe these spaces.

Let $\mathscr{C}$ be a fixed Hilbert space. By a vector we will always mean an element of this space. A special notation is used for the norm and inner product of vectors to avoid confusion with other spaces we shall construct. Let $|c|$ be the norm of a vector c. If b is a vector, let $\bar{b}$ be the corresponding linear functional on vectors so that the inner product takes the form $\langle a,b\rangle = \bar{b}a$. By an operator we mean a bounded linear transformation of vectors into vectors. If A is an operator, let $|A|$ be the operator norm, let $\bar{A}$ be the adjoint of A, and let $\operatorname{Re} A = (A + \bar{A})/2$. We write $A \geqq 0$ if $\bar{c}Ac \geqq 0$ for every vector c, and this inequality implies in particular that $A = \bar{A}$ is self-adjoint. If A and B are operators, the inequality $A \leqq B$ is taken to mean $B - A \geqq 0$. An operator is said to be invertible if it has an everywhere defined and bounded inverse A^{-1}. If a and b are vectors, let $a\bar{b}$ be the corresponding operator defined by $(a\bar{b})c = a(\bar{b}c)$ for every vector c. The theory requires the choice of a fixed invertible operator I such that $\bar{I} = -I = I^{-1}$. Let $\mathscr{C}_+$ be the kernel of $I - i$, and let $\mathscr{C}_-$ be the kernel of $I + i$. Then $\mathscr{C}_-$ and $\mathscr{C}_+$ are orthogonal subspaces of $\mathscr{C}$ which span $\mathscr{C}$. We use τ to denote the finite or infinite trace norm of an operator. Relevant properties of this norm are that $\tau(ABC) = \tau(B)$ whenever A and C are unitary operators, and $\tau(A + B) = \tau(A) + \tau(B)$ whenever $A \geqq 0$ and $B \geqq 0$.

A vector valued function $f(z)$, defined in a region Ω of the complex plane, is said to be analytic in Ω if the complex valued function $\bar{c}f(z)$ is analytic in the region for every choice of vector c. An operator valued function $F(z)$, defined in Ω, is said to be analytic in the region if $\bar{b}F(z)a$ is analytic in the region for every choice of vectors a and b.

If $\mathscr{H}$ is a Hilbert space, whose elements are vector valued entire functions, the axiom (H1) makes perfect sense and the axiom (H2) has an obvious interpretation: the linear transformation of $\mathscr{H}$ into $\mathscr{C}$ defined by $F(z) \to F(w)$ is continuous for all complex w. We shall not here be concerned with a general study of Hilbert spaces of vector valued entire functions which satisfy these axioms. It is sufficient to consider a special case in which the axiom (H1) is strengthened in a way suggested by Lemma 1 of [6]:

(H4) The function $[F(z) - F(w)]/(z - w)$ belongs to $\mathscr{H}$ whenever $F(z)$ belongs to $\mathscr{H}$, for every complex number w, and the identity

$$\begin{aligned}2\pi\bar{G}(\beta)IF(\alpha) = {} & \langle F(t), [G(t) - G(\beta)]/(t - \beta)\rangle \\ & - \langle [F(t) - F(\alpha)]/(t - \alpha), G(t)\rangle \\ & + (\alpha - \bar{\beta})\langle [F(t) - F(\alpha)]/(t - \alpha),\ [G(t) - G(\beta)]/(t - \beta)\rangle\end{aligned}$$

holds for all $F(z)$ and $G(z)$ in $\mathscr{H}$ and all numbers α and β. This identity is a characteristic property of the generalized Hilbert transform of [5].

A general study of such spaces had been completed when we discovered that a wider theory can be constructed with little additional effort. Let Ω be a given region of the complex plane, which is always assumed to contain the origin. If $\mathscr{H}$ is a Hilbert space whose elements are vector valued analytic functions in Ω, the axioms (H2) and (H4) make sense when the numbers α, β, and w are restricted to Ω. A space with these properties is then said to satisfy (H2) and (H4) in Ω. Standard models of linear transformations are constructed from such spaces.

THEOREM I. *If $\mathscr{H}$ is a Hilbert space of vector valued analytic functions which satisfies* (H2) *and* (H4) *in Ω, the transformation*

$$R(w): \ F(z) \to [F(z) - F(w)]/(z - w)$$

is bounded, for all w in Ω, and the resolvent identity

$$(\alpha - \beta)R(\alpha)R(\beta) = R(\alpha) - R(\beta)$$

holds. The spectrum of $R(0)$ is contained in the set of points w such that w^{-1} does not belong to Ω. Let P_+ and P_- be the spectral projections, corresponding to $(0,\infty)$ and $(-\infty,0)$ respectively, for the self-adjoint transformation $i[R(0) - R(0)^*]$. Then the dimension of (the range of) P_+ is no more than the dimension of $\mathscr{C}_+$, and the dimension of P_- is no more than the dimension of $\mathscr{C}_-$.

THEOREM II. *Let T be a bounded linear transformation of a Hilbert space $\mathscr{H}$ into itself. Let P_+ and P_- be the spectral projections, corresponding to $(0,\infty)$ and $(-\infty,0)$ respectively, for the self-adjoint transformation $i(T - T^*)$. We suppose that the dimension of P_+ is no more than the dimension of $\mathscr{C}_+$ and that the dimension of P_- is no more than the dimension of $\mathscr{C}_-$. Then T admits a largest invariant subspace $\mathscr{M}$, which is also invariant under T^* and to which the restriction of T is self-adjoint. If $\mathscr{M}$ is the zero subspace of $\mathscr{H}$, then T is unitarily equivalent to the transformation $R(0)$ in some Hilbert space of vector valued analytic functions which satisfies* (H2) *and* (H4) *in a region Ω (containing the origin), and such that I commutes with the operator m (defined below).*

Such spaces have a simple structure.

THEOREM III. *Let $\mathscr{H}$ be a Hilbert space of vector valued analytic functions which satisfies* (H2) *and* (H4) *in Ω. Then there exists a unique operator valued function $M(z)$, defined and analytic in Ω, with value 1 at the origin, which has this property: if*

$$K(w,z) = [M(z)I\bar{M}(w) - I]/[2\pi(z - \bar{w})],$$

then $K(w,z)c$ belongs to $\mathscr{H}$ as a function of z for every vector c and every number w in Ω, and

$$\bar{c}F(w) = \langle F(t), K(w,t)c\rangle$$

holds for every $F(z)$ in $\mathscr{H}$. The adjoint of the transformation $R(0)$ in $\mathscr{H}$ is

$$R(0)^*: F(z) \to [F(z) - M(z)F(0)]/z.$$

There exists a unique operator valued function $\phi(z)$, defined and analytic for $y > 0$, for $y < 0$, and in a neighborhood of the origin, such that

$$[M(z) + 1]\phi(z) = [M(z) - 1]I.$$

It has value 0 at the origin and satisfies the operator inequality

$$(2) \qquad [\phi(w) - \bar{\phi}(w)]/(w - \bar{w}) \geqq 0.$$

The operator $m = M'(0)I$ is nonnegative and $\phi'(0) = \frac{1}{2}m$.

Since the space $\mathscr{H}$ in question is determined by $M(z)$, we may denote it by $\mathscr{H}(M)$. When an operator valued analytic function $M(z)$ is given, with value 1 at the origin, we shall say that it satisfies (1) if it is of the form

$$(3) \qquad M(z) = [1 - \phi(z)I]/[1 + \phi(z)I],$$

where $\phi(z)$ is defined and analytic for $y > 0$, for $y < 0$, and in a neighborhood of the origin, and if $\phi(z)$ satisfies (2) in its region of analyticity.

THEOREM IV. *Let $M(z)$ be a given operator valued analytic function which satisfies* (1) *in Ω and has value* 1 *at the origin. Then there exists a unique Hilbert space $\mathscr{H}(M)$ of vector valued analytic functions, which satisfies* (H2) *and* (H4) *in Ω and is related to $M(z)$ as in Theorem* III. *If*

$$(4) \qquad \phi(z)I\bar{\phi}(w) = 0,$$

the transformation $R(0)$ is self-adjoint in $\mathscr{H}(M)$.

When $M(z)$ is given, $K(w,z)$ is understood to be defined for the space $\mathscr{H}(M)$ as in Theorem III. When several spaces are present, the distinguishing index, $a,b,\cdots$, is considered a new variable, as in $M(a,z)$, $K(a,w,z)$, $m(a)$, and $\phi(a,z)$. Occasionally it is better to use two indices as in $M(a,b,z)$, $K(a,b,w,z)$, etc. Isometric inclusions of spaces of vector valued analytic functions are related to a factorization of the defining operator valued analytic functions.

THEOREM V. *Let $M(a,z)$ and $M(b,z)$ be operator valued analytic functions which satisfy* (1) *in Ω and have value* 1 *at the origin. The region Ω is supposed to be symmetric about the real axis. If $\mathscr{H}(M(a))$ is contained isometrically in $\mathscr{H}(M(b))$, then*

$$M(b,z) = M(a,z)M(a,b,z),$$

where $M(a,b,z)$ *is a uniquely determined operator valued analytic function which satisfies* (1) *in* Ω *and has value* 1 *at the origin. In this case,* $F(z) \rightarrow M(a,z) F(z)$ *is a linear isometric transformation of* $\mathscr{H}(M(a,b))$ *onto the orthogonal complement of* $\mathscr{H}(M(a))$ *in* $\mathscr{H}(M(b))$.

