EXTENSIONS OF ORDERED GROUPS AND SEQUENCE COMPLETION(1)

BY

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1. **Introduction.** In this paper the relationships between certain extensions of totally (linearly) ordered groups (o-groups) are studied. The additive notation is employed although the groups are not, in general, assumed to be abelian. The four types of extensions considered are defined as follows:

Let $G \subseteq H$ be o-groups.

(a) H is an *a-extension* of G if for every $0 < h \in H$ there exists $g \in G$ and a positive integer n such that $h \leqq g \leqq nh$.

(b) H is a *b-extension* of G if for every $0 < h \in H$ there exists $g \in G$ and a positive integer n such that $g \leqq h \leqq ng$.

(c) H is a *c-extension* of G if for every $0 < h \in H$ there exists $g \in G$ such that for all integers n, $n(h - g) < h$.

(t) H is a *t-extension* of G if for every $h, h', h'' \in H$ with $h < h' < h''$ there exists $g \in G$ such that $h < g < h''$.

If $x \in \{a, b, c, t\}$, G is *x-closed* if G has no proper x-extensions. The connecting notion between the different types of extensions is that of sequences in o-groups; of particular importance are cauchy sequences (definition in §2) and pseudo sequences (definition in §3). Several authors have discussed similar concepts, but in different or more special cases. Cohen and Goffman [3; 4] consider cauchy sequences in abelian groups; Everett and Ulam [8] consider countable cauchy sequences; Banaschewski [1] deals with cauchy filters in partially ordered groups; Gravett [9] relates pseudo sequences to c-extensions in divisible abelian groups.

§2 contains definitions and a basic lemma concerning sequences. It is also shown that every t-extension and every c-extension is a b-extension, and that every b-extension is an a-extension. Some results concerning t-extensions and cauchy sequences ("T" theorems) are stated without proof. In §3 some theorems ("C" theorems) concerning pseudo sequences and c-extensions, including analogues to the "T" theorems, are proved. In §4 are stated the "B" theorems concerning sequences and b-extensions with proofs only sketched, as the proofs

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are mostly similar to those in §3. In §6 are examples which illustrate the limitation of the theory. The system of numbering is such that Theorems Ti, Ci, and Bi are analogues.

2. **Cauchy sequences and t-extensions.** Throughout this paper G will denote an o-group. The pairs (G_γ, G^γ) of convex subgroups of G such that G^γ covers G_γ are called *convex pairs.* G_γ is normal in G^γ and G^γ/G_γ is o-isomorphic to a subgroup of R, the additive group of real numbers. The groups G^γ/G_γ are called the *components* of G. If A and B are subsets of $G, A < B$ means every element of A is less than every element of B. For $g \in G, |g|$ is the max of g and $-g$.

A *sequence* (in G) is a collection $\{s_\alpha \mid \alpha \in \Delta\}$ of elements of G such that Δ is a well-ordered set with no final element. The *breadth* of a sequence $\{s_\alpha\}$ is the intersection of all convex subgroups C of G satisfying:

(1) C is covered by a convex subgroup C'.

(2) For each $g \in C' \setminus C$ there exists $\mu \in \Delta$ such that if $\alpha, \beta > \mu$ then $s_\alpha - s_\beta \in C'$ and $s_\alpha - s_\beta + C < |g| + C$.

Note that the breadth is a convex subgroup. Here and in several places later on the definition is "one-sided" due to lack of commutativity. This would more appropriately be called "left breadth," the "right breadth" being defined similarly. The introduction of this complexity seems, however, to be of no particular aid.

An element s of G is a *limit* of the sequence $\{s_\alpha\}$ if for every convex pair (C,C') such that C contains the breadth of $\{s_\alpha\}$ and for every $g \in C' \setminus C$, there exists $\nu \in \Delta$ such that if $\alpha > \nu$, $s_\alpha - s \in C'$ and $|s_\alpha - s| + C < |g| + C$. We say $\{s_\alpha\}$ converges to s.

LEMMA 1. *If r and s are limits of $\{s_\alpha\}$ then $r - s$ lies in the breadth of $\{s_\alpha\}$.*

Proof. By way of contradiction, if $r-s$ is not in the breadth B then $r - s \in C' \setminus C$ where (C,C') is a convex pair and $B \subseteq C$. Now for all sufficiently large α, $2|s_\alpha - r| + C < |r - s| + C$. For C'/C is o-isomorphic to a subgroup of the real numbers, and if C'/C is not discrete then there exists $g \in G$ such that $2|g| + C < |r - s| + C$ and by assumption, for all α large enough, $|s_\alpha - r| + C < |g| + C$. If C'/C is discrete it is cyclic generated by some $f + C$; then for all α large enough $|s_\alpha - r| + C < |f| + C$ so that $s_\alpha - r \in C$, in which case $2|s_\alpha - r| + C = C < |r - s| + C$. Choose α so that $s_\alpha - r$, $s_\alpha - s \in C'$ and $2|s_\alpha - r| + C < |r - s| + C$ and $2|s_\alpha - s| + C < |r - s| + C$. Then $|r - s| + C = |r - s_\alpha + s_\alpha - s| + C \leqq |s_\alpha - r| + |s_\alpha - s| + C < |r - s| + C$, a contradiction. Hence $r - s \in B$.