Note that $\mathscr{H}(M(a))$ is then an invariant subspace of the transformation $R(0)$ in $\mathscr{H}(M(b))$. Conversely, an invariant subspace of $R(0)$ in $\mathscr{H}(M(b))$ is a Hilbert space of vector valued analytic functions which satisfies (H2) and (H4) in Ω and hence of this form for some $M(a,z)$. For technical reasons the factorization of operator valued analytic functions which satisfy (1) is not equivalent to an isometric inclusion of the corresponding Hilbert spaces.

THEOREM VI. *If* $M(b,z)$ *and* $M(b,d,z)$ *are operator valued analytic functions which satisfy* (1) *in* Ω *and have value* 1 *at the origin, then*

$$M(d,z) = M(b,z)\, M(b,d,z)$$

is an operator valued analytic function which satisfies (1) *in* Ω *and has value* 1 *at the origin. There exist operator valued analytic functions* $M(a, z)$, $M(a,b,z)$, $M(b,c,z)$, $M(c,z)$, *and* $M(c,d,z)$, *which satisfy* (1) *in* Ω *and have value* 1 *at the origin, such that*

$$\begin{aligned} M(b,z) &= M(a,z)\, M(a,b,z), \\ M(c,z) &= M(b,z)\, M(b,c,z), \\ M(d,z) &= M(c,z)\, M(c,d,z), \end{aligned}$$

and

$$\begin{gathered} \mathscr{H}(M(a)) \subset \mathscr{H}(M(b)), \\ \mathscr{H}(M(a)) \subset \mathscr{H}(M(c)) \subset \mathscr{H}(M(d)) \end{gathered}$$

are isometric inclusions, and $M(a,b,z)$, $M(b,c,z)$, *and* $M(a,c,z) = M(a,b,z) M(b,c,z)$ *satisfy* (4).

All known factorization theorems for operator valued analytic functions seem to contain a complete continuity hypothesis. The condition we shall use is first related to the Livšic hypothesis.

THEOREM VII. *Let* $M(z)$ *be a given operator valued analytic function which satisfies* (1) *in* Ω *and has value* 1 *at the origin. A sufficient condition that* $R(0)^* - R(0)$ *be completely continuous is that* m *be a completely continuous operator. If* m *commutes with* I, *this condition is also necessary. The inequality* $\tau[R(0)^* - R(0)] \leqq \tau(m)$ *always holds, with equality when* m *commutes with* I.

THEOREM VIII. *Let* $M(c,z)$ *be a given operator valued analytic function which satisfies* (1) *in* Ω *and has value* 1 *at the origin. We suppose that* Ω *is symmetric about the real axis. If* $m(c)$ *is completely continuous and if* s *is a given number,* $0 \leqq s \leqq \tau(m(c))$, *then there exist operator valued analytic functions* $M(a, z)$, $M(a,b,z)$, *and* $M(b,c,z)$, *which satisfy* (1) *in* Ω *and have value* 1 *at the origin, such that*

$$M(c,z) = M(a,z)\,M(a,b,z)\,M(b,c,z)$$

and $\tau(m(a)) \leqq s \leqq \tau(m(a)) + \tau(m(a,b))$, *and such that* $\mathscr{H}(M(a,b))$ *has dimension* 0 *or* 1.

From this one obtains a useful estimate in the trace norm for operator valued analytic functions which satisfy (1). No similar estimate is known in the operator norm to replace the trace norm when it is infinite.

THEOREM IX. *Let* $M(z)$ *be a given operator valued analytic function which satisfies* (1) *in* Ω *and has value* 1 *at the origin. We suppose that* Ω *is symmetric about the real axis. If* m *is of trace class, then*

$$\log(1 + \tau[M(z) - 1]) \leqq \tau(m)\,\rho(z)^{-1}, \tag{5}$$

where $\rho(z)$ *is the distance from* z^{-1} *to the set of points of the form* w^{-1}, *where* w *is not in* Ω. *The function* $M(z)$ *has an analytic continuation to the half-planes* $y > 0$ *and* $y < 0$, *except for isolated singularities* (w_n) *such that*

$$\sum \left| \bar{w}_n^{-1} - w_n^{-1} \right| \leqq \tau(m). \tag{6}$$

This estimate allows an interesting kind of factorization in which singularities are shifted into preassigned regions.

THEOREM X. *Let* $M(b,z)$ *be a given operator valued analytic function which satisfies* (1) *in* Ω *and has value* 1 *at the origin. We suppose* Ω *symmetric about the real axis. If* $m(b)$ *is of trace class, then*

$$M(b,z) = M(a,z)\,M(a,b,z),$$

where $M(a,z)$ *and* $M(a,b,z)$ *are operator valued analytic functions which satisfy* (1) *in* Ω *and have value* 1 *at the origin,* $M(a,z)$ *has an analytic continuation to the half-plane* $x > 0$, $M(a,b,z)$ *has an analytic continuation to the half-plane* $x < 0$, *and* $\mathscr{H}(M(a))$ *is contained isometrically in* $\mathscr{H}(M(b))$.

A more detailed factorization theorem holds for entire functions.

THEOREM XI. *Let* $m(t)$ *be a continuous, nondecreasing, operator valued function of real* t *which has trace finite increments. Then for each real number* a *and each complex number* w, *there exists a unique continuous, operator valued function* $M(a,t,w)$ *of* $t \geqq a$ *such that*

$$M(a,b,w)\,I - I = w \int_a^b M(a,t,w)\,dm(t) \tag{7}$$

whenever $b \geqq a$. *For each fixed* a *and* b, $M(a,b,z)$ *is an operator valued entire function of* z *which satisfies* (1) *in* Ω *and has value* 1 *at the origin, and*

$$M(a,c,z) = M(a,b,z)\,M(b,c,z)$$

holds whenever $a \leqq b \leqq c$. *Furthermore,* $m(a,b) = m(b) - m(a)$ *is trace finite.*

The integral occurring in (7) is interpreted as a limit of Riemann sums converging in the trace norm.

THEOREM XII. *Let $M(z)$ be a given operator valued entire function which satisfies* (1) *and has value* 1 *at the origin. Let (a,c) be the choice of a finite interval. If m is of trace class, there exists an operator valued function $m(t)$ as in Theorem* XI *such that the solution of* (7) *satisfies $M(a,c,z) = M(z)$.*

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Proof of Theorem I. Because of (H2) and (H4) the transformation $R(w)$ is everywhere defined and has a closed graph when w is in Ω. Boundedness follows by the closed graph theorem. The resolvent identity is a familiar property of difference quotients which is easily verified directly. Since

$$R(w) = R(0)[1 - wR(0)]^{-1}$$

is everywhere defined and bounded for every number w in Ω, for no such point w is w^{-1} in the spectrum of $R(0)$. If $F(z)$ is in $\mathscr{H}$, write $F(0) = F_+(0) + F_-(0)$, where $F_+(0)$ is in $\mathscr{C}_+$ and $F_-(0)$ is in $\mathscr{C}_-$. By (H4),

$$\begin{aligned} i|F_+(0)|^2 - i|F_-(0)|^2 &= \bar{F}(0)\, I\, F(0) \\ &= \langle F(t), [F(t) - F(0)]/t\rangle - \langle [F(t) - F(0)]/t, F(t)\rangle \\ &= \langle [R(0)^* - R(0)]F, F\rangle. \end{aligned}$$

If $F(z)$ is a nonzero element in the range of P_+,

$$|F_+(0)|^2 \geqq \langle i[R(0) - R(0)^*]F, F\rangle > 0.$$

Therefore, $F(z) \to F_+(0)$ is a one-to-one transformation of the range of P_+ into $\mathscr{C}_+$, and the dimension of P_+ cannot exceed the dimension of $\mathscr{C}_+$. A similar argument with the transformation $F(z) \to F_-(0)$ will show that the dimension of P_- is no more that the dimension of $\mathscr{C}_-$.

Proof of Theorem II. Our dimension hypotheses imply the existence of a linear transformation S of $\mathscr{H}$ into $\mathscr{C}$ with these properties: (1) if $Tf = T^*f$, then $Sf = 0$; (2) if $P_+f = f$, then Sf belongs to $\mathscr{C}_+$ and

$$2\pi |Sf|^2 = \langle i(T - T^*)f, f\rangle;$$

(3) if $P_-f = f$, then Sf belongs to $\mathscr{C}_-$ and

$$2\pi |Sf|^2 = \langle -i(T - T^*)f, f\rangle.$$

Corresponding to every f in $\mathscr{H}$, we define a vector valued analytic function $F(z)$ by $F(w) = S(1 - wT)^{-1}f$ whenever w^{-1} belongs to the unbounded component of the region complementary to the spectrum of T. Since

$$(z - w)(1 - zT)^{-1}T(1 - wT)^{-1} = (1 - zT)^{-1} - (1 - wT)^{-1},$$

the identity

$$G(z) = [F(z) - F(w)]/(z - w)$$

holds whenever

$$g = T(1 - wT)^{-1}f.$$

Let us now take $\mathcal{M}$ to be the set of elements f of $\mathcal{H}$ for which the corresponding vector valued analytic function $F(z)$ vanishes identically. If f belongs to $\mathcal{M}$, $T^n f$ is in the kernel of $T - T^*$ for every $n = 0,1,2,\cdots$, and so $T^n f = T^{*n} f$. Conversely, this condition on f clearly implies that it belongs to $\mathcal{M}$. Therefore, $\mathcal{M}$ is indeed an invariant subspace for both T and T^* and the restriction of T to $\mathcal{M}$ is self-adjoint. On the other hand, if f belongs to an invariant subspace for T, which is also invariant for T^* and to which the restriction of T is self-adjoint, then $\langle Tf, g\rangle = \langle f, Tg\rangle$ is valid whether g is in the invariant subspace or orthogonal to it. Therefore it is valid for all g in $\mathcal{H}$ and $Tf = T^*f$. Continuing inductively, one obtains $T^n f = T^{*n} f$ for every n, a condition which implies that f is in $\mathcal{M}$. We have now characterized $\mathcal{M}$ as the largest invariant subspace for T which is invariant under T^* and to which the restriction of T is self-adjoint.