LEMMA 2. *Let H be an a-extension of G. If $\{g_\alpha\}$ is a sequence of elements of G, if N is the breadth of $\{g_\alpha\}$ in G, and if B is the breadth of $\{g_\alpha\}$ in H, then $B \cap G = N$. If $g \in G$ is a limit of $\{g_\alpha\}$ in G then g is a limit of $\{g_\alpha\}$ in H.*

Proof. (K,K') is a convex pair of H if and only if $(K \cap G, K' \cap G)$ is a convex pair of G. In the definition of breadth, it is required that the differences $g_\alpha - g_\beta$ become arbitrarily small in certain components K'/K and $K' \cap G/K \cap G$, respectively. As $K' \cap G/K \cap G \subseteq K'/K$ are subgroups of the real numbers, the differences in question become small in $K' \cap G/K \cap G$ if and only if they be-become small in K'/K. A similar remark holds for the differences $g_\alpha - g$.

A sequence $\{s_\alpha\}$ is a *left cauchy sequence* if its breadth is 0. $\{s_\alpha\}$ is a *cauchy sequence* if both $\{s_\alpha\}$ and $\{-s_\alpha\}$ are left cauchy sequences. Example 1 of §6 contains a left cauchy sequence which is not a cauchy sequence.

In the topology generated by the open intervals, every o-group is a topological group. It is easily seen from the definition that H is a t-extension of G if and only if G is dense in H in the order topology. Banaschewski [1] has shown that the topological completion $t(G)$ of an o-group G is an o-group which extends the order of G. Banaschewski's proof, as well as the earlier and less general ones of Cohen and Goffman [3] and Everett and Ulam [8], proceeds by completing G via "dedekind cuts." It is also possible to construct $t(G)$ by choosing cauchy sequences of appropriate length, showing that they form an o-group under term-wise addition, and continuing in a manner analogous to the cauchy sequence construction of the real numbers. The proof of this, from which the following results easily follow, is notationally involved and sufficiently similar to the classical results that the details are omitted here.

THEOREM T1. *If H is a t-extension of G and $h \in H \backslash G$ then there is a cauchy sequence of elements of G with limit h (and with no limits in G).*

COROLLARY T2. *If every cauchy sequence in G has a limit then G is t-closed.*

Let Γ be an ordered set, and for each $\gamma \in \Gamma$ let R_γ be a subgroup of the real numbers. The group $H(\Gamma, R_\gamma)$ consisting of those functions in the large direct sum of the R_γ whose support is inversely well ordered, is called a Hahn group, and is ordered by calling a function positive if it is positive on the largest element of its support.

LEMMA T4. *If $H(\Gamma, R_\gamma)$ is a Hahn group and either Γ has no smallest element or γ_0 is the smallest element of Γ and R_{γ_0} is the reals or the integers, then $H(\Gamma, R_\gamma)$ is t-closed.*

Proof. Let $\{f_\alpha\}$ be a cauchy sequence in H. It follows from the definition of cauchy sequence that for each $\gamma \in \Gamma$ except the smallest element of Γ, if any, for all sufficiently large α and β, $f_\alpha(\gamma) = f_\beta(\gamma) = f(\gamma)$, say. Moreover, if Γ has a smallest element γ_0 then the sequence $\{f_\alpha(\gamma_0)\}$ is a cauchy sequence in the ordinary sense in R, and hence converges to $f(\gamma_0)$, say. It is easy to verify that the function f is an element of H and the limit of $\{f_\alpha\}$.

THEOREM T6. *G is t-closed if and only if every cauchy sequence in G converges.*

LEMMA 3. *Every t-extension of G is a c-extension of G if and only if G is t-closed or G has no convex subgroup covering* 0.

Proof. If G has no convex subgroup covering 0 and if H is a t-extension of G, let $0 < h \in K' \backslash K$ where (K, K') is a convex pair of H and $h > 0$. Then $K \neq 0$ and hence the interval $h + K$ of H contains an element $g \in G$ (since G is dense in H). Therefore $h - g \in K$; that is, $n(h - g) < h$ for all n.