In the remainder of the proof we suppose $\mathcal{M}$ is the zero subspace of $\mathcal{H}$. Identify each element of $\mathcal{H}$ with the corresponding vector valued analytic function. Then $\mathcal{H}$ becomes a Hilbert space of vector valued analytic functions which clearly satisfies (H2) in Ω. We have seen that

$$R(w) = T(1 - wT)^{-1} : \ F(z) \to [F(z) - F(w)]/(z - w)$$

is a bounded linear transformation of $\mathcal{H}$ into itself when w is in Ω. Our construction was such that

$$\begin{aligned} 2\pi \bar{G}(0) I F(0) &= \langle F(t), \ [G(t) - G(0)]/t\rangle \\ &\quad - \langle [F(t) - F(0)]/t, \ G(t)\rangle \end{aligned}$$

holds whenever $F(z) = G(z)$ is in $\mathcal{H}$. By linearity the formula holds also when $F(z)$ and $G(z)$ are distinct. The axiom (H4) is now easily verified from the resolvent identity for difference quotients.

By this construction the transformation adjoint to S takes $\mathcal{C}_+$ into the range of P_+ and $\mathcal{C}_-$ into the range of P_-. In the notation of Theorem III, this adjoint is of the form $c \to K(0,z)c$, where $2\pi K(0,0) = m$ and

$$\bar{b}K(0,0)a = \langle K(0,t)a, \ K(0,t)b\rangle$$

holds for all vectors a and b. It follows that $\bar{b}ma = 0$ holds whenever a is in $\mathcal{C}_-$ and b is in $\mathcal{C}_+$, and this implies that m commutes with I.

Proof of Theorem III. For each number w in Ω, let $T(w)$ be the bounded linear transformation of $\mathcal{C}$ into $\mathcal{H}$ which is the adjoint of the transformation $F(z) \to F(w)$ of (H2). By linearity and continuity, there exists a unique operator valued analytic function $K(w,z)$ in Ω such that $T(w)c = K(w,z)c$ holds for every vector c. Therefore,

$$\bar{c}F(w) = \langle F(t), T(w)c \rangle = \langle F(t), K(w,t)c \rangle$$

holds for every $F(z)$ in $\mathscr{H}$. If a and b are vectors and if α and β are numbers in Ω, then $[K(\alpha,z) - K(\alpha,w)]a/(z-w)$ and $[K(\beta,z) - K(\beta,w)]b/(z-w)$ belong to $\mathscr{H}$ as functions of z for every complex number w in Ω, by (H4), and

$$\langle K(\alpha,t)a, [K(\beta,t) - K(\beta,\bar{w})]\, b/(t-\bar{w}) \rangle - \langle [K(\alpha,t) - K(\alpha,w)]a/(t-w), K(\beta,t)\, b \rangle$$
$$= 2\pi \bar{b}\bar{K}(\beta,\bar{w}) I K(\alpha,w)a.$$

From the defining property of $K(\alpha,z)$ and $K(\beta,z)$, we obtain

$$\bar{b}[\bar{K}(\beta,\alpha) - \bar{K}(\beta,\bar{w})]a/(\bar{\alpha} - w) - \bar{b}[K(\alpha,\beta) - K(\alpha,w)]a/(\beta - w)$$
$$= 2\pi \bar{b}\bar{K}(\beta,\bar{w})\, I K(\alpha,w)\, a.$$

By the symmetry of an inner product,

$$\bar{b}K(\alpha,\beta)a = \langle K(\alpha,t)a, K(\beta,t)b \rangle$$
$$= \langle K(\beta,t)b, K(\alpha,t)a \rangle^{-} = \bar{b}\bar{K}(\beta,\alpha)a.$$

By the arbitrariness of a and b, an operator identity is obtained which may be put in the form

$$I + 2\pi(z-\bar{w})K(w,z) = [I + 2\pi(z-\gamma)\, K(\bar{\gamma},z)]\, I\, [-I + 2\pi(\bar{w} - \gamma)\bar{K}(\gamma,w)]$$

after a change of variable. The operator valued function $M(z) = 1 - 2\pi z K(0,z)\, I$ is defined and analytic in Ω, has value 1 at the origin, and satisfies

$$K(w,z) = [M(z) I \bar{M}(w) - I]/[2\pi(z-\bar{w})].$$

If $F(z)$ is in $\mathscr{H}$,

$$[F(z) - M(z)F(0)]/z = [F(z) - F(0)]/z + 2\pi K(0,z)\, I\, F(0)$$

belongs to $\mathscr{H}$. The axiom (H4) is now easily used to verify that the transformation

$$F(z) \to [F(z) - M(z)F(0)]/z$$

is the adjoint of $R(0)$. Therefore, if we write $A(z) = \frac{1}{2}[1 + M(z)]$, the transformation

$$H : F(z) \to [F(z) - A(z)F(0)]/z$$

is self-adjoint in $\mathscr{H}$. Since $A(z)$ has value 1 at the origin, it has invertible values in some neighborhood of the origin. When $A(w)$ is invertible, $(1-\bar{w}H)^{-1}$: $F(z) \to [zF(z) - \bar{w}A(z)A(\bar{w})^{-1}F(\bar{w})]/(z-\bar{w})$ is a bounded linear transformation of $\mathscr{H}$ into itself, and

$$(1-\bar{w}H)^{-1}: \quad K(0,z)c \to K(0,z)\bar{A}(w)^{-1}c.$$

Since

$$K(w,z) = A(z)[\phi(z) - \bar{\phi}(w)]\bar{A}(w)/[\pi(z - \bar{w})],$$

we obtain

$$\bar{c}[\phi(\alpha) - \bar{\phi}(\beta)]c/[\pi(\alpha - \bar{\beta})] = \langle(1 - \bar{\beta}H)^{-1}K(0,t)c,(1 - \bar{\alpha}H)^{-1}K(0,t)c\rangle$$

for all vectors c when α and β are in some neighborhood of the origin. This formula may be used to extend $\phi(z)$ analytically to the half-planes $y > 0$ and $y < 0$, and the inequality (2) follows by the positivity of an inner product.

Proof of Theorem IV. To study operator valued analytic functions which satisfy (2), we must use an integration theory for operator valued measures. The construction to be made follows the proof of Lemma 4 of [3] after a conformal mapping of the upper half-plane onto the unit disk. By a non-negative, operator valued measure, we mean a countably additive function, defined for bounded Borel subsets of the real line, whose values are non-negative operators. By countably additive we mean that

$$\mu(\bigcup E_n) = \sum \mu(E_n)$$

holds whenever (E_n) is a sequence of disjoint Borel sets with a bounded union. Convergence is taken in the weak sense:

$$\bar{c}\mu(\bigcup E_n)c = \sum \bar{c}\mu(E_n)c$$

holds for every vector c. Frequently the values of μ are completely continuous, in which case the sum may also be taken in the operator norm. Note that for each fixed vector c, $\bar{c}\mu(E)c$ is a non-negative, numerically valued measure when considered as a function of E. If $f(x)$ is a Borel measurable, complex valued function of real x, an integral $\int f(t)d\mu(t)$ will be defined if the numerical integrals $\int f(t)d[\bar{c}\mu(t)c]$ remain bounded when c is restricted to unit length. The integral is then interpreted as the unique operator such that

$$\bar{c}\left[\int f(t)d\mu(t)\right]c = \int f(t)d[\bar{c}\mu(t)c]$$

holds for every vector c. In present applications measures have mass zero in a neighborhood of the origin and $\int t^{-2}d\mu(t)$ converges. If p is a given non-negative operator, we may associate with μ and p a unique operator valued function $\phi(z)$, defined and analytic for $y > 0$, for $y < 0$, and in a neighborhood of the origin, which has value 0 at the origin and satisfies

$$\pi\phi(z) - \pi\bar{\phi}(w) = p(z - \bar{w}) + (z - \bar{w})\int (t - z)^{-1}(t - \bar{w})^{-1}d\mu(t) \tag{8}$$

for all z and w in this region. Conversely, if a given $\phi(z)$ satisfies (2), it is represented by (8) for some unique measure μ and operator p. This is an operator version of the Poisson representation of a function positive and harmonic in a half-plane. A similar construction has previously been used in the proof of Lemma 4 of [3].

Let $\Omega(\phi)$ be the region of analyticity of the given $\phi(z)$, which includes at least the half-planes $y > 0$ and $y < 0$, as well as a neighborhood of the origin. From the representation (8) we find that if $w_1, \cdots, w_r$ are in $\Omega(\phi)$ and if $c_1, \cdots, c_r$ are corresponding vectors, then

$$\sum \bar{c}_j[\phi(w_j) - \bar{\phi}(w_i)]c_i/(w_j - \bar{w}_i) \geqq 0.$$

Let $\mathscr{L}_0(\phi)$ be the set of finite sums of functions of the form $[\phi(z) - \bar{\phi}(w)]c/(z - \bar{w})$, with c in $\mathscr{C}$ and w in $\Omega(\phi)$. We may define an inner product in $\mathscr{L}_0(\phi)$ by

$$\begin{aligned}\langle \sum[\phi(t) - \bar{\phi}(w_i)]c_i/(t - \bar{w}_i),\ \sum[\phi(t) - \bar{\phi}(w_j)]\, c_j/(t - \bar{w}_j)\rangle \\ = \pi \sum \bar{c}_j\, [\phi(w_j) - \bar{\phi}(w_i)]c_i/(w_j - \bar{w}_i).\end{aligned}$$

The linearity and symmetry of an inner product are obvious from this definition and self inner products are non-negative by what we have shown from the Poisson representation. By the Schwarz inequality, a self inner product is strictly positive unless all the values of the functions are zero. So, $\mathscr{L}_0(\phi)$ is a well-defined inner product space with the property that

$$\bar{c}f(w) = \langle f(t),\ [\phi(t) - \bar{\phi}(w)]c/[\pi(t - \bar{w})]\rangle$$

holds for every vector c and every number w in $\Omega(\phi)$. The completion $\mathscr{L}(\phi)$ of $\mathscr{L}_0(\phi)$ may now be identified with a Hilbert space of vector valued analytic functions in $\Omega(\phi)$ by applying this same formula to every element f of $\mathscr{L}(\phi)$. Analyticity of the functions so defined may be shown directly, but we omit doing this as it is a consequence of a difference-quotient property of the space, now to be established. By this construction the space $\mathscr{L}(\phi)$ obviously satisfies (H2) in $\Omega(\phi)$. For later use in the proof of Theorem VIII, we remark that if μ is supported at a finite number of points, and if p and the values of μ are operators of finite dimensional range, the space $\mathscr{L}_0(\phi)$, and hence also $\mathscr{L}(\phi)$, is finite dimensional.

From the representation (8), it may be verified that if $f(z)$ is in $\mathscr{L}_0(\phi)$ and if w is in $\Omega(\phi)$, then $[f(z) - f(w)]/(z - w)$ is in $\mathscr{L}(\phi)$ and

$$\|[f(t) - f(w)]/(t - w)\| \leqq d(w)^{-1}\|f(t)\|,$$

where $d(w)$ is the distance from w to the support of μ. It follows that $f(z) \to [f(z) - f(w)]/(z - w)$ is a bounded linear transformation of $\mathscr{L}(\phi)$ into itself which satisfies the same inequality. If α and β are in Ω, the identity

$$\begin{aligned}0 = {} & \langle f(t),\ [g(t) - g(\beta)]/(t - \beta)\rangle \\ & - \langle [f(t) - f(\alpha)]/(t - \alpha), g(t)\rangle \\ & + (\alpha - \bar{\beta})\langle [f(t) - f(\alpha)]/(t - \alpha),\ [g(t) - g(\beta)]/(t - \beta)\rangle\end{aligned}$$

may be verified directly from the definition of the inner product if $f(z)$ and $g(z)$ are in $\mathscr{L}_0(\phi)$. The same formula follows by continuity for all elements $f(z)$ and $g(z)$ of $\mathscr{L}(\phi)$, and is equivalent to the self-adjointness of the transformation $R(0)$ in $\mathscr{L}(\phi)$.

Since $\phi(z)$ has value 0 at the origin, $1 + \phi(z)I$ is invertible in a subregion Ω of $\Omega(\phi)$ which contains the origin. Let $\mathscr{H}(M)$ be the Hilbert space of vector valued functions $F(z)$, defined and analytic in Ω, of the form

$$F(z) = [1 + \phi(z)I]^{-1} f(z)$$

for some corresponding $f(z)$ in $\mathscr{L}(\phi)$. We define the inner product in $\mathscr{H}(M)$ so as to make the correspondence $f(z) \to F(z)$ isometric. Since $\mathscr{L}(\phi)$ satisfies (H2) in $\Omega(\phi)$, $\mathscr{H}(M)$ satisfies $(H2)$ in Ω. For each complex number w in Ω,

$$f(z) \to [f(z) - f(w)]/(z - w) + [\phi(z) - \phi(w)]I[1 + \phi(w)I]^{-1} f(w)/(z - w)$$

is a bounded linear transformation of $\mathscr{L}(\phi)$ into itself. It follows that $F(z) \to [F(z) - F(w)]/(z - w)$ is a bounded linear transformation of $\mathscr{H}(M)$ into itself for every w in Ω. The axiom (H4) in $\mathscr{H}(M)$ is now easily verified from the self-adjointness property of difference-quotients in $\mathscr{L}(\phi)$. Our definition of $\mathscr{H}(M)$ is such that

$$\begin{aligned} K(w,z) &= [M(z)I\bar{M}(w) - I]/[2\pi(z - \bar{w})] \\ &= [1 + \phi(z)I]^{-1}[\phi(z) - \bar{\phi}(w)][1 - I\bar{\phi}(w)]^{-1}/[\pi(z - \bar{w})] \end{aligned}$$

has the desired properties: $K(w,z)c$ belongs to $\mathscr{H}(M)$ for every vector c when w is in Ω, and

$$\bar{c}F(w) = \langle F(t), K(w,t)c \rangle$$

holds in $\mathscr{H}(M)$ for every $F(z)$. When (4) holds

$$K(w,z) = [\phi(z) - \bar{\phi}(w)]/[\pi(z - \bar{w})]$$

and $\mathscr{H}(M)$ is equal isometrically to $\mathscr{L}(\phi)$. The transformation $R(0)$, which is always self-adjoint in $\mathscr{L}(\phi)$, becomes self-adjoint also in $\mathscr{H}(M)$ in this special case.

Proof of Theorem V. Since $\phi(a,z)$ satisfies (2), we have $i\bar{\phi}(a,z) - i\phi(a,z) \geqq 0$ for $y > 0$, and by continuity, also on the portion of the real axis where $\phi(a,z)$ remains analytic, whereas the reverse inequality holds for $y \leqq 0$. When z is real, the double inequality implies that $\bar{\phi}(a,\bar{z}) = \phi(a,z)$, an identity which remains valid off the real axis by analytic continuation. It follows that $M(a,z)I\bar{M}(a,\bar{z}) = I$ holds in Ω, which is assumed symmetric about the real axis. Since $M(a,z)$ has value 1 at the origin, it has invertible values in some neighborhood of the origin. For such values of z, we may conclude that $\bar{M}(a,\bar{z})IM(a,z) = I$, an identity which holds in Ω by analytic continuation. It follows that $M(a,z)$ has invertible values in Ω.

Let $\mathscr{M}(a,b)$ be the Hilbert space of vector valued functions $F(z)$, defined and analytic in Ω, such that $M(a,z)F(z)$ belongs to $\mathscr{H}(M(b))$ and is orthogonal to $\mathscr{H}(M(a))$, the norm being defined so as to make the transformation $F(z) \to M(a,z)F(z)$ isometric. Then $\mathscr{M}(a,b)$ is a well-defined Hilbert space of vector valued analytic functions which satisfies (H2) in Ω. If $F(z)$ is in $\mathscr{M}(a,b)$ and if w is in Ω, then

$$M(a,z)[F(z) - F(w)]/(z - w) = [M(a,z)F(z) - M(a,w)F(w)]/(z - w) + 2\pi K(a,\bar{w},z)IM(a,w)IF(w)$$

belongs to $\mathscr{H}(M(b))$ since $M(a,z)F(z)$ belongs to $\mathscr{H}(M(b))$ and since $\mathscr{H}(M(b))$ satisfies (H4). A direct use of the axiom (H4) will show that this function is orthogonal to every element $G(z)$ of $\mathscr{H}(M(a))$. We have shown that $[F(z) - F(w)]/(z-w)$ belongs to $\mathscr{M}(a,b)$ whenever $F(z)$ belongs to $\mathscr{M}(a,b)$, if w is in Ω. The axiom (H4) in $\mathscr{M}(a,b)$ is now verified by a straightforward substitution. By Theorem III, $\mathscr{M}(a,b)$ is equal isometrically to $\mathscr{H}(M(a,b))$ for some operator valued analytic function $M(a,b,z)$ which satisfies (1) in Ω and has value 1 at the origin. For every vector c and every number w in Ω, $K(a,b,w,z)c$ belongs to $\mathscr{H}(M(a,b))$ as a function of z and

$$\bar{c}F(w) = \langle F(t), K(a,b,w,t)\,c\rangle$$

holds in $\mathscr{H}(M(a,b))$ for every $F(z)$. It follows that for every vector c and every number w in Ω, $M(a,z)\,K(a,b,w,z)\bar{M}(a,w)c$ belongs to $\mathscr{H}(M(b))$ and is orthogonal to $\mathscr{H}(M(a))$, and that

$$\bar{c}G(w) = \langle G(t), M(a,t)\,K(a,b,w,t)\bar{M}(a,w)c\rangle$$

holds in $\mathscr{H}(M(b))$ whenever $G(z)$ is orthogonal to $\mathscr{H}(M(a))$. Therefore,

$$M(a,z)\,K(a,b,w,z)\bar{M}(a,w) = K(b,w,z) - K(a,w,z),$$

and $M(b,z) = M(a,z)\,M(a,b,z)$.