Conversely, if every t-extension is a c-extension and if G has a convex subgroup C covering 0, then from the remark at the beginning of the proof of Lemma 2 and the fact (which will be proved presently) that every t-extension is an a-extension, it follows that $t(G)$ has a convex subgroup C' covering 0. As $t(G)$ is by assumption a c-extension of G, for every $0 < h \in C'$ there is a $g \in G$ with $n(g-h) < h$ for all n; that is, $g-h = 0$. Hence $C = C'$. Since G is dense in $t(G)$, for every $h \in t(G)$ the interval $h + C$ of $t(G)$ contains an element of G. Hence $G = t(G)$. Therefore G is t-closed.

No general necessary and sufficient conditions that every c-extension be a t-extension are known. For the abelian case, however, such conditions can be derived from the theorems in §3.

LEMMA 4. *Every t-extension is a b-extension.*

Proof. Let H be a nontrivial t-extension of G and let $0 < h \in H \backslash G$. Then there exists $h' \in H$ with $h' < h < 2h'$ (otherwise H is discrete, which implies $H = G$). Likewise there exists h'' with $h' < h'' < h$. Hence there exists $g \in G$ such that $h' < g < h$. Thus $h < 2h' < 2g$.

LEMMA 5. *Every c-extension is a b-extension.*

Proof. Let H be a c-extension of G and let $0 < h \in H$. Then $h - g = a$ for some $g \in G$ where for all n, $na < h$. Also $a - f = b$ for some $f \in G$ where $nb < |a|$ for all n. Let $x = -|f| + f + g \in G$. It is easily verified that $x < h < 2x$.

LEMMA 6. *Every b-extension is an a-extension.*

Proof. Let H be a b-extension of G. Let $0 < h \in H$. There exists $g \in G$ such that $g \leqq h \leqq ng$. Therefore $h \leqq ng \leqq nh$.

3. **Pseudo sequences and c-extensions.** For every $0 \neq g \in G$ there exists a unique convex pair (G_α, G^α) such that $g \in G^\alpha \backslash G_\alpha$. The mapping $V: G \backslash \{0\} \to \Gamma(G)$, where $\Gamma(G)$ is the set of all convex pairs of G, defined by $V(g) = (G_\alpha, G^\alpha)$ is the *natural valuation* of G. Theorems C1, C2, C5, and C6 which follow are generalizations of the classical valuation theory (Schilling [11]), already generalized by Gravett [9] to vector spaces (and hence to divisible abelian o-groups).

A sequence $\{s_\alpha\}$ is called a *pseudo sequence* if for all $\alpha < \beta < \gamma$, $V(s_\beta - s_\gamma) < V(s_\alpha - s_\beta)$. It follows that for each α there exists $\lambda_\alpha \in \Gamma(G)$

such that for all $\beta > \alpha$, $V(s_\alpha - s_\beta) = \lambda_\alpha$. Moreover, for each limit s of $\{s_\alpha\}$, $V(s_\alpha - s) = \lambda_\alpha$. The breadth of $\{s_\alpha\}$ is the same as that defined in the classical theory.

THEOREM C1. *If H is a c-extension of G and $h \in H \backslash G$ then there is a pseudo sequence of elements of G with limit h and with no limits in G.*

Proof. The empty set is a pseudo sequence with limit h. By Zorn's lemma there is a pseudo sequence of elements of G with limit h of maximal length (since no element can occur twice in the same pseudo sequence). Let $\{g_\alpha\}$ be a pseudo sequence of elements of G with limit h of maximal length. If $\{g_\alpha\}$ has a limit $g \in G$ then by Lemma 1, $g-h$ is in the breadth of $\{g_\alpha\}$. Hence $V(g-h) < \lambda_\alpha$ for all α. Let $g^{(1)} = g$. Define $g^{(i)}$ by induction for $i = 2, 3, \cdots$ as follows: since H is a c-extension of G there exists $f^{(i)} \in G$ with $h - g^{(i-1)} - f^{(i)} = h^{(i)}$ where $V(h^{(i)}) < V(h - g^{(i-1)})$. Let $g^{(i)} = f^{(i)} + g^{(i-1)}$. It is easy to verify that the sequence $\{\cdots, g_\alpha, \cdots, g^{(1)}, g^{(2)}, \cdots\}$ is a pseudo sequence with limit h. This contradicts the maximality of $\{g_\alpha\}$ and shows that $\{g_\alpha\}$ has no limits in G.

COROLLARY C2. *If every pseudo sequence in G converges, then G is c-closed.*

The converse of Corollary C2 is not true, as is shown by Example 2 of §6. For the abelian case, however, Lemma C5, states the converse. The following theorem was proved for abelian G by Ribenboim [10].