Proof of Theorem VI. Let $\mathscr{M}(a)$ be the orthogonal complement in $\mathscr{H}(M(b))$ of those elements which are of the form $M(b,z)\ F(z)$ with $F(z)$ in $\mathscr{H}(M(b,d))$. Clearly, $\mathscr{M}(a)$ is a Hilbert space of vector valued analytic functions which satisfies (H2) in Ω, when considered in the metric of $\mathscr{H}(M(b))$. We shall now show that it satisfies (H4) in Ω. If $G(z)$ is in $\mathscr{M}(a)$ and if w is in Ω, $[G(z) - G(w)]/(z - w)$ certainly belongs to $\mathscr{H}(M(b))$. If $F(z)$ is in $\mathscr{H}(M(b,d))$ and if $M(b,z)\ F(z)$ is in $\mathscr{H}(M(b))$, then $[F(z)-F(\bar{w})]/(z-\bar{w})$ is in $\mathscr{H}(M(b,d))$ and

$$M(b,z)\,[F(z)-F(\bar{w})]/(z - \bar{w})$$

is in $\mathscr{H}(M(b))$, and

$$\langle [G(t) - G(w)]/(t - w),\ M(b,t)\,F(t)\rangle = \langle G(t),\ M(b,t)\ [F(t) - F(\bar{w})]/(t - \bar{w})\rangle = 0.$$

So, $[G(z) - G(w)]/(z - w)$ belongs to $\mathscr{M}(a)$ whenever $G(z)$ belongs to $\mathscr{M}(a)$, if w is in Ω, and the axiom (H4) for $\mathscr{M}(a)$ follows from the axiom (H4) in $\mathscr{H}(M(b))$. By Theorem III, $\mathscr{M}(a)$ is equal isometrically to $\mathscr{H}(M(a))$, where $M(a,z)$ is an operator valued analytic function which satisfies (1) in Ω and has value 1 at the origin. Since $F(z) \rightarrow M(a,z)F(z)$ is a linear isometric transformation of $\mathscr{H}(M(a,b))$ onto the orthogonal complement of $\mathscr{H}(M(a))$ in $\mathscr{H}(M(b))$, the elements of $\mathscr{H}(M(a,b))$ which are of the form $M(a,b,z)\ F(z)$ with $F(z)$ in $\mathscr{H}(M(b,d))$ are dense in $\mathscr{H}(M(a,b))$.

In a similar way we define $\mathscr{M}(b,c)$ as the closure in $\mathscr{H}(M(b,d))$ of those elements $F(z)$ such that $M(b,z)F(z)$ belongs to $\mathscr{H}(M(b))$, or what is equivalent, $M(a,b,z)F(z)$ belongs to $\mathscr{H}(M(a,b))$. Then $\mathscr{M}(b,c)$ is a Hilbert space of vector valued analytic functions which satisfies (H2) and (H4) in Ω. It is equal isometrically to $\mathscr{H}(M(b,c))$ for some operator valued analytic function $M(b,c,z)$ which satisfies (1) in Ω and has value 1 at the origin. We have a factorization

$$M(b,d,z) = M(b,c,z)\, M(c,d,z),$$

where $M(c,d,z)$ is an operator valued analytic function, given by Theorem V, which satisfies (1) in any subregion of Ω symmetric about the real axis, and has value 1 at the origin. We will now show that $M(b,c,z)$ satisfies (4). This condition implies that $M(b,c,z)$ is analytic for $y > 0$ and for $y < 0$, and by the proof of Theorem V, that $M(c,d,z)$ is analytic in Ω.

Let $\mathscr{L}$ be the Hilbert space of vector valued analytic functions $F(z)$ in $\mathscr{H}(M(b,c))$ such that $M(a,b,z)F(z)$ belongs to $\mathscr{H}(M(a,b))$, with the norm

$$\| F(t) \|^2 = \| M(a,b,t)F(t) \|_{ab}^2 + \| F(t) \|_{bc}^2 .$$

Since $\mathscr{H}(M(a,b))$ and $\mathscr{H}(M(b,c))$ satisfy (H4), $R(w)$ is a bounded linear transformation of $\mathscr{L}$ into itself for every number w in Ω, and self-adjointness of the transformation $R(0)$ in $\mathscr{L}$ is verified by an obvious calculation. It follows that $[1 - wR(0)]^{-1}$ is a bounded linear transformation of $\mathscr{L}$ into itself for every nonreal choice of w. When w is in Ω, $F(w)$ may be obtained by substituting 0 for z in

$$[zF(z) - wF(w)]/(z - w) \;=\; [1 - wR(0)]^{-1} F(z).$$

This procedure will now extend $F(z)$ analytically to the half-planes $y > 0$ and $y < 0$. Since (H2) holds, there is a unique operator valued function $\phi(z)$, defined and analytic for $y > 0$, for $y < 0$, and in Ω, with value 0 at the origin, such that $\phi(z)\, u/z$ belongs to $\mathscr{L}$ for every vector u and

$$\pi \bar{u} F(0) \;=\; \langle F(t), \phi(t)\, u/t \rangle$$

holds for every $F(z)$ in $\mathscr{L}$. If w is in Ω or is not real,

$$[\phi(z) - \phi(\bar{w})]u/(z - \bar{w}) \;=\; [1 - \bar{w}R(0)]^{-1} \phi(z)\, u/z$$

belongs to $\mathscr{L}$ and

$$\langle F(t),\ [\phi(t) - \phi(\bar{w})]u/(t - \bar{w})\rangle$$
$$= \langle [1 - wR(0)]^{-1}F(t),\ \phi(t)\, u/t\rangle = \pi\bar{u}F(w).$$

On choosing $F(z) = [\phi(z) - \phi(\bar{w})]\, u/(z - \bar{w})$, we obtain the operator inequality

$$[\phi(w) - \phi(\bar{w})]/(w - \bar{w}) \geqq 0$$

by the positivity of an inner product. In particular, self-adjointness of the operator yields

$$\phi(w) - \bar{\phi}(\bar{w}) = \phi(\bar{w}) - \bar{\phi}(w).$$

Since the expression on the left is analytic and the one on the right is conjugate-analytic, each is a constant, equal to zero by its value at the origin. If follows that $\phi(z) = \bar{\phi}(\bar{z})$ satisfies (2) and that $\mathscr{L}$ is equal isometrically to the space $\mathscr{L}(\phi)$ constructed in the proof of Theorem IV.

Since the inclusion of $\mathscr{L}(\phi)$ in $\mathscr{H}(M(b,c))$ is continuous, there is a bounded adjoint transformation T, which takes $\mathscr{H}(M(b,c))$ into $\mathscr{L}(\phi)$. Because of (H4) for $\mathscr{H}(M(b,c))$ and the self-adjointness of $R(0)$ in $\mathscr{L}(\phi)$, we have

$$T\colon\ [F(z) - M(b,c,z)\, F(0)]/z \ \rightarrow\ [G(z) - G(0)]/z$$

whenever $T\colon\ F(z) \rightarrow G(z)$. We use this formula when $F(z) = K(b,c,w,z)u$, where u is a vector and w is a nonzero element of Ω, in which case

$$G(z) = [\phi(z) - \bar{\phi}(w)]u/[\pi(z - \bar{w})].$$

$\bar{w}[F(z) - M(b,c,z)F(0)]/z = F(z) - K(b,c,0,z)\bar{M}(b,c,w)u$, $\bar{w}[G(z) - G(0)]/z = G(z) - \phi(z)u/(\pi z)$,

$$T\colon\ K(b,c,0,z)\bar{M}(b,c,w)\, u \rightarrow \phi(z)\bar{M}(b,c,w)\, u/(\pi z).$$

It follows that $\phi(z)\bar{M}(b,c,w) = \phi(z)$, an identity which may be written $M(b,c,z)\phi(w) = \phi(w)$ after a change of variable. From this we obtain $M(b,c,z)F(w) = F(w)$ for every $F(z)$ in $\mathscr{L}_0(\phi)$, and hence for every $F(z)$ in $\mathscr{L}(\phi)$ since (H2) holds and $\mathscr{L}_0(\phi)$ is dense in $\mathscr{L}(\phi)$. Since $\mathscr{L}(\phi)$ is dense in $\mathscr{H}(M(b,c))$, $M(b,c,z)F(w) = F(w)$ holds for every $F(z)$ in $\mathscr{H}(M(b,c))$, and in particular when $F(z) = K(b,c,0,z)\, u$ for some vector u. By the arbitrariness of u, we obtain

$$[M(b,c,z) - 1]I[\bar{M}(b,c,w) - 1] = 0,$$

an identity which implies (4) for $M(b,c,z)$.