THEOREM C3. *If for every noncovered convex subgroup C of G, C is normal and G/C is t-closed, then G is c-closed.*

Proof. Suppose that H is a proper c-extension of G and that $h \in H \backslash G$. By Theorem C1 there is a pseudo sequence $\{g_\alpha\}$ with limit h and no limits in G. Let B be the breadth of $\{g_\alpha\}$ in G. Then B is noncovered and hence normal in G. The first aim is to show that $\{g_\alpha + B\}$ is a cauchy sequence in G/B.

As before, for $\beta > \alpha$ let $V(g_\alpha - g_\beta) = \lambda_\alpha = (C'_\alpha, C^{\alpha\prime})$, where $C'_\alpha = C_\alpha \cap G$, $C^{\alpha\prime} = C^\alpha \cap G$, and (C_α, C^α) is a convex pair in H. Then $B = (\bigcap_\alpha C) \cap G$. Denote by ϕ_α the inner automorphism of H induced by g_α, and by ψ the inner automorphism of H induced by h. Since $g_\alpha - h \in C^\alpha$, and for $\beta > \alpha$, $g_\alpha - g_\beta \in C^\alpha$, $C^\alpha\phi_\alpha = C^\alpha\phi_\beta = C^\alpha\psi$. It follows that $\bigcap_\alpha C^\alpha\phi_\alpha$ is a noncovered convex subgroup of H because for $\beta > \alpha$, $C^\beta \subset C^\alpha$ and hence $C^\beta\phi_\beta \subset C^\alpha\phi_\beta = C^\alpha\phi_\alpha$. Hence $(\bigcap_\alpha C^\alpha\phi_\alpha) \cap G$ is a noncovered convex subgroup of G, and by assumption is normal in G.

In particular, $((\bigcap_\alpha C^\alpha\phi_\alpha) \cap G)\phi_\beta = (\bigcap_\alpha C^\alpha\phi_\alpha) \cap G$. It follows that also $(\bigcap_\alpha C^\alpha\phi_\alpha)\phi_\beta = \bigcap_\alpha C^\alpha\phi_\alpha$. For each β, $C^\beta\phi_\beta \supset \bigcap_\alpha C^\alpha\phi_\alpha = (\bigcap_\alpha C^\alpha\phi_\alpha)\phi_\beta$. Hence $C^\beta \supset \bigcap_\alpha C^\alpha\phi_\alpha$. From this follows $\bigcap_\beta C^\beta \supset \bigcap_\alpha C^\alpha\phi_\alpha = \bigcap_\alpha C^\alpha\psi$.

Since $-h + g_\alpha = -h + (g_\alpha - h) + h$, then $-h + g_\alpha \in C^\alpha\psi$. Hence $-g_\alpha + g_\beta = -g_\alpha + h - h + g_\beta \in (C^\alpha\psi) \cup (C^\beta\psi)$. From this and the last statement of the preceding paragraph it follows that $\{g_\alpha + B\}$ is a cauchy sequence in G/B.

By assumption $\{g_\alpha + B\}$ converges to some $g + B$. Then g is a limit of $\{g_\alpha\}$, a contradiction.

LEMMA C4. *In any Hahn group, all pseudo sequences converge. Thus all Hahn groups are c-closed.*

Proof. If H is a Hahn group and $\{h_\alpha\}$ is a pseudo sequence of breadth B, then $\{h_\alpha + B\}$ is a cauchy sequence in H/B, which is also a Hahn group. Thus by Lemma T4 and Theorem T6 $\{h_\alpha + B\}$ has a limit $h + B$. Hence h is a limit of $\{h_\alpha\}$. The last half of the lemma follows from Corollary C2.

LEMMA C5. *If G is abelian and c-closed, then every pseudo sequence in G converges.*

Proof. Suppose that $\{s_\alpha\}$ is a pseudo sequence in G with no limits in G. For each α and all $\beta > \alpha$, $V(s_\alpha - s_\beta) = \lambda_\alpha$, say. Note that for each integer $n \neq 0$, $\{ns_\alpha\}$ is a pseudo sequence. If $\{ns_\alpha\}$ has g as a limit and if $\{ms_\alpha\}$ has h as a limit then it is easily verified that $\{(n + m)s_\alpha)$ has $g + h$ as a limit. Thus the set of all integers p such that $\{ps_\alpha\}$ has a limit in G is a subgroup of the integers, generated by some n_0. It is also easily seen that all $\{ps_\alpha\}$ are pseudo sequences in any extension of G.