If $G(z)$ is in $\mathscr{H}(M(a,b))$ and is of the form $G(z) = M(a,b,z)F(z)$ with $F(z)$ in $\mathscr{H}(M(b,c))$, we now obtain from the self-adjointness of $R(0)$ in the spaces $\mathscr{L}(\phi)$ and $\mathscr{H}(M(b,c))$,

$$\bar{G}(0)IG(0) = \langle G(t), [G(t) - G(0)]/t\rangle_{ab}$$
$$- \langle [G(t) - G(0)]/t,\ G(t)\rangle_{ab} = 0.$$

Since such $G(z)$ are dense in $\mathcal{H}(M(a,b))$, the same identity holds for every element of this space. In particular, we may choose $G(z) = K(a,b,0,z)u$ to obtain

$$[M(a,b,z) - 1]I[\bar{M}(a,b,w) - 1] = 0$$

by the arbitrariness of u, and this implies (4) for $M(a,b,z)$. We have seen that the identity $M(b,c,z)F(w) = F(w)$ holds for every $F(z)$ in $\mathcal{L}(\phi)$. But for each $F(z)$, $G(z) = M(a,b,z)F(z)$ belongs to $\mathcal{H}(M(a,b))$. Since $M(a,b,z)$ satisfies (4), $F(z) = G(z)$. We may now conclude by continuity that $M(b,c,z)F(w) = F(w)$ holds for every $F(z)$ in $\mathcal{H}(M(a,b))$, and hence that

$$[M(b,c,z) - 1]I[\bar{M}(a,b,w) - 1] = 0.$$

It follows that $M(a,c,z) = M(a,b,z)\ M(b,c,z)$ satisfies (1) and (4) and that

$$\phi(a,c,z) = \phi(a,b,z) + \phi(b,c,z).$$

Let us now show that every element $L(z)$ of $\mathcal{H}(M(a,c))$ is of the form $L(z) = F(z) + M(a,b,z)G(z)$, where $F(z)$ is in $\mathcal{H}(M(a,b))$ and $G(z)$ is in $\mathcal{H}(M(b,c))$ and $\| L\|_{ac}^2 = \| F\|_{ab}^2 + \| G\|_{bc}^2$. When $L(z) = \sum K(a,c,w_i,z)c_i$ for some numbers $w_1,\cdots,w_r$ in Ω and corresponding vectors $c_1,\cdots,c_r$ take $F(z) = \sum K(a,b,w_i,z)c_i$ and $G(z) = \sum K(b,c,w_i,z)\ M(a,b,w_i)c_i$. The desired decomposition is obtained by linearity and continuity for other choices of $L(z)$.

From this we will show that the only element $F(z) + M(a,b,z)G(z)$ of $\mathcal{H}(M(a,c))$ such that $M(a,z)[F(z) + M(a,b,z)G(z)] = M(a,z)F(z) + M(b,z)G(z)$ belongs to $\mathcal{H}(M(a))$, vanishes identically. For $M(a,z)F(z)$ belongs to $\mathcal{H}(M(b))$ and is orthogonal to $\mathcal{H}(M(a))$ by Theorem V, whereas $M(b,z)G(z)$ is in $\mathcal{H}(M(b))$ and so is orthogonal to $\mathcal{H}(M(a))$ by the definition of this space. Since

$$M(a,z)[F(z) + M(a,b,z)G(z)]$$

is thereby orthogonal to itself, it vanishes identically. Since $M(a,z)$ has invertible values in a neighborhood of the origin, $F(z) + M(a,z)G(z)$ vanishes identically.

We may now show that $M(c,z) = M(a,z)M(a,c,z)$ satisfies (1) and that $\mathcal{H}(M(a))$ is contained isometrically in $\mathcal{H}(M(c))$. To see this let $\mathcal{M}(c)$ be the Hilbert space of functions of the form $F(z) + M(a,z)G(z)$ with $F(z)$ in $\mathcal{H}(M(a))$ and $G(z)$ in $\mathcal{H}(M(a,c))$ and with $\| F(t) + M(a,t)G(t)\|_c^2 = \| F(t)\|_a^2 + \| G(t)\|_{ac}^2$. This definition is unambiguous by what we have just shown. Since $\mathcal{H}(M(a))$ and $\mathcal{H}(M(a,c))$ are known to satisfy (H2) and (H4) in Ω, it may be verified directly that $\mathcal{M}(c)$ does also. But for every complex number w in Ω and every vector u,

$$[M(c,z)I\bar{M}(c,w) - I]u/[2\pi(z - \bar{w})]$$
$$= K(a,w,z)u + M(a,z)K(a,c,w,z)\bar{M}(a,w)u$$

belongs to $\mathcal{M}(c)$ and

$$\bar{u}F(w) = \langle F(t), [M(c,t)I\bar{M}(c,w) - I]u/[2\pi(t - \bar{w})]\rangle.$$

By Theorem III, $M(c,z)$ satisfies (1) and $\mathscr{M}(c)$ is equal isometrically to $\mathscr{H}(M(c))$, which therefore contains $\mathscr{H}(M(a))$ isometrically. A similar argument will show that $M(a,d,z) = M(a,c,z)M(c,d,z)$ satisfies (1) and that $\mathscr{H}(M(a,c))$ is contained isometrically in $\mathscr{H}(M(a,d))$. It remains to show that $M(d,z) = M(c,z)M(c,d,z)$ satisfies (1) and that $\mathscr{H}(M(c))$ is contained isometrically in $\mathscr{H}(M(d))$.

By the argument just used, we need only show that if $F(z)$ is in $\mathscr{H}(M(c,d))$ and if $M(c,z)F(z)$ is in $\mathscr{H}(M(c))$, then $F(z)$ vanishes identically. We know from Theorem V that $M(b,c,z)F(z) = G(z)$ belongs to $\mathscr{H}(M(b,d))$ and is orthogonal to $\mathscr{H}(M(b,c))$. On the other hand, $M(b,z)G(z) = M(c,z)F(z)$ belongs to $\mathscr{H}(M(c))$. It is known that $\mathscr{H}(M(a))$ is contained isometrically in $\mathscr{H}(M(c))$. By Theorem V there is an element $L(z)$ of $\mathscr{H}(M(a,c))$ such that $M(b,z)G(z) - M(a,z)L(z)$ belongs to $\mathscr{H}(M(a))$. But we have seen that $L(z) = S(z) + M(a,b,z)T(z)$, where $S(z)$ is in $\mathscr{H}(M(a,b))$ and $T(z)$ is in $\mathscr{H}(M(b,c))$. By Theorem V, $M(a,z)S(z)$ is in $\mathscr{H}(M(b))$ and is orthogonal to $\mathscr{H}(M(a))$. On the other hand,

$$M(b,z)G(z) - M(a,z)M(a,b,z)T(z) = M(b,z)[G(z) - T(z)]$$

is orthogonal to $\mathscr{H}(M(a))$ since $G(z) - T(z)$ is in $\mathscr{H}(M(b,d))$. Since $M(b,z)G(z) - M(a,z)L(z)$ is orthogonal to itself, it vanishes identically. We then have

$$L(z) = M(a,b,z)G(z) = M(a,c,z)F(z).$$

Since $\mathscr{H}(M(a,c))$ is known to be contained isometrically in $\mathscr{H}(M(a,d))$, we may conclude from Theorem V that $F(z)$ vanishes identically.

Proof of Theorem VII. If the non-negative operator m is completely continuous, there is an orthonormal set (u_n) of eigenvectors, each for a positive eigenvalue p_n, whose span is dense in the range of m. The corresponding functions

$$F_n(z) = (2\pi/p_n)^{1/2}K(0,z)u_n$$

form an orthonormal set in $\mathscr{H}(M)$ whose closed span is the orthogonal complement of the functions which vanish at the origin. Since m is completely continuous, $\lim p_n = 0$ if the set is infinite. It follows that $F(z) \to F(0)$ is completely continuous as a transformation of $\mathscr{H}(M)$ into $\mathscr{C}$, and that the adjoint transformation $c \to K(0,z)c$ is completely continuous. Therefore,

$$R(0)^* - R(0)\colon\ F(z) \to 2\pi K(0,z)IF(0),$$

as a composition of two completely continuous transformations and a unitary operator, is completely continuous. If m is of trace class, $\tau(m) = \sum p_n$ is finite. The Schmidt norm of the transformation $F(z) \to F(0)$ is $[\tau(m)/(2\pi)]^{1/2}$. The adjoint has the same Schmidt norm. The composition $R(0)^* - R(0)$ of two Schmidt transformations is estimated in the trace norm by

$$\begin{aligned}\tau[R(0)^* - R(0)] &\leqq 2\pi[\tau(m)/(2\pi)]^{1/2}\,[\tau(m)/(2\pi)]^{1/2}\\ &\leqq \tau(m).\end{aligned}$$

Conversely, suppose that $R(0)^* - R(0)$ is completely continuous and that m commutes with I. If $F(z)$ is an eigenvector of this transformation for a nonzero eigenvalue λ, we have $F(z) = K(0,z)c$ for some vector c such that $\lambda c = Imc$. Since m commutes with I, $K(0,z)\frac{1}{2}(1 + iI)c$ and $K(0,z)\frac{1}{2}(1 - iI)c$ are also eigenvectors of $R(0)^* - R(0)$ for the eigenvalue λ. Complete continuity of $R(0)^* - R(0)$ allows us the choice of an orthonormal set $(F_n(z))$ of eigenfunctions, whose closed span is the orthogonal complement of the functions which vanish at the origin. We may choose them so that $R(0)^* - R(0)$: $F_n(z) \to \lambda_n F_n(z)$, where $F_n(z) = K(0,z)c_n$, $\lambda_n c_n = mIc_n$, and $Ic_n = \pm ic_n$. Since m is non-negative, $mc_n = |\lambda_n| c_n$. The vectors (c_n) are now an orthogonal set whose span is dense in the range of m. Since $R(0)^* - R(0)$ is completely continuous, $\lim \lambda_n = 0$, and m is completely continuous. If either $R(0)^* - R(0)$ or m is of trace class, so is the other and

$$\tau[R(0)^* - R(0)] = \sum |\lambda_n| = \tau(m).$$

Proof of Theorem VIII. Let $\mu(c)$ be the non-negative, operator valued measure which appears in the representation (8) of $\phi(c,z)$, and let $p(c)$ be the corresponding non-negative operator. Then,

$$\pi m(c) = 2\pi\phi'(c,0) = 2p(c) + 2\int t^{-2} d\mu(c,t)$$

s a completely continuous operator by hypothesis. Let $(\mu_n(c))$ be a sequence of non-negative, operator valued measures each supported on a finite set and having values of finite dimensional range, such that

$$\int t^{-2} d\mu_n(c,t) \leqq \int t^{-2} d\mu(c,t),$$

and

$$\mu(c,E) = \lim \mu_n(c,E)$$

holds in the operator norm for every bounded Borel set E. The construction of such a sequence is made in the obvious way, approximating μ by measures of finite support and then approximating values by operators of finite dimensional range. We shall also need a sequence $(p_n(c))$ of operators of finite dimensional range, such that $p_n(c) \leqq p(c)$ for every n, and $p(c) = \lim p_n(c)$ in the operator norm. Let $M_n(c,z)$ and $\phi_n(c,z)$ be defined by (3) and (8). The space $\mathscr{H}(M_n(c))$, constructed in Theorem IV, then has finite dimension equal to the dimension of the range of $p_n(c)$ plus the sum of the dimensions of the ranges of the values of $\mu_n(c)$. Let Ω_n be the common region of analyticity for the pair $M_n(c,z)$ and $\bar{M}_n(c,\bar{z})$.

The conclusion of the theorem is easily obtained for a finite dimensional space since invariant subspaces of all possible dimensions exist for $R(0)$. Let (s_n) be a sequence of non-negative numbers such that $s = \lim s_n$ and $s_n \leqq \tau(m_n(c))$ holds for every n. We may grant the existence of operator valued analytic functions

$M_n(a,z)$, $M_n(a,b,z)$, $M_n(b,c,z)$, which satisfy (1) in Ω_n and have value 1 at the origin, such that

$$M_n(c,z) = M_n(a,z)M_n(a,b,z)M_n(b,c,z)$$

and $\tau(m_n(a)) \leqq s \leqq \tau(m_n(a)) + \tau(m_n(a,b))$, and $\mathscr{H}(M_n(a,b))$ has dimension 0 or 1. Observe that $m_n(a) + m_n(a,b) + m_n(b,c) \leqq m(c)$, where $m(c)$ is a fixed completely continuous operator. Since

$$p_n(a) + \int t^{-2} d\mu_n(a,t) = \frac{1}{2}\pi m_n(a)$$

and similarly for $M(a,b,z)$ and $M(b,c,z)$, the sequence may be chosen so that (go to a subsequence if the original sequence does not have this property)

$$\mu(a,E) = \lim \mu_n(a,E), \quad p(a) = \lim p_n(a)$$

converge in the operator norm for every bounded Borel set E, and similarly for $M_n(a,b,z)$ and $M_n(b,c,z)$. The limits $p(a)$, $p(a,b)$, and $p(b,c)$ are non-negative operators. The limits $\mu(a)$, $\mu(a,b)$, and $\mu(b,c)$ are non-negative operator valued measures. They satisfy

$$p(a) + \int t^{-2} d\mu(a,t) \leqq \frac{1}{2}\pi m(c)$$

and similarly for $M(a,b,z)$ and $M(b,c,z)$. The functions $\phi(a,z)$, $\phi(a,b,z)$, $\phi(b,c,z)$ defined by (8) are obtained as limits in the operator norm: $\phi(a,z) = \lim \phi_n(a,z)$, etc., uniformly in some neighborhood of the origin. Therefore, the operator valued analytic functions $M(a,z), M(a,b,z)$, and $M(b,c,z)$, defined by (3), are obtained as limits in the operator norm: $M(a,z) = \lim M_n(a,z)$, etc., uniformly in some neighborhood of the origin. Since $\phi(c,z) = \lim \phi_n(c,z)$ and $M(c,z) = \lim M_n(c,z)$ in the same sense, we have

$$M(c,z) = M(a,z)\,M(a,b,z)\,M(b,c,z).$$

Theorems V and VI now imply that $M(a,z)$, $M(a,b,z)$, and $M(b,c,z)$ have analytic continuations in Ω. It is clear from this construction that $\tau(m(a)) \leqq s \leqq \tau(m(a)) + \tau(m(a,b))$ holds in the limit since operators converge in norm with a completely continuous majorant. Since $\mathscr{H}(M_n(a,b))$ has dimension 0 or 1, an element $f(z)$ of this space is an eigenfunction of $R(0)$ for some eigenvalue λ_n, and so is of the form $f(z) = (1 - \lambda_n z)^{-1} u$ for some vector u. It follows that

$$M_n(a,b,z) = 1 - z(1 - \lambda_n z)^{-1}\, m_n(a,b)I,$$

where the range of $m_n(a,b)$ has dimension 0 or 1. Since $M(a,b,z) = \lim M_n(a,b,z)$ in the operator norm, at least in a neighborhood of the origin,

$$M(a,b,z) = 1 - z(1 - \lambda z)^{-1} m(a,b)I,$$

where $\lambda = \lim \lambda_n$, and $m(a,b) = \lim m_n(a,b)$ has range of dimension 0 or 1. This formula implies that $\mathscr{H}(M(a,b))$ has dimension 0 or 1.

Proof of Theorem IX. By the approximation procedure used in the proof of Theorem VIII, we may restrict explicit proof to the case in which $M(z) = M(r,z)$ has finite dimension r. If $r > 0$, the transformation $R(0)$ in $\mathscr{H}(M(r))$ has an eigenvalue, and hence an invariant subspace of dimension 1. Since this invariant subspace is a Hilbert space of vector valued analytic functions which satisfies (H2) and (H4) in Ω, in the metric of $\mathscr{H}(M(r))$, it is equal isometrically to $\mathscr{H}(M(1))$, for some operator valued analytic function $M(1,z)$ which satisfies (1) in Ω and has value 1 at the origin. Since $\mathscr{H}(M(1))$ is contained isometrically in $\mathscr{H}(M(r))$ and since Ω is symmetric about the real axis, $M(r,z) = M(1,z)M(1,r,z)$, for some operator valued analytic function $M(1,r,z)$ which satisfies (1) in Ω and has value 1 at the origin. Since $F(z) \to M(1,z)F(z)$ is a linear isometric transformation of $\mathscr{H}(M(1,r))$ onto the orthogonal complement of $\mathscr{H}(M(1))$ in $\mathscr{H}(M(r))$, the dimension of $\mathscr{H}(M(1,r))$ is $r - 1$. We may continue inductively to construct spaces $\mathscr{H}(M(n))$ contained isometrically in $\mathscr{H}(M(r))$ for every $n = 1, \cdots, r$, such that

$$M(n + 1,z) = M(n,z)M(n,n + 1,z),$$

where $\mathscr{H}(M(n,n + 1))$ has dimension 1 for $n = 1, \cdots, r - 1$. Let λ_n be the eigenvalue of $R(0)$ in $\mathscr{H}(M(n - 1,n))$ if $n = 2, \cdots, r$, of $R(0)$ in $\mathscr{H}(M(1))$ if $n = 1$. As in the proof of Theorem VIII,

$$M(n - 1,n,z) = 1 - z(1 - \lambda_n z)^{-1} m(n - 1,n)I,$$

when $n > 1$, and similarly for $M(1,z)$ when $n = 1$. From the obvious estimate

$$\tau[M(n - 1,n,z) - 1] \leqq \tau(m(n - 1,n)) \, |\lambda_n - z^{-1}|^{-1},$$

we obtain

$$\log(1 + \tau[M(n - 1,n,z) - 1]) \leqq \tau(m(n - 1,n))|\lambda_n - z^{-1}|^{-1}$$

since $\log(1 + t) \leqq t$ when $t \geqq 0$. Here λ_n^{-1} is a singularity of $M(n-1,n,z)$, and hence does not belong to Ω since this region is assumed symmetric about the real axis. The estimate (5) now follows for $M(z)$ by the additivity of the trace norm for non-negative operators. The identity $M(n - 1,n,z)I\bar{M}(n-1,n,\bar{z}) = I$, which is a consequence of (1), implies that

$$m(n - 1,n)Im(n - 1,n) = (\bar{\lambda}_n - \lambda_n)\, m(n - 1,n).$$

On taking the trace norm of each side, one finds that

$$|\bar{\lambda}_n - \lambda_n| \leqq \tau(m(n - 1,n)),$$

an inequality which implies (6) on summation.

Proof of Theorem X. By the proof of Theorem VIII, it is sufficient to consider the case in which $\mathscr{H}(M(b))$ is finite dimensional. The general case follows by approximation, the regions of analyticity being preserved by estimates from Theorem IX. When $\mathscr{H}(M(b))$ is finite dimensional, the argument is like that for Theorem IX except that we make a choice of eigenvalues so as to have analyticity in the desired half-planes. Details of this procedure can surely be left to the reader.