By Hahn's theorem G can be embedded in a divisible Hahn group H. By Lemma C4 any pseudo sequence in H converges. Choose $s \in H$ as follows: if $\{ns_\alpha\}$ has no limit in G for every integer $n \neq 0$, let s be any limit of $\{s_\alpha\}$ in H. If $\{ns_\alpha\}$ has a limit in G for some integer n, let the group of such integers be generated by n_0, let s' be a limit of $\{n_0 s_\alpha\}$ in G and let $s = s'/n_0$. Then in either case s is a limit of $\{s_\alpha\}$. Now G is properly contained in G', the subgroup of H generated by G and s.

It will now be shown that G' is a c-extension of G. Let $B = \{V(h) \mid h \text{ is in the breadth of } \{s_\alpha\}\}$. That is, B is those elements of $\Gamma(H)$ which are less than every λ_α. Let $g + ns \in G'$, where $g \varepsilon G$.

CASE 1. $V(g + ns) = \gamma \notin B$. Then there exists β such that $V(ns - ns_\beta) < \gamma$. Hence $ns + g = (ns - ns_\beta) + (ns_\beta + g)$ where $ns_\beta + g \in G$.

CASE 2. $V(g + ns) \in B$. In this case $g + ns \in G$ because first, $V(ns_\alpha - (-g)) = V((ns_\alpha - ns) + (ns + g)) = \lambda_\alpha$ so that $-g$ is a limit of $\{ns_\alpha\}$. Hence for some integer m, $n = n_0 m$. Therefore $ns = mn_0 s = ms' \in G$. Hence G' is a c-extension of G.

From Corollary C2 and Lemma C5 follows immediately

THEOREM C6. *If G is abelian, G is c-closed if and only if every pseudo sequence in G converges.*

LEMMA 7. *Suppose that G has no convex subgroup covering* 0. *Then for every element s of $t(G)$ there is a pseudo sequence $\{r_\alpha\}$ in G such that $\{r_\alpha\}$ is also a cauchy sequence and $\{r_\alpha\}$ has limit s.*

Proof. Since G has no smallest convex subgroup, there exists a sequence $\{\theta_\alpha\}$ of elements of G such that for $\alpha>\beta$, $V(\theta_\alpha) < V(\theta_\beta)$, 0 is the greatest lower bound of $\{\theta_\alpha\}$, and for each α, $\theta_\alpha + s \in G$ (since G is dense in $t(G)$). Let $r_\alpha = \theta_\alpha + s$. Then for $\beta > \alpha$, $V(r_\alpha - r_\beta) = V(\theta_\alpha) = V(r_\alpha - s)$. Hence $\{r_\alpha\}$ is a pseudo sequence with limit s. The sequence $\{-s + \theta_\alpha + s\}$ also has greatest lower bound 0. Since $V(-r_\beta + r_\alpha) = V(-s + (s - r_\beta) + s - s + (r_\alpha - s) + s)$ which is less than or equal to the greater of $V(-s + \theta_\alpha + s)$ and $V(-s + \theta_\beta + s)$, $\{r_\alpha\}$ is also a cauchy sequence.

LEMMA C7. *If G is abelian and c-closed, then for every noncovered convex subgroup C of G, G/C is t-closed.*

Proof. For a noncovered convex subgroup C, G/C has no convex subgroup covering 0. If G/C is not t-closed, then by Theorem T6 there is a nonconvergent cauchy sequence $\{s_\alpha + C\}$ in G/C. By Lemma 7 there is a nonconvergent pseudo sequence $\{r_\alpha + C\}$ in G/C. Then $\{r_\alpha\}$ is a nonconvergent pseudo sequence in G. Hence by Theorem C6, G is not c-closed.

THEOREM C8. *If G is abelian, then G is c-closed if and only if G/C is t-closed for every noncovered convex subgroup C of G.*

Proof. Lemma C7 and Theorem C3.

4. Sequences and b-extensions.

THEOREM B1. *If H is a b-extension of G and $h \in H\backslash G$ then there is a sequence of elements of G with limit h and with no limits in G.*

Proof. The proof proceeds as far as possible along the lines of and with the notation of Theorem C1. Let $\{g_\alpha\}$ be a pseudo sequence of elements of G with imit h and of maximal length. If $\{g_\alpha\}$ has no limits in G, the proof is complete. If $\{g_\alpha\}$ has a limit g in G, let $g^{(1)} = g$ and define $g^{(i)}$ as before, as far as possible. It cannot happen that for every $i = 2, 3, \cdots, g^{(i)}$ exists, for then $\{\cdots, g_\alpha, \cdots, g^{(1)}, g^{(2)}, \cdots\}$ is a pseudo sequence with limit h and longer than the maximal such sequence $\{g_\alpha\}$. Hence for some i_0 with $h - g^{(i_0)} \in C' \backslash C$ where (C, C') is a convex pair, there is no $f^{(i_0+1)} \in G$ such that $f^{(i_0+1)} + C = h - g^{(i_0)} + C$. However, in this case it follows from the definition of b-extension that $C' \cap G/C \cap G$ is not discrete. Thus there is a cauchy sequence (in the classical sense) $\{e^{(i)} + C\}$ in C'/C with $h - g^{(i_0)} + C$ as a limit, and with $e^{(i)} \varepsilon\, G$. Then the sequence $\{\cdots, g_\alpha, \cdots, g^{(1)}, g^{(2)}, \cdots, g^{(i_0)}, e^{(1)} + g^{(i_0)}, e^{(2)} + g^{(i_0)}, \cdots\}$ has h as a limit and no limits in G.

COROLLARY B2. *If every sequence in G has a limit then G is b-closed.*

THEOREM B3. *If for every convex subgroup C of G, C is normal and G/C is t-closed, then G is b-closed.*

Proof. If $\{s_\alpha\}$ is a sequence of breadth C in G, and if C is not covered, then in a manner similar to the proof of Theorem C3, it can be shown that $\{s_\alpha + C\}$ is a cauchy sequence in G/C.

If C is covered by C', then there is a μ such that for all $\alpha, \beta > \mu$, $s_\alpha - s_\beta \in C'$. As C'/C is abelian, it is easily verified that for all $x \in C'$ and for all α, $\beta > \mu$, $-s_\alpha + x + s_\alpha + C = -s_\beta + x + s_\beta + C$. Define an automorphism ϕ of C'/C by $(x + C)\phi = -s_\alpha + x + s_\alpha + C$ where $\alpha > \mu$. Then for all $\alpha, \beta > \mu$, $-s_\beta + s_\alpha + C = (s_\alpha - s_\beta + C)\phi$. Now let $0 < g \in C' \backslash C$. For all sufficiently large α and β, $s_\alpha - s_\beta + C < (g + C)\phi^{-1}$; that is, $-s_\beta + s_\alpha < g + C$. Hence $\{s_\alpha + C\}$ is a cauchy sequence in G/C. The rest of the proof is like that of Theorem C3.

Lemma B4. *If H is a Hahn group in which every component is the real numbers or the integers, then all sequences in H converge.*

The proof follows quickly from Lemma T4.

Lemma B5. *If G is abelian and b-closed then every sequence in G converges.*

Proof. G can be embedded in a Hahn group H in which every component is the real numbers. In H every sequence converges by Lemma B4. The rest of the proof follows the model of the proof of Lemma C5.

Theorem B6. *If G is abelian, G is b-closed if and only if every sequence in G converges.*

Proof. Corollary B2 and Lemma B5.

The following is also easily proved.

Theorem B8. *If G is abelian, G is b-closed if and only if for every convex subgroup C, G/C is t-closed.*(3)

Ribenboim [10] calls an abelian o-group G *algebraically complete* if G/C is t-closed for all convex subgroups C. He shows that such a group is c-closed (which follows here Theorem B8 and the fact that every c-extension is a b-extension), and that every abelian o-group can be "algebraically completed." This follows from some remarks in the next section.

5. **Remarks on closure.** Every o-group G has a t-closed t-extension, namely the group $t(G)$ discussed in §2. Every o-group has an a-closed a-extension and a c-closed c-extension, and the cardinality of all a-extensions of a given group is limited (Conrad [6]). Since every b-extension is an a-extension (Lemma 6), it follows from Zorn's lemma that every o-group has a b-closed b-extension.

(3) By way of comparison, Cohen and Goffman [4] prove: An abelian o-group G is a-closed if and only if for every convex subgroup C ($\neq G$), G/C is t-closed and nondiscrete.

The proof of Lemma C5 shows that if G is abelian and has no proper abelian c-extensions then every pseudo sequence in G converges. Hence by Corollary C2, every such G is c-closed. Likewise from Lemma B5, if G is abelian and has no proper abelian b-extensions then every sequence in G converges, so that every such G is b-closed by Corollary B2. Thus by Zorn's lemma, for any abelian G there is a maximal abelian $b(c)$-extension, and by the above argument this extension is $b(c)$-closed. Moreover, it is well known that $t(G)$ is abelian whenever G is, and that the Hahn group $H(\Gamma,R_y)$ where Γ is the set of convex pairs of G and R_y is the real numbers, is an a-closed a-extension of G. Thus:

Let $x \in \{t, c, b, a\}$. G has an x-closed x-extension. If G is abelian, G has an abelian x-closed x-extension.