Proof of Theorem XI. For existence, let

$$M(a,b,z) = \sum M_n(a,b)z^n,$$

where $M_n(a,b)$ is defined inductively by $M_0(a,b) = 1$ and

$$M_{n+1}(a,b) = \int_a^b M_n(a,t)\, dm(t).$$

Since

$$\tau[M_{n+1}(a,t)] \leqq \int_a^b |M_n(a,t)|\; d\tau(m(t)),$$

we obtain inductively

$$\tau[M_{n+1}(a,b)] \leqq \tau(m(a,b))^{n+1}/(n+1)!',$$

which implies convergence of the series defining $M(a,b,z)$ uniformly on bounded sets. Because of this estimate, we may integrate term by term in

$$\begin{aligned} w\int_a^b M(a,t,w)\, dm(t) &= \sum w^{n+1} \int_a^b M_n(a,t)\, dm(t) \\ &= \sum w^{n+1} M_{n+1}(a,b)I \\ &= M(a,b,w)I - I. \end{aligned}$$

Since $M_n(a,c) = \sum M_{n-k}(a,b)\, M_k(b,c)$, we do have $M(a,c,z) = M(a,b,z)\, M(b,c,z)$ when $a \leqq b \leqq c$. The estimate (5) for $M(a,b,z)$ may be verified directly, and from this we see that $M(a,t,w)$ is a continuous function of $t \geqq a$ for each fixed w. We are now justified in rearranging integrals in the following calculations from (7)

$$\begin{aligned} &M(a,b,z)I\bar{M}(a,b,z) - I \\ &\quad = z\bar{w} \int_a^b \int_a^b M(a,s,z)\, dm(s)I\, dm(t),\ \bar{M}(a,t,w) \\ &\qquad + z\int_a^b M(a,s,z)\, dm(s) - \bar{w}\int_a^b dm(t)\bar{M}(a,t,w) \\ &\quad = z\bar{w}\int_a^b [\int_a^t M(a,s,z)\, dm(s)]I dm(t)\bar{M}(a,t,w) \\ &\qquad + z\bar{w}\int_a^b M(a,s,z)dm(s)I[\int_a^s dm(t)\bar{M}(a,t,w)] \\ &\qquad + z\int_a^b M(a,s,z)dm(s) - \bar{w}\int_a^b dm(t)\bar{M}(a,t,w) \\ &\quad = (z-\bar{w})\int_a^b M(a,s,z)dm(t)\,\bar{M}(a,t,w). \end{aligned}$$

From this identity it is clear that $M(a,b,z)$ satisfies (1). It remains to show the uniqueness of solutions of (7). If $M_0(a,t,w)$ is another solution of this equation, the argument just gone through will show that

$$M(a,b,z)\,I\bar{M}_0(a,b,w) - I = (z-\bar{w})\int_a^b M(a,t,z)\,dm(t)\,M_0(a,t,w).$$

The identity $M_0(a,b,w) = M(a,b,w)$ is obtained when $z = \bar{w}$ since

$$M(a,b,w)IM(a,b,\bar{w}) = I$$

by the proof of Theorem V.

Proof of Theorem XII. By Zorn's lemma we may choose a maximal family $(M(t,z))$ of operator valued entire functions, which satisfy (1) and have value 1 at the origin, such the spaces that $\mathscr{H}(M(t))$ are contained isometrically in $\mathscr{H}(M)$ and are totally ordered by inclusion. If $M(s,z)$ and $M(t,z)$ are in this family and if $\mathscr{H}(M(s))$ is contained in $\mathscr{H}(M(t))$, then $M(t,z) = M(s,z)\,M(s,t,z)$, where $M(s,t,z)$ is an operator valued entire function which satisfies (1) and has value 1 at the origin. Since $m(t) = m(s) + m(s,t)$, $\tau(m(s)) \leqq \tau(m(t))$, and equality holds only when $m(s,t) = 0$, a condition which implies that $M(s,t,z) = 1$ identically and that $M(s,z) = M(t,z)$. Therefore, $t = \tau(m(t))$ is a natural choice of parameter for the family. Since $M(z) = M(c,z)$ and $\tau(m)$ is finite by hypothesis, these parameters are well-defined numbers, contained in an interval $[0,c]$. A number t, which corresponds in this way to an element $M(t,z)$ of the family, will be called a regular point. Other points t in $[0,c]$ are said to be singular. The end points 0 and c are obviously regular. We now show that the regular points form a closed set.

If $t = \lim t_n$, where (t_n) is a decreasing sequence of regular points, the intersection of the spaces $\mathscr{H}(M(t_n))$ is a Hilbert space of vector valued entire functions which satisfies (H2) and (H4) and is comparable with every space in our family. By Theorem III and the maximal choice of our family, this intersection coincides with $\mathscr{H}(M(s))$, where s is a regular point, $s \leqq t_n$ for every n, and s is the largest regular point with this property. For every vector u, $K(t_n,0,z)u$ belongs to $\mathscr{H}(M(t_n))$ and has $K(s,0,z)u$ as its projection in $\mathscr{H}(M(s))$. Since $\mathscr{H}(M(s))$ is the intersection of the spaces $\mathscr{H}(M(t_n))$, $K(s,0,z)u = \lim K(t_n,0,z)u$ holds in the metric of $\mathscr{H}(M)$, hence convergence of the norms to yield $\bar{u}m(s)u = \lim \bar{u}m(t_n)u$. It follows that

$$t = \lim t_n = \lim \tau(m(t_n)) = \tau(m(s)) = s$$

is a regular point. A similar argument will show that the limit of an increasing sequence of regular points is regular. The regular points are therefore a closed set.

Consider any one of the possible intervals (r,t), with regular end points, which contains only singular points in its interior. We will show that $\mathscr{H}(M(r,t))$ has dimension 1. This conclusion may be drawn directly from Theorem VIII if

$M(r,t,z)$ cannot be factored. Otherwise, we may write $M(r,t,z) = M(r,s,z)\,M(s,t,z)$, where $M(r,s,z)$ and $M(s,t,z)$ are nonconstant operator valued entire functions which satisfy (1) and have value 1 at the origin. By Theorem VI we obtain a factorization

$$M(r,t,z) = M(r,s_-,z)\,M(s_-,s_+,z)M(s_+,t,z),$$

where $M(r,s_-,z)$, $M(s_-,s_+,z)$, and $M(s_+,t,z)$ are operator valued entire functions which satisfy (1) and have value 1 at the origin, $M(r,s_-,z)$ does not coincide with $M(r,t,z)$, $M(r,s_+,z) = M(r,s_-,z)M(s_-,s_+,z)$ is not constant, $\mathscr{H}(M(r,s_-))$ and $\mathscr{H}(M(r,s_+))$ are contained isometrically in $\mathscr{H}(M(r,t))$, and $M(s_-,s_+,z)$ satisfies (4). By the maximal choice of our family, $M(r,s_-,z) = 1$ and $M(r,s_+,z) = M(r,t,z)$ identically. Therefore, $M(r,t,z) = M(s_-,s_+,z)$ satisfies (4). The maximal choice of our family implies that the transformation $R(0)$ in $\mathscr{H}(M(r,t))$ has no proper invariant subspaces. Since this transformation is self-adjoint, $\mathscr{H}(M(r,t))$ has dimension 1.

When $\mathscr{H}(M(r,t))$ has dimension 1,

$$M(r,t,z) = 1 - z(1-\lambda z)^{-1}m(r,t)I$$

by the proof of Theorem VIII, where $\lambda = 0$ in this case since we have an entire function. In particular, $M(r,t,z)$ satisfies (4). We define $M(r,s,z)$ and $M(s,t,z)$ for $r \leqq s \leqq t$ by

$$\begin{aligned}\phi(r,s,z) &= \phi(r,t,z)(s-r)/(t-r),\\ \phi(s,t,z) &= \phi(r,t,z)(t-s)/(t-r),\end{aligned}$$

so that $M(r,t,z) = M(r,s,z)\,M(s,t,z)$. Then $M(s,z) = M(r,z)\,M(r,s,z)$ is defined for singular points in $[0,c]$, as well as for regular points. But, of course, when s is singular, $\mathscr{H}(M(s))$ is not contained isometrically in $\mathscr{H}(M)$.

To avoid a clumsy but obvious renormalization, we will suppose that the given interval (a,c) is of the form $a = 0$ and $c = \tau(m)$. Let $m(t)$ be the choice of a continuous operator valued function of real t, constant outside of $(0,c)$, such that $m(s,t) = m(t) - m(s)$ whenever $0 \leqq s \leqq t \leqq c$. This function is obviously nondecreasing and has trace finite increments. By Theorem IX, the estimate

$$\log(1 + \tau[M(s,t,z) - 1]) \leqq (t-s)|z|$$

holds whenever $0 \leqq s \leqq t \leqq c$. If

$$M(s,t,z) = \sum M_n(s,t)\,z^n$$

is the power series expansion, we therefore obtain

$$\tau[M_{n+1}(s,t)] \leqq n^{-n}\,e^n(t-s)^{n+1}.$$

But whenever $r \leqq s \leqq t$, $M(r,t,z) = M(r,s,z)\,M(s,t,z)$ and so

$$M_n(r,t) = \sum M_{n-k}(r,s)\,M_k(s,t).$$

This fact may be used to refine our previous estimate in the more elegant form

$$\tau[M_{n+1}(s,t)] \leqq (t-s)^{n+1}/(n+1)!.$$

Since

$$M_{n+1}(t) = \sum M_{n+1-k}(s)\,M_k(s,t),$$

where $M_1(s,t)I = m(s,t)$, we obtain

$$\tau[M_{n+1}(t)I - M_{n+1}(s)I - M_n(s)m(s,t)] \leqq t^{n+1}/(n+1)! - s^{n+1}/(n+1)!$$

when $s \leqq t$. It follows that

$$M_{n+1}(t)I = \int_0^t M_n(s)\,dm(s)$$

and that

$$M(t,w)I - I = w \int_0^t M(s,w)dm(s).$$

The general case of formula (7) follows by similar arguments.

Added in proof. Theorems VI and VIII are incorrectly stated and are revised in the second part of this paper.

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