It is well known that the t-closed t-extension is unique to within equivalence; that is, if H and H' are t-closed t-extensions of G, then there is an o-isomorphism of H onto H' which leaves the elements of G fixed. Also, Conrad [5] showed that for an abelian G the abelian a-closed a-extension is unique to within equivalence. Example 3 of the next section shows, however, that it is possible for an abelian G to have two nonequivalent a-closed a-extensions (one of which is non-abelian). The situation is more chaotic for b- and c-closures as is shown in Example 4 which gives an abelian group with two nonequivalent abelian $b(c)$-closed $b(c)$-extensions.

6. **Examples.** EXAMPLE 1. *A left cauchy sequence which is not a cauchy sequence.* The group used here was first constructed by Clifford [2] for another purpose (and in a slightly different presentation). G is generated by the symbols $g(r)$ where r is a rational number, and subject to the following condition: if $r > s$, $g(r)g(s) = g(2s - r)g(r)$. Each word can be put into normal form $g = (g(r_1))^{m_1}(g(r_2))^{m_2} \cdots (g(r_n))^{m_n}$ where $r_1 < r_2 < \cdots < r_n$. g is to be positive if and only if $m_n > 0$. Then the set of convex pairs of G is isomorphic as an ordered set to the rational numbers. One can easily verify the computational laws, for $r > s$,

$$g(r)^{-1}g(s) = g((r+s)/2)g(r)^{-1}, \quad \text{and} \quad g(r)^{-1}g(s)^{-1} = g((r+s)/2)^{-1}g(r)^{-1}.$$

Now consider the sequence $\{s_i\}$ defined by

$$s_i = g(-i)g(-i+1)\cdots g(-1).$$

For $n, m = 1, 2, \cdots,$ and $n \neq m$,

$$s_n s_m^{-1} = \begin{cases} (g(-m)g(-m+1)\cdots g(-n-1))^{-1} & \text{if } n < m, \\ g(-n)g(-n+1)\cdots g(-m-1) & \text{if } n > m. \end{cases}$$

In any event, $V(s_n s_m^{-1}) = -(\min\{m,n\}) - 1$. Hence $\{s_i\}$ is a left cauchy sequence, and even a pseudo sequence. On the other hand, for $m, n = 2, 3, \cdots,$ and $m \neq n$,

$$s_m^{-1}s_n = \begin{cases} (g(-m)\cdots g(-n-1)s_n)^{-1}s_n & \text{if } n < m, \\ s_m^{-1}(g(-n)\cdots g(-m-1))s_m & \text{if } n > m, \end{cases}$$

$$= \begin{cases} (\cdots g(-n-1+1/2^n+2/2^{n-1}+\cdots+(n-1)/2^2+n/2)^{-1}) \text{ if } n < m, \\ (\cdots g(-m-1+1/2^m+2/2^{m-1}+\cdots+(m-1)/2^2+m/2)) \text{ if } n > m, \end{cases}$$

$$= \begin{cases} (\cdots g(1/2^n-2))^{-1} \text{ if } n < m, \\ (\cdots g(1/2^m-2)) \quad \text{if } n > m. \end{cases}$$

Hence $V(s_m^{-1}s_n) > -2$. Thus $\{s_i\}$ is not a cauchy sequence.

The group G provides an example also of an a-closed (and hence c-closed) group with a nonconvergent pseudo sequence. If there were an a-extension of G in which $\{s_i\}$ converged, the sequence would necessarily have only one limit, as its breadth is 0. But it can be shown that any sequence with exactly one limit is a cauchy sequence. As the sequence $\{s_i\}$ is not a cauchy sequence, this shows that $\{s_i\}$ cannot converge in, for example, an a-closed a-extension of G. The following example gives a much simpler group with the same property.

EXAMPLE 2. *An a-closed group with a nonconvergent pseudo sequence.* The following is a standard device for constructing o-groups. Let A and B be o-groups and let r be a homomorphism from B to $A_0(A)$, the group of order-preserving automorphisms of A. Let S be the set $A \times B$ ordered lexicohrapgically from the right and with addition defined by $(a,b)+(a',b') = (a(b'r)+a',b+b')$. Then S is an o-group, the splitting o-extension of A by B determined by r. Denote S by $A \times_r B(\leftarrow)$ where the arrow indicates the lex order from the right.

Let A be the Hahn group $H(I,R_i)$ where I is the set of integers and for each $i \in I$, R_i is the real numbers. Let B' be the Hahn group $H(N,R_n)$ where N is the set of negative integers and for each $n \in N$, R_n is the real numbers. Let B be that subgroup of B' consisting of those functions which are "initially an integer": that is, $f \in B$ if and only if $f \in B'$ and there exist integers $n_f < 0$ and k_f such that if $i < n_f$ then $f(i) = k_f$. Define an o-automorphism γ of A as follows: $(f\gamma)(i) = f(i-1)$. Define $r: B \to A_0(A)$ by $fr = \gamma^{k_f}$. Then r is a homomorphism. Let $G = A \times_r B(\leftarrow)$.

For each $n \in N$ let $f_n \in B$ be defined by

$$f_n(m) = \begin{cases} 1/2 & \text{if } n \leqq m, \\ 0 & \text{if } n > m. \end{cases}$$

Now consider the pseudo sequence in G, $\{s_i : i = 1,2,\cdots\}$ where $s_i = (0,f_{-i})$. It will now be shown that $\{s_i\}$ cannot have a limit in any a-extension of G.

Supposing, by way of contradiction, that H is an a-extension of G in which $\{s_i\}$ has a limit s, we have first that the breadth of $\{s_i\}$ in G is $A \times 0 \approx A$. As A is a Hahn group with all components the real numbers, A is a-closed, Let $A' = A \times 0$. A' is a convex subgroup of G. Since H is an a-extension of G, and A'

is a-closed, then A' is a convex subgroup of H. Again, as H is an a-extension of G, the set of convex subgroups of H is isomorphic as an ordered set to the set of convex subgroups of G. Hence the set of convex subgroups of H which contain A' is isomorphic to the negative integers. Any inner automorphism induces an order preserving automorphism of the set of convex subgroups, and therefore A' is normal in H, for otherwise there would be an automorphism of H mapping A' onto a convex subgroup which properly contains A', and hence an order-preserving function mapping the negative integers onto a final segment of themselves, which is impossible.

It can be shown that if $\{g_\alpha\}$ is a cauchy sequence (in any o-group) with limit g, then $\{2g_\alpha\}$ is a cauchy sequence with limit $2g$. In H/A', $\{s_i + A'\}$ is a cauchy sequence with limit $s + A'$. Therefore $\{2s_i + A'\}$ is a cauchy sequence with limit $2s + A'$. Hence $2s$ is a limit of the sequence $\{2s_i\}$ in H. Another limit of $\{2s_i\}$ is $f = (0,(\cdots,1,1,1)) \in G$. Hence by Lemma 1, $2s - f \in A'$. Let ϕ be the inner automorphism $x\phi = -2s + x + 2s$. Since A' is normal ϕ induces an o-automorphism of A', and since A' is abelian, for any $y \in A'$, $y\phi = -f + y + f = y\gamma$ (where γ is the automorphism defined in the construction of G). If ψ is the inner automorphism of H, $x\psi = -s + x + s$, then ψ induces an automorphism β of A' such that $\beta^2 = \gamma$. But clearly no such β exists. Hence no such s exists.

Finally, if G' is an a-closed a-extension of G, then the pseudo sequence $\{s_i\}$ can have no limit in G', and this completes the example.

For reasons similar to the above, the equation $2x = f$ cannot have a solution x in G'. This answers a question of Conrad [7] by showing that a group may be a-closed with all components o-isomorphic to the real numbers, without being divisible.

EXAMPLE 3. *An abelian group with two nonequivalent a-closed a-extensions.* Modify Example 2 by letting B be the small direct sum. Then G' is still an a-extension of G. But $G'' = A \oplus B'(\leftarrow)$ is also an a-closed a-extension of G. G'' is abelian and G' is not.

EXAMPLE 4. *An o-group with two nonequivalent abelian b(c)-closed b(c)-extensions.* Let B and B' be as in Example 2. Let I be the o-group of integers and let R be the real numbers. Let $G = I \oplus B(\leftarrow)$ and let $H = R \oplus B'(\leftarrow)$. Consider the elements $x = (0,(\cdots,1/2,1/2))$ and $y = (1/2,(\cdots,1/2,1/2))$ of H. If F' is the subgroup of H generated by G and x, and if E' is the subgroup of H generated by G and y, it is easy to verify that both E' and F' are c-extensions of G. Then E' and F' can be extended to E and F respectively which are maximal c-extensions of G in H. The argument of Lemma C5 shows that E and F are c-closed (H is a Hahn group).

Suppose, by way of contradiction, there were an o-isomorphism σ of E onto F leaving the elements of G fixed. Then since $2x \in G$, $2x\sigma = 2x$ and it follows easily that $x\sigma = x$. Hence $x \in F$. Therefore $y - x = (1/2,(\cdots,0,0)) \in F$. But clearly there is no element of G whose difference with $y - x$ has smaller valuation, since the only thing of smaller valuation is 0 and $y - x \notin G$. This contradiction shows that no such σ exists. Note that every component of E and F is either the integers or the real numbers. Thus since E and F are c-closed, they are b-closed.

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